

CHAPTER 8

Mechanics of Options Markets

Notes for the Instructor

This chapter provides information on how options markets work. I usually go through the chapter fairly quickly leaving students to read the details for themselves. Points I spend time on are the payoffs from the four option positions and how the terms of options change when there are dividends and stock splits. There is less material on employee stock options in this chapter than before. This is because there is now a whole chapter (Chapter 14) devoted to this topic.

Problems 8.24 and 8.26 work well for class discussion. Problems 8.23 and 8.25 can be used as assignment questions.

QUESTIONS AND PROBLEMS

Problem 8.1.

An investor buys a European put on a share for \$3. The stock price is \$42 and the strike price is \$40. Under what circumstances does the investor make a profit? Under what circumstances will the option be exercised? Draw a diagram showing the variation of the investor's profit with the stock price at the maturity of the option.

The investor makes a profit if the price of the stock on the expiration date is less than \$37. In these circumstances the gain from exercising the option is greater than \$3. The option will be exercised if the stock price is less than \$40 at the maturity of the option. The variation of the investor's profit with the stock price in Figure S8.1.

Problem 8.2.

An investor sells a European call on a share for \$4. The stock price is \$47 and the strike price is \$50. Under what circumstances does the investor make a profit? Under what circumstances will the option be exercised? Draw a diagram showing the variation of the investor's profit with the stock price at the maturity of the option.

The investor makes a profit if the price of the stock is below \$54 on the expiration date. If the stock price is below \$50, the option will not be exercised, and the investor makes a profit of \$4. If the stock price is between \$50 and \$54, the option is exercised and the investor makes a profit between \$0 and \$4. The variation of the investor's profit with the stock price is as shown in Figure S8.2.

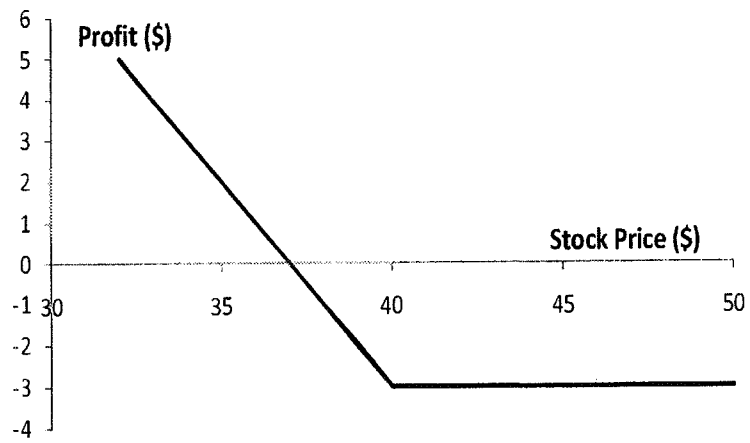


Figure S8.1 Investor's profit in Problem 8.1

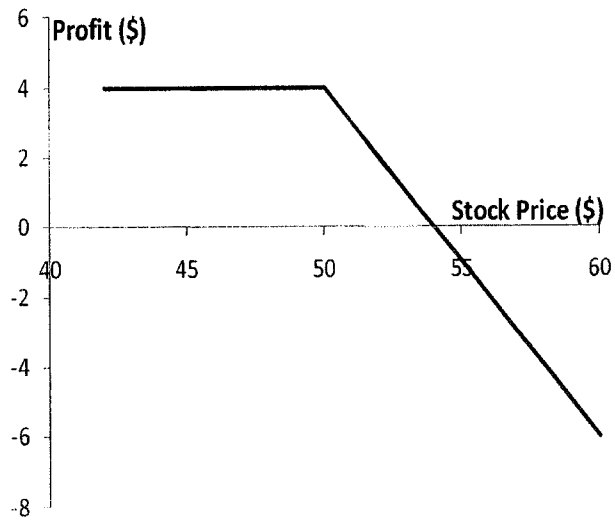


Figure S8.2 Investor's profit in Problem 8.2

Problem 8.3.

An investor sells a European call option with strike price of K and maturity T and buys a put with the same strike price and maturity. Describe the investor's position.

The payoff to the investor is

$$-\max(S_T - K, 0) + \max(K - S_T, 0)$$

This is $K - S_T$ in all circumstances. The investor's position is the same as a short position in a forward contract with delivery price K .

Problem 8.4.

Explain why brokers require margins when clients write options but not when they buy options.

When an investor buys an option, cash must be paid up front. There is no possibility of future liabilities and therefore no need for a margin account. When an investor sells an option, there are potential future liabilities. To protect against the risk of a default, margins are required.

Problem 8.5.

A stock option is on a February, May, August, and November cycle. What options trade on (a) April 1 and (b) May 30?

On April 1 options trade with expiration months of April, May, August, and November. On May 30 options trade with expiration months of June, July, August, and November.

Problem 8.6.

A company declares a 2-for-1 stock split. Explain how the terms change for a call option with a strike price of \$60.

The strike price is reduced to \$30, and the option gives the holder the right to purchase twice as many shares.

Problem 8.7.

"Employee stock options issued by a company are different from regular exchange-traded call options on the company's stock because they can affect the capital structure of the company." Explain this statement.

The exercise of employee stock options usually leads to new shares being issued by the company and sold to the employee. This changes the amount of equity in the capital structure. When a regular exchange-traded option is exercised no new shares are issued and the company's capital structure is not affected.

Problem 8.8.

A corporate treasurer is designing a hedging program involving foreign currency options. What are the pros and cons of using (a) the Philadelphia Stock Exchange and (b) the over-the-counter market for trading?

The Philadelphia Exchange offers European and American options with standard strike prices and times to maturity. Options in the over-the-counter market have the advantage that they can be tailored to meet the precise needs of the treasurer. Their disadvantage is that they expose the treasurer to some credit risk. Exchanges organize their trading so that there is virtually no credit risk.

Problem 8.9.

Suppose that a European call option to buy a share for \$100.00 costs \$5.00 and is held until maturity. Under what circumstances will the holder of the option make a profit? Under what circumstances will the option be exercised? Draw a diagram illustrating how the profit from a long position in the option depends on the stock price at maturity of the option.

Ignoring the time value of money, the holder of the option will make a profit if the stock price at maturity of the option is greater than \$105. This is because the payoff to the holder of the option is, in these circumstances, greater than the \$5 paid for the option. The option will be exercised if the stock price at maturity is greater than \$100. Note that if the stock price is between \$100 and \$105 the option is exercised, but the holder of the option takes a loss overall. The profit from a long position is as shown in Figure S8.3.

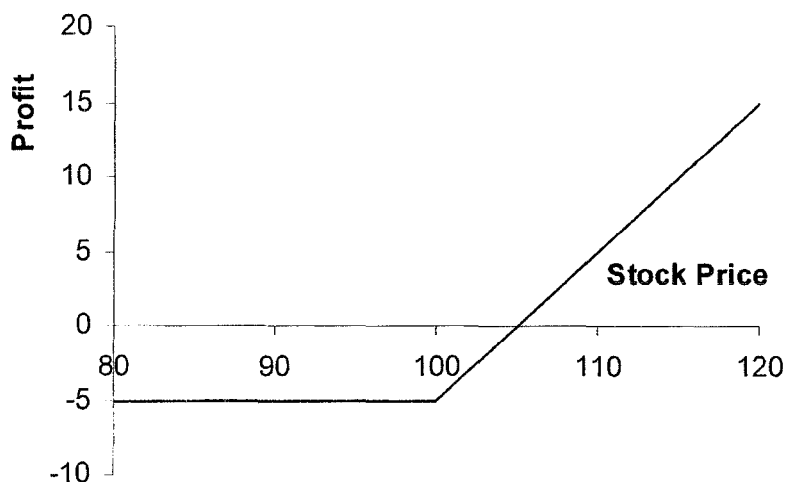


Figure S8.3 Profit from long position in Problem 8.9

Problem 8.10.

Suppose that a European put option to sell a share for \$60 costs \$8 and is held until maturity. Under what circumstances will the seller of the option (the party with the short position) make a profit? Under what circumstances will the option be exercised? Draw a diagram illustrating how the profit from a short position in the option depends on the stock price at maturity of the option.

Ignoring the time value of money, the seller of the option will make a profit if the stock price at maturity is greater than \$52.00. This is because the cost to the seller of the option is in these circumstances less than the price received for the option. The option will be exercised if the stock price at maturity is less than \$60.00. Note that if the stock price is between \$52.00 and \$60.00 the seller of the option makes a profit even though the option is exercised. The profit from the short position is as shown in Figure S8.4.

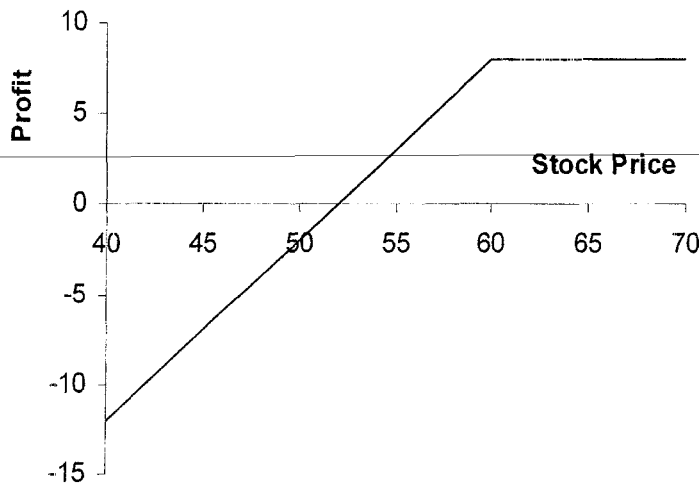


Figure S8.4 Profit from short position in Problem 8.10

Problem 8.11.

Describe the terminal value of the following portfolio: a newly entered-into long forward contract on an asset and a long position in a European put option on the asset with the same maturity as the forward contract and a strike price that is equal to the forward price of the asset at the time the portfolio is set up. Show that the European put option has the same value as a European call option with the same strike price and maturity.

The terminal value of the long forward contract is:

$$S_T - F_0$$

where S_T is the price of the asset at maturity and F_0 is the forward price of the asset at the time the portfolio is set up. (The delivery price in the forward contract is also F_0 .)

The terminal value of the put option is:

$$\max(F_0 - S_T, 0)$$

The terminal value of the portfolio is therefore

$$\begin{aligned} S_T - F_0 + \max(F_0 - S_T, 0) \\ = \max(0, S_T - F_0) \end{aligned}$$

This is the same as the terminal value of a European call option with the same maturity as the forward contract and an exercise price equal to F_0 .

We have shown that the forward contract plus the put is worth the same as a call with the same strike price and time to maturity as the put. The forward contract is worth zero at the time the portfolio is set up. It follows that the put is worth the same as the call at the time the portfolio is set up.

Problem 8.12.

A trader buys a call option with a strike price of \$45 and a put option with a strike price of \$40. Both options have the same maturity. The call costs \$3 and the put costs \$4. Draw a diagram showing the variation of the trader's profit with the asset price.

Figure S8.5 shows the variation of the trader's position with the asset price. We can divide the alternative asset prices into three ranges:

- (a) When the asset price less than \$40, the put option provides a payoff of $40 - S_T$ and the call option provides no payoff. The options cost \$7 and so the total profit is $33 - S_T$.
- (b) When the asset price is between \$40 and \$45, neither option provides a payoff. There is a net loss of \$7.
- (c) When the asset price greater than \$45, the call option provides a payoff of $S_T - 45$ and the put option provides no payoff. Taking into account the \$7 cost of the options, the total profit is $S_T - 52$.

The trader makes a profit (ignoring the time value of money) if the stock price is less than \$33 or greater than \$52. This type of trading strategy is known as a strangle and is discussed in Chapter 10.

Problem 8.13.

Explain why an American option is always worth at least as much as a European option on the same asset with the same strike price and exercise date.

The holder of an American option has all the same rights as the holder of a European option and more. It must therefore be worth at least as much. If it were not, an arbitrageur could short the European option and take a long position in the American option.

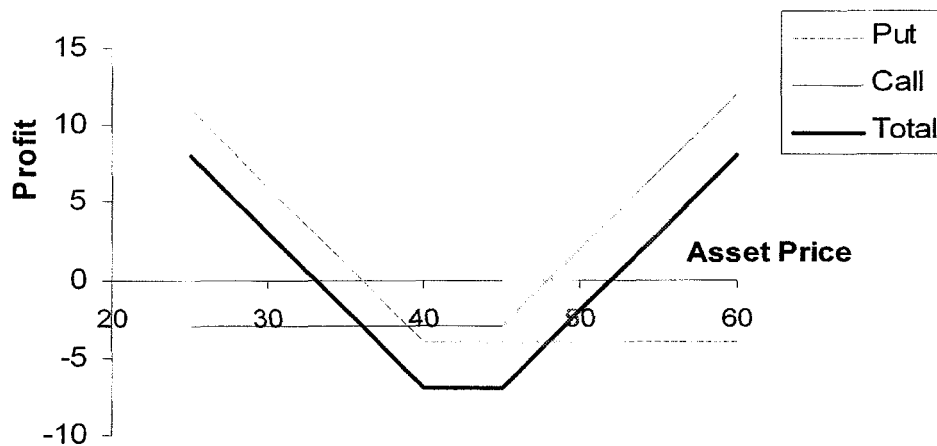


Figure S8.5 Profit from trading strategy in Problem 8.12

Problem 8.14.

Explain why an American option is always worth at least as much as its intrinsic value.

The holder of an American option has the right to exercise it immediately. The American option must therefore be worth at least as much as its intrinsic value. If it were not an arbitrageur could lock in a sure profit by buying the option and exercising it immediately.

Problem 8.15.

Explain carefully the difference between writing a put option and buying a call option.

Writing a put gives a payoff of $\min(S_T - K, 0)$. Buying a call gives a payoff of $\max(S_T - K, 0)$. In both cases the potential payoff is $S_T - K$. The difference is that for a written put the counterparty chooses whether you get the payoff (and will allow you to get it only when it is negative to you). For a long call you decide whether you get the payoff (and you choose to get it when it is positive to you.)

Problem 8.16.

The treasurer of a corporation is trying to choose between options and forward contracts to hedge the corporation's foreign exchange risk. Discuss the advantages and disadvantages of each.

Forward contracts lock in the exchange rate that will apply to a particular transaction in the future. Options provide insurance that the exchange rate will not be worse than some level. The advantage of a forward contract is that uncertainty is eliminated as far as possible. The disadvantage is that the outcome with hedging can be significantly worse

than the outcome with no hedging. This disadvantage is not as marked with options. However, unlike forward contracts, options involve an up-front cost.

Problem 8.17.

Consider an exchange-traded call option contract to buy 500 shares with a strike price of \$40 and maturity in four months. Explain how the terms of the option contract change when there is

- a. A 10% stock dividend
- b. A 10% cash dividend
- c. A 4-for-1 stock split

- (a) The option contract becomes one to buy $500 \times 1.1 = 550$ shares with an exercise price $40/1.1 = 36.36$.
- (b) There is no effect. The terms of an options contract are not normally adjusted for cash dividends.
- (c) The option contract becomes one to buy $500 \times 4 = 2,000$ shares with an exercise price of $40/4 = \$10$.

Problem 8.18.

“If most of the call options on a stock are in the money, it is likely that the stock price has risen rapidly in the last few months.” Discuss this statement.

The exchange has certain rules governing when trading in a new option is initiated. These mean that the option is close-to-the-money when it is first traded. If all call options are in the money it is therefore likely that the stock price has risen since trading in the option began.

Problem 8.19.

What is the effect of an unexpected cash dividend on (a) a call option price and (b) a put option price?

An unexpected cash dividend would reduce the stock price on the ex-dividend date. This stock price reduction would not be anticipated by option holders. As a result there would be a reduction in the value of a call option and an increase the value of a put option. (Note that the terms of an option are adjusted for cash dividends only in exceptional circumstances.)

Problem 8.20.

Options on General Motors stock are on a March, June, September, and December cycle. What options trade on (a) March 1, (b) June 30, and (c) August 5?

- (a) March, April, June and September
 - (b) July, August, September, December
 - (c) August, September, December, March.
- Longer dated options may also trade.

Problem 8.21.

Explain why the market maker's bid-offer spread represents a real cost to options investors.

A "fair" price for the option can reasonably be assumed to be half way between the bid and the offer price quoted by a market maker. An investor typically buys at the market maker's offer and sells at the market maker's bid. Each time he or she does this there is a hidden cost equal to half the bid-offer spread.

Problem 8.22.

A United States investor writes five naked call option contracts. The option price is \$3.50, the strike price is \$60.00, and the stock price is \$57.00. What is the initial margin requirement?

The two calculations are necessary to determine the initial margin. The first gives

$$500 \times (3.5 + 0.2 \times 57 - 3) = 5,950$$

The second gives

$$500 \times (3.5 + 0.1 \times 57) = 4,600$$

The initial margin is the greater of these, or \$5,950. Part of this can be provided by the initial amount of $500 \times 3.5 = \$1,750$ received for the options.

ASSIGNMENT QUESTIONS**Problem 8.23.**

The price of a stock is \$40. The price of a one-year European put option on the stock with a strike price of \$30 is quoted as \$7 and the price of a one-year European call option on the stock with a strike price of \$50 is quoted as \$5. Suppose that an investor buys 100 shares, shorts 100 call options, and buys 100 put options. Draw a diagram illustrating how the investor's profit or loss varies with the stock price over the next year. How does your answer change if the investor buys 100 shares, shorts 200 call options, and buys 200 put options?

Figure M8.1 shows the way in which the investor's profit varies with the stock price in the first case. For stock prices less than \$30 there is a loss of \$1,200. As the stock price increases from \$30 to \$50 the profit increases from -\$1,200 to \$800. Above \$50 the profit is \$800. Students may express surprise that a call which is \$10 out of the money is less expensive than a put which is \$10 out of the money. This could be because of dividends or the crashophobia phenomenon discussed in Chapter 18.

Figure M8.2 shows the way in which the profit varies with stock price in the second case. In this case the profit pattern has a zigzag shape. The problem illustrates how many

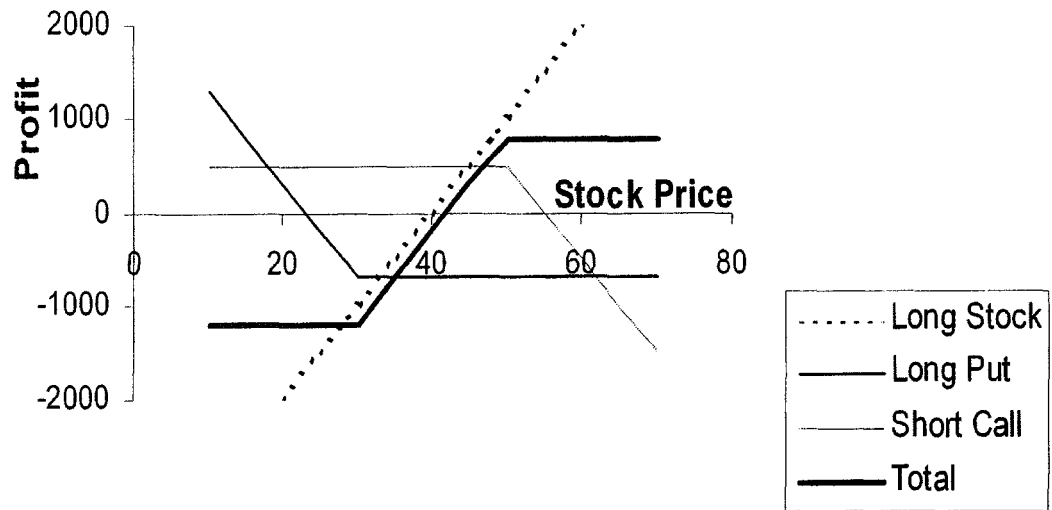


Figure M8.1 Profit in first case considered Problem 8.25

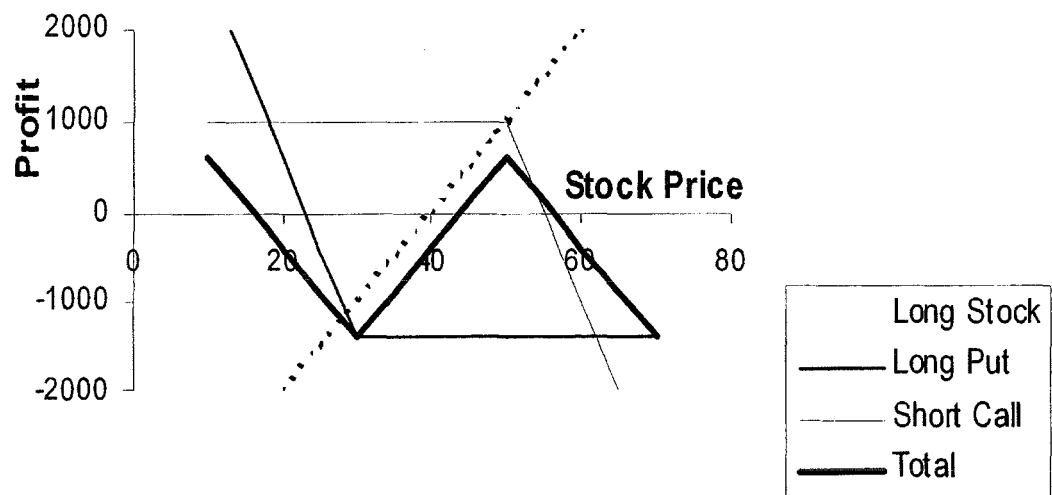


Figure M8.2 Profit for the second case considered Problem 8.25

different patterns can be obtained by including calls, puts, and the underlying asset in a portfolio.

Problem 8.24.

"If a company does not do better than its competitors but the stock market goes

up, executives do very well from their stock options. This makes no sense" Discuss this viewpoint. Can you think of alternatives to the usual executive stock option plan that take the viewpoint into account.

Executive stock option plans account for a high percentage of the total remuneration received by executives. When the market is rising fast (as it was for much of the 1990s) many corporate executives do very well out of their stock option plans — even when their company does worse than its competitors. Large institutional investors have argued that executive stock options should be structured so that the payoff depends how the company has performed relative to an appropriate industry index. In a regular executive stock option the strike price is the stock price at the time the option is issued. In the type of relative-performance stock option favored by institutional investors, the strike price at time t is $S_0 I_t / I_0$ where S_0 is the company's stock price at the time the option is issued, I_0 is the value of an equity index for the industry in which the company operates at the time the option is issued, and I_t is the value of the index at time t . If the company's performance equals the performance of the industry, the options are always at-the-money. If the company outperforms the industry, the options become in the money. If the company underperforms the industry, the options become out of the money. Note that a relative performance stock option can provide a payoff when both the market and the company's stock price decline.

Relative performance stock options clearly provide a better way of rewarding senior management for superior performance. Some companies have argued that, if they introduce relative performance options when their competitors do not, they will lose some of their top management talent.

Problem 8.25.

Use DerivaGem to calculate the value of an American put option on a nondividend paying stock when the stock price is \$30, the strike price is \$32, the risk-free rate is 5%, the volatility is 30%, and the time to maturity is 1.5 years. (Choose "Binomial American" for the "option type" and 50 time steps.)

- What is the option's intrinsic value?
- What is the option's time value?
- What would a time value of zero indicate? What is the value of an option with zero time value?
- Using a trial and error approach calculate how low the stock price would have to be for the time value of the option to be zero.

DerivaGem shows that the value of the option is 4.57. The option's intrinsic value is $32 - 30 = 2.00$. The option's time value is therefore $4.57 - 2.00 = 2.57$. A time value of zero would indicate that it is optimal to exercise the option immediately. In this case the value of the option would equal its intrinsic value. When the stock price is 20, DerivaGem gives the value of the option as 12, which is its intrinsic value. When the stock price is 25, DerivaGem gives the value of the options as 7.54, indicating that the time value is still positive ($= 0.54$). Keeping the number of time steps equal to 50, trial and error indicates the time value disappears when the stock price is reduced to 21.69 or lower. (With 500 time steps this estimate of how low the stock price must become is reduced to 21.35.)

Problem 8.26.

On July 20, 2004 Microsoft surprised the market by announcing a \$3 dividend. The ex-dividend date was November 17, 2004 and the payment date was December 2, 2004. Its stock price at the time was about \$28. It also changed the terms of its employee stock options so that each exercise price was adjusted downward to

$$\text{Pre-dividend Exercise Price} \times \frac{\text{Closing Price} - \$3.00}{\text{Closing Price}}$$

The number of shares covered by each stock option outstanding was adjusted upward to

$$\text{Number of Shares Pre-dividend} \times \frac{\text{Closing Price}}{\text{Closing Price} - \$3.00}$$

“Closing Price” means the official NASDAQ closing price of a share of Microsoft common stock on the last trading day before the ex-dividend date.

Evaluate this adjustment. Compare it with the system used by exchanges to adjust for extraordinary dividends (see Business Snapshot 8.1).

Suppose that the closing stock price is \$28 and an employee has 1000 options with a strike price of \$24. Microsoft’s adjustment involves changing the strike price to $24 \times 25/28 = 21.4286$ and changing the number of options to $1000 \times 28/25 = 1,120$. The system used by exchanges would involve keeping the number of options the same and reducing the strike price by \$3 to \$21.

The Microsoft adjustment is more complicated than that used by the exchange because it requires a knowledge of the Microsoft’s stock price immediately before the stock goes ex-dividend. However, arguably it is a better adjustment than the one used by the exchange. Before the adjustment the employee has the right to pay \$24,000 for Microsoft stock that is worth \$28,000. After the adjustment the employee also has the option to pay \$24,000 for Microsoft stock worth \$28,000. Under the adjustment rule used by exchanges the employee would have the right to buy stock worth \$25,000 for \$21,000. If the volatility of Microsoft remains the same this is a less valuable option.

One complication here is that Microsoft’s volatility does not remain the same. It can be expected to go up because some cash (a zero risk asset) has been transferred to shareholders. The employees therefore have the same basic option as before but the volatility of Microsoft can be expected to increase. The employees are slightly better off because the value of an option increases with volatility.

CHAPTER 9

Properties of Stock Options

Notes for the Instructor

This chapter outlines a number of relationships between a stock option price and the underlying stock price that do not involve any assumptions about the volatility of the stock's price. As will be evident from the slides, I like to present students with sets of numerical data for c , C , p , P , S_0 , K , r , T , and D that violate the relationships in the chapter and ask them what trades they would do. This usually results in good classroom interaction.

I devote most time in class to

1. The $c \geq S_0 - Ke^{-rT}$ result
2. The early exercise arguments for American calls
3. The put-call parity result for European options

When discussing the early exercise of American calls on non-dividend-paying stocks I present students with a situation where the option is deep-in-the-money ($S_0 = 100$; $T = 0.25$; $K = 60$) and ask whether they would exercise early when (a) they want to keep the stock as part of their portfolio and (b) when they think the stock is a “dog”. In the first case, they should delay exercise and thereby delay paying the strike price. In the second case they should sell the option to an investor who does want to keep the stock as part of his or her portfolio. (Such an investor must exist as otherwise the stock price would not be \$100). This investor will pay at least 100 minus the present value of 60 for the option. So, if the possibility of the stock price falling below \$60 is ignored, the option should not be exercised early. When the possibility of the stock price falling below \$60 is recognized, we become even less inclined to exercise early.

Two follow-up questions are “Why does this argument not work for put options?” and “Why are employee stock options frequently exercised early?” The answer to the first question is that the strike price is paid not received in the case of put options and so the time value of money argument does not work. The answer to the second question is that employee stock options cannot be traded.

This can be a good time to introduce the Excel-based software DerivaGem that goes with the book (if the instructor has not already done so). The software can be used to plot the relationship between call/put prices and the variables: S_0 , K , r , σ , and T . It can also be used to check put-call parity and investigate the difference between American and European option prices.

As already mentioned I often use Problem 9.20 in class. Problem 9.22 is a short assignment question. Problems 9.23, 9.24, and 9.25 are more challenging and can be used for assignments or class discussion. Problem 9.26 gets students started with the DerivaGem software.

QUESTIONS AND PROBLEMS

Problem 9.1.

List the six factors affecting stock option prices.

The six factors affecting stock option prices are the stock price, strike price, risk-free interest rate, volatility, time to maturity, and dividends.

Problem 9.2.

What is a lower bound for the price of a four-month call option on a non-dividend-paying stock when the stock price is \$28, the strike price is \$25, and the risk-free interest rate is 8% per annum?

The lower bound is

$$28 - 25e^{-0.08 \times 0.3333} = \$3.66$$

Problem 9.3.

What is a lower bound for the price of a one-month European put option on a non-dividend-paying stock when the stock price is \$12, the strike price is \$15, and the risk-free interest rate is 6% per annum?

The lower bound is

$$15e^{-0.06 \times 0.08333} - 12 = \$2.93$$

Problem 9.4.

Give two reasons that the early exercise of an American call option on a non-dividend-paying stock is not optimal. The first reason should involve the time value of money. The second reason should apply even if interest rates are zero.

Delaying exercise delays the payment of the strike price. This means that the option holder is able to earn interest on the strike price for a longer period of time. Delaying exercise also provides insurance against the stock price falling below the strike price by the expiration date. Assume that the option holder has an amount of cash K and that interest rates are zero. Exercising early means that the option holder's position will be worth S_T at expiration. Delaying exercise means that it will be worth $\max(K, S_T)$ at expiration.

Problem 9.5.

"The early exercise of an American put is a trade-off between the time value of money and the insurance value of a put." Explain this statement.

An American put when held in conjunction with the underlying stock provides insurance. It guarantees that the stock can be sold for the strike price, K . If the put is exercised early, the insurance ceases. However, the option holder receives the strike price immediately and is able to earn interest on it between the time of the early exercise and the expiration date.

Problem 9.6.

Explain why an American call option on a dividend-paying stock is always worth at least as much as its intrinsic value. Is the same true of a European call option? Explain your answer.

An American call option can be exercised at any time. If it is exercised its holder gets the intrinsic value. It follows that an American call option must be worth at least its intrinsic value. A European call option can be worth less than its intrinsic value. Consider, for example, the situation where a stock is expected to provide a very high dividend during the life of an option. The price of the stock will decline as a result of the dividend. Because the European option can be exercised only after the dividend has been paid, its value may be less than the intrinsic value today.

Problem 9.7.

The price of a non-dividend paying stock is \$19 and the price of a three-month European call option on the stock with a strike price of \$20 is \$1. The risk-free rate is 4% per annum. What is the price of a three-month European put option with a strike price of \$20?

In this case $c = 1$, $T = 0.25$, $S_0 = 19$, $K = 20$, and $r = 0.04$. From put-call parity

$$p = c + Ke^{-rT} - S_0$$

or

$$p = 1 + 20e^{-0.04 \times 0.25} - 19 = 1.80$$

so that the European put price is \$1.80.

Problem 9.8.

Explain why the arguments leading to put-call parity for European options cannot be used to give a similar result for American options.

When early exercise is not possible, we can argue that two portfolios that are worth the same at time T must be worth the same at earlier times. When early exercise is possible, the argument falls down. Suppose that $P + S > C + Ke^{-rT}$. This situation does not lead to an arbitrage opportunity. If we buy the call, short the put, and short the stock, we cannot be sure of the result because we do not know when the put will be exercised.

Problem 9.9.

What is a lower bound for the price of a six-month call option on a non-dividend-paying stock when the stock price is \$80, the strike price is \$75, and the risk-free interest rate is 10% per annum?

The lower bound is

$$80 - 75e^{-0.1 \times 0.5} = \$8.66$$

Problem 9.10

What is a lower bound for the price of a two-month European put option on a non-dividend-paying stock when the stock price is \$58, the strike price is \$65, and the risk-free interest rate is 5% per annum?

The lower bound is

$$65e^{-0.05 \times 2/12} - 58 = \$6.46$$

Problem 9.11.

A four-month European call option on a dividend-paying stock is currently selling for \$5. The stock price is \$64, the strike price is \$60, and a dividend of \$0.80 is expected in one month. The risk-free interest rate is 12% per annum for all maturities. What opportunities are there for an arbitrageur?

The present value of the strike price is $60e^{-0.12 \times 4/12} = \57.65 . The present value of the dividend is $0.80e^{-0.12 \times 1/12} = 0.79$. Because

$$5 < 64 - 57.65 - 0.79$$

the condition in equation (9.5) is violated. An arbitrageur should buy the option and short the stock. This generates $64 - 5 = \$59$. The arbitrageur invests \$0.79 of this at 12% for one month to pay the dividend of \$0.80 in one month. The remaining \$58.21 is invested for four months at 12%. Regardless of what happens a profit will materialize.

If the stock price declines below \$60 in four months, the arbitrageur loses the \$5 spent on the option but gains on the short position. The arbitrageur shorts when the stock price is \$64, has to pay dividends with a present value of \$0.79, and closes out the short position when the stock price is \$60 or less. Because \$57.65 is the present value of \$60, the short position generates at least $64 - 57.65 - 0.79 = \$5.56$ in present value terms. The present value of the arbitrageur's gain is therefore at least $5.56 - 5.00 = \$0.56$.

If the stock price is above \$60 at the expiration of the option, the option is exercised. The arbitrageur buys the stock for \$60 in four months and closes out the short position. The present value of the \$60 paid for the stock is \$57.65 and as before the dividend has a present value of \$0.79. The gain from the short position and the exercise of the option is therefore exactly $64 - 57.65 - 0.79 = \$5.56$. The arbitrageur's gain in present value terms is exactly $5.56 - 5.00 = \$0.56$.

Problem 9.12.

A one-month European put option on a non-dividend-paying stock is currently selling for \$2.50. The stock price is \$47, the strike price is \$50, and the risk-free interest rate is 6% per annum. What opportunities are there for an arbitrageur?

In this case the present value of the strike price is $50e^{-0.06 \times 1/12} = 49.75$. Because

$$2.5 < 49.75 - 47.00$$

the condition in equation (9.2) is violated. An arbitrageur should borrow \$49.50 at 6% for one month, buy the stock, and buy the put option. This generates a profit in all circumstances.

If the stock price is above \$50 in one month, the option expires worthless, but the stock can be sold for at least \$50. A sum of \$50 received in one month has a present value of \$49.75 today. The strategy therefore generates profit with a present value of at least \$0.25.

If the stock price is below \$50 in one month the put option is exercised and the stock owned is sold for exactly \$50 (or \$49.75 in present value terms). The trading strategy therefore generates a profit of exactly \$0.25 in present value terms.

Problem 9.13.

Give an intuitive explanation of why the early exercise of an American put becomes more attractive as the risk-free rate increases and volatility decreases.

The early exercise of an American put is attractive when the interest earned on the strike price is greater than the insurance element lost. When interest rates increase, the value of the interest earned on the strike price increases making early exercise more attractive. When volatility decreases, the insurance element is less valuable. Again this makes early exercise more attractive.

Problem 9.14.

The price of a European call that expires in six months and has a strike price of \$30 is \$2. The underlying stock price is \$29, and a dividend of \$0.50 is expected in two months and again in five months. The term structure is flat, with all risk-free interest rates being 10%. What is the price of a European put option that expires in six months and has a strike price of \$30?

Using the notation in the chapter, put-call parity [equation (9.7)] gives

$$c + Ke^{-rT} + D = p + S_0$$

or

$$p = c + Ke^{-rT} + D - S_0$$

In this case

$$p = 2 + 30e^{-0.1 \times 6/12} + (0.5e^{-0.1 \times 2/12} + 0.5e^{-0.1 \times 5/12}) - 29 = 2.51$$

In other words the put price is \$2.51.

Problem 9.15.

Explain carefully the arbitrage opportunities in Problem 9.14 if the European put price is \$3.

If the put price is \$3.00, it is too high relative to the call price. An arbitrageur should buy the call, short the put and short the stock. This generates $-2 + 3 + 29 = \$30$ in cash which is invested at 10%. Regardless of what happens a profit with a present value of $3.00 - 2.51 = \$0.49$ is locked in.

If the stock price is above \$30 in six months, the call option is exercised, and the put option expires worthless. The call option enables the stock to be bought for \$30, or $30e^{-0.10 \times 6/12} = \28.54 in present value terms. The dividends on the short position cost $0.5e^{-0.1 \times 2/12} + 0.5e^{-0.1 \times 5/12} = \0.97 in present value terms so that there is a profit with a present value of $30 - 28.54 - 0.97 = \$0.49$.

If the stock price is below \$30 in six months, the put option is exercised and the call option expires worthless. The short put option leads to the stock being bought for \$30, or $30e^{-0.10 \times 6/12} = \28.54 in present value terms. The dividends on the short position cost $0.5e^{-0.1 \times 2/12} + 0.5e^{-0.1 \times 5/12} = \0.97 in present value terms so that there is a profit with a present value of $30 - 28.54 - 0.97 = \$0.49$.

Problem 9.16.

The price of an American call on a non-dividend-paying stock is \$4. The stock price is \$31, the strike price is \$30, and the expiration date is in three months. The risk-free interest rate is 8%. Derive upper and lower bounds for the price of an American put on the same stock with the same strike price and expiration date.

From equation (9.4)

$$S_0 - K \leq C - P \leq S_0 - Ke^{-rT}$$

In this case

$$31 - 30 \leq 4 - P \leq 31 - 30e^{-0.08 \times 0.25}$$

or

$$1.00 \leq 4.00 - P \leq 1.59$$

or

$$2.41 \leq P \leq 3.00$$

Upper and lower bounds for the price of an American put are therefore \$2.41 and \$3.00.

Problem 9.17.

Explain carefully the arbitrage opportunities in Problem 9.16 if the American put price is greater than the calculated upper bound.

If the American put price is greater than \$3.00 an arbitrageur can sell the American put, short the stock, and buy the American call. This realizes at least $3 + 31 - 4 = \$30$ which can be invested at the risk-free interest rate. At some stage during the 3-month period either the American put or the American call will be exercised. The arbitrageur then pays \$30, receives the stock and closes out the short position. The cash flows to the arbitrageur are +\$30 at time zero and -\$30 at some future time. These cash flows have a positive present value.

Problem 9.18.

Prove the result in equation (9.4). (Hint: For the first part of the relationship consider (a) a portfolio consisting of a European call plus an amount of cash equal to K and (b) a portfolio consisting of an American put option plus one share.)

As in the text we use c and p to denote the European call and put option price, and C and P to denote the American call and put option prices. Because $P \geq p$, it follows from put–call parity that

$$P \geq c + Ke^{-rT} - S_0$$

and since $c = C$,

$$P \geq C + Ke^{-rT} - S_0$$

or

$$C - P \leq S_0 - Ke^{-rT}$$

For a further relationship between C and P , consider

Portfolio I: One European call option plus an amount of cash equal to K .

Portfolio J: One American put option plus one share.

Both options have the same exercise price and expiration date. Assume that the cash in portfolio I is invested at the risk-free interest rate. If the put option is not exercised early portfolio J is worth

$$\max(S_T, K)$$

at time T . Portfolio I is worth

$$\max(S_T - K, 0) + Ke^{rT} = \max(S_T, K) - K + Ke^{rT}$$

at this time. Portfolio I is therefore worth more than portfolio J. Suppose next that the put option in portfolio J is exercised early, say, at time τ . This means that portfolio J is worth K at time τ . However, even if the call option were worthless, portfolio I would be worth $Ke^{r\tau}$ at time τ . It follows that portfolio I is worth at least as much as portfolio J in all circumstances. Hence

$$c + K \geq P + S_0$$

Since $c = C$,

$$C + K \geq P + S_0$$

or

$$C - P \geq S_0 - K$$

Combining this with the other inequality derived above for $C - P$, we obtain

$$S_0 - K \leq C - P \leq S_0 - Ke^{-rT}$$

Problem 9.19.

Prove the result in equation (9.8). (*Hint:* For the first part of the relationship consider (a) a portfolio consisting of a European call plus an amount of cash equal to $D + K$ and (b) a portfolio consisting of an American put option plus one share.)

As in the text we use c and p to denote the European call and put option price, and C and P to denote the American call and put option prices. The present value of the

dividends will be denoted by D . As shown in the answer to Problem 9.18, when there are no dividends

$$C - P \leq S_0 - Ke^{-rT}$$

Dividends reduce C and increase P . Hence this relationship must also be true when there are dividends.

For a further relationship between C and P , consider

Portfolio I: one European call option plus an amount of cash equal to $D + K$

Portfolio J: one American put option plus one share

Both options have the same exercise price and expiration date. Assume that the cash in portfolio I is invested at the risk-free interest rate. If the put option is not exercised early, portfolio J is worth

$$\max(S_T, K) + De^{rT}$$

at time T . Portfolio I is worth

$$\max(S_T - K, 0) + (D + K)e^{rT} = \max(S_T, K) + De^{rT} + Ke^{rT} - K$$

at this time. Portfolio I is therefore worth more than portfolio J. Suppose next that the put option in portfolio J is exercised early, say, at time τ . This means that portfolio J is worth at most $K + De^{r\tau}$ at time τ . However, even if the call option were worthless, portfolio I would be worth $(D + K)e^{r\tau}$ at time τ . It follows that portfolio I is worth more than portfolio J in all circumstances. Hence

$$c + D + K \geq P + S_0$$

Because $C \geq c$

$$C - P \geq S_0 - D - K$$

Problem 9.20.

Consider a 5-year employee stock option on a non-dividend-paying stock. The option can be exercised at any time after the end of the first year. Unlike a regular exchange-traded call option, the employee stock option cannot be sold. What is the likely impact of this restriction on the early exercise decision?

Executive stock options may be exercised early because the executive needs the cash or because he or she is uncertain about the company's future prospects. Regular call options can be sold in the market in either of these two situations, but executive stock options cannot be sold. In theory an executive can short the company's stock as an alternative to exercising. In practice this is not usually encouraged and may even be illegal.

Problem 9.21.

Use the software DerivaGem to verify that Figures 9.1 and 9.2 are correct.

The graphs can be produced from the first worksheet in DerivaGem. Select Equity as the Underlying Type. Select Analytic European as the Option Type. Input stock price

as 50, volatility as 30%, risk-free rate as 5%, time to exercise as 1 year, and exercise price as 50. Leave the dividend table blank because we are assuming no dividends. Select the button corresponding to call. Do not select the implied volatility button. Hit the *Enter* key and click on calculate. DerivaGem will show the price of the option as 7.15562248. Move to the Graph Results on the right hand side of the worksheet. Enter Option Price for the vertical axis and Asset price for the horizontal axis. Choose the minimum strike price value as 10 (software will not accept 0) and the maximum strike price value as 100. Hit *Enter* and click on *Draw Graph*. This will produce Figure 9.1a. Figures 9.1c, 9.1e, 9.2a, and 9.2c can be produced similarly by changing the horizontal axis. By selecting put instead of call and recalculating the rest of the figures can be produced. You are encouraged to experiment with this worksheet. Try different parameter values and different types of options.

ASSIGNMENT QUESTIONS

Problem 9.22.

A European call option and put option on a stock both have a strike price of \$20 and an expiration date in three months. Both sell for \$3. The risk-free interest rate is 10% per annum, the current stock price is \$19, and a \$1 dividend is expected in one month. Identify the arbitrage opportunity open to a trader.

If the call is worth \$3, put-call parity shows that the put should be worth

$$3 + 20e^{-0.10 \times 3/12} + e^{-0.1 \times 1/12} - 19 = 4.50$$

This is greater than \$3. The put is therefore undervalued relative to the call. The correct arbitrage strategy is to buy the put, buy the stock, and short the call. This costs \$19. If the stock price in three months is greater than \$20, the call is exercised. If it is less than \$20, the put is exercised. In either case the arbitrageur sells the stock for \$20 and collects the \$1 dividend in one month. The present value of the gain to the arbitrageur is

$$-3 - 19 + 3 + 20e^{-0.10 \times 3/12} + e^{-0.1 \times 1/12} = 1.50$$

Problem 9.23.

Suppose that c_1 , c_2 , and c_3 are the prices of European call options with strike prices K_1 , K_2 , and K_3 , respectively, where $K_3 > K_2 > K_1$ and $K_3 - K_2 = K_2 - K_1$. All options have the same maturity. Show that

$$c_2 \leq 0.5(c_1 + c_3)$$

(Hint: Consider a portfolio that is long one option with strike price K_1 , long one option with strike price K_3 , and short two options with strike price K_2 .)

Consider a portfolio that is long one option with strike price K_1 , long one option with strike price K_3 , and short two options with strike price K_2 . The value of the portfolio can be worked out in four different situations

$$\begin{aligned}
 S_T \leq K_1 : \text{Portfolio Value} &= 0 \\
 K_1 < S_T \leq K_2 : \text{Portfolio Value} &= S_T - K_1 \\
 K_2 < S_T \leq K_3 : \text{Portfolio Value} &= S_T - K_1 - 2(S_T - K_2) \\
 &= K_2 - K_1 - (S_T - K_2) \geq 0 \\
 S_T > K_3 : \text{Portfolio Value} &= S_T - K_1 - 2(S_T - K_2) + S_T - K_3 \\
 &= K_2 - K_1 - (K_3 - K_2) \\
 &= 0
 \end{aligned}$$

The value is always either positive or zero at the expiration of the option. In the absence of arbitrage possibilities it must be positive or zero today. This means that

$$c_1 + c_3 - 2c_2 \geq 0$$

or

$$c_2 \leq 0.5(c_1 + c_3)$$

Note that students often think they have proved this by writing down

$$\begin{aligned}
 c_1 &\leq S_0 - K_1 e^{-rT} \\
 2c_2 &\leq 2(S_0 - K_2 e^{-rT}) \\
 c_3 &\leq S_0 - K_3 e^{-rT}
 \end{aligned}$$

and subtracting the middle inequality from the sum of the other two. But they are deceiving themselves. Inequality relationships cannot be subtracted. For example, $9 > 8$ and $5 > 2$, but it is not true that $9 - 5 > 8 - 2$

Problem 9.24.

What is the result corresponding to that in Problem 9.23 for European put options?

The corresponding result is

$$p_2 \leq 0.5(p_1 + p_3)$$

where p_1 , p_2 and p_3 are the prices of European put option with the same maturities and strike prices K_1 , K_2 and K_3 respectively. This can be proved from the result in Problem 9.23 using put-call parity. Alternatively we can consider a portfolio consisting of a long position in a put option with strike price K_1 , a long position in a put option with strike price K_3 , and a short position in two put options with strike price K_2 . The value of this portfolio in different situations is given as follows

$$S_T \leq K_1 : \text{Portfolio Value} = K_1 - S_T - 2(K_2 - S_T) + K_3 - S_T$$

$$\begin{aligned}
&= K_3 - K_2 - (K_2 - K_1) \\
&= 0 \\
K_1 < S_T \leq K_2 : \text{ Portfolio Value} &= K_3 - S_T - 2(K_2 - S_T) \\
&= K_3 - K_2 - (K_2 - S_T) \\
&\geq 0 \\
K_2 < S_T \leq K_3 : \text{ Portfolio Value} &= K_3 - S_T \\
S_T > K_3 : \text{ Portfolio Value} &= 0
\end{aligned}$$

Because the portfolio value is always zero or positive at some future time the same must be true today. Hence

$$p_1 + p_3 - 2p_2 \geq 0$$

or

$$p_2 \leq 0.5(p_1 + p_3)$$

Problem 9.25.

Suppose that you are the manager and sole owner of a highly leveraged company. All the debt will mature in one year. If at that time the value of the company is greater than the face value of the debt, you will pay off the debt. If the value of the company is less than the face value of the debt, you will declare bankruptcy and the debt holders will own the company.

- a. Express your position as an option on the value of the company.
- b. Express the position of the debt holders in terms of options on the value of the company.
- c. What can you do to increase the value of your position?

(a) Suppose V is the value of the company and D is the face value of the debt. The value of the manager's position in one year is

$$\max(V - D, 0)$$

This is the payoff from a call option on V with strike price D .

(b) The debt holders get

$$\begin{aligned}
&\min(V, D) \\
&= D - \max(D - V, 0)
\end{aligned}$$

Since $\max(D - V, 0)$ is the payoff from a put option on V with strike price D , the debt holders have in effect made a risk-free loan (worth D at maturity with certainty) and written a put option on the value of the company with strike price D . The position of the debt holders in one year can also be characterized as

$$V - \max(V - D, 0)$$

This is a long position in the assets of the company combined with a short position in a call option on the assets with a strike price of D . The equivalence of the two

characterizations can be presented as an application of put–call parity. (See Business Snapshot 9.1.)

- (c) The manager can increase the value of his or her position by increasing the value of the call option in (a). It follows that the manager should attempt to increase both V and the volatility of V . To see why increasing the volatility of V is beneficial, imagine what happens when there are large changes in V . If V increases, the manager benefits to the full extent of the change. If V decreases, much of the downside is absorbed by the company’s lenders.

Problem 9.26.

Consider an option on a stock when the stock price is \$41, the strike price is \$40, the risk-free rate is 6%, the volatility is 35%, and the time to maturity is 1 year. Assume that a dividend of \$0.50 is expected after six months.

- Use DerivaGem to value the option assuming it is a European call.
- Use DerivaGem to value the option assuming it is a European put.
- Verify that put–call parity holds.
- Explore using DerivaGem what happens to the price of the options as the time to maturity becomes very large. For this purpose assume there are no dividends. Explain the results you get.

DerivaGem shows that the price of the call option is 6.9686 and the price of the put option is 4.1244. In this case

$$c + D + Ke^{-rT} = 6.9686 + 0.5e^{-0.06 \times 0.5} + 40e^{-0.06 \times 1} = 45.1244$$

Also

$$p + S = 4.1244 + 41 = 45.1244$$

As the time to maturity becomes very large and there are no dividends, the price of the call option approaches the stock price of 41. (For example when $T = 100$ it is 40.94.) This is because the call option can be regarded as a position in the stock where the price does not have to be paid for a very long time. The present value of what has to be paid is close to zero. As the time to maturity becomes very large the price of the European put option becomes close to zero. (For example when $T = 100$ it is 0.04.) This is because the present value of what might be received from the put option becomes close to zero.

CHAPTER 10

Trading Strategies Involving Options

Notes for the Instructor

This chapter covers various ways in which traders can form portfolios of calls and puts to get interesting payoff patterns. For ease of exposition, the time value of money is ignored in payoff diagrams and payoff tables.

Students usually enjoy the chapter. As each spread strategy is covered, I like to use put–call parity to relate the cost of the spread created using calls to the cost of a spread created using puts. (See Problems 10.8 and 10.11) This reinforces the Chapter 9 material on put–call parity. It can be useful to cover Business Snapshot 10.1 in class.

Problem 10.19 can be used for class discussion. Problems 10.20, 10.21, 10.22, and 10.23 can be used as hand-in assignments.

QUESTIONS AND PROBLEMS

Problem 10.1.

What is meant by a protective put? What position in call options is equivalent to a protective put?

A protective put consists of a long position in a put option combined with a long position in the underlying shares. It is equivalent to a long position in a call option plus a certain amount of cash. This follows from put–call parity:

$$p + S_0 = c + Ke^{-rT} + D$$

Problem 10.2.

Explain two ways in which a bear spread can be created.

A bear spread can be created using two call options with the same maturity and different strike prices. The investor shorts the call option with the lower strike price and buys the call option with the higher strike price. A bear spread can also be created using two put options with the same maturity and different strike prices. In this case, the investor shorts the put option with the lower strike price and buys the put option with the higher strike price.

Problem 10.3.

When is it appropriate for an investor to purchase a butterfly spread?

A butterfly spread involves a position in options with three different strike prices (K_1, K_2 , and K_3). A butterfly spread should be purchased when the investor considers that the price of the underlying stock is likely to stay close to the central strike price, K_2 .

Problem 10.4.

Call options on a stock are available with strike prices of \$15, $\$17\frac{1}{2}$, and \$20 and expiration dates in three months. Their prices are \$4, \$2, and $\$2\frac{1}{2}$, respectively. Explain how the options can be used to create a butterfly spread. Construct a table showing how profit varies with stock price for the butterfly spread.

An investor can create a butterfly spread by buying call options with strike prices of \$15 and \$20 and selling two call options with strike prices of $\$17\frac{1}{2}$. The initial investment is $4 + \frac{1}{2} - 2 \times 2 = \$\frac{1}{2}$. The following table shows the variation of profit with the final stock price:

Stock Price S_T	Profit
$S_T < 15$	$-\frac{1}{2}$
$15 < S_T < 17\frac{1}{2}$	$(S_T - 15) - \frac{1}{2}$
$17\frac{1}{2} < S_T < 20$	$(20 - S_T) - \frac{1}{2}$
$S_T > 20$	$-\frac{1}{2}$

Problem 10.5.

What trading strategy creates a reverse calendar spread?

A reverse calendar spread is created by buying a short-maturity option and selling a long-maturity option, both with the same strike price.

Problem 10.6.

What is the difference between a strangle and a straddle?

Both a straddle and a strangle are created by combining a long position in a call with a long position in a put. In a straddle the two have the same strike price and expiration date. In a strangle they have different strike prices and the same expiration date.

Problem 10.7.

A call option with a strike price of \$50 costs \$2. A put option with a strike price of \$45 costs \$3. Explain how a strangle can be created from these two options. What is the pattern of profits from the strangle?

A strangle is created by buying both options. The pattern of profits is as follows:

Stock Price S_T	Profit
$S_T < 45$	$(45 - S_T) - 5$
$45 < S_T < 50$	-5
$S_T > 50$	$(S_T - 50) - 5$

Problem 10.8.

Use put-call parity to relate the initial investment for a bull spread created using calls to the initial investment for a bull spread created using puts.

A bull spread using calls provides a profit pattern with the same general shape as a bull spread using puts (see Figures 10.2 and 10.3 in the text). Define p_1 and c_1 as the prices of put and call with strike price K_1 and p_2 and c_2 as the prices of a put and call with strike price K_2 . From put-call parity

$$p_1 + S = c_1 + K_1 e^{-rT}$$

$$p_2 + S = c_2 + K_2 e^{-rT}$$

Hence:

$$p_1 - p_2 = c_1 - c_2 - (K_2 - K_1)e^{-rT}$$

This shows that the initial investment when the spread is created from puts is less than the initial investment when it is created from calls by an amount $(K_2 - K_1)e^{-rT}$. In fact as mentioned in the text the initial investment when the bull spread is created from puts is negative, while the initial investment when it is created from calls is positive.

The profit when calls are used to create the bull spread is higher than when puts are used by $(K_2 - K_1)(1 - e^{-rT})$. This reflects the fact that the call strategy involves an additional risk-free investment of $(K_2 - K_1)e^{-rT}$ over the put strategy. This earns interest of $(K_2 - K_1)e^{-rT}(e^{rT} - 1) = (K_2 - K_1)(1 - e^{-rT})$.

Problem 10.9.

Explain how an aggressive bear spread can be created using put options.

An aggressive bull spread using call options is discussed in the text. Both of the options used have relatively high strike prices. Similarly, an aggressive bear spread can be created using put options. Both of the options should be out of the money (that is, they should have relatively low strike prices). The spread then costs very little to set up because both of the puts are worth close to zero. In most circumstances the spread will provide zero payoff. However, there is a small chance that the stock price will fall fast so that on expiration both options will be in the money. The spread then provides a payoff equal to the difference between the two strike prices, $K_2 - K_1$.

Problem 10.10.

Suppose that put options on a stock with strike prices \$30 and \$35 cost \$4 and \$7, respectively. How can the options be used to create (a) a bull spread and (b) a bear spread? Construct a table that shows the profit and payoff for both spreads.

A bull spread is created by buying the \$30 put and selling the \$35 put. This strategy gives rise to an initial cash inflow of \$3. The outcome is as follows:

Stock Price	Payoff	Profit
$S_T \geq 35$	0	3
$30 \leq S_T < 35$	$S_T - 35$	$S_T - 32$
$S_T < 30$	-5	-2

A bear spread is created by selling the \$30 put and buying the \$35 put. This strategy costs \$3 initially. The outcome is as follows:

Stock Price	Payoff	Profit
$S_T \geq 35$	0	-3
$30 \leq S_T < 35$	$35 - S_T$	$32 - S_T$
$S_T < 30$	5	2

Problem 10.11.

Use put-call parity to show that the cost of a butterfly spread created from European puts is identical to the cost of a butterfly spread created from European calls.

Define c_1 , c_2 , and c_3 as the prices of calls with strike prices K_1 , K_2 and K_3 . Define p_1 , p_2 and p_3 as the prices of puts with strike prices K_1 , K_2 and K_3 . With the usual notation

$$c_1 + K_1 e^{-rT} = p_1 + S$$

$$c_2 + K_2 e^{-rT} = p_2 + S$$

$$c_3 + K_3 e^{-rT} = p_3 + S$$

Hence

$$c_1 + c_3 - 2c_2 + (K_1 + K_3 - 2K_2)e^{-rT} = p_1 + p_3 - 2p_2$$

Because $K_2 - K_1 = K_3 - K_2$, it follows that $K_1 + K_3 - 2K_2 = 0$ and

$$c_1 + c_3 - 2c_2 = p_1 + p_3 - 2p_2$$

The cost of a butterfly spread created using European calls is therefore exactly the same as the cost of a butterfly spread created using European puts.

Problem 10.12.

A call with a strike price of \$60 costs \$6. A put with the same strike price and expiration date costs \$4. Construct a table that shows the profit from a straddle. For what range of stock prices would the straddle lead to a loss?

A straddle is created by buying both the call and the put. This strategy costs \$10. The profit/loss is shown in the following table:

Stock Price	Payoff	Profit
$S_T > 60$	$S_T - 60$	$S_T - 70$
$S_T \leq 60$	$60 - S_T$	$50 - S_T$

This shows that the straddle will lead to a loss if the final stock price is between \$50 and \$70.

Problem 10.13.

Construct a table showing the payoff from a bull spread when puts with strike prices K_1 and K_2 are used ($K_2 > K_1$).

The bull spread is created by buying a put with strike price K_1 and selling a put with strike price K_2 . The payoff is calculated as follows:

Stock Price Range	Payoff from Long Put Option	Payoff from Short Put Option	Total Payoff
$S_T \geq K_2$	0	0	0
$K_1 < S_T < K_2$	0	$S_T - K_2$	$-(K_2 - S_T)$
$S_T \leq K_1$	$K_1 - S_T$	$S_T - K_2$	$-(K_2 - K_1)$

Problem 10.14.

An investor believes that there will be a big jump in a stock price, but is uncertain as to the direction. Identify six different strategies the investor can follow and explain the differences among them.

Possible strategies are:

- Strangle
- Straddle
- Strip
- Strap
- Reverse calendar spread
- Reverse butterfly spread

The strategies all provide positive profits when there are large stock price moves. A strangle is less expensive than a straddle, but requires a bigger move in the stock price in order to provide a positive profit. Strips and straps are more expensive than straddles but provide bigger profits in certain circumstances. A strip will provide a bigger profit when there is a large downward stock price move. A strap will provide a bigger profit when there is a large upward stock price move. In the case of strangles, straddles, strips and straps, the profit increases as the size of the stock price movement increases. By contrast in a reverse calendar spread and a reverse butterfly spread there is a maximum potential profit regardless of the size of the stock price movement.

Problem 10.15.

How can a forward contract on a stock with a particular delivery price and delivery date be created from options?

Suppose that the delivery price is K and the delivery date is T . The forward contract is created by buying a European call and selling a European put when both options have strike price K and exercise date T . This portfolio provides a payoff of $S_T - K$ under all circumstances where S_T is the stock price at time T . Suppose that F_0 is the forward price. If $K = F_0$, the forward contract that is created has zero value. This shows that the price of a call equals the price of a put when the strike price is F_0 .

Problem 10.16.

“A box spread comprises four options. Two can be combined to create a long forward position and two can be combined to create a short forward position.” Explain this statement.

A box spread is a bull spread created using calls and a bear spread created using puts. With the notation in the text it consists of a) a long call with strike K_1 , b) a short call with strike K_2 , c) a long put with strike K_2 , and d) a short put with strike K_1 . a) and d) give a long forward contract with delivery price K_1 ; b) and c) give a short forward contract with delivery price K_2 . The two forward contracts taken together give the payoff of $K_2 - K_1$.

Problem 10.17.

What is the result if the strike price of the put is higher than the strike price of the call in a strangle?

The result is shown in Figure S10.1. The profit pattern from a long position in a call and a put when the put has a higher strike price than a call is much the same as when the call has a higher strike price than the put. Both the initial investment and the final payoff are much higher in the first case.

Problem 10.18.

One Australian dollar is currently worth \$0.64. A one-year butterfly spread is set up using European call options with strike prices of \$0.60, \$0.65, and \$0.70. The risk-free

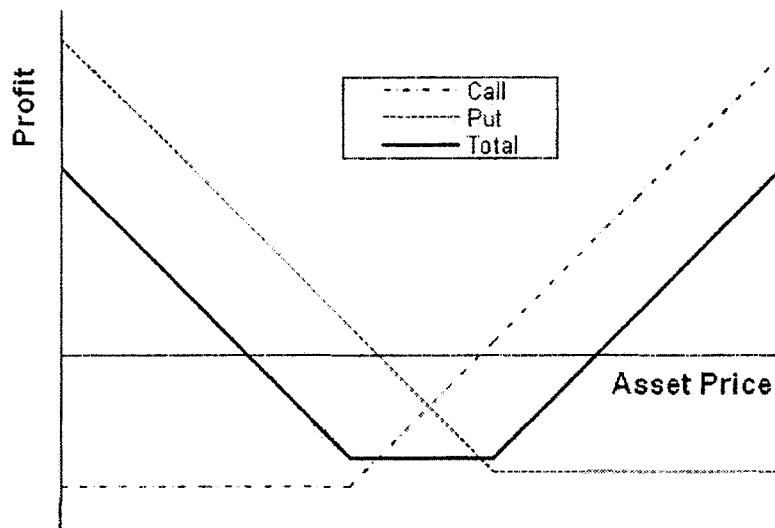


Figure S10.1 Profit Pattern in Problem 10.17

interest rates in the United States and Australia are 5% and 4% respectively, and the volatility of the exchange rate is 15%. Use the DerivaGem software to calculate the cost of setting up the butterfly spread position. Show that the cost is the same if European put options are used instead of European call options.

To use DerivaGem select the first worksheet and choose Currency as the Underlying Type. Select Analytic European as the Option Type. Input exchange rate as 0.64, volatility as 15%, risk-free rate as 5%, foreign risk-free interest rate as 4%, time to exercise as 1 year, and exercise price as 0.60. Select the button corresponding to call. Do not select the implied volatility button. Hit the *Enter* key and click on calculate. DerivaGem will show the price of the option as 0.0618. Change the exercise price to 0.65, hit *Enter*, and click on calculate again. DerivaGem will show the value of the option as 0.0352. Change the exercise price to 0.70, hit *Enter*, and click on calculate. DerivaGem will show the value of the option as 0.0181.

Now select the button corresponding to put and repeat the procedure. DerivaGem shows the values of puts with strike prices 0.60, 0.65, and 0.70 to be 0.0176, 0.0386, and 0.0690, respectively.

The cost of setting up the butterfly spread when calls are used is therefore

$$0.0618 + 0.0181 - 2 \times 0.0352 = 0.0095$$

The cost of setting up the butterfly spread when puts are used is

$$0.0176 + 0.0690 - 2 \times 0.0386 = 0.0094$$

Allowing for rounding errors these two are the same.

ASSIGNMENT QUESTIONS

Problem 10.19.

Three put options on a stock have the same expiration date and strike prices of \$55, \$60, and \$65. The market prices are \$3, \$5, and \$8, respectively. Explain how a butterfly spread can be created. Construct a table showing the profit from the strategy. For what range of stock prices would the butterfly spread lead to a loss?

A butterfly spread is created by buying the \$55 put, buying the \$65 put and selling two of the \$60 puts. This costs $3 + 8 - 2 \times 5 = \$1$ initially. The following table shows the profit/loss from the strategy.

Stock Price	Payoff	Profit
$S_T \geq 65$	0	-1
$60 \leq S_T < 65$	$65 - S_T$	$64 - S_T$
$55 \leq S_T < 60$	$S_T - 55$	$S_T - 56$
$S_T < 55$	0	-1

The butterfly spread leads to a loss when the final stock price is greater than \$64 or less than \$56.

Problem 10.20.

A diagonal spread is created by buying a call with strike price K_2 and exercise date T_2 and selling a call with strike price K_1 and exercise date T_1 ($T_2 > T_1$). Draw a diagram showing the profit when (a) $K_2 > K_1$ and (b) $K_2 < K_1$.

There are two alternative profit patterns for part (a). These are shown in Figures M10.1 and M10.2. In Figure M10.1 the long maturity (high strike price) option is worth more than the short maturity (low strike price) option. In Figure M10.2 the reverse is true. There is no ambiguity about the profit pattern for part (b). This is shown in Figure M10.3.

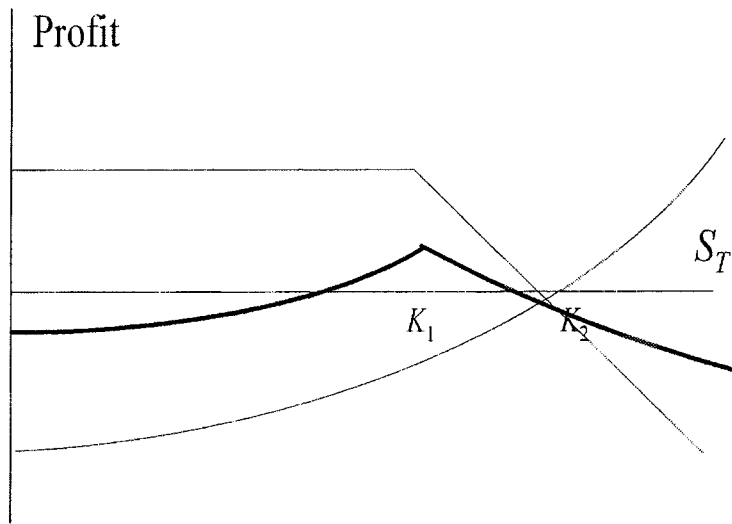


Figure M10.1 Investor's Profit/Loss in Problem 10.20a when long maturity call is worth more than short maturity call

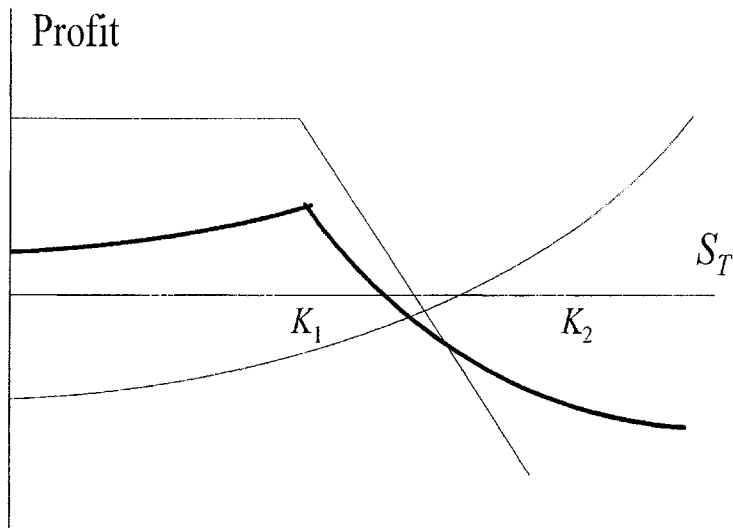


Figure M10.2 Investor's Profit/Loss in Problem 10.20a when short maturity call is worth more than long maturity call

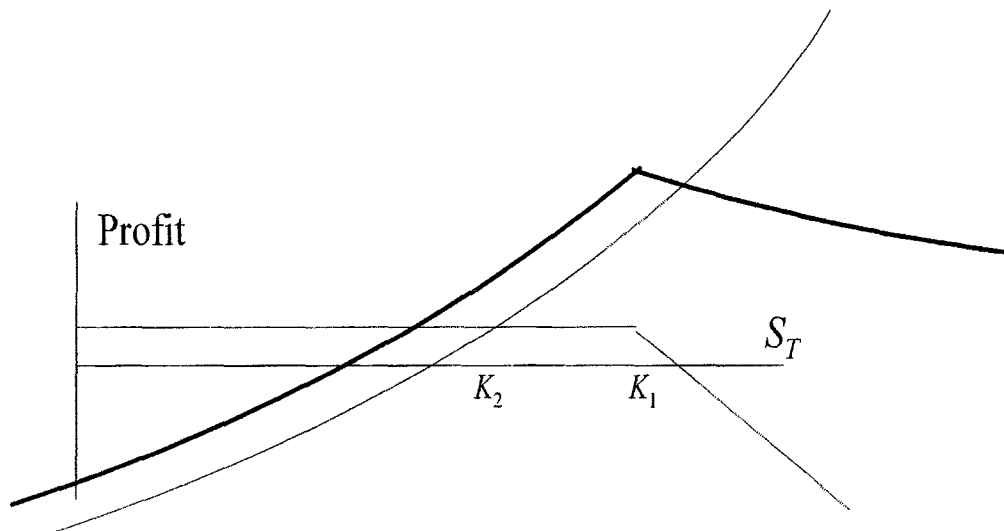


Figure M10.3 Investor's Profit/Loss in Problem 10.20b

Problem 10.21.

Draw a diagram showing the variation of an investor's profit and loss with the terminal stock price for a portfolio consisting of

- a. One share and a short position in one call option
- b. Two shares and a short position in one call option
- c. One share and a short position in two call options
- d. One share and a short position in four call options

In each case, assume that the call option has an exercise price equal to the current stock price.

The variation of an investor's profit/loss with the terminal stock price for each of the four strategies is shown in Figure M10.4. In each case the dotted line shows the profits from the components of the investor's position and the solid line shows the total net profit.

Problem 10.22.

Suppose that the price of a non-dividend-paying stock is \$32, its volatility is 30%, and the risk-free rate for all maturities is 5% per annum. Use DerivaGem to calculate the cost of setting up the following positions. In each case provide a table showing the relationship between profit and final stock price. Ignore the impact of discounting.

- a. A bull spread using European call options with strike prices of \$25 and \$30 and a maturity of six months.
- b. A bear spread using European put options with strike prices of \$25 and \$30 and a maturity of six months
- c. A butterfly spread using European call options with strike prices of \$25, \$30, and \$35 and a maturity of one year.
- d. A butterfly spread using European put options with strike prices of \$25, \$30, and \$35 and a maturity of one year.
- e. A straddle using options with a strike price of \$30 and a six-month maturity.

f. A strangle using options with strike prices of \$25 and \$35 and a six-month maturity.

- (a) A call option with a strike price of 25 costs 7.90 and a call option with a strike price of 30 costs 4.18. The cost of the bull spread is therefore $7.90 - 4.18 = 3.72$. The profits ignoring the impact of discounting are

Stock Price Range	Profit
$S_T \leq 25$	-3.72
$25 < S_T < 30$	$S_T - 28.72$
$S_T \geq 30$	1.28

- (b) A put option with a strike price of 25 costs 0.28 and a put option with a strike price of 30 costs 1.44. The cost of the bear spread is therefore $1.44 - 0.28 = 1.16$. The profits ignoring the impact of discounting are

Stock Price Range	Profit
$S_T \leq 25$	+3.84
$25 < S_T < 30$	$28.84 - S_T$
$S_T \geq 30$	-1.16

- (c) Call options with maturities of one year and strike prices of 25, 30, and 35 cost 8.92, 5.60, and 3.28, respectively. The cost of the butterfly spread is therefore $8.92 + 3.28 - 2 \times 5.60 = 1.00$. The profits ignoring the impact of discounting are

Stock Price Range	Profit
$S_T \leq 25$	-1.00
$25 < S_T < 30$	$S_T - 26.00$
$30 \leq S_T < 35$	$34.00 - S_T$
$S_T \geq 35$	-1.00

- (d) Put options with maturities of one year and strike prices of 25, 30, and 35 cost 0.70, 2.14, 4.57, respectively. The cost of the butterfly spread is therefore $0.70 + 4.57 - 2 \times 2.14 = 0.99$. Allowing for rounding errors, this is the same as in (c). The profits are the same as in (c).
- (e) A call option with a strike price of 30 costs 4.18. A put option with a strike price of 30 costs 1.44. The cost of the straddle is therefore $4.18 + 1.44 = 5.62$. The profits ignoring the impact of discounting are

Stock Price Range	Profit
$S_T \leq 30$	$24.58 - S_T$
$S_T > 30$	$S_T - 35.62$

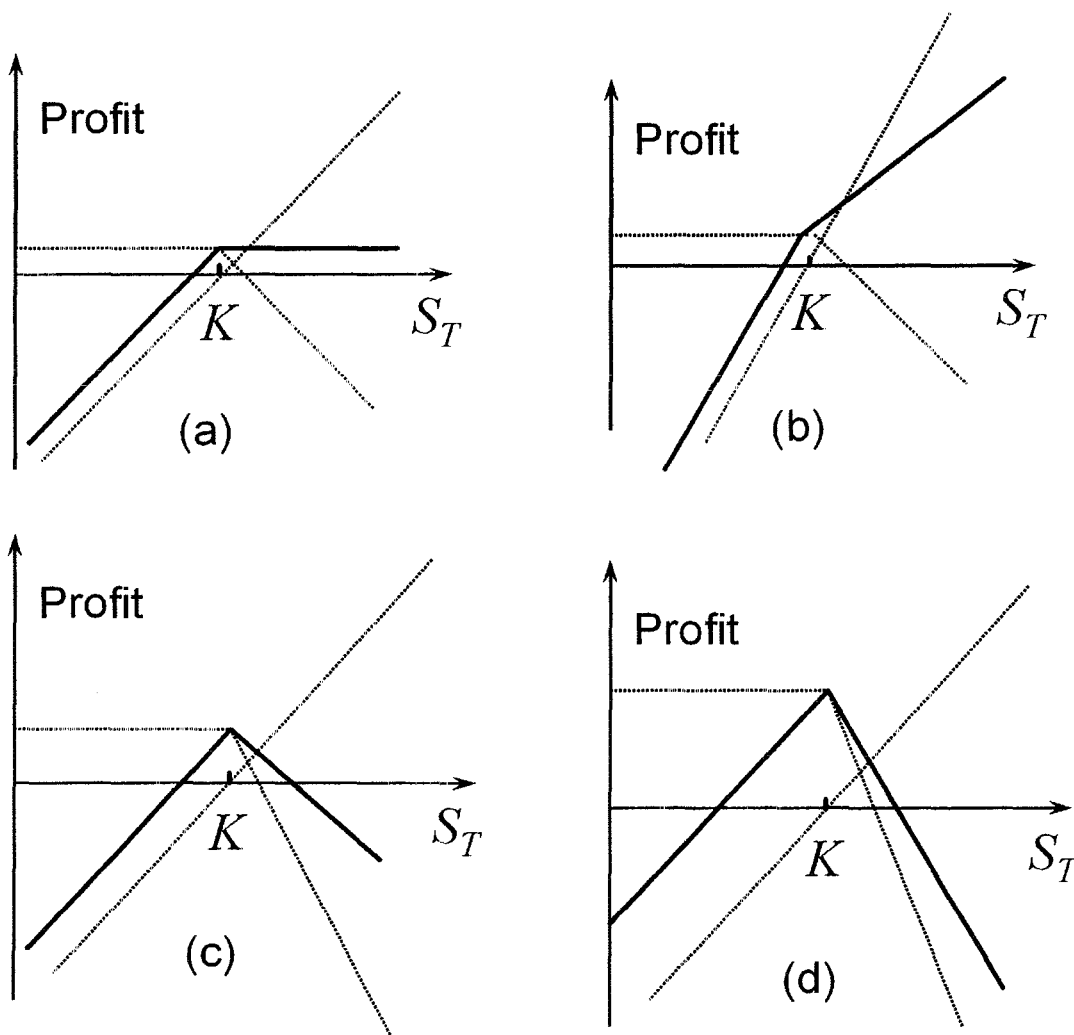


Figure M10.4 Investor's Profit/Loss in Problem 10.21

- (f) A six-month call option with a strike price of 35 costs 1.85. A six-month put option with a strike price of 25 costs 0.28. The cost of the strangle is therefore $1.85 + 0.28 = 2.13$. The profits ignoring the impact of discounting are

Stock Price Range	Profit
$S_T \leq 25$	$22.87 - S_T$
$25 < S_T < 35$	-2.13
$S_T \geq 35$	$S_T - 37.13$

Problem 10.23.

What trading position is created from a long strangle and a short straddle when both have the same time to maturity? Assume that the strike price in the straddle is half way between the two strike prices in the strangle.

A butterfly spread is created.

CHAPTER 11

Binomial Trees

Notes for the Instructor

This chapter discusses binomial trees. It enables some of the key concepts in option valuation to be introduced at a relatively early stage in a course. It includes material on the use of binomial trees for index options, currency options, and futures options (see Section 11.9).

The one-step binomial model can be used to demonstrate both no-arbitrage and risk-neutral valuation arguments. I like to first go through the arguments using the numerical example in the text and then generalize them by introducing some notation. After two- and three-step trees have been covered students should have a good appreciation of the way in which multistep trees are used to value options. DerivaGem provides a convenient way of displaying trees in class. The material on delta serves as an introduction to the hedging material in Chapter 17.

Any of Problems 11.16 to 11.22 can be used as assignments. I usually discuss 11.22 in class.

QUESTIONS AND PROBLEMS

Problem 11.1.

A stock price is currently \$40. It is known that at the end of one month it will be either \$42 or \$38. The risk-free interest rate is 8% per annum with continuous compounding. What is the value of a one-month European call option with a strike price of \$39?

Consider a portfolio consisting of

-1 :	Call option
+ Δ :	Shares

If the stock price rises to \$42, the portfolio is worth $42\Delta - 3$. If the stock price falls to \$38, it is worth 38Δ . These are the same when

$$42\Delta - 3 = 38\Delta$$

or $\Delta = 0.75$. The value of the portfolio in one month is 28.5 for both stock prices. Its value today must be the present value of 28.5, or $28.5e^{-0.08 \times 0.08333} = 28.31$. This means that

$$-f + 40\Delta = 28.31$$

where f is the call price. Because $\Delta = 0.75$, the call price is $40 \times 0.75 - 28.31 = \1.69 . As an alternative approach, we can calculate the probability, p , of an up movement in a risk-neutral world. This must satisfy:

$$42p + 38(1 - p) = 40e^{0.08 \times 0.08333}$$

so that

$$4p = 40e^{0.08 \times 0.08333} - 38$$

or $p = 0.5669$. The value of the option is then its expected payoff discounted at the risk-free rate:

$$[3 \times 0.5669 + 0 \times 0.4331]e^{-0.08 \times 0.08333} = 1.69$$

or \$1.69. This agrees with the previous calculation.

Problem 11.2.

Explain the no-arbitrage and risk-neutral valuation approaches to valuing a European option using a one-step binomial tree.

In the no-arbitrage approach, we set up a riskless portfolio consisting of a position in the option and a position in the stock. By setting the return on the portfolio equal to the risk-free interest rate, we are able to value the option. When we use risk-neutral valuation, we first choose probabilities for the branches of the tree so that the expected return on the stock equals the risk-free interest rate. We then value the option by calculating its expected payoff and discounting this expected payoff at the risk-free interest rate.

Problem 11.3.

What is meant by the delta of a stock option?

The delta of a stock option measures the sensitivity of the option price to the price of the stock when small changes are considered. Specifically, it is the ratio of the change in the price of the stock option to the change in the price of the underlying stock.

Problem 11.4.

A stock price is currently \$50. It is known that at the end of six months it will be either \$45 or \$55. The risk-free interest rate is 10% per annum with continuous compounding. What is the value of a six-month European put option with a strike price of \$50?

Consider a portfolio consisting of

-1 :	Put option
+ Δ :	Shares

If the stock price rises to \$55, this is worth 55Δ . If the stock price falls to \$45, the portfolio is worth $45\Delta - 5$. These are the same when

$$45\Delta - 5 = 55\Delta$$

or $\Delta = -0.50$. The value of the portfolio in six months is -27.5 for both stock prices. Its value today must be the present value of -27.5 , or $-27.5e^{-0.1 \times 0.5} = -26.16$. This means that

$$-f + 50\Delta = -26.16$$

where f is the put price. Because $\Delta = -0.50$, the put price is \$1.16. As an alternative approach we can calculate the probability, p , of an up movement in a risk-neutral world. This must satisfy:

$$55p + 45(1 - p) = 50e^{0.1 \times 0.5}$$

so that

$$10p = 50e^{0.1 \times 0.5} - 45$$

or $p = 0.7564$. The value of the option is then its expected payoff discounted at the risk-free rate:

$$[0 \times 0.7564 + 5 \times 0.2436]e^{-0.1 \times 0.5} = 1.16$$

or \$1.16. This agrees with the previous calculation.

Problem 11.5.

A stock price is currently \$100. Over each of the next two six-month periods it is expected to go up by 10% or down by 10%. The risk-free interest rate is 8% per annum with continuous compounding. What is the value of a one-year European call option with a strike price of \$100?

In this case $u = 1.10$, $d = 0.90$, $\Delta t = 0.5$, and $r = 0.08$, so that

$$p = \frac{e^{0.08 \times 0.5} - 0.90}{1.10 - 0.90} = 0.7041$$

The tree for stock price movements is shown in Figure S11.1. We can work back from the end of the tree to the beginning, as indicated in the diagram, to give the value of the option as \$9.61. The option value can also be calculated directly from equation (11.10):

$$[0.7041^2 \times 21 + 2 \times 0.7041 \times 0.2959 \times 0 + 0.2959^2 \times 0]e^{-2 \times 0.08 \times 0.5} = 9.61$$

or \$9.61.

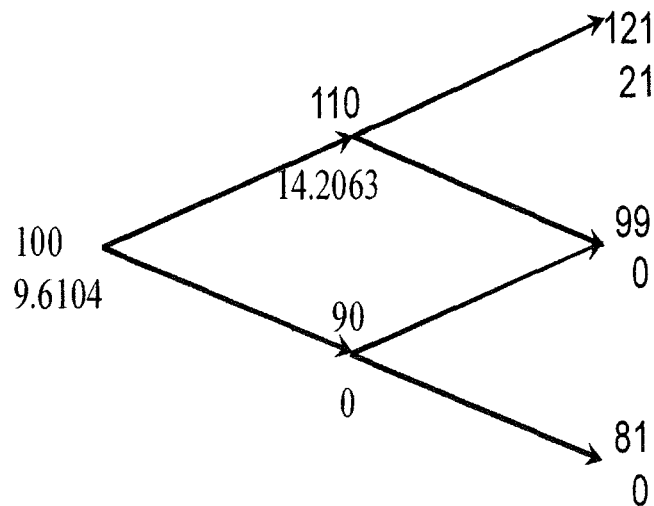


Figure S11.1 Tree for Problem 11.5

Problem 11.6.

For the situation considered in Problem 11.5, what is the value of a one-year European put option with a strike price of \$100? Verify that the European call and European put prices satisfy put-call parity.

Figure S11.2 shows how we can value the put option using the same tree as in Problem 11.5. The value of the option is \$1.92. The option value can also be calculated directly from equation (11.10):

$$e^{-2 \times 0.08 \times 0.5} [0.7041^2 \times 0 + 2 \times 0.7041 \times 0.2959 \times 1 + 0.2959^2 \times 19] = 1.92$$

or \$1.92. The stock price plus the put price is $100 + 1.92 = \$101.92$. The present value of the strike price plus the call price is $100e^{-0.08 \times 1} + 9.61 = \101.92 . These are the same, verifying that put-call parity holds.

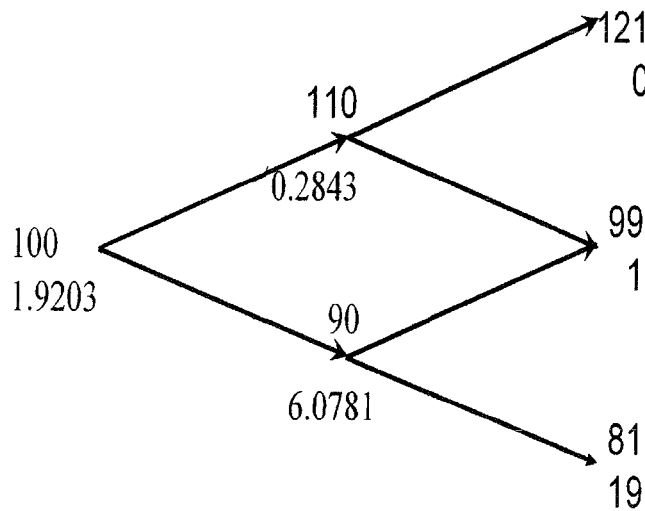


Figure S11.2 Tree for Problem 11.6

Problem 11.7.

What are the formulas for u and d in terms of volatility?

$$u = e^{\sigma\sqrt{\Delta t}} \text{ and } d = e^{-\sigma\sqrt{\Delta t}}$$

Problem 11.8.

Consider the situation in which stock price movements during the life of a European option are governed by a two-step binomial tree. Explain why it is not possible to set up a position in the stock and the option that remains riskless for the whole of the life of the option.

The riskless portfolio consists of a short position in the option and a long position in Δ shares. Because Δ changes during the life of the option, this riskless portfolio must also change.

Problem 11.9.

A stock price is currently \$50. It is known that at the end of two months it will be either \$53 or \$48. The risk-free interest rate is 10% per annum with continuous compounding. What is the value of a two-month European call option with a strike price of \$49? Use no-arbitrage arguments.

At the end of two months the value of the option will be either \$4 (if the stock price is \$53) or \$0 (if the stock price is \$48). Consider a portfolio consisting of:

$$\begin{aligned} +\Delta & : \text{ shares} \\ -1 & : \text{ option} \end{aligned}$$

The value of the portfolio is either 48Δ or $53\Delta - 4$ in two months. If

$$48\Delta = 53\Delta - 4$$

i.e.,

$$\Delta = 0.8$$

the value of the portfolio is certain to be 38.4. For this value of Δ the portfolio is therefore riskless. The current value of the portfolio is:

$$0.8 \times 50 - f$$

where f is the value of the option. Since the portfolio must earn the risk-free rate of interest

$$(0.8 \times 50 - f)e^{0.10 \times 2/12} = 38.4$$

i.e.,

$$f = 2.23$$

The value of the option is therefore \$2.23.

This can also be calculated directly from equations (11.2) and (11.3). $u = 1.06$, $d = 0.96$ so that

$$p = \frac{e^{0.10 \times 2/12} - 0.96}{1.06 - 0.96} = 0.5681$$

and

$$f = e^{-0.10 \times 2/12} \times 0.5681 \times 4 = 2.23$$

Problem 11.10.

A stock price is currently \$80. It is known that at the end of four months it will be either \$75 or \$85. The risk-free interest rate is 5% per annum with continuous compounding. What is the value of a four-month European put option with a strike price of \$80? Use no-arbitrage arguments.

At the end of four months the value of the option will be either \$5 (if the stock price is \$75) or \$0 (if the stock price is \$85). Consider a portfolio consisting of:

$$\begin{aligned} -\Delta & : \text{ shares} \\ +1 & : \text{ option} \end{aligned}$$

(Note: The delta, Δ of a put option is negative. We have constructed the portfolio so that it is +1 option and $-\Delta$ shares rather than -1 option and $+\Delta$ shares so that the initial investment is positive.)

The value of the portfolio is either -85Δ or $-75\Delta + 5$ in four months. If

$$-85\Delta = -75\Delta + 5$$

i.e.,

$$\Delta = -0.5$$

the value of the portfolio is certain to be 42.5. For this value of Δ the portfolio is therefore riskless. The current value of the portfolio is:

$$0.5 \times 80 + f$$

where f is the value of the option. Since the portfolio is riskless

$$(0.5 \times 80 + f)e^{0.05 \times 4/12} = 42.5$$

i.e.,

$$f = 1.80$$

The value of the option is therefore \$1.80.

This can also be calculated directly from equations (11.2) and (11.3). $u = 1.0625$, $d = 0.9375$ so that

$$p = \frac{e^{0.05 \times 4/12} - 0.9375}{1.0625 - 0.9375} = 0.6345$$

$1 - p = 0.3655$ and

$$f = e^{-0.05 \times 4/12} \times 0.3655 \times 5 = 1.80$$

Problem 11.11.

A stock price is currently \$40. It is known that at the end of three months it will be either \$45 or \$35. The risk-free rate of interest with quarterly compounding is 8% per annum. Calculate the value of a three-month European put option on the stock with

an exercise price of \$40. Verify that no-arbitrage arguments and risk-neutral valuation arguments give the same answers.

At the end of three months the value of the option is either \$5 (if the stock price is \$35) or \$0 (if the stock price is \$45).

Consider a portfolio consisting of:

$$\begin{aligned} -\Delta & : \text{ shares} \\ +1 & : \text{ option} \end{aligned}$$

(Note: The delta, Δ , of a put option is negative. We have constructed the portfolio so that it is +1 option and $-\Delta$ shares rather than -1 option and $+\Delta$ shares so that the initial investment is positive.)

The value of the portfolio is either $-35\Delta + 5$ or -45Δ . If:

$$-35\Delta + 5 = -45\Delta$$

i.e.,

$$\Delta = -0.5$$

the value of the portfolio is certain to be 22.5. For this value of Δ the portfolio is therefore riskless. The current value of the portfolio is

$$-40\Delta + f$$

where f is the value of the option. Since the portfolio must earn the risk-free rate of interest

$$(40 \times 0.5 + f) \times 1.02 = 22.5$$

Hence

$$f = 2.06$$

i.e., the value of the option is \$2.06.

This can also be calculated using risk-neutral valuation. Suppose that p is the probability of an upward stock price movement in a risk-neutral world. We must have

$$45p + 35(1 - p) = 40 \times 1.02$$

i.e.,

$$10p = 5.8$$

or:

$$p = 0.58$$

The expected value of the option in a risk-neutral world is:

$$0 \times 0.58 + 5 \times 0.42 = 2.10$$

This has a present value of

$$\frac{2.10}{1.02} = 2.06$$

This is consistent with the no-arbitrage answer.

Problem 11.12.

A stock price is currently \$50. Over each of the next two three-month periods it is expected to go up by 6% or down by 5%. The risk-free interest rate is 5% per annum with continuous compounding. What is the value of a six-month European call option with a strike price of \$51?

A tree describing the behavior of the stock price is shown in Figure S11.3. The risk-neutral probability of an up move, p , is given by

$$p = \frac{e^{0.05 \times 3/12} - 0.95}{1.06 - 0.95} = 0.5689$$

There is a payoff from the option of $56.18 - 51 = 5.18$ for the highest final node (which corresponds to two up moves) zero in all other cases. The value of the option is therefore

$$5.18 \times 0.5689^2 \times e^{-0.05 \times 6/12} = 1.635$$

This can also be calculated by working back through the tree as indicated in Figure S11.3. The value of the call option is the lower number at each node in the figure.

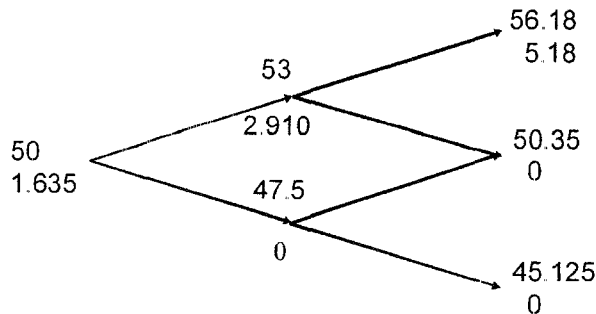


Figure S11.3 Tree for Problem 11.12

Problem 11.13.

For the situation considered in Problem 11.12, what is the value of a six-month European put option with a strike price of \$51? Verify that the European call and European put prices satisfy put-call parity. If the put option were American, would it ever be optimal to exercise it early at any of the nodes on the tree?

The tree for valuing the put option is shown in Figure S11.4. We get a payoff of $51 - 50.35 = 0.65$ if the middle final node is reached and a payoff of $51 - 45.125 = 5.875$ if the lowest final node is reached. The value of the option is therefore

$$(0.65 \times 2 \times 0.5689 \times 0.4311 + 5.875 \times 0.4311^2)e^{-0.05 \times 6/12} = 1.376$$

This can also be calculated by working back through the tree as indicated in Figure S11.4. The value of the put plus the stock price is from Problem 11.12

$$1.376 + 50 = 51.376$$

The value of the call plus the present value of the strike price is

$$1.635 + 51e^{-0.05 \times 6/12} = 51.376$$

This verifies that put-call parity holds

To test whether it worth exercising the option early we compare the value calculated for the option at each node with the payoff from immediate exercise. At node C the payoff from immediate exercise is $51 - 47.5 = 3.5$. Because this is greater than 2.8664, the option should be exercised at this node. The option should not be exercised at either node A or node B.

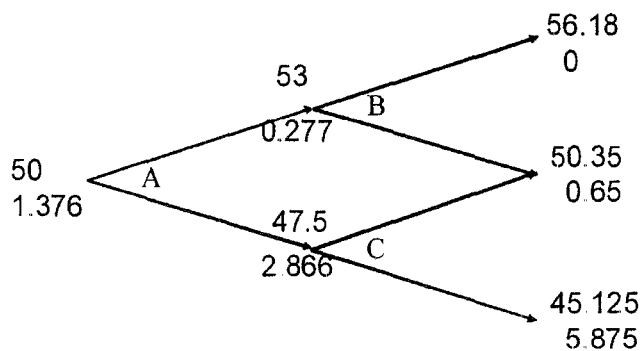


Figure S11.4 Tree for Problem 11.13

Problem 11.14.

A stock price is currently \$25. It is known that at the end of two months it will be either \$23 or \$27. The risk-free interest rate is 10% per annum with continuous compounding. Suppose S_T is the stock price at the end of two months. What is the value of a derivative that pays off S_T^2 at this time?

At the end of two months the value of the derivative will be either 529 (if the stock price is 23) or 729 (if the stock price is 27). Consider a portfolio consisting of:

- $+\Delta$: shares
- -1 : derivative

The value of the portfolio is either $27\Delta - 729$ or $23\Delta - 529$ in two months. If

$$27\Delta - 729 = 23\Delta - 529$$

i.e.,

$$\Delta = 50$$

the value of the portfolio is certain to be 621. For this value of Δ the portfolio is therefore riskless. The current value of the portfolio is:

$$50 \times 25 - f$$

where f is the value of the derivative. Since the portfolio must earn the risk-free rate of interest

$$(50 \times 25 - f)e^{0.10 \times 2/12} = 621$$

i.e.,

$$f = 639.3$$

The value of the option is therefore \$639.3.

This can also be calculated directly from equations (11.2) and (11.3). $u = 1.08$, $d = 0.92$ so that

$$p = \frac{e^{0.10 \times 2/12} - 0.92}{1.08 - 0.92} = 0.6050$$

and

$$f = e^{-0.10 \times 2/12}(0.6050 \times 729 + 0.3950 \times 529) = 639.3$$

Problem 11.15.

Calculate u , d , and p when a binomial tree is constructed to value an option on a foreign currency. The tree step size is one month, the domestic interest rate is 5% per annum, the foreign interest rate is 8% per annum, and the volatility is 12% per annum.

In this case

$$a = e^{(0.05 - 0.08) \times 1/12} = 0.9975$$

$$u = e^{0.12 \sqrt{1/12}} = 1.0352$$

$$d = 1/u = 0.9660$$

$$p = \frac{0.9975 - 0.9660}{1.0352 - 0.9660} = 0.4553$$

ASSIGNMENT QUESTIONS

Problem 11.16.

A stock price is currently \$50. It is known that at the end of six months it will be either \$60 or \$42. The risk-free rate of interest with continuous compounding is 12% per annum. Calculate the value of a six-month European call option on the stock with an exercise price of \$48. Verify that no-arbitrage arguments and risk-neutral valuation arguments give the same answers.

At the end of six months the value of the option will be either \$12 (if the stock price is \$60) or \$0 (if the stock price is \$42). Consider a portfolio consisting of:

$$\begin{aligned} +\Delta & : \text{ shares} \\ -1 & : \text{ option} \end{aligned}$$

The value of the portfolio is either 42Δ or $60\Delta - 12$ in six months. If

$$42\Delta = 60\Delta - 12$$

i.e.,

$$\Delta = 0.6667$$

the value of the portfolio is certain to be 28. For this value of Δ the portfolio is therefore riskless. The current value of the portfolio is:

$$0.6667 \times 50 - f$$

where f is the value of the option. Since the portfolio must earn the risk-free rate of interest

$$(0.6667 \times 50 - f)e^{0.12 \times 0.5} = 28$$

i.e.,

$$f = 6.96$$

The value of the option is therefore \$6.96.

This can also be calculated using risk-neutral valuation. Suppose that p is the probability of an upward stock price movement in a risk-neutral world. We must have

$$60p + 42(1 - p) = 50 \times e^{0.06}$$

i.e.,

$$18p = 11.09$$

or:

$$p = 0.6161$$

The expected value of the option in a risk-neutral world is:

$$12 \times 0.6161 + 0 \times 0.3839 = 7.3932$$

This has a present value of

$$7.3932e^{-0.06} = 6.96$$

Hence the above answer is consistent with risk-neutral valuation.

Problem 11.17.

A stock price is currently \$40. Over each of the next two three-month periods it is expected to go up by 10% or down by 10%. The risk-free interest rate is 12% per annum with continuous compounding.

- a. What is the value of a six-month European put option with a strike price of \$42?
- b. What is the value of a six-month American put option with a strike price of \$42?

(a) A tree describing the behavior of the stock price is shown in Figure M11.1. The risk-neutral probability of an up move, p , is given by

$$p = \frac{e^{0.12 \times 3/12} - 0.90}{1.1 - 0.9} = 0.6523$$

Calculating the expected payoff and discounting, we obtain the value of the option as

$$[2.4 \times 2 \times 0.6523 \times 0.3477 + 9.6 \times 0.3477^2]e^{-0.12 \times 6/12} = 2.118$$

The value of the European option is 2.118. This can also be calculated by working back through the tree as shown in Figure M11.1. The second number at each node is the value of the European option.

(b) The value of the American option is shown as the third number at each node on the tree. It is 2.537. This is greater than the value of the European option because it is optimal to exercise early at node C.

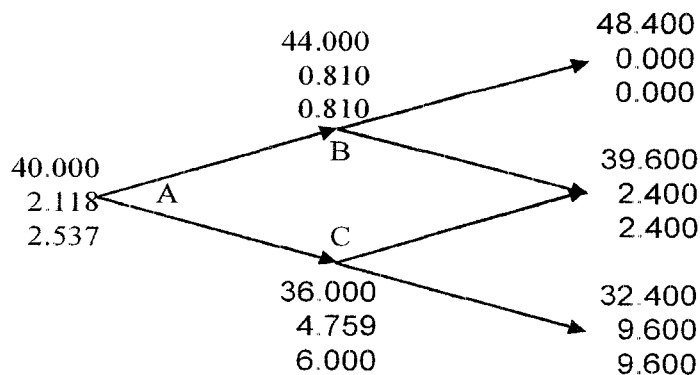


Figure M11.1 Tree to evaluate European and American put options in Problem 11.17. At each node, upper number is the stock price; next number is the European put price; final number is the American put price

Problem 11.18.

Using a “trial-and-error” approach, estimate how high the strike price has to be in Problem 11.17 for it to be optimal to exercise the option immediately.

Trial and error shows that immediate early exercise is optimal when the strike price is above 43.2.

This can be also shown to be true algebraically. Suppose the strike price increases by a relatively small amount q . This increases the value of being at node C by q and the value of being at node B by $0.3477e^{-0.03}q = 0.3374q$. It therefore increases the value of being at node A by

$$(0.6523 \times 0.3374q + 0.3477q)e^{-0.03} = 0.551q$$

For early exercise at node A we require $2.537 + 0.551q < 2 + q$ or $q > 1.196$. This corresponds to the strike price being greater than 43.196.

Problem 11.19.

A stock price is currently \$30. During each two-month period for the next four months it is expected to increase by 8% or reduce by 10%. The risk-free interest rate is 5%. Use a two-step tree to calculate the value of a derivative that pays off $[\max(30 - S_T, 0)]^2$ where S_T is the stock price in four months? If the derivative is American-style, should it be exercised early?

This type of option is known as a power option. A tree describing the behavior of the stock price is shown in Figure M11.2. The risk-neutral probability of an up move, p , is given by

$$p = \frac{e^{0.05 \times 2/12} - 0.9}{1.08 - 0.9} = 0.6020$$

Calculating the expected payoff and discounting, we obtain the value of the option as

$$[0.7056 \times 2 \times 0.6020 \times 0.3980 + 32.49 \times 0.3980^2]e^{-0.05 \times 4/12} = 5.394$$

The value of the European option is 5.394. This can also be calculated by working back through the tree as shown in Figure M11.2. The second number at each node is the value of the European option.

Early exercise at node C would give 9.0 which is less than 13.2449. The option should therefore not be exercised early if it is American.

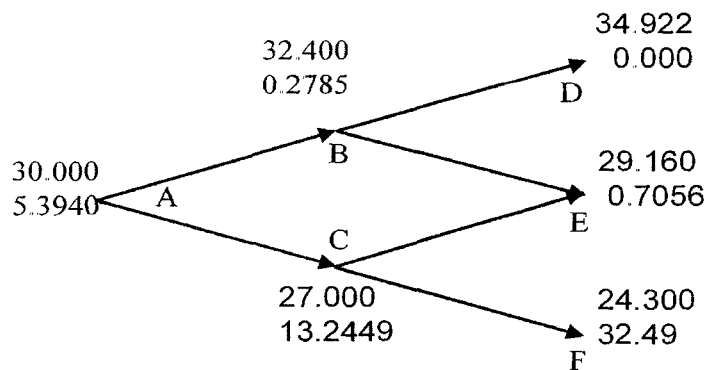


Figure M11.2 Tree to evaluate European power option in Problem 11.19.
At each node, upper number is the stock price; next number is the option price

Problem 11.20.

Consider a European call option on a non-dividend-paying stock where the stock price is \$40, the strike price is \$40, the risk-free rate is 4% per annum, the volatility is 30% per annum, and the time to maturity is six months.

- Calculate u , d , and p for a two step tree
- Value the option using a two step tree.
- Verify that DerivaGem gives the same answer
- Use DerivaGem to value the option with 5, 50, 100, and 500 time steps.

(a) This problem is based on the material in Section 11.8. In this case $\Delta t = 0.25$ so that $u = e^{0.30 \times \sqrt{0.25}} = 1.1618$, $d = 1/u = 0.8607$, and

$$p = \frac{e^{0.04 \times 0.25} - 0.8607}{1.1618 - 0.8607} = 0.4959$$

(b) and (c) The value of the option using a two-step tree as given by DerivaGem is shown in Figure M11.3 to be 3.3739. To use DerivaGem choose the first worksheet, select Equity as the underlying type, and select Binomial European as the Option Type. After carrying out the calculations select Display Tree.

(d) With 5, 50, 100, and 500 time steps the value of the option is 3.9229, 3.7394, 3.7478, and 3.7545, respectively.

At each node:
 Upper value = Underlying Asset Price
 Lower value = Option Price
 Values in red are a result of early exercise.

Strike price = 40
 Discount factor per step = 0.9900
 Time step, dt = 0.2500 years, 91.25 days
 Growth factor per step, a = 1.0101
 Probability of up move, p = 0.4959
 Up step size, u = 1.1618
 Down step size, d = 0.8607

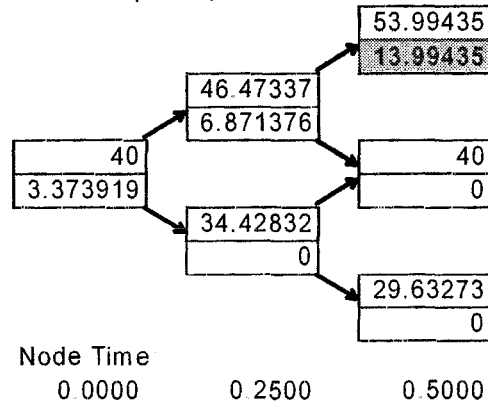


Figure M11.3 Tree produced by DerivaGem to evaluate European option in Problem 11.20.

Problem 11.21.

Repeat Problem 11.20 for an American put option on a futures contract. The strike price and the futures price are \$50, the risk-free rate is 10%, the time to maturity is six months, and the volatility is 40% per annum.

(a) In this case $\Delta t = 0.25$ and $u = e^{0.40 \times \sqrt{0.25}} = 1.2214$, $d = 1/u = 0.8187$, and

$$p = \frac{e^{0.1 \times 0.25} - 0.8187}{1.2214 - 0.8187} = 0.4502$$

(b) and (c) The value of the option using a two-step tree is 4.8604.

(d) With 5, 50, 100, and 500 time steps the value of the option is 5.6858, 5.3869, 5.3981, and 5.4072, respectively.

Problem 11.22.

Footnote 1 shows that the correct discount rate to use for the real world expected payoff in the case of the call option considered in Figure 11.1 is 42.6%. Show that if the option is a put rather than a call the discount rate is -52.5%. Explain why the two real-world discount rates are so different.

The value of the put option is

$$(0.6523 \times 0 + 0.3477 \times 3)e^{-0.12 \times 3/12} = 1.0123$$

The expected payoff in the real world is

$$(0.7041 \times 0 + 0.2959 \times 3) = 0.8877$$

The discount rate R that should be used in the real world is therefore given by solving

$$1.0123 = 0.8877e^{-0.25R}$$

The solution to this is -0.525 or 52.5%.

The call option has a high positive discount rate because it has high positive systematic risk. The put option has a high negative discount rate because it has high negative systematic risk.

CHAPTER 12

Wiener processes and Itô's Lemma

Notes for the Instructor

The chapter provides some basic knowledge about Wiener processes, develops the geometric Brownian motion model of stock price behavior, and covers Itô's lemma. The book has been designed so that this chapter can be skipped if the instructor considers it too technical. For example, a popular way of using the book for a first MBA elective in derivatives is to assign the first 18 chapters, excluding Chapter 12 and Section 13.6 of Chapter 13.

I find that most students have surprisingly little difficulty with the material in Chapter 12. I usually start by discussing the distinction between continuous time and discrete time stochastic processes and the distinction between continuous variable and discrete variable stochastic processes. I do this with simple examples of models of stock price movements. A discrete time, discrete variable model would be one where every day a stock price has a probability p_1 of moving up by \$1, a probability p_2 of remaining the same, and a probability p_3 of moving down by \$1. A continuous time, discrete variable model would be one where price changes of \$1 are generated by a Poisson process. A discrete time, continuous variable model would be one where in each small time interval, stock price movements are sampled from a continuous distribution. The main purpose of the chapter is to develop a continuous time, continuous variable model as a limiting case of this last model.

The nature of Markov processes and the fact that they are consistent with market efficiency needs to be explained carefully. I find it useful to discuss Problems 12.1 and 12.2 in class.

I explain Wiener processes by starting with a discrete time, continuous variable model where values of the variable are observed every year. I assume that the change in the value of the variable during the year is a random sample from the distribution $\phi(0, 1)$. (Note that the second argument of ϕ is the variance not the standard deviation.) I then suppose that values of the variable are observed every 6 months and ask what distribution for 6-month changes is consistent with the distribution of 1-year changes. The answer is $\phi(0, \frac{1}{2})$. When 3-month changes are considered, the distribution is $\phi(0, 0.25)$. When time intervals of length Δt are considered, the distribution is $\phi(0, \Delta t)$. A random sample from $\phi(0, \Delta t)$ is $\epsilon\sqrt{\Delta t}$. This explains where the definition of a Wiener process comes from. In particular it explains why we take the square root of time. Once Wiener processes are understood, generalized Wiener processes and Itô processes should not cause too much difficulty.

In understanding a stochastic process, I find that an explanation of how it can be simulated is useful and usually go through an example such as that in Section 12.3.

The amount of time spent on Itô's lemma will depend on the mathematical backgrounds of students. Mathematically inclined students generally feel quite a sense of achievement when they have managed to work through the material in the appendix to Chapter 12.

All of Problems 12.12 to 12.16 work well as assignment questions.

QUESTIONS AND PROBLEMS

Problem 12.1.

What would it mean to assert that the temperature at a certain place follows a Markov process? Do you think that temperatures do, in fact, follow a Markov process?

Imagine that you have to forecast the future temperature from a) the current temperature, b) the history of the temperature in the last week, and c) a knowledge of seasonal averages and seasonal trends. If temperature followed a Markov process, the history of the temperature in the last week would be irrelevant.

To answer the second part of the question you might like to consider the following scenario for the first week in May:

- (i) Monday to Thursday are warm days; today, Friday, is a very cold day.
- (ii) Monday to Friday are all very cold days.

What is your forecast for the weekend? If you are more pessimistic in the case of the second scenario, temperatures do not follow a Markov process.

Problem 12.2.

Can a trading rule based on the past history of a stock's price ever produce returns that are consistently above average? Discuss.

The first point to make is that any trading strategy can, just because of good luck, produce above average returns. The key question is whether a trading strategy *consistently* outperforms the market when adjustments are made for risk. It is certainly possible that a trading strategy could do this. However, when enough investors know about the strategy and trade on the basis of the strategy, the profit will disappear.

As an illustration of this, consider a phenomenon known as the small firm effect. Portfolios of stocks in small firms appear to have outperformed portfolios of stocks in large firms when appropriate adjustments are made for risk. Papers were published about this in the early 1980s and mutual funds were set up to take advantage of the phenomenon. There is some evidence that this has resulted in the phenomenon disappearing.

Problem 12.3.

A company's cash position, measured in millions of dollars, follows a generalized Wiener process with a drift rate of 0.5 per quarter and a variance rate of 4.0 per quarter. How high does the company's initial cash position have to be for the company to have a less than 5% chance of a negative cash position by the end of one year?

Suppose that the company's initial cash position is x . The probability distribution of the cash position at the end of one year is

$$\phi(x + 4 \times 0.5, 4 \times 4) = \phi(x + 2.0, 16)$$

where $\phi(m, v)$ is a normal probability distribution with mean m and variance v . The probability of a negative cash position at the end of one year is

$$N\left(-\frac{x + 2.0}{4}\right)$$

where $N(x)$ is the cumulative probability that a standardized normal variable (with mean zero and standard deviation 1.0) is less than x . From normal distribution tables

$$N\left(-\frac{x + 2.0}{4}\right) = 0.05$$

when:

$$-\frac{x + 2.0}{4} = -1.6449$$

i.e., when $x = 4.5796$. The initial cash position must therefore be \$4.58 million.

Problem 12.4.

Variables X_1 and X_2 follow generalized Wiener processes with drift rates μ_1 and μ_2 and variances σ_1^2 and σ_2^2 . What process does $X_1 + X_2$ follow if:

(a) The changes in X_1 and X_2 in any short interval of time are uncorrelated?

(b) There is a correlation ρ between the changes in X_1 and X_2 in any short interval of time?

(a) Suppose that X_1 and X_2 equal a_1 and a_2 initially. After a time period of length T , X_1 has the probability distribution

$$\phi(a_1 + \mu_1 T, \sigma_1^2 T)$$

and X_2 has a probability distribution

$$\phi(a_2 + \mu_2 T, \sigma_2^2 T)$$

From the property of sums of independent normally distributed variables, $X_1 + X_2$ has the probability distribution

$$\phi(a_1 + \mu_1 T + a_2 + \mu_2 T, \sigma_1^2 T + \sigma_2^2 T)$$

i.e.,

$$\phi[a_1 + a_2 + (\mu_1 + \mu_2)T, (\sigma_1^2 + \sigma_2^2)T]$$

This shows that $X_1 + X_2$ follows a generalized Wiener process with drift rate $\mu_1 + \mu_2$ and variance rate $\sigma_1^2 + \sigma_2^2$.

(b) In this case the change in the value of $X_1 + X_2$ in a short interval of time Δt has the probability distribution:

$$\phi[(\mu_1 + \mu_2)\Delta t, (\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2)\Delta t]$$

If $\mu_1, \mu_2, \sigma_1, \sigma_2$ and ρ are all constant, arguments similar to those in Section 12.2 show that the change in a longer period of time T is

$$\phi [(\mu_1 + \mu_2)T, (\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2)T]$$

The variable, $X_1 + X_2$, therefore follows a generalized Wiener process with drift rate $\mu_1 + \mu_2$ and variance rate $\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2$.

Problem 12.5.

Consider a variable, S , that follows the process

$$dS = \mu dt + \sigma dz$$

For the first three years, $\mu = 2$ and $\sigma = 3$; for the next three years, $\mu = 3$ and $\sigma = 4$. If the initial value of the variable is 5, what is the probability distribution of the value of the variable at the end of year six?

The change in S during the first three years has the probability distribution

$$\phi(2 \times 3, 9 \times 3) = \phi(6, 27)$$

The change during the next three years has the probability distribution

$$\phi(3 \times 3, 16 \times 3) = \phi(9, 48)$$

The change during the six years is the sum of a variable with probability distribution $\phi(6, 27)$ and a variable with probability distribution $\phi(9, 48)$. The probability distribution of the change is therefore

$$\begin{aligned} \phi(6 + 9, 27 + 48) \\ = \phi(15, 75) \end{aligned}$$

Since the initial value of the variable is 5, the probability distribution of the value of the variable at the end of year six is

$$\phi(20, 75)$$

Problem 12.6.

Suppose that G is a function of a stock price, S and time. Suppose that σ_S and σ_G are the volatilities of S and G . Show that when the expected return of S increases by $\lambda\sigma_S$, the growth rate of G increases by $\lambda\sigma_G$, where λ is a constant.

From Itô's lemma

$$\sigma_G G = \frac{\partial G}{\partial S} \sigma_S S$$

Also the drift of G is

$$\frac{\partial G}{\partial S} \mu S + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial S^2} \sigma^2 S^2$$

where μ is the expected return on the stock. When μ increases by $\lambda\sigma_S$, the drift of G increases by

$$\frac{\partial G}{\partial S} \lambda \sigma_S S$$

or

$$\lambda \sigma_G G$$

The growth rate of G , therefore, increases by $\lambda\sigma_G$.

Problem 12.7.

Stock A and stock B both follow geometric Brownian motion. Changes in any short interval of time are uncorrelated with each other. Does the value of a portfolio consisting of one of stock A and one of stock B follow geometric Brownian motion? Explain your answer.

Define S_A , μ_A and σ_A as the stock price, expected return and volatility for stock A. Define S_B , μ_B and σ_B as the stock price, expected return and volatility for stock B. Define ΔS_A and ΔS_B as the change in S_A and S_B in time Δt . Since each of the two stocks follows geometric Brownian motion,

$$\Delta S_A = \mu_A S_A \Delta t + \sigma_A S_A \epsilon_A \sqrt{\Delta t}$$

$$\Delta S_B = \mu_B S_B \Delta t + \sigma_B S_B \epsilon_B \sqrt{\Delta t}$$

where ϵ_A and ϵ_B are independent random samples from a normal distribution.

$$\Delta S_A + \Delta S_B = (\mu_A S_A + \mu_B S_B) \Delta t + (\sigma_A S_A \epsilon_A + \sigma_B S_B \epsilon_B) \sqrt{\Delta t}$$

This *cannot* be written as

$$\Delta S_A + \Delta S_B = \mu(S_A + S_B) \Delta t + \sigma(S_A + S_B) \epsilon \sqrt{\Delta t}$$

for any constants μ and σ . (Neither the drift term nor the stochastic term correspond.) Hence the value of the portfolio does not follow geometric Brownian motion.

Problem 12.8.

The process for the stock price in equation (12.8) is

$$\Delta S = \mu S \Delta t + \sigma S \epsilon \sqrt{\Delta t}$$

where μ and σ are constant. Explain carefully the difference between this model and each of the following:

$$\Delta S = \mu \Delta t + \sigma \epsilon \sqrt{\Delta t}$$

$$\Delta S = \mu S \Delta t + \sigma \epsilon \sqrt{\Delta t}$$

$$\Delta S = \mu \Delta t + \sigma S \epsilon \sqrt{\Delta t}$$

Why is the model in equation (12.8) a more appropriate model of stock price behavior than any of these three alternatives?

In:

$$\Delta S = \mu S \Delta t + \sigma S \epsilon \sqrt{\Delta t}$$

the expected increase in the stock price and the variability of the stock price are constant when both are expressed as a proportion (or as a percentage) of the stock price

In:

$$\Delta S = \mu \Delta t + \sigma \epsilon \sqrt{\Delta t}$$

the expected increase in the stock price and the variability of the stock price are constant in absolute terms. For example, if the expected growth rate is \$5 per annum when the stock price is \$25, it is also \$5 per annum when it is \$100. If the standard deviation of weekly stock price movements is \$1 when the price is \$25, it is also \$1 when the price is \$100.

In:

$$\Delta S = \mu S \Delta t + \sigma \epsilon \sqrt{\Delta t}$$

the expected increase in the stock price is a constant proportion of the stock price while the variability is constant in absolute terms.

In:

$$\Delta S = \mu \Delta t + \sigma S \epsilon \sqrt{\Delta t}$$

the expected increase in the stock price is constant in absolute terms while the variability of the proportional stock price change is constant.

The model:

$$\Delta S = \mu S \Delta t + \sigma S \epsilon \sqrt{\Delta t}$$

is the most appropriate one since it is most realistic to assume that the expected *percentage return* and the variability of the *percentage return* in a short interval are constant.

Problem 12.9.

It has been suggested that the short-term interest rate, r , follows the stochastic process

$$dr = a(b - r) dt + rc dz$$

where a , b , and c are positive constants and dz is a Wiener process. Describe the nature of this process.

The drift rate is $a(b - r)$. Thus, when the interest rate is above b the drift rate is negative and, when the interest rate is below b , the drift rate is positive. The interest rate is therefore continually pulled towards the level b . The rate at which it is pulled toward this level is a . A volatility equal to c is superimposed upon the "pull" or the drift.

Suppose $a = 0.4$, $b = 0.1$ and $c = 0.15$ and the current interest rate is 20% per annum. The interest rate is pulled towards the level of 10% per annum. This can be regarded as a long run average. The current drift is -4% per annum so that the expected rate at the end of one year is about 16% per annum. (In fact it is slightly greater than this, because as the

interest rate decreases, the “pull” decreases.) Superimposed upon the drift is a volatility of 15% per annum.

Problem 12.10.

Suppose that a stock price, S , follows geometric Brownian motion with expected return μ and volatility σ :

$$dS = \mu S dt + \sigma S dz$$

What is the process followed by the variable S^n ? Show that S^n also follows geometric Brownian motion.

If $G(S, t) = S^n$ then $\partial G/\partial t = 0$, $\partial G/\partial S = nS^{n-1}$, and $\partial^2 G/\partial S^2 = n(n-1)S^{n-2}$. Using Itô's lemma:

$$dG = [\mu nG + \frac{1}{2}n(n-1)\sigma^2 G] dt + \sigma nG dz$$

This shows that $G = S^n$ follows geometric Brownian motion where the expected return is

$$\mu n + \frac{1}{2}n(n-1)\sigma^2$$

and the volatility is $n\sigma$. The stock price S has an expected return of μ and the expected value of S_T is $S_0 e^{\mu T}$. The expected value of S_T^n is

$$S_0^n e^{[\mu n + \frac{1}{2}n(n-1)\sigma^2]T}$$

Problem 12.11.

Suppose that x is the yield to maturity with continuous compounding on a zero-coupon bond that pays off \$1 at time T . Assume that x follows the process

$$dx = a(x_0 - x) dt + sx dz$$

where a , x_0 , and s are positive constants and dz is a Wiener process. What is the process followed by the bond price?

The process followed by B , the bond price, is from Itô's lemma:

$$dB = \left[\frac{\partial B}{\partial x} a(x_0 - x) + \frac{\partial B}{\partial t} + \frac{1}{2} \frac{\partial^2 B}{\partial x^2} s^2 x^2 \right] dt + \frac{\partial B}{\partial x} s x dz$$

Since:

$$B = e^{-x(T-t)}$$

the required partial derivatives are

$$\begin{aligned} \frac{\partial B}{\partial t} &= x e^{-x(T-t)} = xB \\ \frac{\partial B}{\partial x} &= -(T-t)e^{-x(T-t)} = -(T-t)B \\ \frac{\partial^2 B}{\partial x^2} &= (T-t)^2 e^{-x(T-t)} = (T-t)^2 B \end{aligned}$$

Hence:

$$dB = \left[-a(x_0 - x)(T - t) + x + \frac{1}{2}s^2x^2(T - t)^2 \right] Bdt - sx(T - t)Bdz$$

ASSIGNMENT QUESTIONS

Problem 12.12.

Suppose that a stock price has an expected return of 16% per annum and a volatility of 30% per annum. When the stock price at the end of a certain day is \$50, calculate the following:

- The expected stock price at the end of the next day.
- The standard deviation of the stock price at the end of the next day.
- The 95% confidence limits for the stock price at the end of the next day.

With the notation in the text

$$\frac{\Delta S}{S} \sim \phi(\mu\Delta t, \sigma^2\Delta t)$$

In this case $S = 50$, $\mu = 0.16$, $\sigma = 0.30$ and $\Delta t = 1/365 = 0.00274$. Hence

$$\begin{aligned} \frac{\Delta S}{50} &\sim \phi(0.16 \times 0.00274, 0.09 \times 0.00274) \\ &= \phi(0.00044, 0.000247) \end{aligned}$$

and

$$\Delta S \sim \phi(50 \times 0.00044, 50^2 \times 0.000247)$$

that is,

$$\Delta S \sim \phi(0.022, 0.6164)$$

- The expected stock price at the end of the next day is therefore 50.022
- The standard deviation of the stock price at the end of the next day is $\sqrt{0.6164} = 0.785$
- 95% confidence limits for the stock price at the end of the next day are

$$50.022 - 1.96 \times 0.785 \quad \text{and} \quad 50.022 + 1.96 \times 0.785$$

i.e.,

$$48.48 \quad \text{and} \quad 51.56$$

Note that some students may consider one trading day rather than one calendar day. Then $\Delta t = 1/252 = 0.00397$. The answer to (a) is then 50.032. The answer to (b) is 0.945. The answers to part (c) are 48.18 and 51.88.

Problem 12.13.

A company's cash position, measured in millions of dollars, follows a generalized Wiener process with a drift rate of 0.1 per month and a variance rate of 0.16 per month. The initial cash position is 2.0.

- (a) What are the probability distributions of the cash position after one month, six months, and one year?
 (b) What are the probabilities of a negative cash position at the end of six months and one year?
 (c) At what time in the future is the probability of a negative cash position greatest?

(a) The probability distributions are:

$$\phi(2.0 + 0.1, 0.16) = \phi(2.1, 0.16)$$

$$\phi(2.0 + 0.6, 0.16 \times 6) = \phi(2.6, 0.96)$$

$$\phi(2.0 + 1.2, 0.16 \times 12) = \phi(3.2, 1.96)$$

(b) The chance of a random sample from $\phi(2.6, 0.96)$ being negative is

$$N\left(-\frac{2.6}{\sqrt{0.96}}\right) = N(-2.65)$$

where $N(x)$ is the cumulative probability that a standardized normal variable [i.e., a variable with probability distribution $\phi(0, 1)$] is less than x . From normal distribution tables $N(-2.65) = 0.0040$. Hence the probability of a negative cash position at the end of six months is 0.40%.

Similarly the probability of a negative cash position at the end of one year is

$$N\left(-\frac{3.2}{\sqrt{1.96}}\right) = N(-2.30) = 0.0107$$

or 1.07%.

(c) In general the probability distribution of the cash position at the end of x months is

$$\phi(2.0 + 0.1x, 0.16x)$$

The probability of the cash position being negative is maximized when:

$$\frac{2.0 + 0.1x}{\sqrt{0.16x}}$$

is minimized. Define

$$\begin{aligned} y &= \frac{2.0 + 0.1x}{0.4\sqrt{x}} = 5x^{-\frac{1}{2}} + 0.25x^{\frac{1}{2}} \\ \frac{dy}{dx} &= -2.5x^{-\frac{3}{2}} + 0.125x^{-\frac{1}{2}} \\ &= x^{-\frac{3}{2}}(-2.5 + 0.125x) \end{aligned}$$

This is zero when $x = 20$ and it is easy to verify that $d^2y/dx^2 > 0$ for this value of x . It therefore gives a minimum value for y . Hence the probability of a negative cash position is greatest after 20 months.

Problem 12.14.

Suppose that x is the yield on a perpetual government bond that pays interest at the rate of \$1 per annum. Assume that x is expressed with continuous compounding, that interest is paid continuously on the bond, and that x follows the process

$$dx = a(x_0 - x) dt + sx dz$$

where a , x_0 , and s are positive constants and dz is a Wiener process. What is the process followed by the bond price? What is the expected instantaneous return (including interest and capital gains) to the holder of the bond?

The process followed by B , the bond price, is from Itô's lemma:

$$dB = \left[\frac{\partial B}{\partial x} a(x_0 - x) + \frac{\partial B}{\partial t} + \frac{1}{2} \frac{\partial^2 B}{\partial x^2} s^2 x^2 \right] dt + \frac{\partial B}{\partial x} s x dz$$

In this case

$$B = \frac{1}{x}$$

so that:

$$\frac{\partial B}{\partial t} = 0; \quad \frac{\partial B}{\partial x} = -\frac{1}{x^2}; \quad \frac{\partial^2 B}{\partial x^2} = \frac{2}{x^3}$$

Hence

$$\begin{aligned} dB &= \left[-a(x_0 - x) \frac{1}{x^2} + \frac{1}{2} s^2 x^2 \frac{2}{x^3} \right] dt - \frac{1}{x^2} s x dz \\ &= \left[-a(x_0 - x) \frac{1}{x^2} + \frac{s^2}{x} \right] dt - \frac{s}{x} dz \end{aligned}$$

The expected instantaneous rate at which capital gains are earned from the bond is therefore:

$$-a(x_0 - x) \frac{1}{x^2} + \frac{s^2}{x}$$

The expected interest per unit time is 1. The total expected instantaneous return is therefore:

$$1 - a(x_0 - x) \frac{1}{x^2} + \frac{s^2}{x}$$

When expressed as a proportion of the bond price this is:

$$\begin{aligned} &\left(1 - a(x_0 - x) \frac{1}{x^2} + \frac{s^2}{x} \right) / \left(\frac{1}{x} \right) \\ &= x - \frac{a}{x} (x_0 - x) + s^2 \end{aligned}$$

Problem 12.15.

If S follows the geometric Brownian motion process in equation (12.6), what is the process followed by

- a. $y = 2S$
- b. $y = S^2$
- c. $y = e^S$
- d. $y = \frac{e^{r(T-t)}}{S}$

In each case express the coefficients of dt and dz in terms of y rather than S .

- (a) In this case $\partial y/\partial S = 2$, $\partial^2 y/\partial S^2 = 0$, and $\partial y/\partial t = 0$ so that Itô's lemma gives

$$dy = 2\mu S dt + 2\sigma S dz$$

or

$$dy = \mu y dt + \sigma y dz$$

- (b) In this case $\partial y/\partial S = 2S$, $\partial^2 y/\partial S^2 = 2$, and $\partial y/\partial t = 0$ so that Itô's lemma gives

$$dy = (2\mu S^2 + \sigma^2 S^2) dt + 2\sigma S^2 dz$$

or

$$dy = (2\mu + \sigma^2)y dt + 2\sigma y dz$$

- (c) In this case $\partial y/\partial S = e^S$, $\partial^2 y/\partial S^2 = e^S$, and $\partial y/\partial t = 0$ so that Itô's lemma gives

$$dy = (\mu S e^S + \sigma^2 S^2 e^S / 2) dt + \sigma S e^S dz$$

or

$$dy = [\mu y \ln y + \sigma^2 y (\ln y)^2 / 2] dt + \sigma y \ln y dz$$

- (d) In this case $\partial y/\partial S = -e^{r(T-t)}/S^2 = -y/S$, $\partial^2 y/\partial S^2 = 2e^{r(T-t)}/S^3 = 2y/S^2$, and $\partial y/\partial t = -re^{r(T-t)}/S = -ry$ so that Itô's lemma gives

$$dy = (-ry - \mu y + \sigma^2 y) dt - \sigma y dz$$

or

$$dy = -(r + \mu - \sigma^2)y dt - \sigma y dz$$

Problem 12.16.

A stock price is currently 50. Its expected return and volatility are 12% and 30%, respectively. What is the probability that the stock price will be greater than 80 in two years? (Hint $S_T > 80$ when $\ln S_T > \ln 80$.)

The variable $\ln S_T$ is normally distributed with mean $\ln S_0 + (\mu - \sigma^2/2)T$ and standard deviation $\sigma\sqrt{T}$. In this case $S_0 = 50$, $\mu = 0.12$, $T = 2$, and $\sigma = 0.30$ so that the mean and standard deviation of $\ln S_T$ are $\ln 50 + (0.12 - 0.3^2/2)2 = 4.062$ and $0.3\sqrt{2} = 0.424$, respectively. Also, $\ln 80 = 4.382$. The probability that $S_T > 80$ is the same as the probability that $\ln S_T > 4.382$. This is

$$1 - N\left(\frac{4.382 - 4.062}{0.424}\right) = 1 - N(0.754)$$

where $N(x)$ is the probability that a normally distributed variable with mean zero and standard deviation 1 is less than x . From the tables at the back of the book $N(0.754) = 0.775$ so that the required probability is 0.225.

CHAPTER 13

The Black–Scholes–Merton Model

Notes for the Instructor

This chapter covers important material: the lognormality of stock prices, the calculation of volatility from historical data, the Black–Scholes–Merton differential equation, risk-neutral valuation, the Black–Scholes–Merton option pricing formulas, implied volatilities, and the impact of dividends. Section 13.6 should be skipped if Chapter 12 has not already been covered.

The distinction between

μ : the expected rate of return in a short period of time, and

$\mu - \sigma^2/2$: the expected continuously compounded rate of return over any period of time usually causes some problems. I have tried a few different approaches and think that the one that is now in the text works reasonably well.

Business Snapshot 13.2 on the causes of volatility generally leads to a lively discussion. I find that students have an easier time than academics in accepting that trading itself causes volatility!

When presenting Black–Scholes–Merton arguments I point out that in any small interval of time Δt , the stock price and the option price are perfectly correlated. This is the same as saying that the ratio $\Delta c/\Delta S$ is constant where Δc and ΔS are the change in c and S in time Δt respectively. It is possible to set up a portfolio consisting of a position in the derivative and a position in the stock which is, for the next small interval of time Δt , riskless. (For example, if

$$\frac{\Delta c}{\Delta S} = 0.4$$

a short position in 100 of the derivative security when combined with a long position in 40 of the stock is riskless for time Δt .) This is essentially what Black, Scholes, and Merton did to derive their differential equation. After presenting the Black–Scholes–Merton differential equation I like to go through Example 13.5 on forward contracts in class. Later the same example can be used to illustrate risk-neutral valuation.

The risk-neutral valuation argument must be covered carefully. It cannot be emphasized often enough that we are not assuming risk neutrality. It just happens that the value of a derivative security is independent of risk preferences.

When the Black–Scholes–Merton equation for pricing a call option is presented, students sometimes ask for the intuition behind it and are frustrated that they cannot easily derive it. I point out that a European call option holder gets $S_T - K$ whenever $S_T > K$. This means that the option holder is long a security that pays off S_T when $S_T > K$ and short a security that pays off K when $S_T > K$. The first security is known as an asset-or-nothing call. The second security is known as a cash-or-nothing call. The probability that $S_T > K$ in a risk-neutral world is $N(d_2)$. (See Problem 13.22). The expected payoff from the second security in a risk-neutral world is therefore $KN(d_2)$. From risk-neutral

valuation, the value of the security is $KN(d_2)e^{-rT}$. The value of the first security can also be calculated using risk-neutral valuation. It turns out to be $S_0N(d_1)$. (Students have to take this on faith). Putting the two results together we get the Black–Scholes–Merton formula for a European call option.

When calculating the cumulative normal distribution function, most students will choose to use the table at the end of the book or the Excel function NORMSDIST. The polynomial approximation may be useful if they choose to write their own software. I encourage students to develop their own Excel worksheets for option pricing as well as using DerivaGem.

All the assignment questions work well. My favorite is 13.28.

QUESTIONS AND PROBLEMS

Problem 13.1.

What does the Black–Scholes–Merton stock option pricing model assume about the probability distribution of the stock price in one year? What does it assume about the continuously compounded rate of return on the stock during the year?

The Black–Scholes–Merton option pricing model assumes that the probability distribution of the stock price in 1 year (or at any other future time) is lognormal. It assumes that the continuously compounded rate of return on the stock during the year is normally distributed.

Problem 13.2.

The volatility of a stock price is 30% per annum. What is the standard deviation of the percentage price change in one trading day?

The standard deviation of the percentage price change in time Δt is $\sigma\sqrt{\Delta t}$ where σ is the volatility. In this problem $\sigma = 0.3$ and, assuming 252 trading days in one year, $\Delta t = 1/252 = 0.004$ so that $\sigma\sqrt{\Delta t} = 0.3\sqrt{0.004} = 0.019$ or 1.9%.

Problem 13.3.

Explain the principle of risk-neutral valuation.

The price of an option or other derivative when expressed in terms of the price of the underlying stock is independent of risk preferences. Options therefore have the same value in a risk-neutral world as they do in the real world. We may therefore assume that the world is risk neutral for the purposes of valuing options. This simplifies the analysis. In a risk-neutral world all securities have an expected return equal to risk-free interest rate. Also, in a risk-neutral world, the appropriate discount rate to use for expected future cash flows is the risk-free interest rate.

Problem 13.4.

Calculate the price of a three-month European put option on a non-dividend-paying stock with a strike price of \$50 when the current stock price is \$50, the risk-free interest rate is 10% per annum, and the volatility is 30% per annum.

In this case $S_0 = 50$, $K = 50$, $r = 0.1$, $\sigma = 0.3$, $T = 0.25$, and

$$d_1 = \frac{\ln(50/50) + (0.1 + 0.09/2)0.25}{0.3\sqrt{0.25}} = 0.2417$$

$$d_2 = d_1 - 0.3\sqrt{0.25} = 0.0917$$

The European put price is

$$50N(-0.0917)e^{-0.1 \times 0.25} - 50N(-0.2417)$$

$$= 50 \times 0.4634e^{-0.1 \times 0.25} - 50 \times 0.4045 = 2.37$$

or \$2.37.

Problem 13.5.

What difference does it make to your calculations in Problem 13.4 if a dividend of \$1.50 is expected in two months?

In this case we must subtract the present value of the dividend from the stock price before using Black-Scholes. Hence the appropriate value of S_0 is

$$S_0 = 50 - 1.50e^{-0.1667 \times 0.1} = 48.52$$

As before $K = 50$, $r = 0.1$, $\sigma = 0.3$, and $T = 0.25$. In this case

$$d_1 = \frac{\ln(48.52/50) + (0.1 + 0.09/2)0.25}{0.3\sqrt{0.25}} = 0.0414$$

$$d_2 = d_1 - 0.3\sqrt{0.25} = -0.1086$$

The European put price is

$$50N(0.1086)e^{-0.1 \times 0.25} - 48.52N(-0.0414)$$

$$= 50 \times 0.5432e^{-0.1 \times 0.25} - 48.52 \times 0.4835 = 3.03$$

or \$3.03.

Problem 13.6.

What is implied volatility? How can it be calculated?

The implied volatility is the volatility that makes the Black-Scholes price of an option equal to its market price. It is calculated using an iterative procedure.

Problem 13.7.

A stock price is currently \$40. Assume that the expected return from the stock is 15% and that its volatility is 25%. What is the probability distribution for the rate of return (with continuous compounding) earned over a two-year period?

In this case $\mu = 0.15$ and $\sigma = 0.25$. From equation (13.7) the probability distribution for the rate of return over a 2-year period with continuous compounding is:

$$\phi\left(0.15 - \frac{0.25^2}{2}, \frac{0.25^2}{2}\right)$$

i.e.,

$$\phi(0.11875, 0.03125)$$

The expected value of the return is 11.875% per annum and the standard deviation is $\sqrt{0.03125}$ or 17.68% per annum.

Problem 13.8.

A stock price follows geometric Brownian motion with an expected return of 16% and a volatility of 35%. The current price is \$38.

(a) What is the probability that a European call option on the stock with an exercise price of \$40 and a maturity date in six months will be exercised?

(b) What is the probability that a European put option on the stock with the same exercise price and maturity will be exercised?

(a) The required probability is the probability of the stock price being above \$40 in six months' time. Suppose that the stock price in six months is S_T

$$\ln S_T \sim \phi\left(\ln 38 + \left(0.16 - \frac{0.35^2}{2}\right)0.5, 0.35^2 \times 0.5\right)$$

i.e.,

$$\ln S_T \sim \phi(3.687, 0.06125)$$

Since $\ln 40 = 3.689$, the required probability is

$$1 - N\left(\frac{3.689 - 3.687}{\sqrt{0.06125}}\right) = 1 - N(0.008)$$

From normal distribution tables $N(0.008) = 0.5032$ so that the required probability is 0.4968. In general the required probability is $N(d_2)$. (See Problem 13.22).

(b) In this case the required probability is the probability of the stock price being less than \$40 in six months' time. It is

$$1 - 0.4968 = 0.5032$$

Problem 13.9.

Prove that with the notation in the chapter, a 95% confidence interval for S_T is between

$$S_0 e^{(\mu - \sigma^2/2)T - 1.96\sigma\sqrt{T}} \quad \text{and} \quad S_0 e^{(\mu - \sigma^2/2)T + 1.96\sigma\sqrt{T}}$$

From equation (13.3):

$$\ln S_T \sim \phi\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right]$$

95% confidence intervals for $\ln S_T$ are therefore

$$\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T - 1.96\sigma\sqrt{T}$$

and

$$\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T + 1.96\sigma\sqrt{T}$$

95% confidence intervals for S_T are therefore

$$e^{\ln S_0 + (\mu - \sigma^2/2)T - 1.96\sigma\sqrt{T}}$$

and

$$e^{\ln S_0 + (\mu - \sigma^2/2)T + 1.96\sigma\sqrt{T}}$$

i.e.

$$S_0 e^{(\mu - \sigma^2/2)T - 1.96\sigma\sqrt{T}}$$

and

$$S_0 e^{(\mu - \sigma^2/2)T + 1.96\sigma\sqrt{T}}$$

Problem 13.10.

A portfolio manager announces that the average of the returns realized in each year of the last 10 years is 20% per annum. In what respect is this statement misleading?

The statement is misleading in that a certain sum of money, say \$1000, when invested for 10 years in the fund would have realized a return (with annual compounding) of less than 20% per annum.

The average of the returns realized in each year is always greater than the return per annum (with annual compounding) realized over 10 years. The first is an arithmetic average of the returns in each year; the second is a geometric average of these returns.

Problem 13.11.

Assume that a non-dividend-paying stock has an expected return of μ and a volatility of σ . An innovative financial institution has just announced that it will trade a security

that pays off a dollar amount equal to $\ln S_T$ at time T where S_T denotes the value of the stock price at time T .

(a) Use risk-neutral valuation to calculate the price of the security at time t in terms of the stock price, S , at time t .

(b) Confirm that your price satisfies the differential equation (13.16).

(a) At time t , the expected value of $\ln S_T$ is, from equation (13.3)

$$\ln S + \left(\mu - \frac{\sigma^2}{2}\right)(T - t)$$

In a risk-neutral world the expected value of $\ln S_T$ is therefore:

$$\ln S + \left(r - \frac{\sigma^2}{2}\right)(T - t)$$

Using risk-neutral valuation the value of the security at time t is:

$$e^{-r(T-t)} \left[\ln S + \left(r - \frac{\sigma^2}{2}\right)(T - t) \right]$$

(b) If:

$$\begin{aligned} f &= e^{-r(T-t)} \left[\ln S + \left(r - \frac{\sigma^2}{2}\right)(T - t) \right] \\ \frac{\partial f}{\partial t} &= r e^{-r(T-t)} \left[\ln S + \left(r - \frac{\sigma^2}{2}\right)(T - t) \right] - e^{-r(T-t)} \left(r - \frac{\sigma^2}{2}\right) \\ \frac{\partial f}{\partial S} &= \frac{e^{-r(T-t)}}{S} \\ \frac{\partial^2 f}{\partial S^2} &= -\frac{e^{-r(T-t)}}{S^2} \end{aligned}$$

The left-hand side of the Black-Scholes-Merton differential equation is

$$\begin{aligned} &e^{-r(T-t)} \left[r \ln S + r \left(r - \frac{\sigma^2}{2}\right)(T - t) - \left(r - \frac{\sigma^2}{2}\right) + r - \frac{\sigma^2}{2} \right] \\ &= r e^{-r(T-t)} \left[\ln S + \left(r - \frac{\sigma^2}{2}\right)(T - t) \right] \\ &= r f \end{aligned}$$

Hence equation (13.16) is satisfied.

Problem 13.12.

Consider a derivative that pays off S_T^n at time T where S_T is the stock price at that time. When the stock price follows geometric Brownian motion, it can be shown that its price at time t ($t \leq T$) has the form

$$h(t, T) S^n$$

where S is the stock price at time t and h is a function only of t and T .

(a) By substituting into the Black–Scholes–Merton partial differential equation derive an ordinary differential equation satisfied by $h(t, T)$.

(b) What is the boundary condition for the differential equation for $h(t, T)$?

(c) Show that

$$h(t, T) = e^{[0.5\sigma^2 n(n-1) + r(n-1)](T-t)}$$

where r is the risk-free interest rate and σ is the stock price volatility.

This problem is related to Problem 12.10.

(a) If $G(S, t) = h(t, T)S^n$ then $\partial G/\partial t = h_t S^n$, $\partial G/\partial S = hnS^{n-1}$, and $\partial^2 G/\partial S^2 = hn(n-1)S^{n-2}$ where $h_t = \partial h/\partial t$. Substituting into the Black–Scholes–Merton differential equation we obtain

$$h_t + rhn + \frac{1}{2}\sigma^2 hn(n-1) = rh$$

(b) The derivative is worth S^n when $t = T$. The boundary condition for this differential equation is therefore $h(T, T) = 1$

(c) The equation

$$h(t, T) = e^{[0.5\sigma^2 n(n-1) + r(n-1)](T-t)}$$

satisfies the boundary condition since it collapses to $h = 1$ when $t = T$. It can also be shown that it satisfies the differential equation in (a). Alternatively we can solve the differential equation in (a) directly. The differential equation can be written

$$\frac{h_t}{h} = -r(n-1) - \frac{1}{2}\sigma^2 n(n-1)$$

The solution to this is

$$\ln h = [-r(n-1) - \frac{1}{2}\sigma^2 n(n-1)]t + k$$

where k is a constant. Since $\ln h = 0$ when $t = T$ it follows that

$$k = [r(n-1) + \frac{1}{2}\sigma^2 n(n-1)]T$$

so that

$$\ln h = [r(n-1) + \frac{1}{2}\sigma^2 n(n-1)](T-t)$$

or

$$h(t, T) = e^{[0.5\sigma^2 n(n-1) + r(n-1)](T-t)}$$

Problem 13.13.

What is the price of a European call option on a non-dividend-paying stock when the stock price is \$52, the strike price is \$50, the risk-free interest rate is 12% per annum, the volatility is 30% per annum, and the time to maturity is three months?

In this case $S_0 = 52$, $K = 50$, $r = 0.12$, $\sigma = 0.30$ and $T = 0.25$.

$$d_1 = \frac{\ln(52/50) + (0.12 + 0.3^2/2)0.25}{0.30\sqrt{0.25}} = 0.5365$$

$$d_2 = d_1 - 0.30\sqrt{0.25} = 0.3865$$

The price of the European call is

$$\begin{aligned} & 52N(0.5365) - 50e^{-0.12 \times 0.25}N(0.3865) \\ &= 52 \times 0.7042 - 50e^{-0.03} \times 0.6504 \\ &= 5.06 \end{aligned}$$

or \$5.06.

Problem 13.14.

What is the price of a European put option on a non-dividend-paying stock when the stock price is \$69, the strike price is \$70, the risk-free interest rate is 5% per annum, the volatility is 35% per annum, and the time to maturity is six months?

In this case $S_0 = 69$, $K = 70$, $r = 0.05$, $\sigma = 0.35$ and $T = 0.5$.

$$d_1 = \frac{\ln(69/70) + (0.05 + 0.35^2/2) \times 0.5}{0.35\sqrt{0.5}} = 0.1666$$

$$d_2 = d_1 - 0.35\sqrt{0.5} = -0.0809$$

The price of the European put is

$$\begin{aligned} & 70e^{-0.05 \times 0.5}N(0.0809) - 69N(-0.1666) \\ &= 70e^{-0.025} \times 0.5323 - 69 \times 0.4338 \\ &= 6.40 \end{aligned}$$

or \$6.40.

Problem 13.15.

Consider an American call option on a stock. The stock price is \$70, the time to maturity is eight months, the risk-free rate of interest is 10% per annum, the exercise price is \$65, and the volatility is 32%. A dividend of \$1 is expected after three months and again after six months. Show that it can never be optimal to exercise the option on either of the two dividend dates. Use DerivaGem to calculate the price of the option.

Using the notation of Section 13.12, $D_1 = D_2 = 1$, $K(1 - e^{-r(T-t_2)}) = 65(1 - e^{-0.1 \times 0.1667}) = 1.07$, and $K(1 - e^{-r(t_2-t_1)}) = 65(1 - e^{-0.1 \times 0.25}) = 1.60$. Since

$$D_1 < K(1 - e^{-r(T-t_2)})$$

and

$$D_2 < K(1 - e^{-r(t_2 - t_1)})$$

It is never optimal to exercise the call option early. DerivaGem shows that the value of the option is 10.94.

Problem 13.16.

A call option on a non-dividend-paying stock has a market price of $\$2\frac{1}{2}$. The stock price is $\$15$, the exercise price is $\$13$, the time to maturity is three months, and the risk-free interest rate is 5% per annum. What is the implied volatility?

In the case $c = 2.5$, $S_0 = 15$, $K = 13$, $T = 0.25$, $r = 0.05$. The implied volatility must be calculated using an iterative procedure.

A volatility of 0.2 (or 20% per annum) gives $c = 2.20$. A volatility of 0.3 gives $c = 2.32$. A volatility of 0.4 gives $c = 2.507$. A volatility of 0.39 gives $c = 2.487$. By interpolation the implied volatility is about 0.397 or 39.7% per annum.

Problem 13.17.

With the notation used in this chapter

(a) What is $N'(x)$?

(b) Show that $SN'(d_1) = Ke^{-r(T-t)}N'(d_2)$, where S is the stock price at time t

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}$$

$$d_2 = \frac{\ln(S/K) + (r - \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}$$

(c) Calculate $\partial d_1/\partial S$ and $\partial d_2/\partial S$.

(d) Show that when

$$c = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$

$$\frac{\partial c}{\partial t} = -rKe^{-r(T-t)}N(d_2) - SN'(d_1)\frac{\sigma}{2\sqrt{T-t}}$$

where c is the price of a call option on a non-dividend-paying stock.

(e) Show that $\partial c/\partial S = N(d_1)$.

(f) Show that the c satisfies the Black-Scholes-Merton differential equation.

(g) Show that c satisfies the boundary condition for a European call option, i.e., that $c = \max(S - K, 0)$ as $t \rightarrow T$

(a) Since $N(x)$ is the cumulative probability that a variable with a standardized normal distribution will be less than x , $N'(x)$ is the probability density function for a standardized normal distribution, that is,

$$N'(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

(b)

$$\begin{aligned} N'(d_1) &= N'(d_2 + \sigma\sqrt{T-t}) \\ &= \frac{1}{\sqrt{2\pi}} \exp \left[-\frac{d_2^2}{2} - \sigma d_2 \sqrt{T-t} - \frac{1}{2} \sigma^2 (T-t) \right] \\ &= N'(d_2) \exp \left[-\sigma d_2 \sqrt{T-t} - \frac{1}{2} \sigma^2 (T-t) \right] \end{aligned}$$

Because

$$d_2 = \frac{\ln(S/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

it follows that

$$\exp \left[-\sigma d_2 \sqrt{T-t} - \frac{1}{2} \sigma^2 (T-t) \right] = \frac{Ke^{-r(T-t)}}{S}$$

As a result

$$SN'(d_1) = Ke^{-r(T-t)} N'(d_2)$$

which is the required result.

(c)

$$\begin{aligned} d_1 &= \frac{\ln \frac{S}{K} + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \\ &= \frac{\ln S - \ln K + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \end{aligned}$$

Hence

$$\frac{\partial d_1}{\partial S} = \frac{1}{S\sigma\sqrt{T-t}}$$

Similarly

$$d_2 = \frac{\ln S - \ln K + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

and

$$\frac{\partial d_2}{\partial S} = \frac{1}{S\sigma\sqrt{T-t}}$$

Therefore:

$$\frac{\partial d_1}{\partial S} = \frac{\partial d_2}{\partial S}$$

(d)

$$\begin{aligned} c &= SN(d_1) - Ke^{-r(T-t)} N(d_2) \\ \frac{\partial c}{\partial t} &= SN'(d_1) \frac{\partial d_1}{\partial t} - rKe^{-r(T-t)} N(d_2) - Ke^{-r(T-t)} N'(d_2) \frac{\partial d_2}{\partial t} \end{aligned}$$

From (b):

$$SN'(d_1) = Ke^{-r(T-t)}N'(d_2)$$

Hence

$$\frac{\partial c}{\partial t} = -rKe^{-r(T-t)}N(d_2) + SN'(d_1) \left(\frac{\partial d_1}{\partial t} - \frac{\partial d_2}{\partial t} \right)$$

Since

$$\begin{aligned} d_1 - d_2 &= \sigma\sqrt{T-t} \\ \frac{\partial d_1}{\partial t} - \frac{\partial d_2}{\partial t} &= \frac{\partial}{\partial t}(\sigma\sqrt{T-t}) \\ &= -\frac{\sigma}{2\sqrt{T-t}} \end{aligned}$$

Hence

$$\frac{\partial c}{\partial t} = -rKe^{-r(T-t)}N(d_2) - SN'(d_1)\frac{\sigma}{2\sqrt{T-t}}$$

(e) From differentiating the Black-Scholes-Merton formula for a call price we obtain

$$\frac{\partial c}{\partial S} = N(d_1) + SN'(d_1)\frac{\partial d_1}{\partial S} - Ke^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial S}$$

From the results in (b) and (c) it follows that

$$\frac{\partial c}{\partial S} = N(d_1)$$

(f) Differentiating the result in (e) and using the result in (c), we obtain

$$\begin{aligned} \frac{\partial^2 c}{\partial S^2} &= N'(d_1)\frac{\partial d_1}{\partial S} \\ &= N'(d_1)\frac{1}{S\sigma\sqrt{T-t}} \end{aligned}$$

From the results in d) and e)

$$\begin{aligned} \frac{\partial c}{\partial t} + rS\frac{\partial c}{\partial S} + \frac{1}{2}\sigma^2S^2\frac{\partial^2 c}{\partial S^2} &= -rKe^{-r(T-t)}N(d_2) - SN'(d_1)\frac{\sigma}{2\sqrt{T-t}} \\ &\quad + rSN(d_1) + \frac{1}{2}\sigma^2S^2N'(d_1)\frac{1}{S\sigma\sqrt{T-t}} \\ &= r[SN(d_1) - Ke^{-r(T-t)}N(d_2)] \\ &= rc \end{aligned}$$

This shows that the Black-Scholes formula for a call option does indeed satisfy the Black-Scholes-Merton differential equation

- (g) Consider what happens in the formula for c in part (d) as t approaches T . If $S > K$, d_1 and d_2 tend to infinity and $N(d_1)$ and $N(d_2)$ tend to 1. If $S < K$, d_1 and d_2 tend to zero. It follows that the formula for c tends to $\max(S - K, 0)$.

Problem 13.18.

Show that the Black-Scholes formulas for call and put options satisfy put-call parity.

From the Black-Scholes equations

$$p + S_0 = Ke^{-rT}N(-d_2) - S_0N(-d_1) + S_0$$

Because $1 - N(-d_1) = N(d_1)$ this is

$$Ke^{-rT}N(-d_2) + S_0N(d_1)$$

Also:

$$c + Ke^{-rT} = S_0N(d_1) - Ke^{-rT}N(d_2) + Ke^{-rT}$$

Because $1 - N(d_2) = N(-d_2)$, this is also

$$Ke^{-rT}N(-d_2) + S_0N(d_1)$$

The Black-Scholes equations are therefore consistent with put-call parity.

Problem 13.19.

A stock price is currently \$50 and the risk-free interest rate is 5%. Use the DerivaGem software to translate the following table of European call options on the stock into a table of implied volatilities, assuming no dividends. Are the option prices consistent with the assumptions underlying Black-Scholes?

Strike Price (\$)	Maturity (months)		
	3	6	12
45	7.0	8.3	10.5
50	3.7	5.2	7.5
55	1.6	2.9	5.1

This problem naturally leads on to the material in Chapter 18 on volatility smiles. Using DerivaGem we obtain the following table of implied volatilities:

Strike Price (\$)	Maturity (months)		
	3	6	12
45	37.78	34.99	34.02
50	34.15	32.78	32.03
55	31.98	30.77	30.45

The option prices are not exactly consistent with Black–Scholes. If they were, the implied volatilities would be all the same. We usually find in practice that low strike price options on a stock have significantly higher implied volatilities than high strike price options on the same stock.

Problem 13.20.

Explain carefully why Black’s approach to evaluating an American call option on a dividend-paying stock may give an approximate answer even when only one dividend is anticipated. Does the answer given by Black’s approach understate or overstate the true option value? Explain your answer.

Black’s approach in effect assumes that the holder of option must decide at time zero whether it is a European option maturing at time t_n (the final ex-dividend date) or a European option maturing at time T . In fact the holder of the option has more flexibility than this. The holder can choose to exercise at time t_n if the stock price at that time is above some level but not otherwise. Furthermore, if the option is not exercised at time t_n , it can still be exercised at time T .

It appears from this argument that Black’s approach understates the true option value. However, the way in which volatility is applied can lead to Black’s approach overstating the option value. Black applies the volatility to the option price. The binomial model, as we will see in Chapter 19, applies the volatility to the stock price less the present value of the dividend. This issue is also discussed in Example 13.10.

Problem 13.21.

Consider an American call option on a stock. The stock price is \$50, the time to maturity is 15 months, the risk-free rate of interest is 8% per annum, the exercise price is \$55, and the volatility is 25%. Dividends of \$1.50 are expected in 4 months and 10 months. Show that it can never be optimal to exercise the option on either of the two dividend dates. Calculate the price of the option.

With the notation in the text

$$D_1 = D_2 = 1.50, \quad t_1 = 0.3333, \quad t_2 = 0.8333, \quad T = 1.25, \quad r = 0.08 \quad \text{and} \quad K = 55$$

$$K \left[1 - e^{-r(T-t_2)} \right] = 55(1 - e^{-0.08 \times 0.4167}) = 1.80$$

Hence

$$D_2 < K \left[1 - e^{-r(T-t_2)} \right]$$

Also:

$$K \left[1 - e^{-r(t_2-t_1)} \right] = 55(1 - e^{-0.08 \times 0.5}) = 2.16$$

Hence:

$$D_1 < K \left[1 - e^{-r(t_2-t_1)} \right]$$

It follows from the conditions established in Section 13.12 that the option should never be exercised early.

The present value of the dividends is

$$1.5e^{-0.3333 \times 0.08} + 1.5e^{-0.8333 \times 0.08} = 2.864$$

The option can be valued using the European pricing formula with:

$$S_0 = 50 - 2.864 = 47.136, \quad K = 55, \quad \sigma = 0.25, \quad r = 0.08, \quad T = 1.25$$

$$d_1 = \frac{\ln(47.136/55) + (0.08 + 0.25^2/2)1.25}{0.25\sqrt{1.25}} = -0.0545$$

$$d_2 = d_1 - 0.25\sqrt{1.25} = -0.3340$$

$$N(d_1) = 0.4783, \quad N(d_2) = 0.3692$$

and the call price is

$$47.136 \times 0.4783 - 55e^{-0.08 \times 1.25} \times 0.3692 = 4.17$$

or \$4.17.

Problem 13.22.

Show that the probability that a European call option will be exercised in a risk-neutral world is, with the notation introduced in this chapter, $N(d_2)$. What is an expression for the value of a derivative that pays off \$100 if the price of a stock at time T is greater than K ?

The probability that the call option will be exercised is the probability that $S_T > K$ where S_T is the stock price at time T . In a risk neutral world

$$\ln S_T \sim \phi[\ln S_0 + (r - \sigma^2/2)T, \sigma^2 T]$$

The probability that $S_T > K$ is the same as the probability that $\ln S_T > \ln K$. This is

$$\begin{aligned} 1 - N \left[\frac{\ln K - \ln S_0 - (r - \sigma^2/2)T}{\sigma\sqrt{T}} \right] \\ = N \left[\frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} \right] \\ = N(d_2) \end{aligned}$$

The expected value at time T in a risk neutral world of a derivative security which pays off \$100 when $S_T > K$ is therefore

$$100N(d_2)$$

From risk neutral valuation the value of the security at time t is

$$100e^{-rT}N(d_2)$$

Problem 13.23.

Show that S^{-2r/σ^2} could be the price of a traded security.

If $f = S^{-2r/\sigma^2}$ then

$$\frac{\partial f}{\partial S} = -\frac{2r}{\sigma^2}S^{-2r/\sigma^2-1}$$

$$\frac{\partial^2 f}{\partial S^2} = \left(\frac{2r}{\sigma^2}\right)\left(\frac{2r}{\sigma^2} + 1\right)S^{-2r/\sigma^2-2}$$

$$\frac{\partial f}{\partial t} = 0$$

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2S^2\frac{\partial^2 f}{\partial S^2} = rS^{-2r/\sigma^2} = rf$$

This shows that the Black-Scholes equation is satisfied. S^{-2r/σ^2} could therefore be the price of a traded security.

Problem 13.24.

A company has an issue of executive stock options outstanding. Should dilution be taken into account when the options are valued? Explain your answer.

The answer is no. If markets are efficient they have already taken potential dilution into account in determining the stock price. This argument is explained in Business Snapshot 13.3.

Problem 13.25.

A company's stock price is \$50 and 10 million shares are outstanding. The company is considering giving its employees three million at-the-money five-year call options. Option exercises will be handled by issuing more shares. The stock price volatility is 25%, the five-year risk-free rate is 5% and the company does not pay dividends. Estimate the cost to the company of the employee stock option issue.

The Black-Scholes price of the option is given by setting $S_0 = 50$, $K = 50$, $r = 0.05$, $\sigma = 0.25$, and $T = 5$. It is 16.252. From an analysis similar to that in Section 13.10 the cost to the company of the options is

$$\frac{10}{10 + 3} \times 16.252 = 12.5$$

or about \$12.5 per option. The total cost is therefore 3 million times this or \$37.5 million. If the market perceives no benefits from the options the stock price will fall by \$3.75.

ASSIGNMENT QUESTIONS

Problem 13.26.

A stock price is currently \$50. Assume that the expected return from the stock is 18% and its volatility is 30%. What is the probability distribution for the stock price in two years? Calculate the mean and standard deviation of the distribution. Determine the 95% confidence interval.

In this case $S_0 = 50$, $\mu = 0.18$ and $\sigma = 0.30$. The probability distribution of the stock price in two years, S_T , is lognormal and is, from equation (13.3), given by:

$$\ln S_T \sim \phi\left[\ln 50 + \left(0.18 - \frac{0.09}{2}\right)2, 0.3^2 \times 2\right]$$

i.e.,

$$\ln S_T \sim \phi(4.18, 0.18)$$

The mean stock price is from equation (13.4)

$$50e^{2 \times 0.18} = 50e^{0.36} = 71.67$$

and the standard deviation is, from equation (13.5),

$$50e^{2 \times 0.18} \sqrt{e^{0.09 \times 2} - 1} = 31.83$$

95% confidence intervals for $\ln S_T$ are

$$4.18 - 1.96 \times 0.42 \quad \text{and} \quad 4.18 + 1.96 \times 0.42$$

i.e.,

$$3.35 \quad \text{and} \quad 5.01$$

These correspond to 95% confidence limits for S_T of

$$e^{3.35} \quad \text{and} \quad e^{5.01}$$

i.e.,

$$28.52 \quad \text{and} \quad 150.44$$

Problem 13.27.

Suppose that observations on a stock price (in dollars) at the end of each of 15 consecutive weeks are as follows:

30.2, 32.0, 31.1, 30.1, 30.2, 30.3, 30.6, 33.0, 32.9, 33.0, 33.5, 33.5, 33.7, 33.5, 33.2

Estimate the stock price volatility. What is the standard error of your estimate?

The calculations are shown in the table below

$$\sum u_i = 0.09471 \quad \sum u_i^2 = 0.01145$$

and an estimate of standard deviation of weekly returns is:

$$\sqrt{\frac{0.01145}{13} - \frac{0.09471^2}{14 \times 13}} = 0.02884$$

The volatility per annum is therefore $0.02884\sqrt{52} = 0.2079$ or 20.79%. The standard error of this estimate is

$$\frac{0.2079}{\sqrt{2 \times 14}} = 0.0393$$

or 3.9% per annum.

Problem 13.27
Computation of Volatility

Week	Closing Stock Price (\$)	Price Relative $= S_i/S_{i-1}$	Daily Return $u_i = \ln(S_i/S_{i-1})$
1	30.2		
2	32.0	1.05960	0.05789
3	31.1	0.97188	-0.02853
4	30.1	0.96785	-0.03268
5	30.2	1.00332	0.00332
6	30.3	1.00331	0.00331
7	30.6	1.00990	0.00985
8	33.0	1.07843	0.07551
9	32.9	0.99697	-0.00303
10	33.0	1.00304	0.00303
11	33.5	1.01515	0.01504
12	33.5	1.00000	0.00000
13	33.7	1.00597	0.00595
14	33.5	0.99407	-0.00595
15	33.2	0.99104	-0.00900

Problem 13.28.

A financial institution plans to offer a security that pays off a dollar amount equal to S_T^2 at time T .

- (a) Use risk-neutral valuation to calculate the price of the security at time t in terms of the stock price, S , at time t . (Hint: The expected value of S_T^2 can be calculated from the mean and variance of S_T given in section 13.1.)
 (b) Confirm that your price satisfies the differential equation (13.16).

- (a) The expected value of the security is $E[(S_T)^2]$ From equations (13.4) and (13.5), at time t :

$$E(S_T) = S e^{\mu(T-t)}$$

$$\text{var}(S_T) = S^2 e^{2\mu(T-t)} [e^{\sigma^2(T-t)} - 1]$$

Since $\text{var}(S_T) = E[(S_T)^2] - [E(S_T)]^2$, it follows that $E[(S_T)^2] = \text{var}(S_T) + [E(S_T)]^2$ so that

$$\begin{aligned} E[(S_T)^2] &= S^2 e^{2\mu(T-t)} [e^{\sigma^2(T-t)} - 1] + S^2 e^{2\mu(T-t)} \\ &= S^2 e^{(2\mu + \sigma^2)(T-t)} \end{aligned}$$

In a risk-neutral world $\mu = r$ so that

$$\hat{E}[(S_T)^2] = S^2 e^{(2r + \sigma^2)(T-t)}$$

Using risk-neutral valuation, the value of the derivative security at time t is

$$\begin{aligned} e^{-r(T-t)} \hat{E}[(S_T)^2] \\ &= S^2 e^{(2r + \sigma^2)(T-t)} e^{-r(T-t)} \\ &= S^2 e^{(r + \sigma^2)(T-t)} \end{aligned}$$

(b) If:

$$\begin{aligned} f &= S^2 e^{(r + \sigma^2)(T-t)} \\ \frac{\partial f}{\partial t} &= -S^2 (r + \sigma^2) e^{(r + \sigma^2)(T-t)} \\ \frac{\partial f}{\partial S} &= 2S e^{(r + \sigma^2)(T-t)} \\ \frac{\partial^2 f}{\partial S^2} &= 2e^{(r + \sigma^2)(T-t)} \end{aligned}$$

The left-hand side of the Black-Scholes-Merton differential equation is:

$$\begin{aligned} -S^2 (r + \sigma^2) e^{(r + \sigma^2)(T-t)} + 2r S^2 e^{(r + \sigma^2)(T-t)} + \sigma^2 S^2 e^{(r + \sigma^2)(T-t)} \\ = r S^2 e^{(r + \sigma^2)(T-t)} \\ = r f \end{aligned}$$

Hence the Black-Scholes equation is satisfied.

Problem 13.29.

Consider an option on a non-dividend-paying stock when the stock price is \$30, the exercise price is \$29, the risk-free interest rate is 5%, the volatility is 25% per annum, and the time to maturity is 4 months.

- What is the price of the option if it is a European call?
- What is the price of the option if it is an American call?
- What is the price of the option if it is a European put?
- Verify that put-call parity holds.

In this case $S_0 = 30$, $K = 29$, $r = 0.05$, $\sigma = 0.25$ and $T = 0.3333$

$$d_1 = \frac{\ln(30/29) + (0.05 + 0.25^2/2) \times 0.3333}{0.25\sqrt{0.3333}} = 0.4225$$

$$d_2 = \frac{\ln(30/29) + (0.05 - 0.25^2/2) \times 0.3333}{0.25\sqrt{0.3333}} = 0.2782$$

$$N(0.4225) = 0.6637, \quad N(0.2782) = 0.6096$$

$$N(-0.4225) = 0.3363, \quad N(-0.2782) = 0.3904$$

(a) The European call price is

$$30 \times 0.6637 - 29e^{-0.05 \times 0.3333} \times 0.6096 = 2.52$$

or \$2.52.

(b) The American call price is the same as the European call price. It is \$2.52.

(c) The European put price is

$$29e^{-0.05 \times 0.3333} \times 0.3904 - 30 \times 0.3363 = 1.05$$

or \$1.05.

(d) Put-call parity states that:

$$p + S_0 = c + Ke^{-rT}$$

In this case $c = 2.52$, $S_0 = 30$, $K = 29$, $p = 1.05$ and $e^{-rT} = 0.9835$ and it is easy to verify that the relationship is satisfied.

Problem 13.30.

Assume that the stock in Problem 13.29 is due to go ex-dividend in $1\frac{1}{2}$ months. The expected dividend is 50 cents.

(a) What is the price of the option if it is a European call?

(b) What is the price of the option if it is a European put?

(c) If the option is an American call, are there any circumstances under which it will be exercised early?

(a) The present value of the dividend must be subtracted from the stock price. This gives a new stock price of:

$$30 - 0.5e^{-0.125 \times 0.05} = 29.5031$$

and

$$d_1 = \frac{\ln(29.5031/29) + (0.05 + 0.25^2/2) \times 0.3333}{0.25\sqrt{0.3333}} = 0.3068$$

$$d_2 = \frac{\ln(29.5031/29) + (0.05 - 0.25^2/2) \times 0.3333}{0.25\sqrt{0.3333}} = 0.1625$$

$$N(d_1) = 0.6205; \quad N(d_2) = 0.5645$$

The price of the option is therefore

$$29.5031 \times 0.6205 - 29e^{-0.3333 \times 0.05} \times 0.5645 = 2.21$$

or \$2.21.

(b) Since

$$N(-d_1) = 0.3795, \quad N(-d_2) = 0.4355$$

the value of the option when it is a European put is

$$29e^{-0.3333 \times 0.05} \times 0.4355 - 29.5031 \times 0.3795 = 1.22$$

or \$1.22.

(c) If t_1 denotes the time when the dividend is paid:

$$K[1 - e^{-r(T-t_1)}] = 29(1 - e^{-0.05 \times 0.2083}) = 0.3005$$

This is less than the dividend. Hence the option should be exercised immediately before the ex-dividend date for a sufficiently high value of the stock price.

Problem 13.31.

Consider an American call option when the stock price is \$18, the exercise price is \$20, the time to maturity is six months, the volatility is 30% per annum, and the risk-free interest rate is 10% per annum. Two equal dividends are expected during the life of the option with ex-dividend dates at the end of two months and five months. Assume the dividends are 40 cents. Use Black's approximation and the DerivaGem software to value the option. How high can the dividends be without the American option being worth more than the corresponding European option?

We first value the option assuming that it is not exercised early, we set the time to maturity equal to 0.5. There is a dividend of 0.4 in 2 months and 5 months. Other parameters are $S_0 = 18$, $K = 20$, $r = 10\%$, $\sigma = 30\%$. DerivaGem gives the price as 0.7947. We next value the option assuming that it is exercised at the five-month point just before the final dividend. DerivaGem gives the price as 0.7668. The price given by Black's approximation is therefore 0.7947. DerivaGem also shows that the correct American option price calculated with 100 time steps is 0.8243.

It is never optimal to exercise the option immediately before the first ex-dividend date when

$$D_1 \leq K[1 - e^{-r(t_2-t_1)}]$$

where D_1 is the size of the first dividend, and t_1 and t_2 are the times of the first and second dividend respectively. Hence we must have:

$$D_1 \leq 20[1 - e^{-(0.1 \times 0.25)}]$$

that is,

$$D_1 \leq 0.494$$

It is never optimal to exercise the option immediately before the second ex-dividend date when:

$$D_2 \leq K(1 - e^{-r(T-t_2)})$$

where D_2 is the size of the second dividend. Hence we must have:

$$D_2 \leq 20(1 - e^{-0.1 \times 0.0833})$$

that is,

$$D_2 \leq 0.166$$

It follows that the dividend can be as high as 16.6 cents per share without the American option being worth more than the corresponding European option.

CHAPTER 14

Employee Stock Options

Notes for the Instructor

This chapter is new to the seventh edition. Employee stock options have been much in the news in recent years and I find students enjoy talking about them. Many students hope to become rich one day by exercising such options!

The chapter covers how the options typically work, whether they align the interests of senior executives and shareholders, their accounting treatment, alternative valuation approaches, and backdating scandals. Many instructors will want to spend time on the academic research of Yermack, Lie, and Heron which was largely responsible for exposing the backdating scandals (see Section 14.5). Others may want to focus on how employee stock options can be designed to better align the interests of shareholders and senior managers. As described in Section 14.1 the traditional stock option plan is one where at-the-money options are issued periodically. For many years, companies were reluctant to move away from this type of plan because they would then be required to expense the options. Now the accounting treatment of employee stock options has now changed (with expensing being mandatory) and so there is no reason for companies not to consider nontraditional plans such as those mentioned in Section 14.3.

I recommend spending some time talking about the fact that employee stock options (unlike regular call options) cannot be sold. This leads to the situation where they tend to be exercised much earlier than regular call options (see Section 14.1). A discussion of this should reinforce a student's understanding of the arguments concerning early exercise of calls in Chapter 9.

The three assignment questions test whether students can use some of the approaches for valuing employee stock options. If 14.14 is assigned it is a good idea to suggest to students that they calculated the expected life using a tree.

QUESTIONS AND PROBLEMS

Problem 14.1.

Why was it attractive for companies to grant at-the-money stock options prior to 2005? What changed in 2005?

Prior to 2005 companies did not have to expense at-the-money options on the income statement. They merely had to report the value of the options in notes to the accounts. FAS 123 and IAS 2 required the fair value of the options to be reported as a cost on the income statement starting in 2005.

Problem 14.2.

What are the main differences between a typical employee stock option and a call option traded on an exchange or in the over-the-counter market?

The main differences are a) employee stock options last much longer than the typical exchange-traded or over-the-counter option, b) there is usually a vesting period during which they cannot be exercised, c) the options cannot be sold by the employee, d) if the employee leaves the company the options usually either expire worthless or have to be exercised immediately, and e) exercise of the options usually leads to the company issuing more shares.

Problem 14.3.

Explain why employee stock options on a non-dividend-paying stock are frequently exercised before the end of their lives whereas an exchange-traded call option on such a stock is never exercised early.

It is always better for the option holder to sell a call option on a non-dividend-paying stock rather than exercise it. Employee stock options cannot be sold and so the only way an employee can monetize the option is to exercise the option and sell the stock.

Problem 14.4.

“Stock option grants are good because they motivate executives to act in the best interests of shareholders.” Discuss this viewpoint.

This is questionable. Executives benefit from share price increases but do not bear the costs of share price decreases. Employee stock options are liable to encourage executives to take decisions that boost the value of the stock in the short term at the expense of the long term health of the company. It may even be the case that executives are encouraged to take high risks so as to maximize the value of their options.

Problem 14.5.

“Granting stock options to executives is like allowing a professional footballer to bet on the outcome of games.” Discuss this viewpoint.

Professional footballers are not allowed to bet on the outcomes of games because they themselves influence the outcomes. Arguably, an executive should not be allowed to bet on the future stock price of her company because her actions influence that price. However, it could be argued that there is nothing wrong with a professional footballer betting that his team will win (but everything wrong with betting that it will lose). Similarly there is nothing wrong with an executive betting that her company will do well.

Problem 14.6.

Why did some companies backdate stock option grants in the US prior to 2002? What changed in 2002?

Backdating allowed the company to issue employee stock options with a strike price equal to the price at some previous date and claim that they were at the money. At

the money options did not lead to an expense on the income statement until 2005. The amount recorded for the value of the options in the notes to the income was less than the actual cost on the true grant date. In 2002 the SEC required companies to report stock option grants within two business days of the grant date. This eliminated the possibility of backdating for companies that complied with this rule.

Problem 14.7.

In what way would the benefits of backdating be reduced if a stock option grant had to be revalued at the end of each quarter?

If a stock option grant had to be revalued each quarter the value of the option of the grant date (true or fabricated) would become less important. Stock price movements following the reported grant date would be incorporated in the next revaluation. The total cost of the options would be independent of the stock price on the grant date.

Problem 14.8.

Explain how you would do the analysis to produce a chart such as the one in Figure 15.2.

It would be necessary to look at returns on each stock in the sample (possibly adjusted for the returns on the market and the beta of the stock) around the reported employee stock option grant date. One could designate Day 0 as the grant date and look at returns on each stock each day from Day -30 to Day +30. The returns would then be averaged across the stocks.

Problem 14.9.

On May 31 a company's stock price is \$70. One million shares are outstanding. An executive exercises 100,000 stock options with a strike price of \$50. What is the impact of this on the stock price?

There should be no impact on the stock price because the stock price will already reflect the dilution expected from the executive's exercise decision.

Problem 14.10.

The notes accompanying a company's financial statements say: "Our executive stock options last 10 years and vest after four years. We valued the options granted this year using the Black-Scholes model with an expected life of 5 years and a volatility of 20%." What does this mean? Discuss the modeling approach used by the company.

The notes indicate that the Black-Scholes model was used to produce the valuation with T the option life being set equal to 5 years and the stock price volatility being set equal to 20%.

Problem 14.11.

In a Dutch auction of 10,000 options, bids are as follows
A bids \$30 for 3,000
B bids \$33 for 2,500

C bids \$29 for 5,000

D bids \$40 for 1,000

E bids \$22 for 8,000

F bids \$35 for 6,000

What is the result of the auction? Who buys how many at what price?

The price at which 10,000 options can be sold is \$30. B, D, and F get their order completely filled at this price. A buys 500 options (out of its total bid for 3,000 options) at this price.

Problem 14.12.

A company has granted 500,000 options to its executives. The stock price and strike price are both \$40. The options last for 12 years and vest after four years. The company decides to value the options using an expected life of five years and a volatility of 30% per annum. The company pays no dividends and the risk-free rate is 4%. What will the company report as an expense for the options on its income statement?

The options are valued using Black–Scholes with $S_0 = 40$, $K = 40$, $T = 5$, $\sigma = 0.3$ and $r = 0.04$. The value of each option is \$4.488. The total expense reported is $500,000 \times \$4.488$ or \$2.244 million.

Problem 14.13.

A company's CFO says: "The accounting treatment of stock options is crazy. We granted 10,000,000 at-the-money stock options to our employees last year when the stock price was \$30. We estimated the value of each option on the grant date to be \$5. At our year end the stock price had fallen to \$4, but we were still stuck with a \$50 million charge to the P&L." Discuss.

The problem is that under the current rules the options are valued only once—on the grant date. Arguably it would make sense to treat the options in the same way as other derivatives entered into by the company and revalue them on each reporting date. However, this does not happen under the current rules in the United States unless the options are settled in cash.

ASSIGNMENT QUESTIONS

Problem 14.14.

What is the (risk-neutral) expected life for the employee stock option in Example 14.2? What is the value of the option obtained by using this expected life in Black–Scholes?

The expected life at time zero can be calculated by rolling back through the tree asking the question at each node: "What is the expected life if the node is reached." This is what has been done in Figure M14.1. For example at node G (time 6 years) there is a 81% chance that the option will be exercised and a 19% chance that it will last an extra two years. The expected life if node G is reached is therefore $0.81 \times 6 + 0.19 \times 8 = 6.38$

years. Similarly, the expected life if node H is reached is $0.335 \times 6 + 0.665 \times 8 = 7.33$ years. The expected life if node I or J is reached is $0.05 \times 6 + 0.95 \times 8 = 7.90$ years. The expected life if node D is reached is

$$0.43 \times 4 + 0.57 \times (0.5158 \times 6.38 + 0.4842 \times 7.33) = 5.62$$

Continuing in this way the expected life at time zero is 6.86 years. (As in Example 14.2 we assume that no employees leave at time zero.)

The value of the option assuming an expected life of 6.86 years is given by Black-Scholes with $S_0 = 40$, $K = 40$, $r = 0.05$, $\sigma = 0.3$ and $T = 6.86$. It is 17.17. Using a four-step tree it is 16.51.

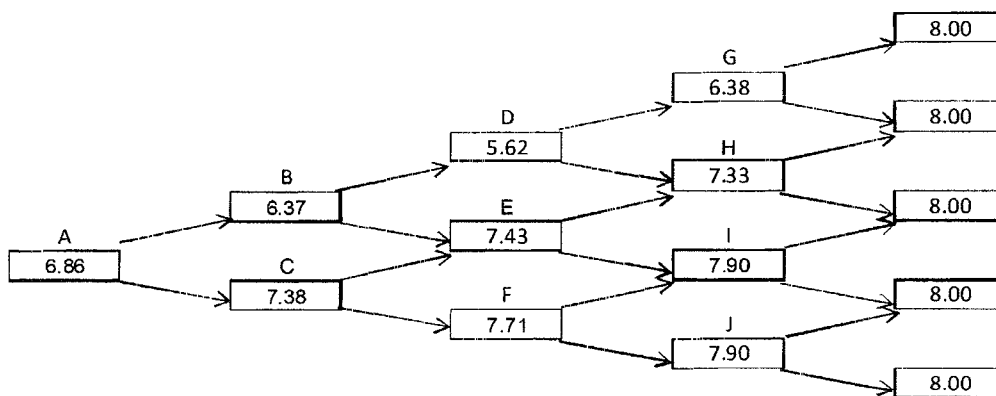


Figure M14.1 Tree for calculating expected life in Problem 14.14

Problem 14.15.

A company has granted 2,000,000 options to its employees. The stock price and strike price are both \$60. The options last for 8 years and vest after two years. The company decides to value the options using an expected life of six years and a volatility of 22% per annum. The dividend on the stock is \$1, payable half way through each year, and the risk-free rate is 5%. What will the company report as an expense for the options on its income statement.

The options are valued using Black-Scholes with $K = 60$, $T = 6$, $\sigma = 0.22$, $r = 0.05$. The present value of the dividends during the six years assumed life are

$$1 \times e^{-0.05 \times 0.5} + 1 \times e^{-0.05 \times 1.5} + 1 \times e^{-0.05 \times 2.5} + 1 \times e^{-0.05 \times 3.5} + 1 \times e^{-0.05 \times 4.5} + 1 \times e^{-0.05 \times 5.5}$$

$$= 5.183$$

The stock price, S_0 , adjusted for dividend is therefore $60 - 5.183 = 54.817$. The Black-Scholes model gives the price of one option as \$16.492. The company will therefore report as an expense $2,000,000 \times \$16.492$ or \$32.984 million.

Problem 14.16.

A company has granted 1,000,000 options to its employees. The stock price and strike price are both \$20. The options last 10 years and vest after three years. The stock price volatility is 30%, the risk-free rate is 5%, and the company pays no dividends. Use a four-step tree to value the options. Assume that there is a probability of 4% that an employee leaves the company at the beginning of each the time steps on your tree. Assume also that the probability of voluntary early exercise at a node, conditional on no prior exercise, when a) the option has vested and b) the option is in the money, is

$$1 - \exp[-a(S/K - 1)/T]$$

where S is the stock price, K is the strike price, T is the time to maturity and $a = 2$.

The valuation is shown in Figure M14.2. The tree is similar to Figure 14.1 in the text. The upper number at each node is the stock price and the lower number is the value of the option. In this case $u = 1.6070$ and $p = 0.5188$. The probability of voluntary exercise at nodes A, B, and C are 0.4690, 0.9195, and 0.3846, respectively. The total probability of exercise at these nodes (including the impact of employees leaving the company) is 0.4902, 0.9227, and 0.4093. The value of each option is \$8.54 and the value of the option grant is \$8.54 million. This problem and Example 14.2 in the text specify that employees are assumed to leave at the beginning of each time period. It is questionable whether this includes time zero. Both my answer to this question and the answer to Example 14.1 assume that it does not include time zero. (On reflection, it would have been better for both questions to say that employees leave at the end of each time period.) If in this question it is assumed that 4% of employees leave the company at the initial node the answer is reduced by 4% to \$8.20 million.

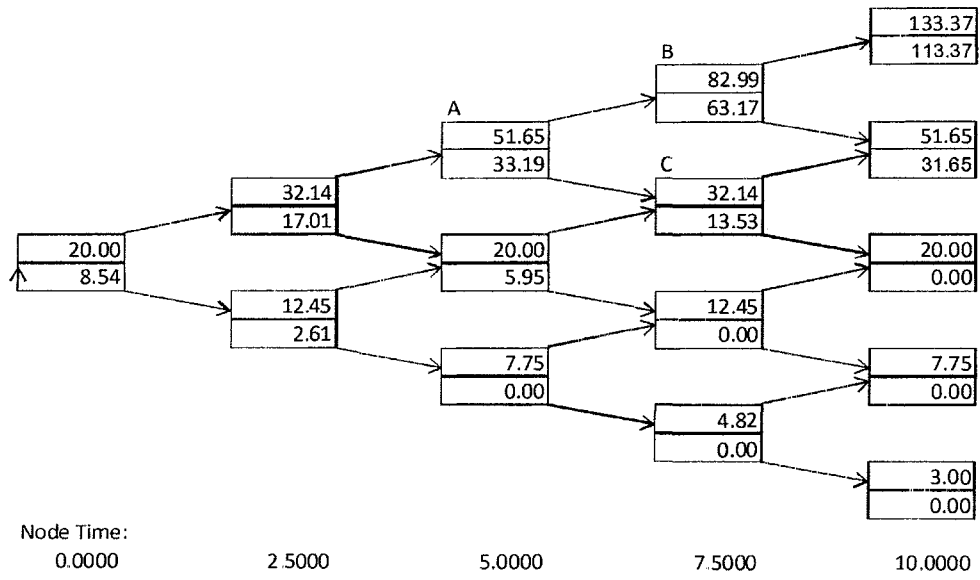


Figure M14.2 Valuation of employee stock option in Problem 14.16

CHAPTER 15

Options on Stock Indices and Currencies

Notes for the Instructor

The chapter concerned with options on stock indices, currencies, and futures in the sixth edition has been split into two chapters (15 and 16) in the seventh edition. Chapter 15 is concerned with options on stock indices and currencies; Chapter 16 is concerned with options on futures.

The material on options on stock indices and currencies has also been restructured for the seventh edition to make it more interesting. Instead of starting with valuation, it now starts with examples of how options on stock indices and options on foreign currencies are used. Range forwards are discussed here rather than later in the book.

For students who have a good knowledge of Chapter 13, the valuation material in this chapter should present few problems. The key argument is in Section 15.3 and shows how the Black-Scholes formulas can be modified to provide valuations of European call and put options on a stock paying a known dividend yield. Stock indices and currencies are analogous to stocks paying known dividend yields.

Any of Problems 15.23 to 13.28 make good assignment questions. Problem 15.22 requires Ito's lemma to have been covered.

QUESTIONS AND PROBLEMS

Problem 15.1.

A portfolio is currently worth \$10 million and has a beta of 1.0. An index is currently standing at 800. Explain how a put option on the index with a strike of 700 can be used to provide portfolio insurance.

When the index goes down to 700, the value of the portfolio can be expected to be $10 \times (700/800) = \$8.75$ million. (This assumes that the dividend yield on the portfolio equals the dividend yield on the index.) Buying put options on $10,000,000/800 = 12,500$ times the index with a strike of 700 therefore provides protection against a drop in the value of the portfolio below \$8.75 million. If each contract is on 100 times the index a total of 125 contracts would be required.

Problem 15.2.

“Once we know how to value options on a stock paying a dividend yield, we know how to value options on stock indices, currencies, and futures.” Explain this statement.

A stock index is analogous to a stock paying a continuous dividend yield, the dividend yield being the dividend yield on the index. A currency is analogous to a stock paying a continuous dividend yield, the dividend yield being the foreign risk-free interest rate.

Problem 15.3.

A stock index is currently 300, the dividend yield on the index is 3% per annum, and the risk-free interest rate is 8% per annum. What is a lower bound for the price of a six-month European call option on the index when the strike price is 290?

The lower bound is given by equation 15.1 as

$$300e^{-0.03 \times 0.5} - 290e^{-0.08 \times 0.5} = 16.90$$

Problem 15.4.

A currency is currently worth \$0.80 and has a volatility of 12%. The domestic and foreign risk-free interest rates are 6% and 8%, respectively. Use a two-step binomial tree to value a) a European four-month call option with a strike price of \$0.79 and b) an American four-month call option with the same strike price

In this case $u = 1.0502$ and $p = 0.4538$. The tree is shown in Figure S15.1. The value of the option if it is European is \$0.0235. the value of the option if it is American is \$0.0250.

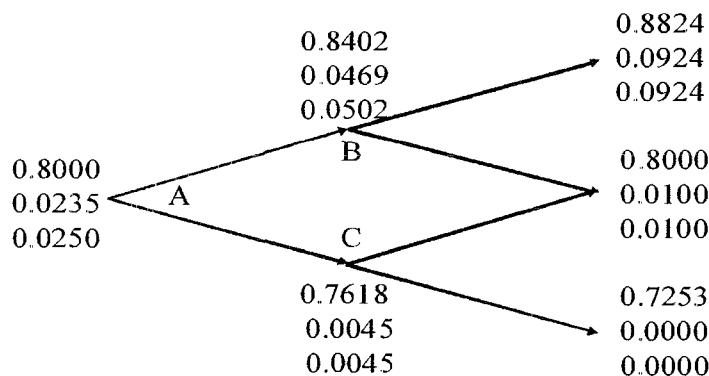


Figure S15.1 Tree to evaluate European and American put options in Problem 15.4.

At each node, upper number is the stock price; next number is the European put price; final number is the American put price

Problem 15.5.

Explain how corporations use range forward contracts to hedge their foreign exchange risk.

A range forward contract allows a corporation to ensure that the exchange rate applicable to a transaction will not be worse than one exchange rate and will not be better than another exchange rate. Depending on the exposure being hedged a range forward contract

is created by either a) buying a put with the lower exchange rate and selling a call with the higher exchange rate or b) selling a put with the lower exchange rate and buying a call with the higher exchange rate.

Problem 15.6.

Calculate the value of a three-month at-the-money European call option on a stock index when the index is at 250, the risk-free interest rate is 10% per annum, the volatility of the index is 18% per annum, and the dividend yield on the index is 3% per annum.

In this case, $S_0 = 250$, $K = 250$, $r = 0.10$, $\sigma = 0.18$, $T = 0.25$, $q = 0.03$ and

$$d_1 = \frac{\ln(250/250) + (0.10 - 0.03 + 0.18^2/2)0.25}{0.18\sqrt{0.25}} = 0.2394$$

$$d_2 = d_1 - 0.18\sqrt{0.25} = 0.1494$$

and the call price is

$$250N(0.2394)e^{-0.03 \times 0.25} - 250N(0.1494)e^{-0.10 \times 0.25}$$

$$= 250 \times 0.5946e^{-0.03 \times 0.25} - 250 \times 0.5594e^{-0.10 \times 0.25}$$

or 11.15.

Problem 15.7.

Calculate the value of an eight-month European put option on a currency with a strike price of 0.50. The current exchange rate is 0.52, the volatility of the exchange rate is 12%, the domestic risk-free interest rate is 4% per annum, and the foreign risk-free interest rate is 8% per annum.

In this case $S_0 = 0.52$, $K = 0.50$, $r = 0.04$, $r_f = 0.08$, $\sigma = 0.12$, $T = 0.6667$, and

$$d_1 = \frac{\ln(0.52/0.50) + (0.04 - 0.08 + 0.12^2/2)0.6667}{0.12\sqrt{0.6667}} = 0.1771$$

$$d_2 = d_1 - 0.12\sqrt{0.6667} = 0.0791$$

and the put price is

$$0.50N(-0.0791)e^{-0.04 \times 0.6667} - 0.52N(-0.1771)e^{-0.08 \times 0.6667}$$

$$= 0.50 \times 0.4685e^{-0.04 \times 0.6667} - 0.52 \times 0.4297e^{-0.08 \times 0.6667}$$

$$= 0.0162$$

Problem 15.8.

Show that the formula in equation (15.12) for a put option to sell one unit of currency A for currency B at strike price K gives the same value as equation (15.11) for a call option to buy K units of currency B for currency A at a strike price of $1/K$.

A put option to sell one unit of currency A for K units of currency B is worth

$$Ke^{-r_B T} N(-d_2) - S_0 e^{-r_A T} N(-d_1)$$

where

$$d_1 = \frac{\ln(S_0/K) + (r_A - r_B + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0/K) + (r_A - r_B - \sigma^2/2)T}{\sigma\sqrt{T}}$$

and r_A and r_B are the risk-free rates in currencies A and B, respectively. The value of the option is measured in units of currency B. Defining $S_0^* = 1/S_0$ and $K^* = 1/K$

$$d_1 = \frac{-\ln(S_0^*/K^*) - (r_A - r_B - \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{-\ln(S_0^*/K^*) - (r_A - r_B + \sigma^2/2)T}{\sigma\sqrt{T}}$$

The put price is therefore

$$S_0 K [S_0^* e^{-r_B T} N(d_1^*) - K^* e^{-r_A T} N(d_2^*)]$$

where

$$d_1^* = \frac{\ln(S_0^*/K^*) + (r_B - r_A - \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2^* = \frac{\ln(S_0^*/K^*) + (r_B - r_A + \sigma^2/2)T}{\sigma\sqrt{T}}$$

This shows that put option is equivalent to KS_0 call options to buy 1 unit of currency A for $1/K$ units of currency B. In this case the value of the option is measured in units of currency A. To obtain the call option value in units of currency B (the same units as the value of the put option was measured in) we must divide by S_0 . This proves the result.

Problem 15.9.

A foreign currency is currently worth \$1.50. The domestic and foreign risk-free interest rates are 5% and 9%, respectively. Calculate a lower bound for the value of a six-month call option on the currency with a strike price of \$1.40 if it is (a) European and (b) American.

Lower bound for European option is

$$S_0 e^{-r_f T} - K e^{-r T} = 1.5 e^{-0.09 \times 0.5} - 1.4 e^{-0.05 \times 0.5} = 0.069$$

Lower bound for American option is

$$S_0 - K = 0.10$$

Problem 15.10.

Consider a stock index currently standing at 250. The dividend yield on the index is 4% per annum, and the risk-free rate is 6% per annum. A three-month European call option on the index with a strike price of 245 is currently worth \$10. What is the value of a three-month put option on the index with a strike price of 245?

In this case $S_0 = 250$, $q = 0.04$, $r = 0.06$, $T = 0.25$, $K = 245$, and $c = 10$. Using put-call parity

$$c + Ke^{-rT} = p + S_0e^{-qT}$$

or

$$p = c + Ke^{-rT} - S_0e^{-qT}$$

Substituting:

$$p = 10 + 245e^{-0.25 \times 0.06} - 250e^{-0.25 \times 0.04} = 3.84$$

The put price is 3.84.

Problem 15.11.

An index currently stands at 696 and has a volatility of 30% per annum. The risk-free rate of interest is 7% per annum and the index provides a dividend yield of 4% per annum. Calculate the value of a three-month European put with an exercise price of 700.

In this case $S_0 = 696$, $K = 700$, $r = 0.07$, $\sigma = 0.3$, $T = 0.25$ and $q = 0.04$. The option can be valued using equation (15.5).

$$d_1 = \frac{\ln(696/700) + (0.07 - 0.04 + 0.09/2) \times 0.25}{0.3\sqrt{0.25}} = 0.0868$$

$$d_2 = d_1 - 0.3\sqrt{0.25} = -0.0632$$

and

$$N(-d_1) = 0.4654, \quad N(-d_2) = 0.5252$$

The value of the put, p , is given by:

$$p = 700e^{-0.07 \times 0.25} \times 0.5252 - 696e^{-0.04 \times 0.25} \times 0.4654 = 40.6$$

i.e., it is \$40.6.

Problem 15.12.

Show that if C is the price of an American call with exercise price K and maturity T on a stock paying a dividend yield of q , and P is the price of an American put on the same stock with the same strike price and exercise date,

$$S_0e^{-qT} - K < C - P < S_0 - Ke^{-rT}$$

where S_0 is the stock price, r is the risk-free rate, and $r > 0$. (Hint: To obtain the first half of the inequality, consider possible values of:

Portfolio A: a European call option plus an amount K invested at the risk-free rate

Portfolio B: an American put option plus e^{-qT} of stock with dividends being reinvested in the stock

To obtain the second half of the inequality, consider possible values of:

Portfolio C: an American call option plus an amount Ke^{-rT} invested at the risk-free rate

Portfolio D: a European put option plus one stock with dividends being reinvested in the stock)

Following the hint, we first consider

Portfolio A: A European call option plus an amount K invested at the risk-free rate

Portfolio B: An American put option plus e^{-qT} of stock with dividends being reinvested in the stock.

Portfolio A is worth $c + K$ while portfolio B is worth $P + S_0e^{-qT}$. If the put option is exercised at time τ ($0 \leq \tau < T$), portfolio B becomes:

$$K - S_\tau + S_\tau e^{-q(T-\tau)} \leq K$$

where S_τ is the stock price at time τ . Portfolio A is worth

$$c + Ke^{r\tau} \geq K$$

Hence portfolio A is worth at least as much as portfolio B. If both portfolios are held to maturity (time T), portfolio A is worth

$$\begin{aligned} & \max(S_T - K, 0) + Ke^{rT} \\ &= \max(S_T, K) + K(e^{rT} - 1) \end{aligned}$$

Portfolio B is worth $\max(S_T, K)$. Hence portfolio A is worth more than portfolio B.

Because portfolio A is worth at least as much as portfolio B in all circumstances

$$P + S_0e^{-qT} \leq c + K$$

Because $c \leq C$:

$$P + S_0e^{-qT} \leq C + K$$

or

$$S_0e^{-qT} - K \leq C - P$$

This proves the first part of the inequality.

For the second part consider:

Portfolio C: An American call option plus an amount Ke^{-rT} invested at the risk-free rate

Portfolio D: A European put option plus one stock with dividends being reinvested in the stock.

Portfolio C is worth $C + Ke^{-rT}$ while portfolio D is worth $p + S_0$. If the call option is exercised at time τ ($0 \leq \tau < T$) portfolio C becomes:

$$S_\tau - K + Ke^{-r(T-\tau)} < S_\tau$$

while portfolio D is worth

$$p + S_\tau e^{q(\tau-t)} \geq S_\tau$$

Hence portfolio D is worth more than portfolio C. If both portfolios are held to maturity (time T), portfolio C is worth $\max(S_T, K)$ while portfolio D is worth

$$\begin{aligned} & \max(K - S_T, 0) + S_T e^{qT} \\ &= \max(S_T, K) + S_T(e^{qT} - 1) \end{aligned}$$

Hence portfolio D is worth at least as much as portfolio C.

Since portfolio D is worth at least as much as portfolio C in all circumstances:

$$C + Ke^{-rT} \leq p + S_0$$

Since $p \leq P$:

$$C + Ke^{-rT} \leq P + S_0$$

or

$$C - P \leq S_0 - Ke^{-rT}$$

This proves the second part of the inequality. Hence:

$$S_0 e^{-qT} - K \leq C - P \leq S_0 - Ke^{-rT}$$

Problem 15.13.

Show that a European call option on a currency has the same price as the corresponding European put option on the currency when the forward price equals the strike price.

This follows from put-call parity and the relationship between the forward price, F_0 , and the spot price, S_0 .

$$c + Ke^{-rT} = p + S_0 e^{-r_f T}$$

and

$$F_0 = S_0 e^{(r-r_f)T}$$

so that

$$c + Ke^{-rT} = p + F_0 e^{-rT}$$

If $K = F_0$ this reduces to $c = p$. The result that $c = p$ when $K = F_0$ is true for options on all underlying assets, not just options on currencies. An at-the-money option is frequently defined as one where $K = F_0$ (or $c = p$) rather than one where $K = S_0$.

Problem 15.14.

Would you expect the volatility of a stock index to be greater or less than the volatility of a typical stock? Explain your answer.

The volatility of a stock index can be expected to be less than the volatility of a typical stock. This is because some risk (i.e., return uncertainty) is diversified away when a portfolio of stocks is created. In capital asset pricing model terminology, there exists systematic and unsystematic risk in the returns from an individual stock. However, in a stock index, unsystematic risk has been diversified away and only the systematic risk contributes to volatility.

Problem 15.15.

Does the cost of portfolio insurance increase or decrease as the beta of a portfolio increases? Explain your answer.

The cost of portfolio insurance increases as the beta of the portfolio increases. This is because portfolio insurance involves the purchase of a put option on the portfolio. As beta increases, the volatility of the portfolio increases causing the cost of the put option to increase. When index options are used to provide portfolio insurance, both the number of options required and the strike price increase as beta increases.

Problem 15.16.

Suppose that a portfolio is worth \$60 million and the S&P 500 is at 1200. If the value of the portfolio mirrors the value of the index, what options should be purchased to provide protection against the value of the portfolio falling below \$54 million in one year's time?

If the value of the portfolio mirrors the value of the index, the index can be expected to have dropped by 10% when the value of the portfolio drops by 10%. Hence when the value of the portfolio drops to \$54 million the value of the index can be expected to be 1080. This indicates that put options with an exercise price of 1080 should be purchased. The options should be on:

$$\frac{60,000,000}{1200} = \$50,000$$

times the index. Each option contract is for \$100 times the index. Hence 500 contracts should be purchased.

Problem 15.17.

Consider again the situation in Problem 15.16. Suppose that the portfolio has a beta of 2.0, the risk-free interest rate is 5% per annum, and the dividend yield on both the portfolio and the index is 3% per annum. What options should be purchased to provide protection against the value of the portfolio falling below \$54 million in one year's time?

When the value of the portfolio falls to \$54 million the holder of the portfolio makes a capital loss of 10%. After dividends are taken into account the loss is 7% during the year. This is 12% below the risk-free interest rate. According to the capital asset pricing model:

$$\begin{array}{l} \text{Excess expected return of portfolio} \\ \text{above riskless interest rate} \end{array} = \beta \times \begin{array}{l} \text{Excess expected return of market} \\ \text{above riskless interest rate} \end{array}$$

Therefore, when the portfolio provides a return 12% below the risk-free interest rate, the market's expected return is 6% below the risk-free interest rate. As the index can be assumed to have a beta of 1.0, this is also the excess expected return (including dividends) from the index. The expected return from the index is therefore -1% per annum. Since the index provides a 3% per annum dividend yield, the expected movement in the index is -4% . Thus when the portfolio's value is \$54 million the expected value of the index $0.96 \times 1200 = 1152$. Hence European put options should be purchased with an exercise price of 1152. Their maturity date should be in one year.

The number of options required is twice the number required in Problem 15.16. This is because we wish to protect a portfolio which is twice as sensitive to changes in market conditions as the portfolio in Problem 15.16. Hence options on \$100,000 (or 1,000 contracts) should be purchased. To check that the answer is correct consider what happens when the value of the portfolio declines by 20% to \$48 million. The return including dividends is -17% . This is 22% less than the risk-free interest rate. The index can be expected to provide a return (including dividends) which is 11% less than the risk-free interest rate, i.e. a return of -6% . The index can therefore be expected to drop by 9% to 1092. The payoff from the put options is $(1152 - 1092) \times 100,000 = \6 million. This is exactly what is required to restore the value of the portfolio to \$54 million.

Problem 15.18.

An index currently stands at 1,500. European call and put options with a strike price of 1,400 and time to maturity of six months have market prices of 154.00 and 34.25, respectively. The six-month risk-free rate is 5%. What is the implied dividend yield?

The implied dividend yield is the value of q that satisfies the put-call parity equation. It is the value of q that solves

$$154 + 1400e^{-0.05 \times 0.5} = 34.25 + 1500e^{-0.5q}$$

This is 1.99%.

Problem 15.19.

A total return index tracks the return, including dividends, on a certain portfolio. Explain how you would value (a) forward contracts and (b) European options on the index.

A total return index behaves like a stock paying no dividends. In a risk-neutral world it can be expected to grow on average at the risk-free rate. Forward contracts and options on total return indices should be valued in the same way as forward contracts and options on non-dividend-paying stocks.

Problem 15.20.

What is the put-call parity relationship for European currency options

The put-call parity relationship for European currency options is

$$c + Ke^{-rT} = p + Se^{-r_f T}$$

To prove this result, the two portfolios to consider are:

Portfolio A: one call option plus one discount bond which will be worth K at time T

Portfolio B: one put option plus $e^{-r_f T}$ of foreign currency invested at the foreign risk-free interest rate.

Both portfolios are worth $\max(S_T, K)$ at time T . They must therefore be worth the same today. The result follows.

Problem 15.21.

Can an option on the yen-euro exchange rate be created from two options, one on the dollar-euro exchange rate, and the other on the dollar-yen exchange rate? Explain your answer.

There is no way of doing this. A natural idea is to create an option to exchange K euros for one yen from an option to exchange Y dollars for 1 yen and an option to exchange K euros for Y dollars. The problem with this is that it assumes that either both options are exercised or that neither option is exercised. There are always some circumstances where the first option is in-the-money at expiration while the second is not and vice versa.

Problem 15.22.

Prove the results in equation (15.1), (15.2), and (15.3) using the portfolios indicated.

In portfolio A, the cash, if it is invested at the risk-free interest rate, will grow to K at time T . If $S_T > K$, the call option is exercised at time T and portfolio A is worth S_T . If $S_T < K$, the call option expires worthless and the portfolio is worth K . Hence, at time T , portfolio A is worth

$$\max(S_T, K)$$

Because of the reinvestment of dividends, portfolio B becomes one share at time T . It is, therefore, worth S_T at this time. It follows that portfolio A is always worth as much as, and is sometimes worth more than, portfolio B at time T . In the absence of arbitrage opportunities, this must also be true today. Hence,

$$c + Ke^{-rT} \geq S_0e^{-qT}$$

or

$$c \geq S_0e^{-qT} - Ke^{-rT}$$

This proves equation (15.1)

In portfolio C, the reinvestment of dividends means that the portfolio is one put option plus one share at time T . If $S_T < K$, the put option is exercised at time T and portfolio C is worth K . If $S_T > K$, the put option expires worthless and the portfolio is worth S_T . Hence, at time T , portfolio C is worth

$$\max(S_T, K)$$

Portfolio D is worth K at time T . It follows that portfolio C is always worth as much as, and is sometimes worth more than, portfolio D at time T . In the absence of arbitrage opportunities, this must also be true today. Hence,

$$p + S_0 e^{-qT} \geq K e^{-rT}$$

or

$$p \geq K e^{-rT} - S_0 e^{-qT}$$

This proves equation (15.2)

Portfolios A and C are both worth $\max(S_T, K)$ at time T . They must, therefore, be worth the same today, and the put-call parity result in equation (15.3) follows.

ASSIGNMENT QUESTIONS

Problem 15.23.

The Dow Jones Industrial Average on January 12, 2007 was 12,556 and the price of the March 126 call was \$2.25. Use the DerivaGem software to calculate the implied volatility of this option. Assume that the risk-free rate was 5.3% and the dividend yield was 3%. The option expires on March 20, 2007. Estimate the price of a March 126 put. What is the volatility implied by the price you estimate for this option? (Note that options are on the Dow Jones index divided by 100.)

Options on the DJIA are European. There are 47 trading days between January 12, 2007 and March 20, 2007. Setting the time to maturity equal to $47/252 = 0.1865$, DerivaGem gives the implied volatility as 10.23%. (If instead we use calendar days the time to maturity is $67/365 = 0.1836$ and the implied volatility is 10.33%.)

From put call parity (equation 13.3) the price of the put, p , (using trading time) is given by

$$2.25 + 126e^{-0.053 \times 0.1865} = p + 125.56e^{-0.03 \times 0.1865}$$

so that $p = 2.1512$. DerivaGem shows that the implied volatility is 10.23% (as for the call). (If calendar time is used the price of the put is 2.1597 and the implied volatility is 10.33% as for the call.)

A European call has the same implied volatility as a European put when both have the same strike price and time to maturity. This is formally proved in the appendix to Chapter 17.

Problem 15.24.

A stock index currently stands at 300 and has a volatility of 20%. The risk-free interest rate is 8% and the dividend yield on the index is 3%. Use a three-step binomial tree to value a six-month put option on the index with a strike price of 300 if it is (a) European and (b) American?

As shown by DerivaGem the value of the European option is 14.39 and the value of the American option is 14.97.

Problem 15.25.

Suppose that the spot price of the Canadian dollar is U.S. \$0.85 and that the Canadian dollar/U.S. dollar exchange rate has a volatility of 4% per annum. The risk-free rates of interest in Canada and the United States are 4% and 5% per annum, respectively. Calculate the value of a European call option to buy one Canadian dollar for U.S. \$0.85 in nine months. Use put-call parity to calculate the price of a European put option to sell one Canadian dollar for U.S. \$0.85 in nine months. What is the price of a call option to buy U.S. \$0.85 with one Canadian dollar in nine months?

In this case $S_0 = 0.85$, $K = 0.85$, $r = 0.05$, $r_f = 0.04$, $\sigma = 0.04$ and $T = 0.75$. The option can be valued using equation (15.11)

$$d_1 = \frac{\ln(0.85/0.85) + (0.05 - 0.04 + 0.0016/2) \times 0.75}{0.04\sqrt{0.75}} = 0.2338$$

$$d_2 = d_1 - 0.04\sqrt{0.75} = 0.1992$$

and

$$N(d_1) = 0.5924, \quad N(d_2) = 0.5789$$

The value of the call, c , is given by

$$c = 0.85e^{-0.04 \times 0.75} \times 0.5924 - 0.85e^{-0.05 \times 0.75} \times 0.5789 = 0.0147$$

i.e., it is 1.47 cents. From put-call parity

$$p + S_0e^{-r_f T} = c + Ke^{-rT}$$

so that

$$p = 0.0147 + 0.85e^{-0.05 \times 9/12} - 0.85e^{-0.04 \times 9/12} = 0.00854$$

The option to buy US\$0.85 with C\$1.00 is the same as the same as an option to sell one Canadian dollar for US\$0.85. This means that it is a put option on the Canadian dollar and its price is US\$0.00854.

Problem 15.26.

A mutual fund announces that the salaries of its fund managers will depend on the performance of the fund. If the fund loses money, the salaries will be zero. If the fund makes a profit, the salaries will be proportional to the profit. Describe the salary of a fund manager as an option. How is a fund manager motivated to behave with this type of remuneration package?

Suppose that K is the value of the fund at the beginning of the year and S_T is the value of the fund at the end of the year.

The salary of a fund manager is

$$\alpha \max(S_T - K, 0)$$

where α is a constant.

This shows that a fund manager has a call option on the value of the fund at the end of the year. All of the parameters determining the value of this call option are outside the control of the fund manager except the volatility of the fund. The fund manager has an incentive to make the fund as volatile as possible! This may not correspond with the desires of the investors. One way of making the fund highly volatile would be by investing only in high-beta stocks. Another would be by using the whole fund to buy call options on a market index.

It might be argued that a fund manager would not do this because of the risk which the manager faces. If the fund earns a negative return the manager's salary is zero. However, a fund manager could hedge the risk of a negative return by, on his or her own account, taking a short position in call options on a stock market index.

The position could be chosen so that if the market goes up, the gain on salary more than offsets the losses on the call options.

If the market goes down the fund manager ends up with the price received for the call options. It is easy to see that the strategy becomes more attractive as the riskiness of the fund's portfolio increases.

To summarize, the (superficially attractive) remuneration package is open to abuse and does not necessarily motivate the fund managers to act in the best interests of the fund's investors.

Problem 15.27.

Assume that the price of currency A expressed in terms of the price of currency B follows the process

$$dS = (r_B - r_A)S dt + \sigma S dz$$

where r_A is the risk-free interest rate in currency A and r_B is the risk-free interest rate in currency B. What is the process followed by the price of currency B expressed in terms of currency A?

The price of currency B expressed in terms of currency A is $1/S$. From Ito's lemma the process followed by $X = 1/S$ is

$$dX = [(r_B - r_A)S \times (-1/S^2) + 0.5\sigma^2 S^2 \times (2/S^3)]dt + \sigma S \times (-1/S^2)dz$$

or

$$dX = [r_A - r_B + \sigma^2]Xdt - \sigma Xdz$$

This is Siegel's paradox and is discussed further in Business Snapshot 29.1.

Problem 15.28.

The three-month forward USD/euro exchange rate is 1.3000. The exchange rate volatility is 15%. A US company will have to pay 1 million euros in three months. The

euro and USD risk-free rates are 5% and 4%, respectively. The company decides to use a range forward contract with the lower strike price equal to 1.2500.

(a) What should the higher strike price be to create a zero-cost contract.

(b) What position in calls and puts should the company take.

(c) Does your answer depend on the euro risk-free rate? Explain.

(d) Does your answer depend on the USD risk-free rate? Explain.

(a) A put with a strike price of 1.25 is worth \$0.019. By trial and error DerivaGem can be used to show that the strike price of a call that leads to a call having a price of \$0.019 is 1.3477. This is the higher strike price to create a zero cost contract.

(b) The company should sell a put with strike price 1.25 and buy a call with strike price 1.3477. This ensures that the exchange rate it pays for the euros is between 1.2500 and 1.3477.

(c) The answer does depend on the euro risk-free rate because the forward exchange rate depends on this rate

(d) The answer does depend on the dollar risk-free rate because the forward exchange rate depends on this rate. However, if the interest rates change so that the spread between the dollar and euro interest rates remains the same, the upper strike price is unchanged at 1.3477. This can be seen from equations (15.13) and (15.14). The forward exchange rate, F_0 is unchanged and changing r has the same percentage effect on both the call and the put.

CHAPTER 16

Futures Options

Notes for the Instructor

The chapter concerned with options on stock indices, currencies, and futures in the sixth edition has been split into two chapters (15 and 16) in the seventh edition. Chapter 15 is concerned with options on stock indices and currencies; Chapter 16 is concerned with options on futures.

The material on futures options has been restructured for the seventh edition. The chapter now spends more time discussing how Black's model can be used to price European options in terms of forward or futures prices. This is important material because in practice it is usually the case that practitioners use Black's model rather than Black-Scholes model for European options. By doing this they avoid the need to estimate the income on the underlying asset explicitly. (The Black's model material in this chapter is extended to the stochastic interest rate case in Section 27.6.) This chapter also discusses futures style options which are becoming popular at some exchanges. A futures style option is a futures contract on the payoff from an option.

The way I approach the material in the chapter is indicated by the slides. I like to use Problem 16.23 as a hand-in assignment because it provides practice using DerivaGem and links in with the material on volatility smiles in Chapter 18 and the material on American options in Chapter 19.

QUESTIONS AND PROBLEMS

Problem 16.1

Explain the difference between a call option on yen and a call option on yen futures.

A call option on yen gives the holder the right to buy yen in the spot market at an exchange rate equal to the strike price. A call option on yen futures gives the holder the right to receive the amount by which the futures price exceeds the strike price. If the yen futures option is exercised, the holder also obtains a long position in the yen futures contract.

Problem 16.2.

Why are options on bond futures more actively traded than options on bonds?

The main reason is that a bond futures contract is a more liquid instrument than a bond. The price of a Treasury bond futures contract is known immediately from trading on CBOT. The price of a bond can be obtained only by contacting dealers.

Problem 16.3.

“A futures price is like a stock paying a dividend yield.” What is the dividend yield?

A futures price behaves like a stock paying a dividend yield at the risk-free interest rate.

Problem 16.4.

A futures price is currently 50. At the end of six months it will be either 56 or 46. The risk-free interest rate is 6% per annum. What is the value of a six-month European call option with a strike price of 50?

In this case $u = 1.12$ and $d = 0.92$. The probability of an up movement in a risk-neutral world is

$$\frac{1 - 0.92}{1.12 - 0.92} = 0.4$$

From risk-neutral valuation, the value of the call is

$$e^{-0.06 \times 0.5} (0.4 \times 6 + 0.6 \times 0) = 2.33$$

Problem 16.5.

How does the put–call parity formula for a futures option differ from put–call parity for an option on a non-dividend-paying stock?

The put–call parity formula for futures options is the same as the put–call parity formula for stock options except that the stock price is replaced by $F_0 e^{-rT}$, where F_0 is the current futures price, r is the risk-free interest rate, and T is the life of the option.

Problem 16.6.

Consider an American futures call option where the futures contract and the option contract expire at the same time. Under what circumstances is the futures option worth more than the corresponding American option on the underlying asset?

The American futures call option is worth more than the corresponding American option on the underlying asset when the futures price is greater than the spot price prior to the maturity of the futures contract. This is the case when the risk-free rate is greater than the income on the asset plus the convenience yield.

Problem 16.7.

Calculate the value of a five-month European put futures option when the futures price is \$19, the strike price is \$20, the risk-free interest rate is 12% per annum, and the volatility of the futures price is 20% per annum.

In this case $F_0 = 19$, $K = 20$, $r = 0.12$, $\sigma = 0.20$, and $T = 0.4167$. The value of the European put futures option is

$$20N(-d_2)e^{-0.12 \times 0.4167} - 19N(-d_1)e^{-0.12 \times 0.4167}$$

where

$$d_1 = \frac{\ln(19/20) + (0.04/2)0.4167}{0.2\sqrt{0.4167}} = -0.3327$$

$$d_2 = d_1 - 0.2\sqrt{0.4167} = -0.4618$$

This is

$$\begin{aligned} & e^{-0.12 \times 0.4167} [20N(0.4618) - 19N(0.3327)] \\ &= e^{-0.12 \times 0.4167} (20 \times 0.6778 - 19 \times 0.6303) \\ &= 1.50 \end{aligned}$$

or \$1.50.

Problem 16.8.

Suppose you buy a put option contract on October gold futures with a strike price of \$700 per ounce. Each contract is for the delivery of 100 ounces. What happens if you exercise when the October futures price is \$680?

An amount $(700 - 680) \times 100 = \$2,000$ is added to your margin account and you acquire a short futures position giving you the right to sell 100 ounces of gold in October. This position is marked to market in the usual way until you choose to close it out.

Problem 16.9.

Suppose you sell a call option contract on April live cattle futures with a strike price of 90 cents per pound. Each contract is for the delivery of 40,000 pounds. What happens if the contract is exercised when the futures price is 95 cents?

In this case an amount $(0.95 - 0.90) \times 40,000 = \$2,000$ is subtracted from your margin account and you acquire a short position in a live cattle futures contract to sell 40,000 pounds of cattle in April. This position is marked to market in the usual way until you choose to close it out.

Problem 16.10.

Consider a two-month call futures option with a strike price of 40 when the risk-free interest rate is 10% per annum. The current futures price is 47. What is a lower bound for the value of the futures option if it is (a) European and (b) American?

Lower bound if option is European is

$$(F_0 - K)e^{-rT} = (47 - 40)e^{-0.1 \times 2/12} = 6.88$$

Lower bound if option is American is

$$F_0 - K = 7$$

Problem 16.11.

Consider a four-month put futures option with a strike price of 50 when the risk-free interest rate is 10% per annum. The current futures price is 47. What is a lower bound for the value of the futures option if it is (a) European and (b) American?

Lower bound if option is European is

$$(K - F_0)e^{-rT} = (50 - 47)e^{-0.1 \times 4/12} = 2.90$$

Lower bound if option is American is

$$K - F_0 = 3$$

Problem 16.12.

A futures price is currently 60 and its volatility is 30%. The risk-free interest rate is 8% per annum. Use a two-step binomial tree to calculate the value of a six-month call option on the futures with a strike price of 60. If the call were American, would it ever be worth exercising it early?

In this case $u = e^{0.3 \times \sqrt{0.25}} = 1.1618$ and $d = 1/u = 0.8607$ the risk-neutral probability of an up move is

$$p = \frac{1 - 0.8607}{1.1618 - 0.8607} = 0.4626$$

In the tree shown in Figure S16.1 the middle number at each node is the price of the European option and the lower number is the price of the American option. The tree shows that the price of the European option is 4.3155 and the price of the American option is 4.4026. The American option should sometimes be exercised early.

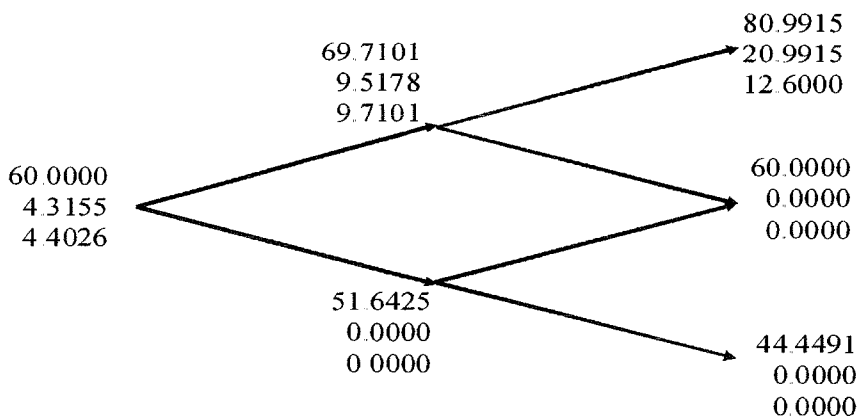


Figure S16.1 Tree to evaluate European and American call options in Problem 16.12.

Problem 16.13.

In Problem 16.12 what is the value of a six-month European put option on futures with a strike price of 60? If the put were American, would it ever be worth exercising it early? Verify that the call prices calculated in Problem 16.12 and the put prices calculated here satisfy put-call parity relationships.

The parameters u , d , and p are the same as in Problem 16.12. The tree in Figure S16.2 shows that the price of the European option is 3.0265 while the price of the American option is 3.0847.

Because $c = p$ and $F_0 = K$ the put-call parity relationship in equation (16.1) clearly holds. For the American option prices we have:

$$C - P = 0; \quad F_0 e^{-rT} - K = -2.353; \quad F_0 - K e^{-rT} = 2.353$$

The put-call inequalities for American options in equation (16.2) are therefore satisfied

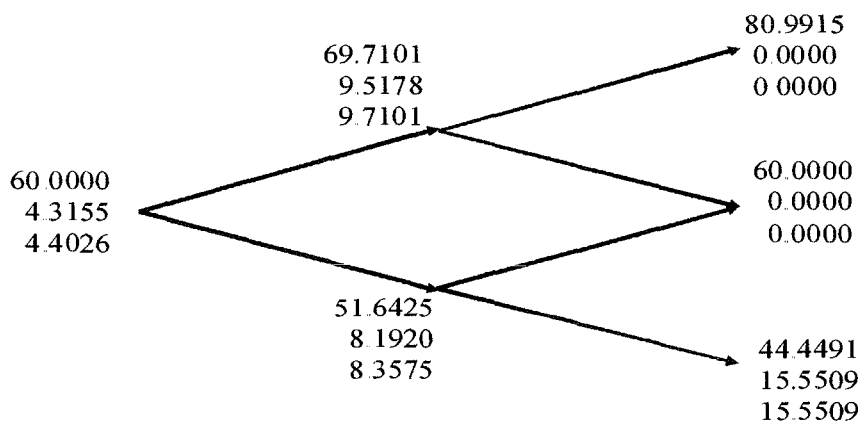


Figure S16.2 Tree to evaluate European and American put options in Problem 16.13.

Problem 16.14.

A futures price is currently 25, its volatility is 30% per annum, and the risk-free interest rate is 10% per annum. What is the value of a nine-month European call on the futures with a strike price of 26?

In this case $F_0 = 25$, $K = 26$, $\sigma = 0.3$, $r = 0.1$, $T = 0.75$

$$d_1 = \frac{\ln(F_0/K) + \sigma^2 T/2}{\sigma \sqrt{T}} = -0.0211$$

$$d_2 = \frac{\ln(F_0/K) - \sigma^2 T/2}{\sigma \sqrt{T}} = -0.2809$$

$$c = e^{-0.075} [25N(-0.0211) - 26N(-0.2809)]$$

$$= e^{-0.075} [25 \times 0.4916 - 26 \times 0.3894] = 2.01$$

Problem 16.15.

A futures price is currently 70, its volatility is 20% per annum, and the risk-free interest rate is 6% per annum. What is the value of a five-month European put on the futures with a strike price of 65?

In this case $F_0 = 70$, $K = 65$, $\sigma = 0.2$, $r = 0.06$, $T = 0.4167$

$$d_1 = \frac{\ln(F_0/K) + \sigma^2 T/2}{\sigma\sqrt{T}} = 0.6386$$

$$d_2 = \frac{\ln(F_0/K) - \sigma^2 T/2}{\sigma\sqrt{T}} = 0.5095$$

$$p = e^{-0.025} [65N(-0.5095) - 70N(-0.6386)]$$

$$= e^{-0.025} [65 \times 0.3052 - 70 \times 0.2615] = 1.495$$

Problem 16.16.

Suppose that a one-year futures price is currently 35. A one-year European call option and a one-year European put option on the futures with a strike price of 34 are both priced at 2 in the market. The risk-free interest rate is 10% per annum. Identify an arbitrage opportunity.

In this case

$$c + Ke^{-rT} = 2 + 34e^{-0.1 \times 1} = 32.76$$

$$p + F_0e^{-rT} = 2 + 35e^{-0.1 \times 1} = 33.67$$

Put-call parity shows that we should buy one call, short one put and short a futures contract. This costs nothing up front. In one year, either we exercise the call or the put is exercised against us. In either case, we buy the asset for 34 and close out the futures position. The gain on the short futures position is $35 - 34 = 1$.

Problem 16.17.

“The price of an at-the-money European call futures option always equals the price of a similar at-the-money European put futures option.” Explain why this statement is true.

The put price is

$$e^{-rT} [KN(-d_2) - F_0N(-d_1)]$$

Because $N(-x) = 1 - N(x)$ for all x the put price can also be written

$$e^{-rT} [K - KN(d_2) - F_0 + F_0N(d_1)]$$

Because $F_0 = K$ this is the same as the call price:

$$e^{-rT} [F_0N(d_1) - KN(d_2)]$$

This result can also be proved from put–call parity showing that it is not model dependent.

Problem 16.18.

Suppose that a futures price is currently 30. The risk-free interest rate is 5% per annum. A three-month American call futures option with a strike price of 28 is worth 4. Calculate bounds for the price of a three-month American put futures option with a strike price of 28.

From equation (16.2), $C - P$ must lie between

$$30e^{-0.05 \times 3/12} - 28 = 1.63$$

and

$$30 - 28e^{-0.05 \times 3/12} = 2.35$$

Because $C = 4$ we must have $1.63 < 4 - P < 2.35$ or

$$1.65 < P < 2.37$$

Problem 16.19.

Show that if C is the price of an American call option on a futures contract when the strike price is K and the maturity is T , and P is the price of an American put on the same futures contract with the same strike price and exercise date,

$$F_0 e^{-rT} - K < C - P < F_0 - K e^{-rT}$$

where F_0 is the futures price and r is the risk-free rate. Assume that $r > 0$ and that there is no difference between forward and futures contracts. (Hint: Use an analogous approach to that indicated for Problem 15.12.)

In this case we consider

Portfolio A: A European call option on futures plus an amount K invested at the risk-free interest rate

Portfolio B: An American put option on futures plus an amount $F_0 e^{-rT}$ invested at the risk-free interest rate plus a long futures contract maturing at time T .

Following the arguments in Chapter 5 we will treat all futures contracts as forward contracts. Portfolio A is worth $c + K$ while portfolio B is worth $P + F_0 e^{-rT}$. If the put option is exercised at time τ ($0 \leq \tau < T$), portfolio B is worth

$$\begin{aligned} & K - F_\tau + F_0 e^{-r(T-\tau)} + F_\tau - F_0 \\ & = K + F_0 e^{-r(T-\tau)} - F_0 < K \end{aligned}$$

at time τ where F_τ is the futures price at time τ . Portfolio A is worth

$$c + K e^{r\tau} \geq K$$

Hence Portfolio A more than Portfolio B. If both portfolios are held to maturity (time T), Portfolio A is worth

$$\begin{aligned} & \max(F_T - K, 0) + Ke^{rT} \\ &= \max(F_T, K) + K(e^{rT} - 1) \end{aligned}$$

Portfolio B is worth

$$\max(K - F_T, 0) + F_0 + F_T - F_0 = \max(F_T, K)$$

Hence portfolio A is worth more than portfolio B.

Because portfolio A is worth more than portfolio B in all circumstances:

$$P + F_0e^{-r(T-t)} < c + K$$

Because $c \leq C$ it follows that

$$P + F_0e^{-rT} < C + K$$

or

$$F_0e^{-rT} - K < C - P$$

This proves the first part of the inequality.

For the second part of the inequality consider:

Portfolio C: An American call futures option plus an amount Ke^{-rT} invested at the risk-free interest rate

Portfolio D: A European put futures option plus an amount F_0 invested at the risk-free interest rate plus a long futures contract.

Portfolio C is worth $C + Ke^{-rT}$ while portfolio D is worth $p + F_0$. If the call option is exercised at time τ ($0 \leq \tau < T$) portfolio C becomes:

$$F_\tau - K + Ke^{-r(T-\tau)} < F_\tau$$

while portfolio D is worth

$$\begin{aligned} & p + F_0e^{r\tau} + F_\tau - F_0 \\ &= p + F_0(e^{r\tau} - 1) + F_\tau \geq F_\tau \end{aligned}$$

Hence portfolio D is worth more than portfolio C. If both portfolios are held to maturity (time T), portfolio C is worth $\max(F_T, K)$ while portfolio D is worth

$$\begin{aligned} & \max(K - F_T, 0) + F_0e^{rT} + F_T - F_0 \\ &= \max(K, F_T) + F_0(e^{rT} - 1) \\ &> \max(K, F_T) \end{aligned}$$

Hence portfolio D is worth more than portfolio C.

Because portfolio D is worth more than portfolio C in all circumstances

$$C + Ke^{-rT} < p + F_0$$

Because $p \leq P$ it follows that

$$C + Ke^{-rT} < P + F_0$$

or

$$C - P < F_0 - Ke^{-rT}$$

This proves the second part of the inequality. The result:

$$F_0e^{-rT} - K < C - P < F_0 - Ke^{-rT}$$

has therefore been proved.

Problem 16.20.

Calculate the price of a three-month European call option on the spot price of silver. The three-month futures price is \$12, the strike price is \$13, the risk-free rate is 4%, and the volatility of the price of silver is 25%.

This has the same value as a three-month call option on silver futures where the futures contract expires in three months. It can therefore be valued using equation (16.9) with $F_0 = 12$, $K = 13$, $r = 0.04$, $\sigma = 0.25$ and $T = 0.25$. The value is 0.244.

Problem 16.21.

A corporation knows that in three months it will have \$5 million to invest for 90 days at LIBOR minus 50 basis points and wishes to ensure that the rate obtained will be at least 6.5%. What position in exchange-traded interest-rate options should the corporation take?

The rate received will be less than 6.5% when LIBOR is less than 7%. The corporation requires a three-month call option on a Eurodollar futures option with a strike price of 93. If three-month LIBOR is greater than 7% at the option maturity, the Eurodollar futures quote at option maturity will be less than 93 and there will be no payoff from the option. If the three-month LIBOR is less than 7%, one Eurodollar futures options provide a payoff of \$25 per 0.01%. Each 0.01% of interest costs the corporation \$500 ($= 5,000,000 \times 0.0001$). A total of $500/25 = 20$ contracts are therefore required.

ASSIGNMENT QUESTIONS

Problem 16.22.

A futures price is currently 40. It is known that at the end of three months the price will be either 35 or 45. What is the value of a three-month European call option on the futures with a strike price of 42 if the risk-free interest rate is 7% per annum?

In this case $u = 1.125$ and $d = 0.875$. The risk-neutral probability of an up move is

$$(1 - .875)/(1.125 - 0.875) = 0.5$$

The value of the option is

$$e^{-0.07 \times 0.25} [0.5 \times 3 + 0.5 \times 0] = 1.474$$

Problem 16.23.

It is February 4. July call options on corn futures with strike prices of 260, 270, 280, 290, and 300 cost 26.75, 21.25, 17.25, 14.00, and 11.375, respectively. July put options with these strike prices cost 8.50, 13.50, 19.00, 25.625, and 32.625, respectively. The options mature on June 19, the current July corn futures price is 278.25, and the risk-free interest rate is 1.1%. Calculate implied volatilities for the options using DerivaGem. Comment on the results you get.

There are 135 days to maturity (assuming this is not a leap year). Using DerivaGem with $F_0 = 278.25$, $r = 1.1\%$, $T = 135/365$, and 500 time steps gives the implied volatilities shown in the table below.

Strike Price	Call Price	Put Price	Call Imp Vol	Put Imp Vol
260	26.75	8.50	24.69	24.59
270	21.25	13.50	25.40	26.14
280	17.25	19.00	26.85	26.86
290	14.00	25.625	28.11	27.98
300	11.375	32.625	29.24	28.57
310	9.25		34.32	

We do not expect put-call parity to hold exactly for American options and so there is no reason why the implied volatility of a call should be exactly the same as the implied volatility of a put. Nevertheless it is reassuring that they are close.

There is a tendency for high strike price options to have a higher implied volatility. As explained in Chapter 18, this is an indication that the probability distribution for corn futures prices in the future has a heavier right tail and less heavy left tail than the lognormal distribution.

Problem 16.24.

Calculate the implied volatility of soybean futures prices from the following information concerning a European put on soybean futures:

Current futures price	525
Exercise price	525
Risk-free rate	6% per annum
Time to maturity	5 months
Put price	20

In this case $F_0 = 525$, $K = 525$, $r = 0.06$, $T = 0.4167$. We wish to find the value of σ for which $p = 20$ where:

$$p = Ke^{-rT} N(-d_2) - F_0 e^{-rT} N(-d_1)$$

This must be done by trial and error. When $\sigma = 0.2$, $p = 26.36$. When $\sigma = 0.15$, $p = 19.78$. When $\sigma = 0.155$, $p = 20.44$. When $\sigma = 0.152$, $p = 20.04$. These calculations show that the implied volatility is approximately 15.2% per annum.

Problem 16.25.

Calculate the price of a six-month European put option on the spot value of the S&P 500. The six-month forward price of the index is 1,400, the strike price is 1,450, the risk-free rate is 5%, and the volatility of the index is 15%.

The price of the option is the same as the price of a European put option on the forward price of the index where the forward contract has a maturity of six months. It is given by equation (16.10) with $F_0 = 1400$, $K = 1450$, $r = 0.05$, $\sigma = 0.15$, and $T = 0.5$. It is 86.35.

CHAPTER 17

The Greek Letters

Notes for the Instructor

This chapter covers the way in which traders working for financial institutions and market makers on the floor of an exchange hedge portfolio of derivatives. Students generally enjoy the chapter. The software, DerivaGem for Excel, can be used to demonstrate the relationships between any of the Greek letters and variables such as S_0 , K , r , σ , and T .

The chapter has been restructured for the seventh edition. Up to Section 17.12 the presentation now focuses on the calculation of Greek letters for stocks. Section 17.12 then extends the results to other underlying assets (stock indices, currencies and futures). Section 17.12 also covers the difference between the delta of futures and forward contracts. This restructuring, suggested by an instructor who adopted the book, creates a significant improvement in the way the material is presented.

It is important to make sure that students understand what is meant by hedging and in particular what constitutes a good hedge. A financial institution is well hedged with respect to an underlying variable if its wealth position is largely unaffected by changes in the value of the variable. The naked positions and covered positions described in Section 17.2 are clearly not perfect hedges. The deceptively simple stop-loss rule in Section 17.3 is also far from perfect. Delta hedging works better. In fact, it works perfectly if volatility is constant and the position in the underlying asset is changed continuously. In practice of course positions cannot be changed continuously and volatility is not constant so that delta hedging is than less than perfect (See Tables 17.2, 17.3 and 17.4 for the impact of discrete rebalancing.)

I spend some time on Figure 17.7. It shows that the error in delta hedging depends on the curvature of the relationship between the derivative's price and the price of the underlying asset. This observation provides a lead in to gamma, which measures curvature. I find it worth going through a numerical example to show how a portfolio that is both gamma-neutral and delta-neutral can be constructed.

Theta is not the same type of hedge statistic as delta and gamma because there is no uncertainty about the rate at which time will pass. It is an interesting description of one aspect of a portfolio of derivatives. When delta is zero, equation (17.4) shows that theta is a proxy for gamma. When gamma is large and negative theta is large and positive, and vice versa.

Whereas gamma hedging protects the hedger against the fact that the hedge can only be adjusted discretely (e.g. every day or every week), vega hedging protects against volatility changes. Generally gamma is more important for short-life options while vega is more important for long-life options. This is clear from the equations for them. Gamma is proportional to $1/\sqrt{T}$; vega is proportional to \sqrt{T} .

Portfolio insurance usually generates a lively discussion—particularly if students are familiar with the details of the October 19, 1987 crash. It is important to explain that

portfolio insurance involves creating a long position in an option synthetically. By contrast, hedging a long option position involves creating a short position in the option synthetically.

Problems 17.24, 17.25, 17.26, 17.27 (more difficult), and 17.30 (more difficult) all make good assignment questions.

QUESTIONS AND PROBLEMS

Problem 17.1.

Explain how a stop-loss hedging scheme can be implemented for the writer of an out-of-the-money call option. Why does it provide a relatively poor hedge?

Suppose the strike price is 10.00. The option writer aims to be fully covered whenever the option is in the money and naked whenever it is out of the money. The option writer attempts to achieve this by buying the assets underlying the option as soon as the asset price reaches 10.00 from below and selling as soon as the asset price reaches 10.00 from above. The trouble with this scheme is that it assumes that when the asset price moves from 9.99 to 10.00, the next move will be to a price above 10.00. (In practice the next move might back to 9.99.) Similarly it assumes that when the asset price moves from 10.01 to 10.00, the next move will be to a price below 10.00. (In practice the next move might be back to 10.01.) The scheme can be implemented by buying at 10.01 and selling at 9.99. However, it is not a good hedge. The cost of the trading strategy is zero if the asset price never reaches 10.00 and can be quite high if it reaches 10.00 many times. A good hedge has the property that its cost is always very close the value of the option.

Problem 17.2.

What does it mean to assert that the delta of a call option is 0.7? How can a short position in 1,000 options be made delta neutral when the delta of each option is 0.7?

A delta of 0.7 means that, when the price of the stock increases by a small amount, the price of the option increases by 70% of this amount. Similarly, when the price of the stock decreases by a small amount, the price of the option decreases by 70% of this amount. A short position in 1,000 options has a delta of -700 and can be made delta neutral with the purchase of 700 shares.

Problem 17.3.

Calculate the delta of an at-the-money six-month European call option on a non-dividend-paying stock when the risk-free interest rate is 10% per annum and the stock price volatility is 25% per annum.

In this case $S_0 = K$, $r = 0.1$, $\sigma = 0.25$, and $T = 0.5$. Also,

$$d_1 = \frac{\ln(S_0/K) + (0.1 + 0.25^2/2)0.5}{0.25\sqrt{0.5}} = 0.3712$$

The delta of the option is $N(d_1)$ or 0.64.

Problem 17.4.

What does it mean to assert that the theta of an option position is -0.1 when time is measured in years? If a trader feels that neither a stock price nor its implied volatility will change, what type of option position is appropriate?

A theta of -0.1 means that if Δt units of time pass with no change in either the stock price or its volatility, the value of the option declines by $0.1\Delta t$. A trader who feels that neither the stock price nor its implied volatility will change should write an option with as high a negative theta as possible. Relatively short-life at-the-money options have the most negative thetas.

Problem 17.5.

What is meant by the gamma of an option position? What are the risks in the situation where the gamma of a position is large and negative and the delta is zero?

The gamma of an option position is the rate of change of the delta of the position with respect to the asset price. For example, a gamma of 0.1 would indicate that when the asset price increases by a certain small amount delta increases by 0.1 of this amount. When the gamma of an option writer's position is large and negative and the delta is zero, the option writer will lose significant amounts of money if there is a large movement (either an increase or a decrease) in the asset price.

Problem 17.6.

"The procedure for creating an option position synthetically is the reverse of the procedure for hedging the option position." Explain this statement.

To hedge an option position it is necessary to create the opposite option position synthetically. For example, to hedge a long position in a put it is necessary to create a short position in a put synthetically. It follows that the procedure for creating an option position synthetically is the reverse of the procedure for hedging the option position.

Problem 17.7.

Why did portfolio insurance not work well on October 19, 1987?

Portfolio insurance involves creating a put option synthetically. It assumes that as soon as a portfolio's value declines by a small amount the portfolio manager's position is rebalanced by either (a) selling part of the portfolio, or (b) selling index futures. On October 19, 1987, the market declined so quickly that the sort of rebalancing anticipated in portfolio insurance schemes could not be accomplished.

Problem 17.8.

The Black-Scholes price of an out-of-the-money call option with an exercise price of \$40 is \$4. A trader who has written the option plans to use a stop-loss strategy. The trader's plan is to buy at \$40.10 and to sell at \$39.90. Estimate the expected number of times the stock will be bought or sold.

The strategy costs the trader 0.10 each time the stock is bought or sold. The total expected cost of the strategy, in present value terms, must be \$4. This means that the

expected number of times the stock will be bought or sold is approximately 40. The expected number of times it will be bought is approximately 20 and the expected number of times it will be sold is also approximately 20. The buy and sell transactions can take place at any time during the life of the option. The above numbers are therefore only approximately correct because of the effects of discounting. Also the estimate is of the number of times the stock is bought or sold in the risk-neutral world, not the real world.

Problem 17.9.

Suppose that a stock price is currently \$20 and that a call option with an exercise price of \$25 is created synthetically using a continually changing position in the stock. Consider the following two scenarios:

- a. Stock price increases steadily from \$20 to \$35 during the life of the option.
- b. Stock price oscillates wildly, ending up at \$35.

Which scenario would make the synthetically created option more expensive? Explain your answer.

The holding of the stock at any given time must be $N(d_1)$. Hence the stock is bought just after the price has risen and sold just after the price has fallen. (This is the buy high sell low strategy referred to in the text.) In the first scenario the stock is continually bought. In second scenario the stock is bought, sold, bought again, sold again, etc. The final holding is the same in both scenarios. The buy, sell, buy, sell... situation clearly leads to higher costs than the buy, buy, buy... situation. This problem emphasizes one disadvantage of creating options synthetically. Whereas the cost of an option that is purchased is known up front and depends on the forecasted volatility, the cost of an option that is created synthetically is not known up front and depends on the volatility actually encountered.

Problem 17.10.

What is the delta of a short position in 1,000 European call options on silver futures? The options mature in eight months, and the futures contract underlying the option matures in nine months. The current nine-month futures price is \$8 per ounce, the exercise price of the options is \$8, the risk-free interest rate is 12% per annum, and the volatility of silver is 18% per annum.

The delta of a European futures call option is usually defined as the rate of change of the option price with respect to the futures price (not the spot price). It is

$$e^{-rT} N(d_1)$$

In this case $F_0 = 8$, $K = 8$, $r = 0.12$, $\sigma = 0.18$, $T = 0.6667$

$$d_1 = \frac{\ln(8/8) + (0.18^2/2) \times 0.6667}{0.18\sqrt{0.6667}} = 0.0735$$

$N(d_1) = 0.5293$ and the delta of the option is

$$e^{-0.12 \times 0.6667} \times 0.5293 = 0.4886$$

The delta of a short position in 1,000 futures options is therefore -488.6 .

Problem 17.11.

In Problem 17.10, what initial position in nine-month silver futures is necessary for delta hedging? If silver itself is used, what is the initial position? If one-year silver futures are used, what is the initial position? Assume no storage costs for silver.

In order to answer this problem it is important to distinguish between the rate of change of the option with respect to the futures price and the rate of change of its price with respect to the spot price.

The former will be referred to as the futures delta; the latter will be referred to as the spot delta. The futures delta of a nine-month futures contract to buy one ounce of silver is by definition 1.0. Hence, from the answer to Problem 17.10, a long position in nine-month futures on 488.6 ounces is necessary to hedge the option position.

The spot delta of a nine-month futures contract is $e^{0.12 \times 0.75} = 1.094$ assuming no storage costs. (This is because silver can be treated in the same way as a non-dividend-paying stock when there are no storage costs. $F_0 = S_0 e^{rT}$ so that the spot delta is the futures delta times e^{rT}) Hence the spot delta of the option position is $-488.6 \times 1.094 = -534.6$. Thus a long position in 534.6 ounces of silver is necessary to hedge the option position.

The spot delta of a one-year silver futures contract to buy one ounce of silver is $e^{0.12} = 1.1275$. Hence a long position in $e^{-0.12} \times 534.6 = 474.1$ ounces of one-year silver futures is necessary to hedge the option position.

Problem 17.12.

A company uses delta hedging to hedge a portfolio of long positions in put and call options on a currency. Which of the following would give the most favorable result?

- a. A virtually constant spot rate
- b. Wild movements in the spot rate

Explain your answer.

A long position in either a put or a call option has a positive gamma. From Figure 17.8, when gamma is positive the hedger gains from a large change in the stock price and loses from a small change in the stock price. Hence the hedger will fare better in case (b).

Problem 17.13.

Repeat Problem 17.12 for a financial institution with a portfolio of short positions in put and call options on a currency.

A short position in either a put or a call option has a negative gamma. From Figure 17.8, when gamma is negative the hedger gains from a small change in the stock price and loses from a large change in the stock price. Hence the hedger will fare better in case (a).

Problem 17.14.

A financial institution has just sold 1,000 seven-month European call options on the Japanese yen. Suppose that the spot exchange rate is 0.80 cent per yen, the exercise price

is 0.81 cent per yen, the risk-free interest rate in the United States is 8% per annum, the risk-free interest rate in Japan is 5% per annum, and the volatility of the yen is 15% per annum. Calculate the delta, gamma, vega, theta, and rho of the financial institution's position. Interpret each number.

In this case $S_0 = 0.80$, $K = 0.81$, $r = 0.08$, $r_f = 0.05$, $\sigma = 0.15$, $T = 0.5833$

$$d_1 = \frac{\ln(0.80/0.81) + (0.08 - 0.05 + 0.15^2/2) \times 0.5833}{0.15\sqrt{0.5833}} = 0.1016$$

$$d_2 = d_1 - 0.15\sqrt{0.5833} = -0.0130$$

$$N(d_1) = 0.5405; \quad N(d_2) = 0.4998$$

The delta of one call option is $e^{-r_f T} N(d_1) = e^{-0.05 \times 0.5833} \times 0.5405 = 0.5250$.

$$N'(d_1) = \frac{1}{\sqrt{2\pi}} e^{-d_1^2/2} = \frac{1}{\sqrt{2\pi}} e^{-0.00516} = 0.3969$$

so that the gamma of one call option is

$$\frac{N'(d_1)e^{-r_f T}}{S_0\sigma\sqrt{T}} = \frac{0.3969 \times 0.9713}{0.80 \times 0.15 \times \sqrt{0.5833}} = 4.206$$

The vega of one call option is

$$S_0\sqrt{T}N'(d_1)e^{-r_f T} = 0.80\sqrt{0.5833} \times 0.3969 \times 0.9713 = 0.2355$$

The theta of one call option is

$$\begin{aligned} & -\frac{S_0N'(d_1)\sigma e^{-r_f T}}{2\sqrt{T}} + r_f S_0N(d_1)e^{-r_f T} - rKe^{-rT}N(d_2) \\ &= -\frac{0.8 \times 0.3969 \times 0.15 \times 0.9713}{2\sqrt{0.5833}} \\ & \quad + 0.05 \times 0.8 \times 0.5405 \times 0.9713 - 0.08 \times 0.81 \times 0.9544 \times 0.4948 \\ &= -0.0399 \end{aligned}$$

The rho of one call option is

$$\begin{aligned} & KTe^{-rT}N(d_2) \\ &= 0.81 \times 0.5833 \times 0.9544 \times 0.4948 \\ &= 0.2231 \end{aligned}$$

Delta can be interpreted as meaning that, when the spot price increases by a small amount (measured in cents), the value of an option to buy one yen increases by 0.525 times that amount. Gamma can be interpreted as meaning that, when the spot price increases

by a small amount (measured in cents), the delta increases by 4.206 times that amount. Vega can be interpreted as meaning that, when the volatility (measured in decimal form) increases by a small amount, the option's value increases by 0.2355 times that amount. When volatility increases by 1% (= 0.01) the option price increases by 0.002355. Theta can be interpreted as meaning that, when a small amount of time (measured in years) passes, the option's value decreases by 0.0399 times that amount. In particular when one calendar day passes it decreases by $0.0399/365 = 0.000109$. Finally, rho can be interpreted as meaning that, when the interest rate (measured in decimal form) increases by a small amount the option's value increases by 0.2231 times that amount. When the interest rate increases by 1% (= 0.01), the options value increases by 0.002231.

Problem 17.15.

Under what circumstances is it possible to make a European option on a stock index both gamma neutral and vega neutral by adding a position in one other European option?

Assume that S_0, K, r, σ, T, q are the parameters for the option held and $S_0, K^*, r, \sigma, T^*, q$ are the parameters for another option. Suppose that d_1 has its usual meaning and is calculated on the basis of the first set of parameters while d_1^* is the value of d_1 calculated on the basis of the second set of parameters. Suppose further that w of the second option are held for each of the first option held. The gamma of the portfolio is:

$$\alpha \left[\frac{N'(d_1)e^{-qT}}{S_0\sigma\sqrt{T}} + w \frac{N'(d_1^*)e^{-qT^*}}{S_0\sigma\sqrt{T^*}} \right]$$

where α is the number of the first option held.

Since we require gamma to be zero:

$$w = - \frac{N'(d_1)e^{-q(T-T^*)}}{N'(d_1^*)} \sqrt{\frac{T^*}{T}}$$

The vega of the portfolio is:

$$\alpha \left[S_0\sqrt{T}N'(d_1)e^{-q(T)} + wS_0\sqrt{T^*}N'(d_1^*)e^{-q(T^*)} \right]$$

Since we require vega to be zero:

$$w = - \sqrt{\frac{T}{T^*}} \frac{N'(d_1)e^{-q(T-T^*)}}{N'(d_1^*)}$$

Equating the two expressions for w

$$T^* = T$$

Hence the maturity of the option held must equal the maturity of the option used for hedging.

Problem 17.16.

A fund manager has a well-diversified portfolio that mirrors the performance of the S&P 500 and is worth \$360 million. The value of the S&P 500 is 1,200, and the portfolio manager would like to buy insurance against a reduction of more than 5% in the value of the portfolio over the next six months. The risk-free interest rate is 6% per annum. The dividend yield on both the portfolio and the S&P 500 is 3%, and the volatility of the index is 30% per annum.

- If the fund manager buys traded European put options, how much would the insurance cost?
- Explain carefully alternative strategies open to the fund manager involving traded European call options, and show that they lead to the same result.
- If the fund manager decides to provide insurance by keeping part of the portfolio in risk-free securities, what should the initial position be?
- If the fund manager decides to provide insurance by using nine-month index futures, what should the initial position be?

The fund is worth \$300,000 times the value of the index. When the value of the portfolio falls by 5% (to \$342 million), the value of the S&P 500 also falls by 5% to 1140. The fund manager therefore requires European put options on 300,000 times the S&P 500 with exercise price 1140.

(a) $S_0 = 1200$, $K = 1140$, $r = 0.06$, $\sigma = 0.30$, $T = 0.50$ and $q = 0.03$. Hence:

$$d_1 = \frac{\ln(1200/1140) + (0.06 - 0.03 + 0.3^2/2) \times 0.5}{0.3\sqrt{0.5}} = 0.4186$$

$$d_2 = d_1 - 0.3\sqrt{0.5} = 0.2064$$

$$N(d_1) = 0.6622; \quad N(d_2) = 0.5818$$

$$N(-d_1) = 0.3378; \quad N(-d_2) = 0.4182$$

The value of one put option is

$$\begin{aligned} & 1140e^{-rT}N(-d_2) - 1200e^{-qT}N(-d_1) \\ &= 1140e^{-0.06 \times 0.5} \times 0.4182 - 1200e^{-0.03 \times 0.5} \times 0.3378 \\ &= 63.40 \end{aligned}$$

The total cost of the insurance is therefore

$$300,000 \times 63.40 = \$19,020,000$$

(b) From put-call parity

$$S_0e^{-qT} + p = c + Ke^{-rT}$$

or:

$$p = c - S_0e^{-qT} + Ke^{-rT}$$

This shows that a put option can be created by selling (or shorting) e^{-qT} of the index, buying a call option and investing the remainder at the risk-free rate of interest. Applying this to the situation under consideration, the fund manager should:

- 1) Sell $360e^{-0.03 \times 0.5} = \354.64 million of stock
- 2) Buy call options on 300,000 times the S&P 500 with exercise price 1140 and maturity in six months.
- 3) Invest the remaining cash at the risk-free interest rate of 6% per annum.

This strategy gives the same result as buying put options directly.

(c) The delta of one put option is

$$\begin{aligned} & e^{-qT} [N(d_1) - 1] \\ &= e^{-0.03 \times 0.5} (0.6622 - 1) \\ &= -0.3327 \end{aligned}$$

This indicates that 33.27% of the portfolio (i.e., \$119.77 million) should be initially sold and invested in risk-free securities.

(d) The delta of a nine-month index futures contract is

$$e^{(r-q)T} = e^{0.03 \times 0.75} = 1.023$$

The spot short position required is

$$\frac{119,770,000}{1200} = 99,808$$

times the index. Hence a short position in

$$\frac{99,808}{1.023 \times 250} = 390$$

futures contracts is required.

Problem 17.17.

Repeat Problem 17.16 on the assumption that the portfolio has a beta of 1.5. Assume that the dividend yield on the portfolio is 4% per annum.

When the value of the portfolio goes down 5% in six months, the total return from the portfolio, including dividends, in the six months is

$$-5 + 2 = -3\%$$

i.e., -6% per annum. This is 12% per annum less than the risk-free interest rate. Since the portfolio has a beta of 1.5 we would expect the market to provide a return of 8% per annum less than the risk-free interest rate, i.e., we would expect the market to provide a return of -2% per annum. Since dividends on the market index are 3% per annum, we would expect the market index to have dropped at the rate of 5% per annum or 2.5% per six months; i.e.,

we would expect the market to have dropped to 1170. A total of $450,000 = (1.5 \times 300,000)$ put options on the S&P 500 with exercise price 1170 and exercise date in six months are therefore required.

(a) $S_0 = 1200$, $K = 1170$, $r = 0.06$, $\sigma = 0.3$, $T = 0.5$ and $q = 0.03$. Hence

$$d_1 = \frac{\ln(1200/1170) + (0.06 - 0.03 + 0.09/2) \times 0.5}{0.3\sqrt{0.5}} = 0.2961$$

$$d_2 = d_1 - 0.3\sqrt{0.5} = 0.0840$$

$$N(d_1) = 0.6164; \quad N(d_2) = 0.5335$$

$$N(-d_1) = 0.3836; \quad N(-d_2) = 0.4665$$

The value of one put option is

$$\begin{aligned} & Ke^{-rT}N(-d_2) - S_0e^{-qT}N(-d_1) \\ &= 1170e^{-0.06 \times 0.5} \times 0.4665 - 1200e^{-0.03 \times 0.5} \times 0.3836 \\ &= 76.28 \end{aligned}$$

The total cost of the insurance is therefore

$$450,000 \times 76.28 = \$34,326,000$$

Note that this is significantly greater than the cost of the insurance in Problem 17.16.

- (b) As in Problem 17.16 the fund manager can 1) sell \$354.64 million of stock, 2) buy call options on 450,000 times the S&P 500 with exercise price 1170 and exercise date in six months and 3) invest the remaining cash at the risk-free interest rate.
- (c) The portfolio is 50% more volatile than the S&P 500. When the insurance is considered as an option on the portfolio the parameters are as follows: $S_0 = 360$, $K = 342$, $r = 0.06$, $\sigma = 0.45$, $T = 0.5$ and $q = 0.04$

$$d_1 = \frac{\ln(360/342) + (0.06 - 0.04 + 0.45^2/2) \times 0.5}{0.45\sqrt{0.5}} = 0.3517$$

$$N(d_1) = 0.6374$$

The delta of the option is

$$\begin{aligned} & e^{-qT}[N(d_1) - 1] \\ &= e^{-0.04 \times 0.5}(0.6374 - 1) \\ &= -0.355 \end{aligned}$$

This indicates that 35.5% of the portfolio (i.e., \$127.8 million) should be sold and invested in riskless securities.

- (d) We now return to the situation considered in (a) where put options on the index are required. The delta of each put option is

$$\begin{aligned} & e^{-qT}(N(d_1) - 1) \\ &= e^{-0.03 \times 0.5}(0.6164 - 1) \\ &= -0.3779 \end{aligned}$$

The delta of the total position required in put options is $-450,000 \times 0.3779 = -170,000$. The delta of a nine month index futures is (see Problem 17.16) 1.023. Hence a short position in

$$\frac{170,000}{1.023 \times 250} = 665$$

index futures contracts.

Problem 17.18.

Show by substituting for the various terms in equation (17.4) that the equation is true for:

- A single European call option on a non-dividend-paying stock
 - A single European put option on a non-dividend-paying stock
 - Any portfolio of European put and call options on a non-dividend-paying stock
- (a) For a call option on a non-dividend-paying stock

$$\begin{aligned} \Delta &= N(d_1) \\ \Gamma &= \frac{N'(d_1)}{S_0 \sigma \sqrt{T}} \\ \Theta &= -\frac{S_0 N'(d_1) \sigma}{2\sqrt{T}} - rK e^{-rT} N(d_2) \end{aligned}$$

Hence the left-hand side of equation (17.4) is:

$$\begin{aligned} &= -\frac{S_0 N'(d_1) \sigma}{2\sqrt{T}} - rK e^{-rT} N(d_2) + rS_0 N(d_1) + \frac{1}{2} \sigma S_0 \frac{N'(d_1)}{\sqrt{T}} \\ &= r[S_0 N(d_1) - K e^{-rT} N(d_2)] \\ &= r\Pi \end{aligned}$$

- (b) For a put option on a non-dividend-paying stock

$$\begin{aligned} \Delta &= N(d_1) - 1 = -N(-d_1) \\ \Gamma &= \frac{N'(d_1)}{S_0 \sigma \sqrt{T}} \\ \Theta &= -\frac{S_0 N'(d_1) \sigma}{2\sqrt{T}} + rK e^{-rT} N(-d_2) \end{aligned}$$

Hence the left-hand side of equation (17.4) is:

$$\begin{aligned} & -\frac{S_0 N'(d_1) \sigma}{2\sqrt{T}} + rK e^{-rT} N(-d_2) - rS_0 N(-d_1) + \frac{1}{2} \sigma S_0 \frac{N'(d_1)}{\sqrt{T}} \\ & = r[K e^{-rT} N(-d_2) - S_0 N(-d_1)] \\ & = r\Pi \end{aligned}$$

- (c) For a portfolio of options, Π , Δ , Θ and Γ are the sums of their values for the individual options in the portfolio. It follows that equation (17.4) is true for any portfolio of European put and call options.

Problem 17.19

What is the equation corresponding to equation (17.4) for (a) a portfolio of derivatives on a currency and (b) a portfolio of derivatives on a futures contract?

A currency is analogous to a stock paying a continuous dividend yield at rate r_f . The differential equation for a portfolio of derivatives dependent on a currency is (see equation 15.6)

$$\frac{\partial \Pi}{\partial t} + (r - r_f) S \frac{\partial \Pi}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 \Pi}{\partial S^2} = r\Pi$$

Hence

$$\Theta + (r - r_f) S \Delta + \frac{1}{2} \sigma^2 S^2 \Gamma = r\Pi$$

Similarly, for a portfolio of derivatives dependent on a futures price (see equation 16.8)

$$\Theta + \frac{1}{2} \sigma^2 S^2 \Gamma = r\Pi$$

Problem 17.20.

Suppose that \$70 billion of equity assets are the subject of portfolio insurance schemes. Assume that the schemes are designed to provide insurance against the value of the assets declining by more than 5% within one year. Making whatever estimates you find necessary, use the DerivaGem software to calculate the value of the stock or futures contracts that the administrators of the portfolio insurance schemes will attempt to sell if the market falls by 23% in a single day.

We can regard the position of all portfolio insurers taken together as a single put option. The three known parameters of the option, before the 23% decline, are $S_0 = 70$, $K = 66.5$, $T = 1$. Other parameters can be estimated as $r = 0.06$, $\sigma = 0.25$ and $q = 0.03$. Then:

$$d_1 = \frac{\ln(70/66.5) + (0.06 - 0.03 + 0.25^2/2)}{0.25} = 0.4502$$

$$N(d_1) = 0.6737$$

The delta of the option is

$$\begin{aligned} & e^{-qT}[N(d_1) - 1] \\ & = e^{-0.03}(0.6737 - 1) \\ & = -0.3167 \end{aligned}$$

This shows that 31.67% or \$22.17 billion of assets should have been sold before the decline. These numbers can also be produced from DerivaGem by selecting Underlying Type and Index and Option Type as Analytic European.

After the decline, $S_0 = 53.9$, $K = 66.5$, $T = 1$, $r = 0.06$, $\sigma = 0.25$ and $q = 0.03$.

$$\begin{aligned} d_1 & = \frac{\ln(53.9/66.5) + (0.06 - 0.03 + 0.25^2/2)}{0.25} = -0.5953 \\ N(d_1) & = 0.2758 \end{aligned}$$

The delta of the option has dropped to

$$\begin{aligned} & e^{-0.03 \times 0.5}(0.2758 - 1) \\ & = -0.7028 \end{aligned}$$

This shows that cumulatively 70.28% of the assets originally held should be sold. An additional 38.61% of the original portfolio should be sold. The sales measured at pre-crash prices are about \$27.0 billion. At post crash prices they are about 20.8 billion.

Problem 17.21.

Does a forward contract on a stock index have the same delta as the corresponding futures contract? Explain your answer.

With our usual notation the value of a forward contract on the asset is $S_0e^{-qT} - Ke^{-rT}$. When there is a small change, ΔS , in S_0 the value of the forward contract changes by $e^{-qT}\Delta S$. The delta of the forward contract is therefore e^{-qT} . The futures price is $S_0e^{(r-q)T}$. When there is a small change, ΔS , in S_0 the futures price changes by $\Delta Se^{(r-q)T}$. Given the daily settlement procedures in futures contracts, this is also the immediate change in the wealth of the holder of the futures contract. The delta of the futures contract is therefore $e^{(r-q)T}$. We conclude that the deltas of a futures and forward contract are not the same. The delta of the futures is greater than the delta of the corresponding forward by a factor of e^{rT} .

Problem 17.22.

A bank's position in options on the dollar-euro exchange rate has a delta of 30,000 and a gamma of -80,000. Explain how these numbers can be interpreted. The exchange rate (dollars per euro) is 0.90. What position would you take to make the position delta neutral? After a short period of time, the exchange rate moves to 0.93. Estimate the new delta. What additional trade is necessary to keep the position delta neutral? Assuming the bank did set up a delta-neutral position originally, has it gained or lost money from the exchange-rate movement?

The delta indicates that when the value of the euro exchange rate increases by \$0.01, the value of the bank's position increases by $0.01 \times 30,000 = \$300$. The gamma indicates that when the euro exchange rate increases by \$0.01 the delta of the portfolio decreases by $0.01 \times 80,000 = 800$. For delta neutrality 30,000 euros should be shorted. When the exchange rate moves up to 0.93, we expect the delta of the portfolio to decrease by $(0.93 - 0.90) \times 80,000 = 2,400$ so that it becomes 27,600. To maintain delta neutrality, it is therefore necessary for the bank to unwind its short position 2,400 euros so that a net 27,600 have been shorted. As shown in the text (see Figure 17.8), when a portfolio is delta neutral and has a negative gamma, a loss is experienced when there is a large movement in the underlying asset price. We can conclude that the bank is likely to have lost money.

Problem 17.23.

Use the put-call parity relationship to derive, for a non-dividend-paying stock, the relationship between:

- (a) The delta of a European call and the delta of a European put.
- (b) The gamma of a European call and the gamma of a European put.
- (c) The vega of a European call and the vega of a European put.
- (d) The theta of a European call and the theta of a European put.

For a non-dividend paying stock, put-call parity gives at a general time t :

$$p + S = c + Ke^{-r(T-t)}$$

- (a) Differentiating with respect to S :

$$\frac{\partial p}{\partial S} + 1 = \frac{\partial c}{\partial S}$$

or

$$\frac{\partial p}{\partial S} = \frac{\partial c}{\partial S} - 1$$

This shows that the delta of a European put equals the delta of the corresponding European call less 1.0.

- (b) Differentiating with respect to S again

$$\frac{\partial^2 p}{\partial S^2} = \frac{\partial^2 c}{\partial S^2}$$

Hence the gamma of a European put equals the gamma of a European call.

- (c) Differentiating the put-call parity relationship with respect to σ

$$\frac{\partial p}{\partial \sigma} = \frac{\partial c}{\partial \sigma}$$

showing that the vega of a European put equals the vega of a European call.

- (d) Differentiating the put-call parity relationship with respect to T

$$\frac{\partial p}{\partial t} = rKe^{-r(T-t)} + \frac{\partial c}{\partial t}$$

This is in agreement with the thetas of European calls and puts given in Section 17.5 since $N(d_2) = 1 - N(-d_2)$.

ASSIGNMENT QUESTIONS

Problem 17.24.

Consider a one-year European call option on a stock when the stock price is \$30, the strike price is \$30, the risk-free rate is 5%, and the volatility is 25% per annum. Use the DerivaGem software to calculate the price, delta, gamma, vega, theta, and rho of the option. Verify that delta is correct by changing the stock price to \$30.1 and recomputing the option price. Verify that gamma is correct by recomputing the delta for the situation where the stock price is \$30.1. Carry out similar calculations to verify that vega, theta, and rho are correct. Use the DerivaGem software to plot the option price, delta, gamma, vega, theta, and rho against the stock price for the stock option.

The price, delta, gamma, vega, theta, and rho of the option are 3.7008, 0.6274, 0.050, 0.1135, -0.00596 , and 0.1512. When the stock price increases to 30.1, the option price increases to 3.7638. The change in the option price is $3.7638 - 3.7008 = 0.0630$. Delta predicts a change in the option price of $0.6274 \times 0.1 = 0.0627$ which is very close. When the stock price increases to 30.1, delta increases to 0.6324. The size of the increase in delta is $0.6324 - 0.6274 = 0.005$. Gamma predicts an increase of $0.050 \times 0.1 = 0.005$ which is the same. When the volatility increases from 25% to 26%, the option price increases by 0.1136 from 3.7008 to 3.8144. This is consistent with the vega value of 0.1135. When the time to maturity is changed from 1 to $1 - 1/365$ the option price reduces by 0.006 from 3.7008 to 3.6948. This is consistent with a theta of -0.00596 . Finally when the interest rate increases from 5% to 6% the value of the option increases by 0.1527 from 3.7008 to 3.8535. This is consistent with a rho of 0.1512.

Problem 17.25.

A financial institution has the following portfolio of over-the-counter options on sterling:

Type	Position	Delta of Option	Gamma of Option	Vega of Option
Call	-1,000	0.50	2.2	1.8
Call	-500	0.80	0.6	0.2
Put	-2,000	-0.40	1.3	0.7
Call	-500	0.70	1.8	1.4

A traded option is available with a delta of 0.6, a gamma of 1.5, and a vega of 0.8.

- What position in the traded option and in sterling would make the portfolio both gamma neutral and delta neutral?

b. What position in the traded option and in sterling would make the portfolio both vega neutral and delta neutral?

The delta of the portfolio is

$$-1,000 \times 0.50 - 500 \times 0.80 - 2,000 \times (-0.40) - 500 \times 0.70 = -450$$

The gamma of the portfolio is

$$-1,000 \times 2.2 - 500 \times 0.6 - 2,000 \times 1.3 - 500 \times 1.8 = -6,000$$

The vega of the portfolio is

$$-1,000 \times 1.8 - 500 \times 0.2 - 2,000 \times 0.7 - 500 \times 1.4 = -4,000$$

(a) A long position in 4,000 traded options will give a gamma-neutral portfolio since the long position has a gamma of $4,000 \times 1.5 = +6,000$. The delta of the whole portfolio (including traded options) is then:

$$4,000 \times 0.6 - 450 = 1,950$$

Hence, in addition to the 4,000 traded options, a short position in £1,950 is necessary so that the portfolio is both gamma and delta neutral.

(b) A long position in 5,000 traded options will give a vega-neutral portfolio since the long position has a vega of $5,000 \times 0.8 = +4,000$. The delta of the whole portfolio (including traded options) is then

$$5,000 \times 0.6 - 450 = 2,550$$

Hence, in addition to the 5,000 traded options, a short position in £2,550 is necessary so that the portfolio is both vega and delta neutral.

Problem 17.26.

Consider again the situation in Problem 17.25. Suppose that a second traded option with a delta of 0.1, a gamma of 0.5, and a vega of 0.6 is available. How could the portfolio be made delta, gamma, and vega neutral?

Let w_1 be the position in the first traded option and w_2 be the position in the second traded option. We require:

$$6,000 = 1.5w_1 + 0.5w_2$$

$$4,000 = 0.8w_1 + 0.6w_2$$

The solution to these equations can easily be seen to be $w_1 = 3,200$, $w_2 = 2,400$. The whole portfolio then has a delta of

$$-450 + 3,200 \times 0.6 + 2,400 \times 0.1 = 1,710$$

Therefore the portfolio can be made delta, gamma and vega neutral by taking a long position in 3,200 of the first traded option, a long position in 2,400 of the second traded option and a short position in £1,710.

Problem 17.27.

A deposit instrument offered by a bank guarantees that investors will receive a return during a six-month period that is the greater of (a) zero and (b) 40% of the return provided by a market index. An investor is planning to put \$100,000 in the instrument. Describe the payoff as an option on the index. Assuming that the risk-free rate of interest is 8% per annum, the dividend yield on the index is 3% per annum, and the volatility of the index is 25% per annum, is the product a good deal for the investor?

The product provides a six-month return equal to

$$\max(0, 0.4R)$$

where R is the return on the index. Suppose that S_0 is the current value of the index and S_T is the value in six months.

When an amount A is invested, the return received at the end of six months is:

$$\begin{aligned} & A \max\left(0, 0.4 \frac{S_T - S_0}{S_0}\right) \\ &= \frac{0.4A}{S_0} \max(0, S_T - S_0) \end{aligned}$$

This is $0.4A/S_0$ of at-the-money European call options on the index. With the usual notation, they have value:

$$\begin{aligned} & \frac{0.4A}{S_0} [S_0 e^{-qT} N(d_1) - S_0 e^{-rT} N(d_2)] \\ &= 0.4A [e^{-qT} N(d_1) - e^{-rT} N(d_2)] \end{aligned}$$

In this case $r = 0.08$, $\sigma = 0.25$, $T = 0.50$ and $q = 0.03$

$$\begin{aligned} d_1 &= \frac{(0.08 - 0.03 + 0.25^2/2) 0.50}{0.25\sqrt{0.50}} = 0.2298 \\ d_2 &= d_1 - 0.25\sqrt{0.50} = 0.0530 \end{aligned}$$

$$N(d_1) = 0.5909; \quad N(d_2) = 0.5212$$

The value of the European call options being offered is

$$\begin{aligned} & 0.4A(e^{-0.03 \times 0.5} \times 0.5909 - e^{-0.08 \times 0.5} \times 0.5212) \\ &= 0.0325A \end{aligned}$$

This is the present value of the payoff from the product. If an investor buys the product he or she avoids having to pay $0.0325A$ at time zero for the underlying option. The cash flows to the investor are therefore

Time 0: $-A + 0.0325A = 0.9675A$

After six months: $+A$

The return with continuous compounding is $2 \ln(1/0.9675) = 0.066$ or 6.6% per annum. The product is therefore slightly less attractive than a risk-free investment.

Problem 17.28.

The formula for the price of a European call futures option in terms of the futures price, F_0 , is given in Chapter 16 as

$$c = e^{-rT} [F_0 N(d_1) - KN(d_2)]$$

where

$$d_1 = \frac{\ln(F_0/K) + \sigma^2 T/2}{\sigma \sqrt{T}}$$

$$d_2 = d_1 - \sigma \sqrt{T}$$

and K , r , T , and σ are the strike price, interest rate, time to maturity, and volatility, respectively.

(a) Prove that $F_0 N'(d_1) = KN'(d_2)$

(b) Prove that the delta of the call price with respect to the futures price is $e^{-rT} N(d_1)$.

(c) Prove that the vega of the call price is $F_0 \sqrt{T} N'(d_1) e^{-rT}$

(d) Prove the formula for the rho of a call futures option given in Section 17.12. The delta, gamma, theta, and vega of a call futures option are the same as those for a call option on a stock paying dividends at rate q with q replaced by r and S_0 replaced by F_0 . Explain why the same is not true of the rho of a call futures option.

(a)

$$FN'(d_1) = \frac{F}{\sqrt{2\pi}} e^{-d_1^2/2}$$

$$KN'(d_2) = KN'(d_1 - \sigma \sqrt{T}) = \frac{K}{\sqrt{2\pi}} e^{-(d_1^2/2) + d_1 \sigma \sqrt{T} - \sigma^2 T/2}$$

Because $d_1 \sigma \sqrt{T} = \ln(F/K) + \sigma^2 T/2$ the second equation reduces to

$$KN'(d_2) = \frac{K}{\sqrt{2\pi}} e^{-(d_1^2/2) + \ln(F/K)} = \frac{F}{\sqrt{2\pi}} e^{-d_1^2/2}$$

The result follows.

(b)

$$\frac{\partial c}{\partial F} = e^{-rT} N(d_1) + e^{-rT} FN'(d_1) \frac{\partial d_1}{\partial F} - e^{-rT} KN'(d_2) \frac{\partial d_2}{\partial F}$$

Because

$$\frac{\partial d_1}{\partial F} = \frac{\partial d_2}{\partial F}$$

it follows from the result in (a) that

$$\frac{\partial c}{\partial F} = e^{-rT} N(d_1)$$

(c)

$$\frac{\partial c}{\partial \sigma} = e^{-rT} FN'(d_1) \frac{\partial d_1}{\partial \sigma} - e^{-rT} KN'(d_2) \frac{\partial d_2}{\partial \sigma}$$

Because $d_1 = d_2 + \sigma\sqrt{T}$

$$\frac{\partial d_1}{\partial \sigma} = \frac{\partial d_2}{\partial \sigma} + \sqrt{T}$$

From the result in (a) it follows that

$$\frac{\partial c}{\partial \sigma} = e^{-rT} FN'(d_1) \sqrt{T}$$

(d) Rho is given by

$$\frac{\partial c}{\partial r} = -Te^{-rT} [FN(d_1) - KN(d_2)]$$

or $-cT$. Because $q = r$ in the case of a futures option there are two components to rho. One arises from differentiation with respect to r , the other from differentiation with respect to q .

Problem 17.29.

Use DerivaGem to check that equation (17.4) is satisfied for the option considered in Section 17.1. (Note: DerivaGem produces a value of theta “per calendar day.” The theta in equation (17.7) is “per year.”)

For the option considered in Section 17.1, $S_0 = 49$, $K = 50$, $r = 0.05$, $\sigma = 0.20$, and $T = 20/52$. DerivaGem shows that $\Theta = -0.011795 \times 365 = -4.305$, $\Delta = 0.5216$, $\Gamma = 0.065544$, $\Pi = 2.4005$. The left hand side of equation (17.7)

$$-4.305 + 0.05 \times 49 \times 0.5216 + \frac{1}{2} \times 0.2^2 \times 49^2 \times 0.065544 = 0.120$$

The right hand side is

$$0.05 \times 2.4005 = 0.120$$

This shows that the result in equation (17.4) is satisfied.

Problem 17.30.

Use the DerivaGem Application Builder functions to reproduce Table 17.2. (Note that in Table 17.2 the stock position is rounded to the nearest 100 shares.) Calculate the gamma and theta of the position each week. Calculate the change in the value of the portfolio each week and check whether equation (17.3) is approximately satisfied. (Note: DerivaGem produces a value of theta “per calendar day.” The theta in equation (17.3) is “per year.”)

Consider the first week. The portfolio consists of a short position in 100,000 options and a long position in 52,200 shares. The value of the option changes from \$240,053 at the beginning of the week to \$188,760 at the end of the week for a gain of \$51,293. The value of the shares change from $52,200 \times 49 = \$2,557,800$ to $52,200 \times 48.12 = \$2,511,864$ for a loss of \$45,936. The net gain is $51,293 - 45,936 = \$5,357$. The gamma and theta (per year) of the portfolio are -6554.4 and $430,533$ so that equation (17.3) predicts the gain as

$$430533 \times \frac{1}{52} - \frac{1}{2} \times 6554.4 \times (48.12 - 49)^2 = 5742$$

The results for all 20 weeks are shown in the following table.

Week	Actual Gain	Predicted Gain
1	5,357	5,742
2	5,689	6,093
3	-19,742	-21,084
4	1,941	1,572
5	3,706	3,652
6	9,320	9,191
7	6,249	5,936
8	9,491	9,259
9	961	870
10	-23,380	-18,992
11	1,643	2,497
12	2,645	1,356
13	11,365	10,923
14	-2,876	-3,342
15	12,936	12,302
16	7,566	8,815
17	-3,880	-2,763
18	6,764	6,899
19	4,295	5,205
20	4,804	4,805

CHAPTER 18

Volatility Smiles

Notes for the Instructor

This chapter covers volatility smiles and how they are used in practice. The approach is to start with the volatility smiles that are observed in the equity and foreign currency markets and then show what the implied distributions look like. A number of improvements have been made to the chapter. Section 18.1 reads more easily. Different ways used by practitioners to quantify the volatility smile are covered.

I find that many students are interested in the details of how one goes from a volatility smile to an implied distribution. The appendix to the chapter now has more information on this and includes a numerical example.

I focus on foreign exchange and equity markets when covering this chapter, but futures markets can also be mentioned. (Problem 16.23 from Chapter 16 derives a volatility for corn futures.)

It is not difficult to construct interesting assignments based on the material. For example, students can be asked to calculate a volatility smile for options on the S&P 500 using data obtained from a newspaper or a live data feed. Problems 18.19 and 18.26 work well for class discussion. The others make good assignment questions.

QUESTIONS AND PROBLEMS

Problem 18.1.

What volatility smile is likely to be observed when

- a. *Both tails of the stock price distribution are less heavy than those of the lognormal distribution?*
- b. *The right tail is heavier, and the left tail is less heavy, than that of a lognormal distribution?*

A downward sloping volatility smile is usually observed for equities.

Problem 18.2.

What volatility smile is observed for equities?

A downward sloping volatility smile is usually observed for equities.

Problem 18.3.

What volatility smile is likely to be caused by jumps in the underlying asset price? Is the pattern likely to be more pronounced for a two-year option than for a three-month option?

Jumps tend to make both tails of the stock price distribution heavier than those of the lognormal distribution. This creates a volatility smile similar to that in Figure 18.1. The volatility smile is likely to be more pronounced for the three-month option.

Problem 18.4.

A European call and put option have the same strike price and time to maturity. The call has an implied volatility of 30% and the put has an implied volatility of 25%. What trades would you do?

The put has a price that is too low relative to the call's price. The correct trading strategy is to buy the put, buy the stock, and sell the call.

Problem 18.5.

Explain carefully why a distribution with a heavier left tail and less heavy right tail than the lognormal distribution gives rise to a downward sloping volatility smile.

The heavier left tail should lead to high prices, and therefore high implied volatilities, for out-of-the-money (low-strike-price) puts. Similarly the less heavy right tail should lead to low prices, and therefore low volatilities for out-of-the-money (high-strike-price) calls. A volatility smile where volatility is a decreasing function of strike price results.

Problem 18.6.

The market price of a European call is \$3.00 and its price given by Black-Scholes model with a volatility of 30% is \$3.50. The price given by this Black-Scholes model for a European put option with the same strike price and time to maturity is \$1.00. What should the market price of the put option be? Explain the reasons for your answer.

With the notation in the text

$$c_{bs} + Ke^{-rT} = p_{bs} + Se^{-qT}$$

$$c_{mkt} + Ke^{-rT} = p_{mkt} + Se^{-qT}$$

It follows that

$$c_{bs} - c_{mkt} = p_{bs} - p_{mkt}$$

In this case $c_{mkt} = 3.00$; $c_{bs} = 3.50$; and $p_{bs} = 1.00$. It follows that p_{mkt} should be 0.50.

Problem 18.7.

Explain what is meant by crashophobia.

The crashophobia argument is an attempt to explain the pronounced volatility skew in equity markets since 1987. (This was the year equity markets shocked everyone by crashing more than 20% in one day). The argument is that traders are concerned about another crash and as a result increase the price of out-of-the-money puts. This creates the volatility skew.

Problem 18.8.

A stock price is currently \$20. Tomorrow, news is expected to be announced that will either increase the price by \$5 or decrease the price by \$5. What are the problems in using Black–Scholes to value one-month options on the stock?

The probability distribution of the stock price in one month is not lognormal. Possibly it consists of two lognormal distributions superimposed upon each other and is bimodal. Black–Scholes is clearly inappropriate, because it assumes that the stock price at any future time is lognormal.

Problem 18.9.

What volatility smile is likely to be observed for six-month options when the volatility is uncertain and positively correlated to the stock price?

When the asset price is positively correlated with volatility, the volatility tends to increase as the asset price increases, producing less heavy left tails and heavier right tails. Implied volatility then increases with the strike price.

Problem 18.10.

What problems do you think would be encountered in testing a stock option pricing model empirically?

There are a number of problems in testing an option pricing model empirically. These include the problem of obtaining synchronous data on stock prices and option prices, the problem of estimating the dividends that will be paid on the stock during the option's life, the problem of distinguishing between situations where the market is inefficient and situations where the option pricing model is incorrect, and the problems of estimating stock price volatility.

Problem 18.11.

Suppose that a central bank's policy is to allow an exchange rate to fluctuate between 0.97 and 1.03. What pattern of implied volatilities for options on the exchange rate would you expect to see?

In this case the probability distribution of the exchange rate has a thin left tail and a thin right tail relative to the lognormal distribution. We are in the opposite situation to that described for foreign currencies in Section 18.1. Both out-of-the-money and in-the-money calls and puts can be expected to have lower implied volatilities than at-the-money calls and puts. The pattern of implied volatilities is likely to be similar to Figure 18.7.

Problem 18.12.

Option traders sometimes refer to deep-out-of-the-money options as being options on volatility. Why do you think they do this?

A deep-out-of-the-money option has a low value. Decreases in its volatility reduce its value. However, this reduction is small because the value can never go below zero. Increases in its volatility, on the other hand, can lead to significant percentage increases

in the value of the option. The option does, therefore, have some of the same attributes as an option on volatility.

Problem 18.13.

A European call option on a certain stock has a strike price of \$30, a time to maturity of one year, and an implied volatility of 30%. A European put option on the same stock has a strike price of \$30, a time to maturity of one year, and an implied volatility of 33%. What is the arbitrage opportunity open to a trader? Does the arbitrage work only when the lognormal assumption underlying Black–Scholes holds? Explain the reasons for your answer carefully.

As explained in the appendix to the chapter, put–call parity implies that European put and call options have the same implied volatility. If a call option has an implied volatility of 30% and a put option has an implied volatility of 33%, the call is priced too low relative to the put. The correct trading strategy is to buy the call, sell the put and short the stock. This does not depend on the lognormal assumption underlying Black–Scholes. Put–call parity is true for any set of assumptions.

Problem 18.14.

Suppose that the result of a major lawsuit affecting a company is due to be announced tomorrow. The company’s stock price is currently \$60. If the ruling is favorable to the company, the stock price is expected to jump to \$75. If it is unfavorable, the stock is expected to jump to \$50. What is the risk-neutral probability of a favorable ruling? Assume that the volatility of the company’s stock will be 25% for six months after the ruling if the ruling is favorable and 40% if it is unfavorable. Use DerivaGem to calculate the relationship between implied volatility and strike price for six-month European options on the company today. The company does not pay dividends. Assume that the six-month risk-free rate is 6%. Consider call options with strike prices of \$30, \$40, \$50, \$60, \$70, and \$80.

Suppose that p is the probability of a favorable ruling. The expected price of the company’s stock tomorrow is

$$75p + 50(1 - p) = 50 + 25p$$

This must be the price of the stock today. (We ignore the expected return to an investor over one day.) Hence

$$50 + 25p = 60$$

or $p = 0.4$.

If the ruling is favorable, the volatility, σ , will be 25%. Other option parameters are $S_0 = 75$, $r = 0.06$, and $T = 0.5$. For a value of K equal to 50, DerivaGem gives the value of a European call option price as 26.502.

If the ruling is unfavorable, the volatility, σ will be 40%. Other option parameters are $S_0 = 50$, $r = 0.06$, and $T = 0.5$. For a value of K equal to 50, DerivaGem gives the value of a European call option price as 6.310.

The value today of a European call option with a strike price today is the weighted average of 26.502 and 6.310 or:

$$0.4 \times 26.502 + 0.6 \times 6.310 = 14.387$$

DerivaGem can be used to calculate the implied volatility when the option has this price. The parameter values are $S_0 = 60$, $K = 50$, $T = 0.5$, $r = 0.06$ and $c = 14.387$. The implied volatility is 47.76%.

These calculations can be repeated for other strike prices. The results are shown in the table below. The pattern of implied volatilities is shown in Figure S18.1.

Strike Price	Call Option Price Favorable Outcome	Call Option Price Unfavorable Outcome	Weighted Price	Implied Volatility (%)
30	45.887	21.001	30.955	46.67
40	36.182	12.437	21.935	47.78
50	26.502	6.310	14.387	47.76
60	17.171	2.826	8.564	46.05
70	9.334	1.161	4.430	43.22
80	4.159	0.451	1.934	40.36



Figure S18.1 Implied Volatilities in Problem 18.14

Problem 18.15.

An exchange rate is currently 0.8000. The volatility of the exchange rate is quoted as 12% and interest rates in the two countries are the same. Using the lognormal assumption, estimate the probability that the exchange rate in three months will be (a) less than 0.7000, (b) between 0.7000 and 0.7500, (c) between 0.7500 and 0.8000, (d) between 0.8000 and 0.8500, (e) between 0.8500 and 0.9000, and (f) greater than 0.9000. Based on the volatility smile usually observed in the market for exchange rates, which of these estimates would you expect to be too low and which would you expect to be too high?

As pointed out in Chapters 5 and 13 an exchange rate behaves like a stock that provides a dividend yield equal to the foreign risk-free rate. Whereas the growth rate in a non-dividend-paying stock in a risk-neutral world is r , the growth rate in the exchange rate in a risk-neutral world is $r - r_f$. Exchange rates have low systematic risks and so we can reasonably assume that this is also the growth rate in the real world. In this case the foreign risk-free rate equals the domestic risk-free rate ($r = r_f$). The expected growth rate in the exchange rate is therefore zero. If S_T is the exchange rate at time T its probability distribution is given by equation (12.2) with $\mu = 0$:

$$\ln S_T \sim \phi(\ln S_0 - \sigma^2 T/2, \sigma^2 T)$$

where S_0 is the exchange rate at time zero and σ is the volatility of the exchange rate. In this case $S_0 = 0.8000$ and $\sigma = 0.12$, and $T = 0.25$ so that

$$\ln S_T \sim \phi(\ln 0.8 - 0.12^2 \times 0.25/2, 0.12^2 \times 0.25)$$

or

$$\ln S_T \sim \phi(-0.2249, 0.0036)$$

- (a) $\ln 0.70 = -0.3567$. The probability that $S_T < 0.70$ is the same as the probability that $\ln S_T < -0.3567$. It is

$$N\left(\frac{-0.3567 + 0.2249}{0.06}\right) = N(-2.1955)$$

This is 1.41%.

- (b) $\ln 0.75 = -0.2877$. The probability that $S_T < 0.75$ is the same as the probability that $\ln S_T < -0.2877$. It is

$$N\left(\frac{-0.2877 + 0.2249}{0.06}\right) = N(-1.0456)$$

This is 14.79%. The probability that the exchange rate is between 0.70 and 0.75 is therefore $14.79 - 1.41 = 13.38\%$.

- (c) $\ln 0.80 = -0.2231$. The probability that $S_T < 0.80$ is the same as the probability that $\ln S_T < -0.2231$. It is

$$N\left(\frac{-0.2231 + 0.2249}{0.06}\right) = N(0.0300)$$

This is 51.20%. The probability that the exchange rate is between 0.75 and 0.80 is therefore $51.20 - 14.79 = 36.41\%$.

- (d) $\ln 0.85 = -0.1625$. The probability that $S_T < 0.85$ is the same as the probability that $\ln S_T < -0.1625$. It is

$$N\left(\frac{-0.1625 + 0.2249}{0.06}\right) = N(1.0404)$$

This is 85.09%. The probability that the exchange rate is between 0.80 and 0.85 is therefore $85.09 - 51.20 = 33.89\%$.

- (e) $\ln 0.90 = -0.1054$. The probability that $S_T < 0.90$ is the same as the probability that $\ln S_T < -0.1054$. It is

$$N\left(\frac{-0.1054 + 0.2249}{0.06}\right) = N(1.9931)$$

This is 97.69%. The probability that the exchange rate is between 0.85 and 0.90 is therefore $97.69 - 85.09 = 12.60\%$.

- (f) The probability that the exchange rate is greater than 0.90 is $100 - 97.69 = 2.31\%$

The volatility smile encountered for foreign exchange options is shown in Figure 18.1 of the text and implies the probability distribution in Figure 18.2. Figure 18.2 suggests that we would expect the probabilities in (a), (c), (d), and (f) to be too low and the probabilities in (b) and (e) to be too high.

Problem 18.16.

The price of a stock is \$40. A six-month European call option on the stock with a strike price of \$30 has an implied volatility of 35%. A six month European call option on the stock with a strike price of \$50 has an implied volatility of 28%. The six-month risk-free rate is 5% and no dividends are expected. Explain why the two implied volatilities are different. Use DerivaGem to calculate the prices of the two options. Use put-call parity to calculate the prices of six-month European put options with strike prices of \$30 and \$50. Use DerivaGem to calculate the implied volatilities of these two put options.

The difference between the two implied volatilities is consistent with Figure 18.3 in the text. For equities the volatility smile is downward sloping. A high strike price option has a lower implied volatility than a low strike price option. The reason is that traders consider that the probability of a large downward movement in the stock price is higher than that predicted by the lognormal probability distribution. The implied distribution assumed by traders is shown in Figure 18.4.

To use DerivaGem to calculate the price of the first option, proceed as follows. Select Equity as the Underlying Type in the first worksheet. Select Analytic European as the Option Type. Input the stock price as 40, volatility as 35%, risk-free rate as 5%, time to exercise as 0.5 year, and exercise price as 30. Leave the dividend table blank because we are assuming no dividends. Select the button corresponding to call. Do not select the implied volatility button. Hit the *Enter* key and click on calculate. DerivaGem will show

the price of the option as 11.155. Change the volatility to 28% and the strike price to 50. Hit the *Enter* key and click on calculate. DerivaGem will show the price of the option as 0.725.

Put–call parity is

$$c + Ke^{-rT} = p + S_0$$

so that

$$p = c + Ke^{-rT} - S_0$$

For the first option, $c = 11.155$, $S_0 = 40$, $r = 0.054$, $K = 30$, and $T = 0.5$ so that

$$p = 11.155 + 30e^{-0.05 \times 0.5} - 40 = 0.414$$

For the second option, $c = 0.725$, $S_0 = 40$, $r = 0.06$, $K = 50$, and $T = 0.5$ so that

$$p = 0.725 + 50e^{-0.05 \times 0.5} - 40 = 9.490$$

To use DerivaGem to calculate the implied volatility of the first put option, input the stock price as 40, the risk-free rate as 5%, time to exercise as 0.5 year, and the exercise price as 30. Input the price as 0.414 in the second half of the Option Data table. Select the buttons for a put option and implied volatility. Hit the *Enter* key and click on calculate. DerivaGem will show the implied volatility as 34.99%.

Similarly, to use DerivaGem to calculate the implied volatility of the first put option, input the stock price as 40, the risk-free rate as 5%, time to exercise as 0.5 year, and the exercise price as 50. Input the price as 9.490 in the second half of the Option Data table. Select the buttons for a put option and implied volatility. Hit the *Enter* key and click on calculate. DerivaGem will show the implied volatility as 27.99%.

These results are what we would expect. DerivaGem gives the implied volatility of a put with strike price 30 to be almost exactly the same as the implied volatility of a call with a strike price of 30. Similarly, it gives the implied volatility of a put with strike price 50 to be almost exactly the same as the implied volatility of a call with a strike price of 50.

Problem 18.17.

“The Black–Scholes model is used by traders as an interpolation tool.” Discuss this view.

When plain vanilla call and put options are being priced, traders do use the Black–Scholes model as an interpolation tool. They calculate implied volatilities for the options whose prices they can observe in the market. By interpolating between strike prices and between times to maturity, they estimate implied volatilities for other options. These implied volatilities are then substituted into Black–Scholes to calculate prices for these options. In practice much of the work in producing a table such as Table 18.2 in the over-the-counter market is done by brokers. Brokers often act as intermediaries between participants in the over-the-counter market and usually have more information on the trades taking place than any individual financial institution. The brokers provide a table such as Table 18.2 to their clients as a service.

Problem 18.18

Using Table 18.2 calculate the implied volatility a trader would use for an 8-month option with $K/S_0 = 1.04$.

13.45%. We get the same answer by (a) interpolating between strike prices of 1.00 and 1.05 and then between maturities six months and one year and (b) interpolating between maturities of six months and one year and then between strike prices of 1.00 and 1.05.

ASSIGNMENT QUESTIONS

Problem 18.19.

A company's stock is selling for \$4. The company has no outstanding debt. Analysts consider the liquidation value of the company to be at least \$300,000 and there are 100,000 shares outstanding. What volatility smile would you expect to see?

In liquidation the company's stock price must be at least $300,000/100,000 = \$3$. The company's stock price should therefore always be at least \$3. This means that the stock price distribution that has a thinner left tail and fatter right tail than the lognormal distribution. An upward sloping volatility smile can be expected.

Problem 18.20.

A company is currently awaiting the outcome of a major lawsuit. This is expected to be known within one month. The stock price is currently \$20. If the outcome is positive, the stock price is expected to be \$24 at the end of one month. If the outcome is negative, it is expected to be \$18 at this time. The one-month risk-free interest rate is 8% per annum.

- What is the risk-neutral probability of a positive outcome?
- What are the values of one-month call options with strike prices of \$19, \$20, \$21, \$22, and \$23?
- Use DerivaGem to calculate a volatility smile for one-month call options.
- Verify that the same volatility smile is obtained for one-month put options.

- (a) If p is the risk-neutral probability of a positive outcome (stock price rises to \$24), we must have

$$24p + 18(1 - p) = 20e^{0.08 \times 0.0833}$$

so that $p = 0.356$

- (b) The price of a call option with strike price K is $(24 - K)pe^{-0.08 \times 0.0833}$ when $K < 24$. Call options with strike prices of 19, 20, 21, 22, and 23 therefore have prices 1.766, 1.413, 1.060, 0.707, and 0.353, respectively.
- (c) From DerivaGem the implied volatilities of the options with strike prices of 19, 20, 21, 22, and 23 are 49.8%, 58.7%, 61.7%, 60.2%, and 53.4%, respectively. The volatility smile is therefore a "frown" with the volatilities for deep-out-of-the-money and deep-in-the-money options being lower than those for close-to-the-money options.
- (d) The price of a put option with strike price K is $(K - 18)(1 - p)e^{-0.08 \times 0.0833}$. Put options with strike prices of 19, 20, 21, 22, and 23 therefore have prices of 0.640, 1.280,

1.920, 2.560, and 3.200. DerivaGem gives the implied volatilities as 49.81%, 58.68%, 61.69%, 60.21%, and 53.38%. Allowing for rounding errors these are the same as the implied volatilities for put options.

Problem 18.21.

A futures price is currently \$40. The risk-free interest rate is 5%. Some news is expected tomorrow that will cause the volatility over the next three months to be either 10% or 30%. There is a 60% chance of the first outcome and a 40% chance of the second outcome. Use DerivaGem to calculate a volatility smile for three-month options.

The calculations are shown in the following table. For example, when the strike price is 34, the price of a call option with a volatility of 10% is 5.926, and the price of a call option when the volatility is 30% is 6.312. When there is a 60% chance of the first volatility and 40% of the second, the price is $0.6 \times 5.926 + 0.4 \times 6.312 = 6.080$. The implied volatility given by this price is 23.21. The table shows that the uncertainty about volatility leads to a classic volatility smile similar to that in Figure 18.1 of the text. In general when volatility is stochastic with the stock price and volatility uncorrelated we get a pattern of implied volatilities similar to that observed for currency options.

Strike Price	Call Option Price 10% Volatility	Call Option Price 30% Volatility	Weighted Price	Implied Volatility (%)
34	5.926	6.312	6.080	23.21
36	3.962	4.749	4.277	21.03
38	2.128	3.423	2.646	18.88
40	0.788	2.362	1.418	18.00
42	0.177	1.560	0.730	18.80
44	0.023	0.988	0.409	20.61
46	0.002	0.601	0.242	22.43

Problem 18.22.

*Data for a number of foreign currencies are provided on the author's Web site:
<http://www.rotman.utoronto.ca/~hull/data>*

Choose a currency and use the data to produce a table similar to Table 18.1.

The following table shows the percentage of daily returns greater than 1, 2, 3, 4, 5, and 6 standard deviations for each currency. The pattern is similar to that in Table 18.1.

	> 1sd	> 2sd	> 3sd	> 4sd	> 5sd	> 6sd
AUD	24.8	5.3	1.3	0.2	0.1	0.0
BEF	24.3	5.7	1.3	0.6	0.2	0.0
CHF	26.1	4.2	1.3	0.6	0.1	0.0
DEM	23.9	5.0	1.4	0.6	0.1	0.0
DKK	26.7	5.8	1.3	0.3	0.0	0.0
ESP	28.2	5.1	0.9	0.3	0.1	0.0
FRF	26.0	5.4	1.4	0.2	0.0	0.0
GBP	23.9	6.4	1.1	0.4	0.1	0.0
ITL	25.4	6.6	1.1	0.2	0.0	0.0
NLG	25.6	5.7	1.7	0.2	0.0	0.0
SEK	28.2	5.2	1.0	0.0	0.0	0.0
Normal	31.7	4.6	0.3	0.0	0.0	0.0

Problem 18.23.

Data for a number of stock indices are provided on the author's Web site:

<http://www.rotman.utoronto.ca/~hull/data>

Choose an index and test whether a three standard deviation down movement happens more often than a three standard deviation up movement.

The results are shown in the table below.

	> 3sd down	> 3sd up
TSE	0.88	0.22
S&P	0.55	0.44
FTSE	0.55	0.66
CAC	0.33	0.33
Nikkei	0.55	0.66
Total	0.57	0.46

Problem 18.24.

Consider a European call and a European put with the same strike price and time to maturity. Show that they change in value by the same amount when the volatility increases from a level, σ_1 , to a new level, σ_2 within a short period of time. (Hint Use put-call parity.)

Define c_1 and p_1 as the values of the call and the put when the volatility is σ_1 . Define c_2 and p_2 as the values of the call and the put when the volatility is σ_2 . From put-call

parity

$$\begin{aligned} p_1 + S_0 e^{-qT} &= c_1 + K e^{-rT} \\ p_2 + S_0 e^{-qT} &= c_2 + K e^{-rT} \end{aligned}$$

If follows that

$$p_1 - p_2 = c_1 - c_2$$

Problem 18.25.

An exchange rate is currently 1.0 and the implied volatilities of six-month European options with strike prices 0.7, 0.8, 0.9, 1.0, 1.1, 1.2, and 1.3 are 13%, 12%, 11%, 10%, 11%, 12%, and 13%. The domestic and foreign risk free rates are both 2.5%. Calculate the implied probability distribution using an approach similar to that used in the appendix for Example 18.2. Compare it with the implied distribution where all the implied volatilities are 11.5%.

Define:

$$\begin{aligned} g(S_T) &= g_1 \text{ for } 0.7 \leq S_T < 0.8 \\ g(S_T) &= g_2 \text{ for } 0.8 \leq S_T < 0.9 \\ g(S_T) &= g_3 \text{ for } 0.9 \leq S_T < 1.0 \\ g(S_T) &= g_4 \text{ for } 1.0 \leq S_T < 1.1 \\ g(S_T) &= g_5 \text{ for } 1.1 \leq S_T < 1.2 \\ g(S_T) &= g_6 \text{ for } 1.2 \leq S_T < 1.3 \end{aligned}$$

The value of g_1 can be calculated by interpolating to get the implied volatility for a six-month option with a strike price of 0.75 as 12.5%. This means that options with strike prices of 0.7, 0.75, and 0.8 have implied volatilities of 13%, 12.5% and 12%, respectively. From DerivaGem their prices are \$0.2963, \$0.2469, and \$0.1976, respectively. Using equation (18A.1) with $K = 0.75$ and $\delta = 0.05$ we get

$$g_1 = \frac{e^{0.025 \times 0.5} (0.2963 + 0.1976 - 2 \times 0.2469)}{0.05^2} = 0.0315$$

Similar calculations show that $g_2 = 0.7241$, $g_3 = 4.0788$, $g_4 = 3.6766$, $g_5 = 0.07285$, and $g_6 = 0.0898$. The total probability between 0.7 and 1.3 is the sum of these numbers multiplied by 0.1 or 0.9329. If the volatility had been flat at 11.5% the values of g_1 , g_2 , g_3 , g_4 , g_5 , and g_6 would have been 0.0239, 0.9328, 4.2248, 3.7590, 0.9613, and 0.0938. The total probability between 0.7 and 1.3 is in this case 0.9996. This shows that the volatility smile gives rise to heavy tails for the distribution.

Problem 18.26.

Use Table 18.2 to calculate the implied volatility a trader would use for an 11-month option with $K/S_0 = 0.98$

Interpolation gives the volatility for a six-month option with a strike price of 98 as 12.82%. Interpolation also gives the volatility for a 12-month option with a strike price of 98 as 13.7%. A final interpolation gives the volatility of an 11-month option with a strike price of 98 as 13.55%. The same answer is obtained if the sequence in which the interpolations is done is reversed.