ELASTIC PLATE THEORY APPLIED TO PCC PAVEMENTS

I Summary of elastic plate theory

Elastic plates in bending can be characterized as:

1) Thick Plates for which transverse shear is considered;

2) Thin plates for which bending and membrane stresses are considered;

3) Medium thick plates for which only bending is considered; (thickness between 1/20 to 1/100 of span length)

If it is assumed that transverse deflections are small so that in-plane forces produced by stretching of the middle plane can be ignored, then the medium thick plate theory is applicable.

Three basic assumptions:

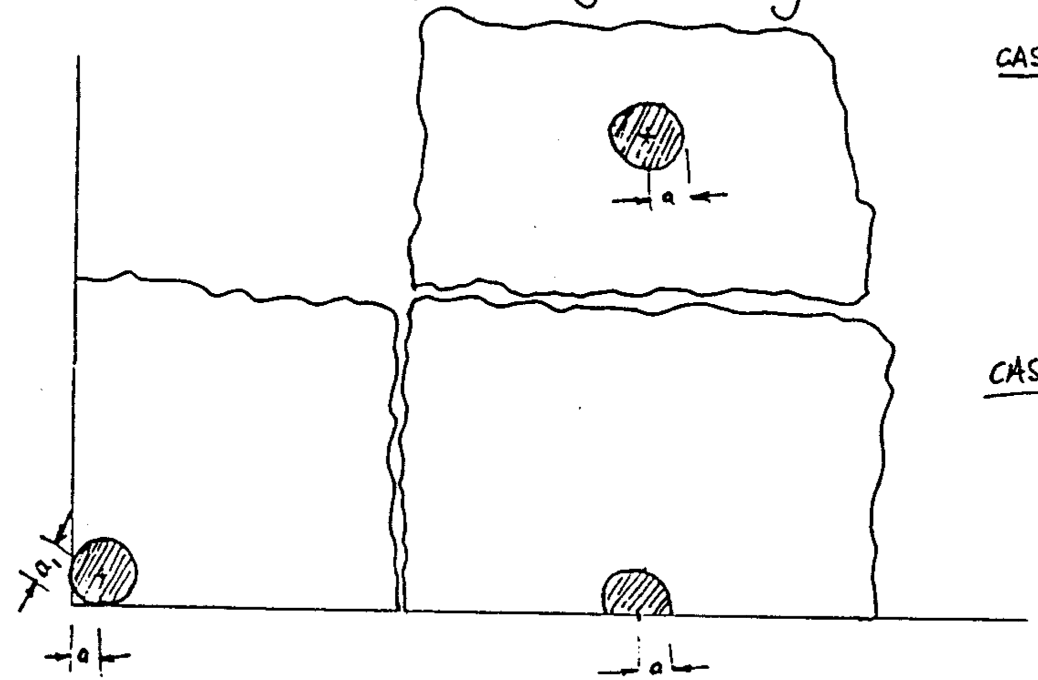
1. Planes normal to the middle plane of the plate before bending remain normal after bending (this is equivalent to an assumption of no shear deformation, $\gamma_{xz} = \gamma_{yz} = 0$);

2. Forces normal to the plate surfaces are small when compared with bending stresses and can be ignored ($\sigma_z = 0$);

3. Stresses normal to a plane normal to the middle plane of the plate are small and can be idnored ($\Sigma N_x = \Sigma N_y = 0$).

All stresses are defined per unit of length of the plate mid-surface.

- Westergaard (1926) :
  - Three cases of loading Investigated



CASE 1 : INTERIOR : critical tension occurs at bottom of slab under center of circle

CASE 2 : EDGE : critical tension occurs at bottom of slab under center of semi-circle.

CASE 3 : CORNER : critical tension occurs at the top of slab. The resultant of applied pressure is at center of circle ; $a_1 = a\sqrt{2}$

TABLE 1

WESTERGAARD EQUATIONS FOR INTERIOR LOADING

(a) Maximum Bending Stress, σ_I

ORDINARY THEORY:	$BS1OT = \frac{3P(1+\mu)}{2\pi h^2}\,[\ln(2\ell/a) + 0.5 - \gamma] + BS12OT$	(I-1)	Theoretically rigorous: USE!
SPECIAL THEORY:	$BS1ST = \frac{3P(1+\mu)}{2\pi h^2}\,[\ln(2\ell/b) + 0.5 - \gamma] + BS12ST$	(I-2)	Crude adjustment for neglected shear stresses: AVOID!
FOR SQUARE:	$BS1SQ = \frac{3P(1+\mu)}{2\pi h^2}\,[\ln(2\ell/c') + 0.5 - \gamma] + BS12SQ$	(I-3)	For comparison w/ FEM, etc.
SUPPLEMENTARY, σ_2 (Ordinary Theory)	$BS12OT = \frac{3P(1+\mu)}{64h^2}\,[(a/\ell)^2] = \frac{3P(1+\mu)}{2\pi h^2}\left\{\left(\frac{\pi}{32}\right)(a/\ell)^2\right\}$	(I-4)	Additional terms to truncated infinite series (usually negligible)
SUPPLEMENTARY, σ_2 (Special Theory)	$BS12ST = \frac{3P(1+\mu)}{64h^2}\,[(b/\ell)^2]$	(I-5)	
SUPPLEMENTARY, σ_2 (For Square)	$BS12SQ = \frac{3P(1+\mu)}{64h^2}\,[(c'/\ell)^2]$	(I-6)	

(b) Maximum Deflection, δ_I

CIRCLE: $DEFIC = \frac{P}{8k\ell^2}\,[1 + (1/2\pi)\{\ln(a/2\ell) + \gamma - 5/4\}(a/\ell)^2]$ (I-7)

P : total applied load;
E : slab Young's modulus;
μ : slab Poisson's ratio;
h : slab thickness;
k : modulus of subgrade reaction;

a : radius of circular load;
c : side length of square load;
$\ell^4 = \frac{Eh^3}{12(1-\mu^2)k}$: radius of relative stiffness;

$b = \sqrt{1.6a^2 + h^2} - 0.675h$, if $a < 1.724h$
$= a$ if $a > 1.724h$
$c' = \frac{e^{\pi/4} - 1}{\sqrt{2}}\,c$
γ : Euler's Constant (=0.577 215 664 90)

Ref: "Westergaard Solutions Reconsidered," by A.M. Ioannides, M.R. Thompson and E.J. Barenberg (1985)

TABLE 2

WESTERGAARD EQUATIONS FOR EDGE LOADING

(a) Maximum Bending Stress, σ_e

ORDINARY THEORY: (Semi-circle)	$BSEWOT = 0.529\ (1 + 0.54u)\ \frac{P}{h^2}\ [\log_{10} (\frac{Eh^3}{k a_2^4}) - 0.71\ ]$	(E-1)
SPECIAL THEORY: (Semi-circle) *← AVOID*	$BSEWST = 0.529\ (1 + 0.54u)\ \frac{P}{h^2}\ [\log_{10} (\frac{Eh^3}{k b_2^4}) - 0.71\ ]$	(E-2)
"NEW" FORMULA: (Circle)	$BSEIC = \frac{3(1+u)P}{\pi(3+u)h^2}\ [\ln \frac{Eh^3}{100ka^4} + 1.84 - 4u/3 + \frac{(1-u)}{2} + 1.18(1+2u)(a/\ell)\ ]$	(E-3)
"NEW" FORMULA: (Semi-circle)	$BSEIS = \frac{3(1+u)P}{\pi(3+u)h^2}\ [\ln \frac{Eh^3}{100ka_2^4} + 3.84 - 4u/3 + 0.5(1+2u)(a_2/\ell)\ ]$	(E-4)
SIMPLIFIED "NEW" FORMULA: (Semi-circle)	$BSELS = \frac{-6P}{h^2}\ (1+0.5u)\ [0.489\ \log_{10}\ (a_2/\ell) - 0.091 - 0.027(a_2/\ell)\ ]$	(E-5)
SIMPLIFIED "NEW" FORMULA: (Circle)	$BSELC = \frac{-6P}{h^2}\ (1+0.5u)\ [0.489\ \log_{10}\ (a/\ell) - 0.012 - 0.063(a/\ell)\ ]$	(E-6)

Westerg. 1948 (E-3, E-4); *Losberg, 1960* (E-5, E-6); *only about 2% different from full expressions*

(b) Maximum Deflection, δ_e

ORIGINAL FORMULA: *ignores load distribution*	$DEFEW = \frac{1}{\sqrt{6}}\ (1 + 0.4u)\ \frac{P}{k\ell^2}$	(E-7)
"NEW" FORMULA: (Circle)	$DEFEIC = \frac{P\sqrt{2+1.2u}}{\sqrt{Eh^3k}}\ [1 - (0.76 + 0.4u)(a/\ell)\ ]$	(E-8)
"NEW" FORMULA: (Semi-circle)	$DEFEIS = \frac{P\sqrt{2+1.2u}}{\sqrt{Eh^3k}}\ [1 - (0.323 + 0.17u)(a_2/\ell)\ ]$	(E-9)
SIMPLIFIED "NEW" FORMULA: (Semi-circle)	$DEFELS = \frac{1}{\sqrt{6}}\ (1 + 0.4u)\ \frac{P}{k\ell^2}\ [1 - 0.323(1+0.5u)(a_2/\ell)\ ]$	(E-10)
SIMPLIFIED "NEW" FORMULA: (Circle)	$DEFELC = \frac{1}{\sqrt{6}}\ (1 + 0.4u)\ \frac{P}{k\ell^2}\ [1 - 0.760(1+0.5u)(a/\ell)\ ]$	(E-11)

a_2 : radius of semi-circle;

$b_2 = \sqrt{1.6\ a_2^2 + h^2} - 0.675h$, if $a_2 < 1.724h$

$\quad = a_2$ if $a_2 > 1.724h$

See Table 1 for other symbols

TABLE 5

EQUATIONS PROPOSED FOR THE CORNER LOADING CONDITION

Deflection:

$$\delta_c = \frac{P}{k\ell^2} \{1.1 - 0.88 \frac{a_1}{\ell}\}$$ Westergaard [5] (C-1)

Stress:

$$\sigma_c = \frac{3P}{h^2}$$ Early simplistic analysis — Goldbeck [25] ; Older [26] (C-2)

$$\sigma_c = \frac{3P}{h^2} [1 - (\frac{a_1}{\ell})^{0.6}]$$ Westergaard [5] (C-3)

Crude numerical analysis

Empirical adjustments to Westergaard's equation (C-3):

$$\sigma_c = \frac{3P}{h^2} [1 - (\frac{a}{\ell})^{0.6}]$$ Bradbury [15] (C-4)

$$\sigma_c = \frac{3P}{h^2} [1 - (\frac{a_1}{\ell})^{1.2}]$$ Kelley [16] , Teller and Sutherland [13] (C-5)

$$\sigma_c = \frac{3.2P}{h^2} [1 - (\frac{a_1}{\ell})]$$ Spangler [29] (C-6)

$$\sigma_c = \frac{4.2P}{h^2} [1 - \frac{\sqrt{(a/\ell)}}{0.925 + 0.22(\frac{a}{\ell})}]$$ Pickett [23] (C-7)

Distance to point of max. stress along corner angle bisector:

$$X_1 = 2\sqrt{(a_1 \ell)}$$ Westergaard [5] (C-8)

where a : radius of circular load tangent to both edges at corner
a_1 : distance to point of action of resultant along corner angle bisector
$= \sqrt{2}\, a$

(See Table 1 for other symbols.)

Ioannides (1984) [based on FEM]:

USE!

$$\delta_c = \frac{P}{k\ell^2} \{1.205 - 0.69(\frac{c}{\ell})\} = \frac{P}{k\ell^2} [1.2 - 0.88(\frac{a_1}{\ell})]$$

$$\sigma_c = \frac{3P}{h^2} \{1.0 - (\frac{c}{\ell})^{0.72}\} = \frac{3P}{h^2} [1 - (\frac{a_1}{\ell})^{0.64}]$$

$$X_1 = 1.80\, c^{0.32} \ell^{0.59}$$

c: side length of square load

Radii of Relative Stiffness						
	Concrete Modulus = 4 million psi, Poisson's ratio = .15					
Thickness	Subgrade Support (k)					
inches	50 psi/in	100 psi/in	200 psi/in	300 psi/in	400 psi/in	500 psi/in
4.0	25.70	21.61	18.18	16.42	15.28	14.45
4.5	28.08	23.61	19.85	17.94	16.69	15.79
5.0	30.39	25.55	21.49	19.42	18.07	17.09
5.5	32.64	27.44	23.08	20.85	19.41	18.35
6.0	34.84	29.30	24.63	22.26	20.72	19.59
6.5	36.99	31.11	26.16	23.64	22.00	20.80
7.0	39.11	32.89	27.65	24.99	23.25	21.99
7.5	41.19	34.63	29.12	26.32	24.49	23.16
8.0	43.23	36.35	30.57	27.62	25.70	24.31
8.5	45.24	38.04	31.99	28.91	26.90	25.44
9.0	47.22	39.71	33.39	30.17	28.08	26.55
9.5	49.17	41.35	34.77	31.42	29.24	27.65
10.0	51.10	42.97	36.14	32.65	30.39	28.74
10.5	53.01	44.57	37.48	33.87	31.52	29.81
11.0	54.89	46.16	38.81	35.07	32.64	30.87
11.5	56.75	47.72	40.13	36.26	33.74	31.91
12.0	58.59	49.27	41.43	37.44	34.84	32.95
12.5	60.41	50.80	42.72	38.60	35.92	33.97
13.0	62.22	52.32	43.99	39.75	36.99	34.99
13.5	64.00	53.82	45.26	40.89	38.06	35.99
14.0	65.77	55.31	46.51	42.02	39.11	36.99
14.5	67.53	56.78	47.75	43.15	40.15	37.97
15.0	69.27	58.25	48.98	44.26	41.19	38.95
15.5	70.99	59.70	50.20	45.36	42.21	39.92
16.0	72.70	61.13	51.41	46.45	43.23	40.88
16.5	74.40	62.56	52.61	47.54	44.24	41.84
17.0	76.08	63.98	53.80	48.61	45.24	42.78
17.5	77.75	65.38	54.98	49.68	46.23	43.72
18.0	79.41	66.78	56.15	50.74	47.22	44.66
18.5	81.06	68.17	57.32	51.80	48.20	45.59
19.0	82.70	69.54	58.48	52.84	49.17	46.51
19.5	84.33	70.91	59.63	53.88	50.14	47.42
20.0	85.94	72.27	60.77	54.91	51.10	48.33

Radius of Relative Stiffness = l

$l = [E*h^3/(12*(1-\mu^2))]^{1/4}$

E v k

According to Boussinesq; for $\mu = .5$

$$w = \frac{1.5p*a}{E}$$

where

p = unit pressure over circular area

a = radius of circular loaded area

E = modulus of the elastic mass

By definition; $k = \frac{p}{w}$ (Westergaard)

Rearranging the Boussinesq Equation

$$\frac{p}{w} = k = \frac{E}{1.5*a}$$

Therefore, either E or k must be a function of the loaded area "a"

E v k Continued

From Slab Theory:

$$w_k = \frac{P * l_k^2}{8 * D} \quad \text{for slab on dense liquid}$$

$$w_e = \frac{P * l_e^2}{3\sqrt{3} * D} \quad \text{for slab on elastic solid}$$

but $w_k = w_e$

Therefore:

$$\frac{P * l_k^2}{8 * D} = \frac{P * l_e^2}{3\sqrt{3} * D}$$

or

$$\frac{l_k^2}{8} = \frac{l_e^2}{3\sqrt{3}} \qquad = .6495\, l_k^2 = l_e^2$$

$$l_k = \left[\frac{D}{k}\right]^{1/4}$$

$$l_e = \left[\frac{2 * D}{E_s / (1 - \mu_s^2)}\right]^{1/3}$$

E v k Continued

$$0.6495 * l_k^2 = l_e^2$$

Where:

$$l_k = \left[\frac{E * h^3}{12 * (1 - \mu_c^2) * k} \right]^{1/4} \quad \text{and} \quad l_e = \left[\frac{2 * E * h^3 * (1 - \mu_s^2)}{12 * (1 - \mu_c^2) * E_s} \right]^{1/3}$$

Substituting

Poisson's Ratio of Soil = 0.5

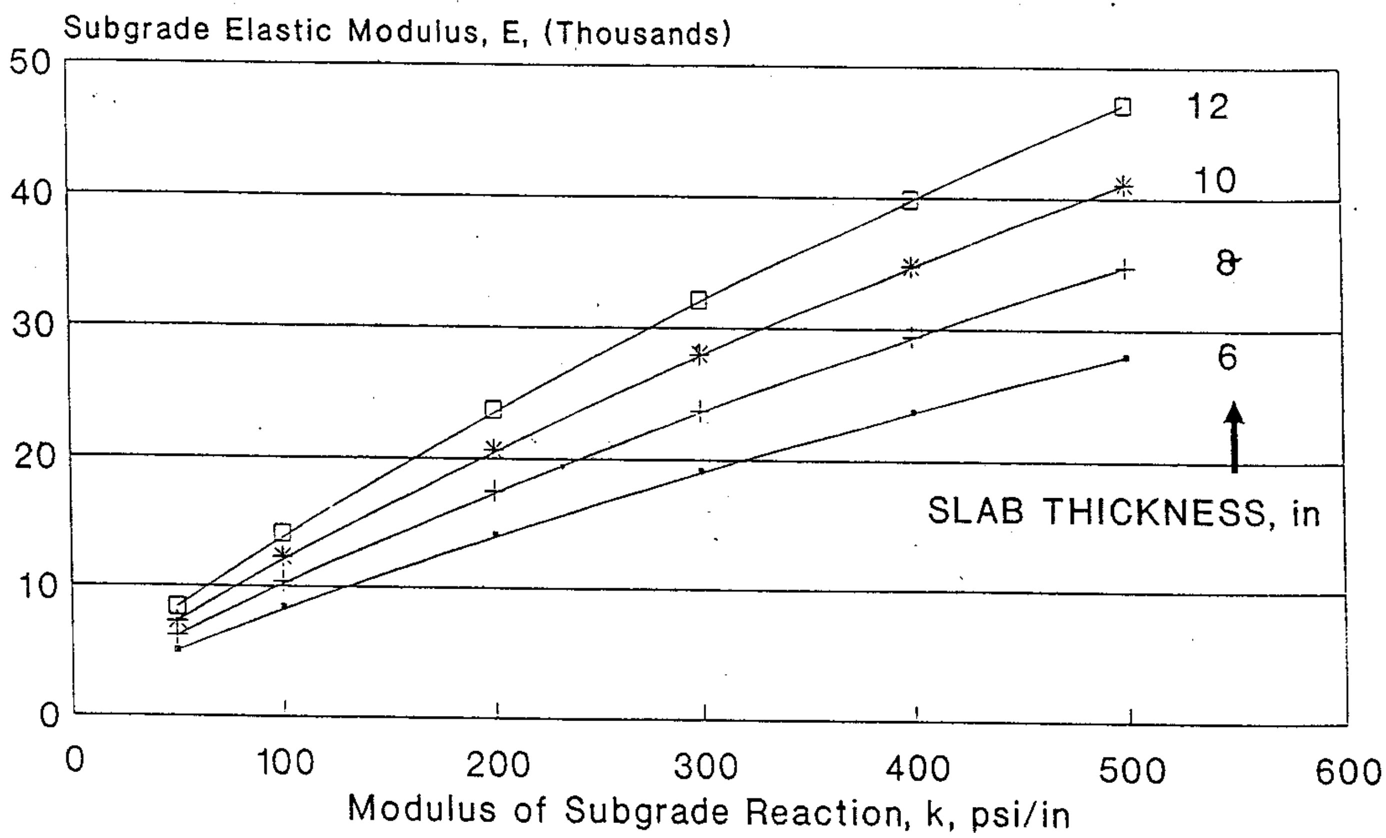
Poisson's Ratio of Concrete = 0.15

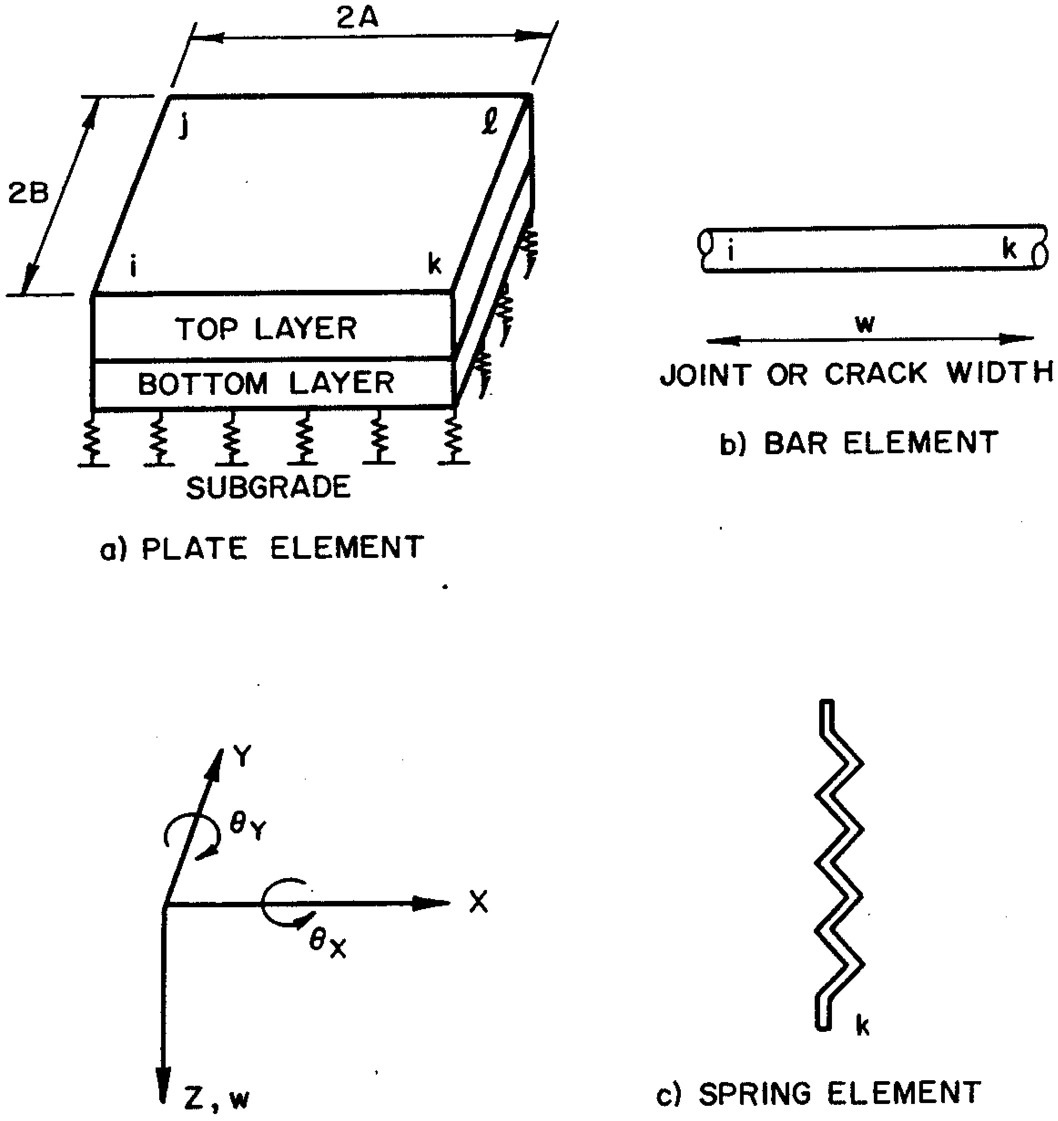
Modulus of Concrete = 4,000,000 psi

and Clearing the Equation Yields

$$E_s^{4/3} = 283.7 * h * k$$

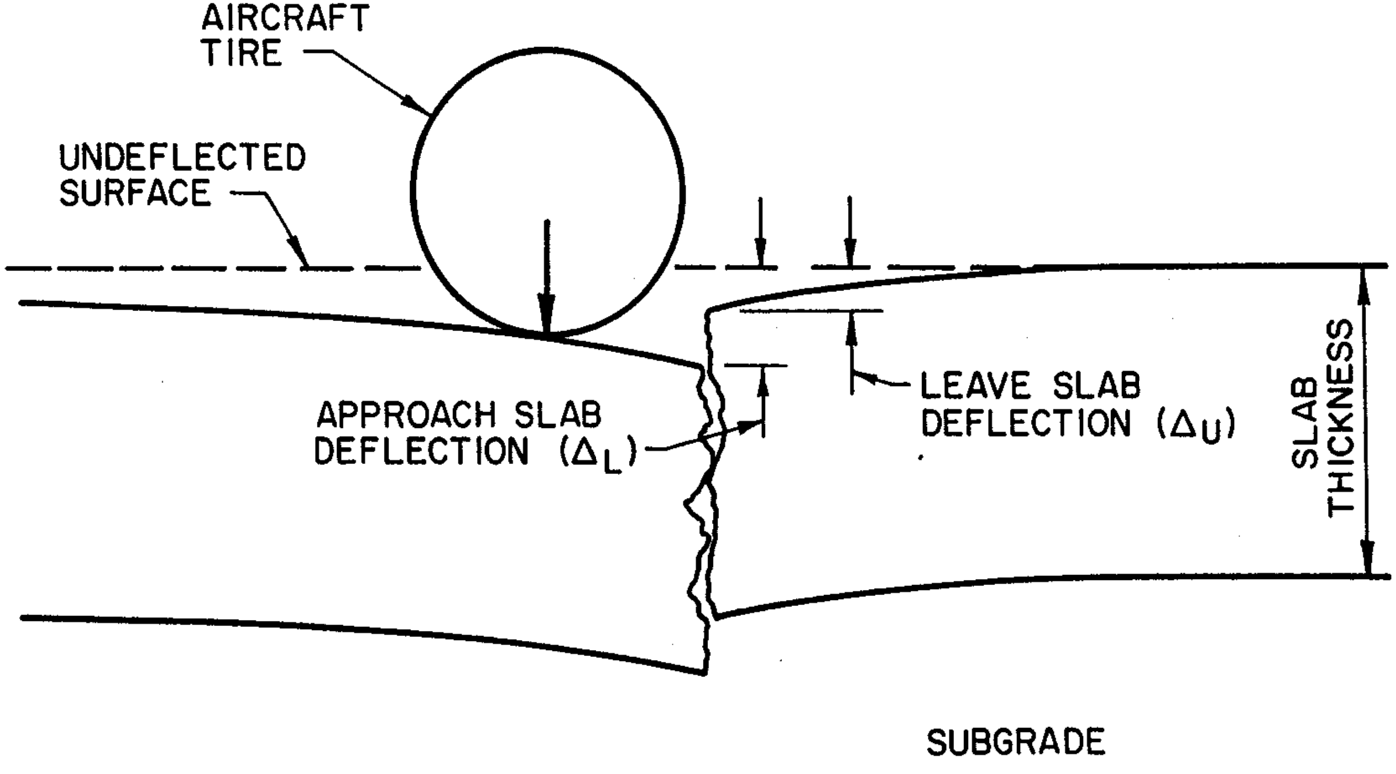
E soil versus k soil
For Various Slab Thicknesses





FINITE-ELEMENT MODEL OF PAVEMENT SYSTEM

$$\text{LOAD TRANSFER EFFICIENCY (\%)} = \frac{\Delta_U}{\Delta_L} \times 100$$



THE CONCEPT OF JOINT LOAD TRANSFER EFFICIENCY

MAXIMUM EDGE STRESS vs. JOINT EFFICIENCY

A/C	LOAD PER TIRE	TIRE PRESSURE	WHEEL CONFIGURATI
DC-10	52.6K	172 PSI	64" [illegible]
B-727	40.0K	170 PSI	34"
B-747	45.5K	207 PSI	58" 44"