+series	: positive series (正級數)
conv. abs.	: converges absolutely (絕對收斂)
conv. cond.	: converges conditionally (條件收斂)

正級數審斂法縮寫: **I**-*ntegral*, **N**$^{\text{th}}$-*term*, **P**-*series*, *ordinary* **C**-*omparision*, **L**-*imit comparision*, **G**-*eometric series*, **R**-*atio*, *roo*-**T** test.
C(w/P) 表示 *"ordinary comparision test with some p-series"*.

如果你的正級數 $\sum_n a_n$ 收斂, 則 $a_n \overset{n\to\infty}{\longrightarrow} 0$ 是必然的。請你自己另想辦法證明, 因爲了解數列 a_n 的收斂/發散性是最基本的問題, 我們一定會考的.

Example 1 **L(w/P):**

- $\sum_n \frac{1}{(\ln n)^2}$: $\lim\limits_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{(\ln n)^2}} = \lim\limits_{n\to\infty} \frac{(\ln n)^2}{n} \overset{\frac{\infty}{\infty}}{=} \lim\limits_{n\to\infty} \frac{2(\ln n)\frac{1}{n}}{1} \overset{\frac{\infty}{\infty}}{=} \lim\limits_{n\to\infty} \frac{2}{n} = 0\,(\star)$, $\therefore \sum_n \frac{1}{(\ln n)^2}$ div.
- $\sum_n \frac{(\ln n)^2}{n^3}$: $\lim\limits_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{(\ln n)^2}{n^3}} = \lim\limits_{n\to\infty} \frac{n}{(\ln n)^2} = \infty$ by $(\star)$, $\therefore \sum_n \frac{(\ln n)^2}{n^3}$ conv.
- $\sum_n \frac{(\ln n)^3}{n^3}$: $\lim\limits_{n\to\infty} \frac{\frac{(\ln n)^3}{n^3}}{\frac{1}{n^2}} = \lim\limits_{n\to\infty} \frac{(\ln n)^3}{n} \overset{\frac{\infty}{\infty}}{=} \lim\limits_{n\to\infty} \frac{3(\ln n)^2\frac{1}{n}}{1} = 0$ by $(\star)$, $\therefore \sum_n \frac{(\ln n)^3}{n^3}$ conv.
- $\sum_n \frac{1}{\sqrt{n}\ln n}$: $\lim\limits_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{\sqrt{n}\ln n}} = \lim\limits_{n\to\infty} \frac{\ln n}{\sqrt{n}} \overset{\frac{\infty}{\infty}}{=} \lim\limits_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{2\sqrt{n}}} = \lim\limits_{n\to\infty} \frac{2}{\sqrt{n}} = 0$, $\therefore \sum_n \frac{1}{\sqrt{n}\ln n}$ div.

Example 2 $\sum_n \frac{n!}{n^n} = \sum_n a_n$, **R:** $\frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+1)^n}}{\frac{1}{n^n}} = \frac{1}{(1+\frac{1}{n})^n} \overset{n\to\infty}{\longrightarrow} \frac{1}{e} < 1$, so 正的 $\sum_n a_n$ conv.

Example 3 $\sum_n \frac{(-1)^n(n+1)^n}{(2n)^n} = \sum_n (-1)^n \left(\frac{n+1}{2n}\right)^n$:

- **C(w/ conv. G):** $\because \frac{n+1}{2n} \searrow \frac{1}{2}, \Rightarrow \frac{n+1}{2n} \le 0.6$ for $n \ge 5, \therefore$ 正的 $\sum\limits_{n\ge5}(\frac{n+1}{2n})^n \le \sum\limits_{n\ge5}(0.6)^n$, $\sum_n \frac{(-1)^n(n+1)^n}{(2n)^n}$ conv. abs.
- **L(w/ conv. G):** $\because \frac{n+1}{2n} \searrow \frac{1}{2}$, $\therefore$ 正的 $\sum_n(\frac{n+1}{2n})^n$ 和 $\sum_n(\frac{1}{2})^n$ 比較: $\lim\limits_{n\to\infty} \frac{(\frac{n+1}{2n})^n}{(\frac{1}{2})^n} = \lim\limits_{n\to\infty}(1+\frac{1}{n})^n = e \Longrightarrow \sum_n \frac{(-1)^n(n+1)^n}{(2n)^n}$ conv. abs.

Example 4 $\sum_n \frac{(-1)^{n+1}(n!)^2}{(2n)!} = \sum_n (-1)^{n+1} a_n$:

- **R:** $\frac{a_{n+1}}{a_n} = \frac{(n+1)!(n+1)!/(2n+2)!}{n!n!/(2n)!} = \frac{(n+1)^2}{(2n+1)(2n+2)} \overset{n\to\infty}{\longrightarrow} \frac{1}{4} < 1$, so $\sum_n (-1)^{n+1}a_n$ conv. abs.
- **C(w/ conv. G):** $\because a_n = \underbrace{\frac{1}{n+1}\frac{2}{n+2}\cdots\frac{n}{n+n}}_{n\text{ 項}} \le (\frac{1}{2})^n$ (因爲 $\frac{1}{n+1} < \frac{2}{n+2} < \cdots < \frac{n}{n+n} = \frac{1}{2}$),
  $\therefore$ 正的 $\sum_n a_n \le \sum_n (\frac{1}{2})^n$, 即 $\sum_n (-1)^{n+1}a_n$ conv. abs.

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