Vector Algebra

General Inner Product Space

Let $\mathcal{X}$ be a vector space over a field $\mathbb{F}$ (here our vector space $\mathcal{X}$ denotes $\mathbb{R}^n$ or $\mathbb{C}^n$ and $\mathbb{F}$ denotes either real field $\mathbb{R}$ or complex field $\mathbb{C}$ throughout this course).

Definition 1 *A semi-inner product is a binary operation $\langle \cdot, \cdot \rangle : \mathcal{X} \times \mathcal{X} \to \mathbb{F}$ such that for all $\alpha, \beta \in \mathbb{F}$ and $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{X}$, the followings are satisfied:*

1. $\langle \alpha\mathbf{x} + \beta\mathbf{y}, \mathbf{z} \rangle = \alpha\langle \mathbf{x}, \mathbf{z} \rangle + \beta\langle \mathbf{y}, \mathbf{z} \rangle$
2. $\langle \mathbf{x}, \alpha\mathbf{y} + \beta\mathbf{z} \rangle = \overline{\alpha}\langle \mathbf{x}, \mathbf{y} \rangle + \overline{\beta}\langle \mathbf{x}, \mathbf{z} \rangle$
3. $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$
4. $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$

An inner product on $\mathcal{X}$ is a semi-inner product that also satisfies

5. *If $\langle \mathbf{x}, \mathbf{x} \rangle = 0$, then $\mathbf{x} = \mathbf{0}$.*

Theorem 1 (Cauchy-Schwarz Inequality) *If $\langle \cdot, \cdot \rangle$ is a semi-inner product on $\mathcal{X}$, then*

$$|\langle \mathbf{x}, \mathbf{y} \rangle|^2 \leq \langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle \text{ for all } \mathbf{x}, \mathbf{y} \in \mathcal{X}.$$

Moreover, the equality occurs iff $\exists\, \alpha, \beta \in \mathbb{F}$, both not 0, such that $\langle \alpha\mathbf{x} + \beta\mathbf{y}, \alpha\mathbf{x} + \beta\mathbf{y} \rangle = 0$.

Corollary 1 *If $\langle \cdot, \cdot \rangle$ is a semi-inner product on $\mathcal{X}$ and $\|\mathbf{x}\| \stackrel{\text{def}}{=} \langle \mathbf{x}, \mathbf{x} \rangle^{\frac{1}{2}}$ for all $\mathbf{x} \in \mathcal{X}$, then*

$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ *for all* $\mathbf{x}, \mathbf{y} \in \mathcal{X}$ (**Triangle Inequality**),

$\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$ *for* $\alpha \in \mathbb{F}$ *and* $\mathbf{x} \in \mathcal{X}$.

If $\langle \cdot, \cdot \rangle$ is an inner product, then, $\|\mathbf{x}\| = 0$ implies $\mathbf{x} = \mathbf{0}$.

The quantity $\|\mathbf{x}\| \stackrel{\text{def}}{=} \langle \mathbf{x}, \mathbf{x} \rangle^{\frac{1}{2}}$ for an inner product is called the *norm* of $\mathbf{x}$, said it's the norm induced by the inner product.

By definitions of $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$,

$$\begin{aligned} \langle \mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y} \rangle &= \langle \mathbf{x}, \mathbf{x} \rangle + 2\,\mathrm{Re}(\langle \mathbf{x}, \mathbf{y} \rangle) + \langle \mathbf{y}, \mathbf{y} \rangle, \\ \text{i.e.} \qquad \|\mathbf{x} + \mathbf{y}\|^2 &= \|\mathbf{x}\|^2 + 2\,\mathrm{Re}(\langle \mathbf{x}, \mathbf{y} \rangle) + \|\mathbf{y}\|^2. \end{aligned}$$

Abstract vector algebra on Hilbert spaces

Exercise 1 Look up *metric space* and *complete* metric space.

Remark 1 *A Hilbert space is a vector space $\mathcal{H}$ over $\mathbb{F}$ together with an inner product $\langle \cdot, \cdot \rangle$ such that relative to the metric $d(\mathbf{x}, \mathbf{y}) \stackrel{\text{def}}{=} \|\mathbf{x} - \mathbf{y}\|$ induced by the norm, $\mathcal{H}$ is a complete metric space. (for the continuity issue).*

Definition 2 (Orthgonality) If $\mathcal{H}$ is a Hilbert space and $\mathbf{x}, \mathbf{y} \in \mathcal{H}$, then $\mathbf{x}$ and $\mathbf{y}$ are *orthogonal (perpendicular)* to each other if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$, in symbol, $\mathbf{x} \perp \mathbf{y}$. If $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{H}$, then $\mathcal{X} \perp \mathcal{Y}$ provided that $\mathbf{x} \perp \mathbf{y}$ for every $\mathbf{x} \in \mathcal{X}$ and $\mathbf{y} \in \mathcal{Y}$.

Theorem 2 (The Pythagorean Theorem) *If $\mathbf{x}_1, \cdots, \mathbf{x}_n$ are orthogonal to one another in $\mathcal{H}$, then*

$$\|\mathbf{x}_1 + \cdots + \mathbf{x}_n\|^2 = \|\mathbf{x}_1\|^2 + \cdots + \|\mathbf{x}_n\|^2.$$

Theorem 3 (Parallelogram Law) *If $\mathcal{H}$ is a Hilbert space and $\mathbf{x}, \mathbf{y} \in \mathcal{H}$, then*

$$\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2 = 2(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2).$$

Theorem 4 *If $\mathcal{M} \subseteq \mathcal{H}$ is a closed linear subspace and $\mathbf{h} \in \mathcal{H}$, let $P\mathbf{h} \in \mathcal{M}$ be the unique point such that $\mathbf{h} - P\mathbf{h} \perp \mathcal{M}$. Then*

1. *P is a linear transformation on $\mathcal{H}$,*
2. *$\|P\mathbf{h}\| \leq \|\mathbf{h}\|$ for every $\mathbf{h} \in \mathcal{H}$,*
3. *$P^2 = P$,*
4. *$\ker P = \mathcal{M}^\perp$ and $\operatorname{ran} P = \mathcal{M}$.*

Such P is called the orthogonal projection of $\mathcal{H}$ onto subspace $\mathcal{M}$.

Exercise 2 Prove the *Cauchy-Schwarz inequality.*

Exercise 3 Prove the *triangle inequality.*

Exercise 4 Prove the *Parallelogram Law.*

A Hilbert Space $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$

We will focus on $\mathbb{R}^n$, specially $\mathbb{R}^3$ from now on.

It can be shown that $\mathbb{R}^n$ together with the inner product defined this way

$$\langle \mathbf{x}, \mathbf{y}\rangle \stackrel{\text{def}}{=} \sum_{i=1}^n x_i y_i \text{ for any } \mathbf{x} = (x_1, \cdots, x_n),\ \mathbf{y} = (y_1, \cdots, y_n) \text{ in } \mathbb{R}^n$$

is a Hilbert space.

The projection of vector $\mathbf{x}$ onto vector $\mathbf{y}$ is a vector denoted by $\operatorname{proj}_{\mathbf{y}}\mathbf{x} \stackrel{\text{def}}{=} \underbrace{\langle \mathbf{x}, \frac{\mathbf{y}}{\|\mathbf{y}\|}\rangle}_{\text{scalar}} \underbrace{\frac{\mathbf{y}}{\|\mathbf{y}\|}}_{\text{unit vector}} = \frac{\langle \mathbf{x}, \mathbf{y}\rangle}{\|\mathbf{y}\|^2}$.

Note that usually $\langle \mathbf{x}, \mathbf{y}\rangle$ is not equal to the magnitude of the projection of one onto the other.

Let θ be the angle between vectors $\mathbf{x}$ and $\mathbf{y}$. By the law of cosine, $\cos\theta = \dfrac{\langle \mathbf{x}, \mathbf{y}\rangle}{\|\mathbf{x}\|\,\|\mathbf{y}\|}$.

Exercise 5 State the *law of cosine* and prove it.

Definition 3 *Cross Product (in $\mathbb{R}^3$)* is a binary operation between two vectors:

$$\mathbf{x} \times \mathbf{y} \stackrel{\text{def}}{=} \|\mathbf{x}\|\,\|\mathbf{y}\| \sin\theta\, \mathbf{z} \ \in \mathbb{R}^3$$

where $\mathbf{z}$ is a unit vector in the direction of a right-hand screw as $\mathbf{x}$ rotating toward $\mathbf{y}$ through angle θ. The alternative definition of cross product is

$$\mathbf{x} \times \mathbf{y} \stackrel{\text{def}}{=} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

Exercise 6

1. Show that the two definitions of cross product are equivalent.

2. Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^3$, $\lambda \in \mathbb{R}$. Show that cross product has the following properties:

   (a) $\mathbf{x} \times \mathbf{y} = -\mathbf{y} \times \mathbf{x}$ (skew-symmetry)

   (b) $\mathbf{x} \times (\mathbf{y} + \mathbf{z}) = \mathbf{x} \times \mathbf{y} + \mathbf{x} \times \mathbf{z}$ (distributive law)

   (c) $\lambda(\mathbf{x} \times \mathbf{y}) = \mathbf{x} \times (\lambda \mathbf{y}) = (\lambda \mathbf{x}) \times \mathbf{y}$

   (d) $\mathbf{x} \times \mathbf{x} = 0$

3. Prove the *Lagrange's identity*: $\|\mathbf{x} \times \mathbf{y}\|^2 = \|\mathbf{x}\|^2 \|\mathbf{y}\|^2 - (\mathbf{x} \bullet \mathbf{y})^2$.

4. State the law of sines and prove it.

Notice that cross product does not have associativity, i.e. $\mathbf{x} \times (\mathbf{y} \times \mathbf{z}) \neq (\mathbf{x} \times \mathbf{y}) \times \mathbf{z}$!

Exercise 7 Namely, $\mathbf{x} \bullet \mathbf{y} \times \mathbf{z}$ means $\mathbf{x} \bullet (\mathbf{y} \times \mathbf{z})$ and $\mathbf{x} \times \mathbf{y} \bullet \mathbf{z}$ means $(\mathbf{x} \times \mathbf{y}) \bullet \mathbf{z}$.

1. Show that $\mathbf{x} \bullet \mathbf{y} \times \mathbf{z} = \mathbf{x} \times \mathbf{y} \bullet \mathbf{z}$. This sometimes is called *scalar triple product*.

2. Let $[\mathbf{x}\,\mathbf{y}\,\mathbf{z}] \stackrel{\text{def}}{=} \mathbf{x} \bullet \mathbf{y} \times \mathbf{z}$. Show that $[\mathbf{x}\,\mathbf{y}\,\mathbf{z}] = \det(\mathbf{x}, \mathbf{y}, \mathbf{z})$ and $[\mathbf{x}\,\mathbf{y}\,\mathbf{z}] = [\mathbf{y}\,\mathbf{z}\,\mathbf{x}] = [\mathbf{z}\,\mathbf{x}\,\mathbf{y}] = -[\mathbf{x}\,\mathbf{z}\,\mathbf{y}] = [\mathbf{z}\,\mathbf{y}\,\mathbf{x}] = [\mathbf{y}\,\mathbf{x}\,\mathbf{z}]$. Geometrically, $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are co-planar iff $[\mathbf{x}\,\mathbf{y}\,\mathbf{z}] = 0$.

3. Show that *vector triple products* $\begin{array}{l} \mathbf{x} \times (\mathbf{y} \times \mathbf{z}) = (\mathbf{x} \bullet \mathbf{z})\,\mathbf{y} - (\mathbf{x} \bullet \mathbf{y})\,\mathbf{z}, \text{ and} \\ (\mathbf{x} \times \mathbf{y}) \times \mathbf{z} = (\mathbf{x} \bullet \mathbf{z})\,\mathbf{y} - (\mathbf{y} \bullet \mathbf{z})\,\mathbf{x}. \end{array}$

4. Show that $(\mathbf{u} \times \mathbf{v}) \times (\mathbf{x} \times \mathbf{y}) = [\mathbf{u}\,\mathbf{v}\,\mathbf{y}]\,\mathbf{x} - [\mathbf{u}\,\mathbf{v}\,\mathbf{x}]\,\mathbf{y} = [\mathbf{x}\,\mathbf{y}\,\mathbf{u}]\,\mathbf{v} - [\mathbf{x}\,\mathbf{y}\,\mathbf{v}]\,\mathbf{u}$. and implies that any vector can be expressed as a linear combination of any non-co-planar vectors.

5. Show the *extended Lagrange identity*: $(\mathbf{u} \times \mathbf{v}) \bullet (\mathbf{x} \times \mathbf{y}) = (\mathbf{u} \bullet \mathbf{x})(\mathbf{v} \bullet \mathbf{y}) - (\mathbf{v} \bullet \mathbf{x})(\mathbf{u} \bullet \mathbf{y})$.

6. Show the *Jacobi identity*: $\mathbf{x} \times (\mathbf{y} \times \mathbf{z}) + \mathbf{y} \times (\mathbf{z} \times \mathbf{x}) + \mathbf{z} \times (\mathbf{x} \times \mathbf{y}) = \mathbf{0}$.

7. Show that $(\mathbf{x} \times \mathbf{y}) \bullet (\mathbf{y} \times \mathbf{z}) \times (\mathbf{z} \times \mathbf{x}) = [\mathbf{x}\,\mathbf{y}\,\mathbf{z}]^2$.

Definition 4 (Orthonormal Set) Let $X = \{\mathbf{x}_1, \cdots, \mathbf{x}_k\} \subset \mathbb{R}^n$. X is called *orthonormal* if $\mathbf{x}_i \bullet \mathbf{x}_j = \delta_{ij}$ for any $i, j = 1, \cdots, k$. If $k = \#X = n$, then X is called an *orthonormal basis* of $\mathbb{R}^n$.

Definition 5 (Reciprocal Sets of Vectors) Let $X = \{\mathbf{x}_1, \cdots, \mathbf{x}_k\}$, $Y = \{\mathbf{y}_1, \cdots, \mathbf{y}_k\} \subset \mathbb{R}^n$. X and Y are said *reciprocal* to each other if $\mathbf{x}_i \bullet \mathbf{y}_j = \delta_{ij}$ for any $i, j = 1, \cdots, k$.

Exercise 8 Show that if $X = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ and $Y = \{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ are reciprocal sets in $\mathbb{R}^3$, then

1. $[\mathbf{x}_1\,\mathbf{x}_2\,\mathbf{x}_3] \neq 0$ and $[\mathbf{y}_1\,\mathbf{y}_2\,\mathbf{y}_3] \neq 0$.
2. $\mathbf{x}_1 = \dfrac{\mathbf{y}_2 \times \mathbf{y}_3}{[\mathbf{y}_1\,\mathbf{y}_2\,\mathbf{y}_3]}, \quad \mathbf{x}_2 = \dfrac{\mathbf{y}_3 \times \mathbf{y}_1}{[\mathbf{y}_1\,\mathbf{y}_2\,\mathbf{y}_3]}, \quad \mathbf{x}_3 = \dfrac{\mathbf{y}_1 \times \mathbf{y}_2}{[\mathbf{y}_1\,\mathbf{y}_2\,\mathbf{y}_3]}$, and
   $\mathbf{y}_1 = \dfrac{\mathbf{x}_2 \times \mathbf{x}_3}{[\mathbf{x}_1\,\mathbf{x}_2\,\mathbf{x}_3]}, \quad \mathbf{y}_2 = \dfrac{\mathbf{x}_3 \times \mathbf{x}_1}{[\mathbf{x}_1\,\mathbf{x}_2\,\mathbf{x}_3]}, \quad \mathbf{y}_3 = \dfrac{\mathbf{x}_1 \times \mathbf{x}_2}{[\mathbf{x}_1\,\mathbf{x}_2\,\mathbf{x}_3]}$.