- $\lim\limits_{x\to0}\frac{\sin x-\tan x}{x^2\sin x}$ 原式 ($\frac{0}{0}$式) $=\lim\limits_{x\to0}\frac{1-\sec x}{x^2}$ ($\frac{0}{0}$式)
  $\overset{\pounds}{=}\lim\limits_{x\to0}\frac{-\sec x\tan x}{2x}$ ($\frac{0}{0}$式)
  $\overset{\pounds}{=}\lim\limits_{x\to0}\frac{-\sec x\tan^2-\sec^3 x}{2}=\frac{-1}{2}$

- $\lim\limits_{x\to0}\frac{e^x-\ln(1+x)-1}{x^2}$ 原式 ($\frac{0}{0}$式) $\overset{\pounds}{=}\lim\limits_{x\to0}\frac{e^x-\frac{1}{1+x}}{2x}$ ($\frac{0}{0}$式)
  $\overset{\pounds}{=}\lim\limits_{x\to0}\frac{e^x+\frac{1}{(1+x)^2}}{2}=1$

- $\lim\limits_{x\to0^+}\frac{1-\cos x-x\sin x}{2-2\cos x-\sin^2 x}$ 原式 ($\frac{0}{0}$式) $\overset{\pounds}{=}\lim\limits_{x\to0^+}\frac{\sin x-(\sin x+x\cos x)}{2\sin x-2\sin x\cos x}$ ($\frac{0}{0}$式)
  $\overset{\pounds}{=}\lim\limits_{x\to0^+}\frac{-\cos x+x\sin x}{2\cos x-2\cos 2x}$ ($\frac{-1}{0^+}$) $=-\infty$

- $\lim\limits_{x\to0^-}\frac{\sin x+\tan x}{e^x+e^{-x}-2}$ 原式 ($\frac{0}{0}$式) $\overset{\pounds}{=}\lim\limits_{x\to0^-}\frac{\cos x+\sec^2 x}{e^x-e^{-x}}$ ($\frac{2}{0^-}$) $=-\infty$

- $\lim\limits_{x\to0}\frac{\sin x(1-\cos x)}{x-\sin x\cos x}=\lim\limits_{x\to0}\frac{\sin x-\frac{1}{2}\sin 2x}{x-\frac{1}{2}\sin 2x}$ ($\frac{0}{0}$式)
  $\overset{\pounds}{=}\lim\limits_{x\to0}\frac{\cos x-\cos 2x}{1-\cos 2x}$ ($\frac{0}{0}$式)
  $\overset{\pounds}{=}\lim\limits_{x\to0}\frac{-\sin x+2\sin 2x}{2\sin 2x}=\frac{3}{4}$

- $\lim\limits_{x\to0}\frac{\sin x\cos x-x\cos^2 x}{x-\sin x\cos x}$ ($\frac{0}{0}$式) $\overset{\pounds}{=}\lim\limits_{x\to0}\frac{\cos 2x-\cos^2 x+x\sin 2x}{1-\cos 2x}$ ($\frac{0}{0}$式)
  $\overset{\pounds}{=}\lim\limits_{x\to0}\frac{\cancel{-2\sin 2x+\sin 2x+\sin 2x}+2x\cos 2x}{2\sin 2x}$ 可£但不£ $=\frac{1}{2}$

- $\lim\limits_{x\to0}\frac{1-\cos(x^2)}{x^3\sin x}$ ($\frac{0}{0}$式) $\overset{\pounds}{=}\lim\limits_{x\to0}\frac{\sin(x^2)2x}{3x^2\sin x+x^3\cos x}=\lim\limits_{x\to0}\frac{2\sin(x^2)}{3x\sin x+x^2\cos x}$ 可£但不£
  $=\frac{2}{\lim\limits_{x\to0}\left(\frac{3x^2\sin x}{\sin(x^2)x}+\frac{x^2\cos x}{\sin(x^2)}\right)}=\frac{2}{3+1}=\frac{1}{2}$

- $\lim\limits_{x\to0}\frac{\tan x-x}{\arcsin x-x}$ ($\frac{0}{0}$式) $\overset{\pounds}{=}\lim\limits_{x\to0}\frac{\sec^2 x-1}{\frac{1}{\sqrt{1-x^2}}-1}$ ($\frac{0}{0}$式)
  $\overset{\pounds}{=}\lim\limits_{x\to0}\frac{2\sec^2 x\tan x}{\frac{x}{(1-x^2)^{\frac{3}{2}}}}$ 可£但不£
  $\overset{\pounds}{=}\lim\limits_{x\to0}2(1-x^2)^{\frac{3}{2}}\sec^2 x\frac{\sin x}{x\cos x}=2$

- $\lim\limits_{x\to\infty}\frac{(\ln x)^2}{2^x}=\left(\lim\limits_{x\to\infty}\frac{\ln x}{\sqrt{2}^x}\right)^2$ ($\frac{\infty}{\infty}$式)
  $=\left(\lim\limits_{x\to\infty}\frac{\frac{1}{x}}{\sqrt{2}^x\ln\sqrt{2}}\right)^2=0$

- $\lim\limits_{x\to\infty}\frac{3x}{\ln(100x+e^x)}$ ($\frac{\infty}{\infty}$式) $\lim\limits_{x\to\infty}\frac{3(100x+e^x)}{(100+e^x)}\overset{\pounds}{=}\cdots\overset{\pounds}{=}\cdots=3$

- $\lim\limits_{x\to\frac{-1}{2}}\frac{\ln(4-8x)}{\tan(\pi x)}\overset{t=\pi(\frac{1}{2}-x)}{=}\lim\limits_{t\to0^+}\frac{\ln(\frac{8}{\pi}t)}{\tan(\frac{\pi}{2}-t)}$
  $=\lim\limits_{t\to0^+}\frac{\ln t+c}{\cot t}$ ($\frac{\infty}{\infty}$式)
  $\overset{\pounds}{=}\lim\limits_{t\to0^+}\frac{\frac{1}{t}}{-\csc^2 t}$
  $=\lim\limits_{t\to0^+}\frac{-\sin^2 t}{t}=0$

*Copyright © by Dr. Mengnien Wu :* `mwu@mail.tku.edu.tw`

- $\displaystyle\lim_{x\to\frac{-1}{2}}\frac{(\ln(4-8x))^2}{\tan(\pi x)}$ (即上一題的 0 乘 $\infty$, 仍爲不定式

$$\overset{t=\pi(\frac{1}{2}-x)}{=}\lim_{t\to0^+}\frac{(\ln(\frac{8}{\pi}t))^2}{\tan(\frac{\pi}{2}-t)}$$

$$=\lim_{t\to0^+}\frac{(\ln t+c)^2}{\cot t}\ (\tfrac{\infty}{\infty}\text{式})$$

$$\overset{\pounds}{=}\lim_{t\to0^+}\frac{2(\ln t+c)\frac{1}{t}}{-\csc^2 t}$$

$$=\lim_{t\to0^+}-2\frac{\ln t+c}{\csc t}\frac{\sin t}{t}$$

$$\overset{\pounds}{=}\lim_{t\to0^+}-2\frac{\frac{1}{t}}{-\csc t\cot t}$$

$$=\lim_{t\to0^+}2\frac{\sin}{t}\tan t=0$$

- $\displaystyle\lim_{x\to0}(\cos x)^{\csc x}=e^{\lim_{x\to0}\csc x\cdot\ln(\cos x)}$ $(\infty\cdot0)$

$$=e^{\lim_{x\to0}\frac{\ln(\cos x)}{\sin x}}\ (\tfrac{0}{0}\text{式})$$

$$\overset{\pounds}{=}e^{\lim_{x\to0}\frac{\frac{-\sin x}{\cos x}}{\cos x}}=e^0=1$$

- $\displaystyle\lim_{x\to0}(\csc^2 x-\frac{1}{x^2})=\lim_{x\to0}\frac{(x-\sin x)\overbrace{(1+\frac{\sin x}{x})}^{2}}{x\sin^2 x}$

$$=2\lim_{x\to0}\frac{x-\sin x}{x\sin^2 x}\ (\tfrac{0}{0}\text{式})=2\lim_{x\to0}\frac{1-\cos x}{\sin^2 x+x\sin 2x}\ (\tfrac{0}{0}\text{式})$$

$$\overset{\pounds}{=}2\lim_{x\to0}\frac{\sin x}{\sin 2x+\sin 2x+2x\cos 2x}\ (\tfrac{0}{0}\text{式})$$

$$\overset{\pounds}{=}2\lim_{x\to0}\frac{\cos x}{4\cos 2x+2\cos 2x-4x\sin 2x}=\frac{1}{3}$$

如果知道 $\sin x$ 在 0 的展開式 就容易多了: $\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$ ,

$$\frac{(x-\sin x)(x+\sin x)}{x^2\sin^2 x}=\frac{(\frac{x^3}{3!}+\mathcal{O}(x^5))(2x+\mathcal{O}(x^3))}{x^2(x^2+\mathcal{O}(x^4))}\xrightarrow{x\to0^+}\frac{1}{3}$$

- $\displaystyle\lim_{x\to0}(x+e^{x/3})^{3/x}=e^{\lim_{x\to0}\frac{3}{x}\ln(x+e^{x/3})}$ $(\frac{0}{0}\text{式})$

$$\overset{\pounds}{=}e^{\lim_{x\to0}\frac{3(1+e^{x/3}/3)}{x+e^{x/3}}}=e^{\frac{3(1+1/3)}{0+1}}=e^4$$

- $\displaystyle\lim_{x\to\infty}x^{\frac{1}{x}}=e^{\lim_{x\to\infty}\frac{\ln x}{x}}=e^0=1$

- $\displaystyle\lim_{x\to0^+}x^x=e^{\lim_{x\to0^+}x\ln x}=e^0=1$

- $\displaystyle\lim_{x\to0^+}x^{x^x}=e^{\lim_{x\to0^+}x^x\ln x}=e^{1\cdot(-\infty)}=0$

- $\displaystyle\lim_{x\to0^+}\left(\frac{x^{x^x}}{x}\right)$ $(\frac{0}{0}\text{式})$ $\displaystyle\overset{\pounds}{=}\lim_{x\to0^+}\frac{(x^{x^x})'}{(x)'}$

$$=\cdots$$

$$=\lim_{x\to0^+}\left(\frac{x^{x^x}}{x}\right)\underbrace{x^x\Big(x(\ln x)^2+x\ln x+1\Big)}_{1},$$

原式 $(\frac{\infty}{\infty}\text{式})$ $\displaystyle\overset{\pounds}{=}\lim_{x\to0^+}\frac{(\frac{1}{x})'}{(x^{-x^x})'}$

$$=\cdots$$

$$=\lim_{x\to0^+}\left(\frac{x^{x^x}}{x}\right)\underbrace{\frac{1}{x^x\Big(x(\ln x)^2+x\ln x+1\Big)}}_{1},$$

若堅持此二路線, 則無法判定。

*Copyright © by Dr. Mengnien Wu :* `mwu@mail.tku.edu.tw`

- $$\begin{aligned}\lim_{x\to0^+} x^{x^{x^x}} &= e^{\lim\limits_{x\to0^+} x^{x^x}\ln x} \quad \left(0^+\cdot(-\infty)\right)\\ &= e^{\lim\limits_{x\to0^+}\frac{\ln x}{x^{-x^x}}} \quad \left(\tfrac{\infty}{\infty}\text{式}\right)\\ &\overset{£}{=} e^{\lim\limits_{x\to0^+}\frac{\frac{1}{x}}{-x^{-x^x}x^x\left((\ln x)^2+\ln x+\frac{1}{x}\right)}}\\ &= e^{\lim\limits_{x\to0^+}\frac{-x^{x^x}}{x^x\left(x(\ln x)^2+x\ln x+1\right)}} = e^{\frac{0}{1}} = 1\end{aligned}$$

- 如果知道 $e^x$ 在 0 的展開式 就容易多了:

$$\begin{aligned}\because e^x &= e^x = 1+\tfrac{x}{1!}+\tfrac{x^2}{2!}+\tfrac{x^3}{3!}+\cdots,\\ x &= e^{\ln x} = 1+\frac{(\ln x)^1}{1!}+\frac{(\ln x)^2}{2!}+\frac{(\ln x)^3}{3!}+\frac{(\ln x)^4}{4!}+\cdots, && (x)\\ \Rightarrow x^x &= e^{(x\ln x)} = 1+\frac{(x\ln x)}{1!}+\frac{(x\ln x)^2}{2!}+\frac{(x\ln x)^3}{3!}+\cdots, && (x^x)\\ \Rightarrow x^{x^x} &= e^{(x^x\ln x)} = 1+\frac{(x^x\ln x)}{1!}+\frac{(x^x\ln x)^2}{2!}+\frac{(x^x\ln x)^3}{3!}+\cdots\\ &= 1\\ &\quad+\frac{\ln x}{1!}\left(1+\frac{(x\ln x)}{1!}+\frac{(x\ln x)^2}{2!}+\frac{(x\ln x)^3}{3!}+\cdots\right)^1\\ &\quad+\frac{(\ln x)^2}{2!}\left(1+\frac{(x\ln x)}{1!}+\frac{(x\ln x)^2}{2!}+\frac{(x\ln x)^3}{3!}+\cdots\right)^2\\ &\quad+\frac{(\ln x)^3}{3!}\left(1+\frac{(x\ln x)}{1!}+\frac{(x\ln x)^2}{2!}+\frac{(x\ln x)^3}{3!}+\cdots\right)^3+\cdots\\ &= x+\tfrac{x^2}{1!}(\ln x)^2+\tfrac{x^3}{2!}\left((\ln x)^3+(\ln x)^4\right)+\tfrac{x^4}{3!}\left((\ln x)^4+3(\ln x)^5+(\ln x)^6\right)+\cdots,\end{aligned}$$

$$\therefore \lim_{x\to0^+}\frac{x^{x^x}}{x} = 1$$

下學期就會學到展開式了。不管會不會過, 好好唸, 慢慢就會融會貫通了。

Copyright © by Dr. Mengnien Wu : mwu@mail.tku.edu.tw