Gradient, Divergence, and Curl

Definitions

Let $\mathbf{x} = (x_1, x_2, x_3)$,
scalar $f(\mathbf{x}) = f(x_1, x_2, x_3)$,
vector $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), f_3(\mathbf{x}))$, regarded as *flux* (velocity of fluid),
operator $\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$.

Definition	Value	Physical meaning
$\operatorname{grad} f = \nabla f = (\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3})$	vector	the direction in which f changes most rapidly
$\operatorname{div} \mathbf{f} = \nabla \bullet \mathbf{f} = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}$	scalar	density of flux, i.e. the fluid velocity per unit volume
$\operatorname{curl} \mathbf{f} = \nabla \times \mathbf{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ f_1 & f_2 & f_3 \end{vmatrix}$	vector	spining flux (wheelpool) affecting on the virtural surface

These three are all *linear operators*.

Further explanations *(intuitive approaches)*

Gradient: By *total differentiation*, $df = \frac{\partial f}{\partial x_3}dx_1 + \frac{\partial f}{\partial x_3}dx_3 + \frac{\partial f}{\partial x_3}dx_3 = \operatorname{grad} f \bullet (dx_1, dx_2, dx_3)$, in order to maximize $|df|$, $d\mathbf{x} = (dx_1, dx_2, dx_3)$ has to be parallel to $\operatorname{grad} f$.

Divergence: Consider a tiny rectangular box $\mathbf{S}$ centered at point $\mathbf{x}$ with dimension $(\Delta x_1, \Delta x_2, \Delta x_3)$. Then

the flux $\mathbf{f}$ thru facet $\mathbf{S_1}$ of outer normal $(1, 0, 0)$ is approximately equal to

$$\mathbf{f} \bullet \mathbf{S_1} \approx (f_1 + \frac{\partial f_1}{\partial x_1}\frac{\Delta x_1}{2})\Delta x_2 \Delta x_3,$$

the flux $\mathbf{f}$ thru facet $\mathbf{S_2}$ of outer normal $(-1, 0, 0)$ is approximately equal to

$$\mathbf{f} \bullet \mathbf{S_2} \approx -(f_1 + \frac{\partial f_1}{\partial x_1}\frac{-\Delta x_1}{2})\Delta x_2 \Delta x_3,$$

the flux $\mathbf{f}$ thru facet $\mathbf{S_3}$ of outer normal $(0, 1, 0)$ is approximately equal to

$$\mathbf{f} \bullet \mathbf{S_3} \approx (f_2 + \frac{\partial f_2}{\partial x_2}\frac{\Delta x_2}{2})\Delta x_3 \Delta x_1,$$

the flux $\mathbf{f}$ thru facet $\mathbf{S_4}$ of outer normal $(0, -1, 0)$ is approximately equal to

$$\mathbf{f} \bullet \mathbf{S_4} \approx -(f_2 + \frac{\partial f_2}{\partial x_2}\frac{-\Delta x_2}{2})\Delta x_3 \Delta x_1,$$

the flux $\mathbf{f}$ thru facet $\mathbf{S_5}$ of outer normal $(0, 0, 1)$ is approximately equal to

$$\mathbf{f} \bullet \mathbf{S_5} \approx (f_3 + \frac{\partial f_3}{\partial x_3}\frac{\Delta x_3}{2})\Delta x_1 \Delta x_2,$$

the flux $\mathbf{f}$ thru facet $\mathbf{S_6}$ of outer normal $(0, 0, -1)$ is approximately equal to

$$\mathbf{f} \bullet \mathbf{S_6} \approx -(f_3 + \frac{\partial f_3}{\partial x_3}\frac{-\Delta x_3}{2})\Delta x_1 \Delta x_2.$$

Sum them up, and the total flux thru $\mathbf{S}$ is roughly $(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3})\Delta x_1 \Delta x_2 \Delta x_3$,
i.e. $(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3})$ times the volume of $\mathbf{S}$.
$\Longrightarrow \operatorname{div} \mathbf{f} \overset{\text{def}}{=} \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}$ is the flux in a unit volume.

Curl: Consider a rectangular facet $\mathbf{S_1}$ parallel to x_2-x_3 plane centered at point $\mathbf{x}$ with dimension $(\Delta x_2, \Delta x_3$ and its boundary $C_1 = \partial \mathbf{S}$ consists of 4 edges $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4$ in the counterclockwise direction, i.e. $(0,1,0),(0,0,1),(0,-1,0),(0,0,-1)$ respectively. Then

the flux $\mathbf{f}$ along $\mathbf{r}_1$:

$$\mathbf{f} \bullet d\mathbf{r}_1 \approx (f_2 + \frac{\partial f_2}{\partial x_3}\frac{-\Delta x_3}{2})\Delta x_2,$$

the flux $\mathbf{f}$ along $\mathbf{r}_2$:

$$\mathbf{f} \bullet d\mathbf{r}_2 \approx (f_3 + \frac{\partial f_3}{\partial x_2}\frac{\Delta x_2}{2})\Delta x_2,$$

the flux $\mathbf{f}$ along $\mathbf{r}_3$:

$$\mathbf{f} \bullet d\mathbf{r}_3 \approx -(f_2 + \frac{\partial f_2}{\partial x_3}\frac{-\Delta x_3}{2})\Delta x_2,$$

the flux $\mathbf{f}$ along $\mathbf{r}_4$:

$$\mathbf{f} \bullet d\mathbf{r}_4 \approx -(f_3 + \frac{\partial f_3}{\partial x_2}\frac{-\Delta x_2}{2})\Delta x_2.$$

Sum them up, and the flux $\mathbf{f}$ along C_1 is roughly

$$\underline{(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3})\Delta x_2 \Delta x_3}.$$

Similarly,

if $C_2 = \partial \mathbf{S_2}$ is parallel to x_3-x_1 plane then flux $\mathbf{f}$ along C_2 is roughly

$$\underline{(\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1})\Delta x_3 \Delta x_1},$$

if $C_3 = \partial \mathbf{S_3}$ is parallel to x_1-x_2 plane then flux $\mathbf{f}$ along C_3 is roughly

$$\underline{(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2})\Delta x_1 \Delta x_2},$$

These highlighted three are components of

$$\operatorname{curl}\mathbf{f} \overset{\text{def}}{=} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ f_1 & f_2 & f_3 \end{vmatrix} = (\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3})\hat{i} + (\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1})\hat{j} + (\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2})\hat{k}$$

and the underlined three are the influences of $\mathbf{f}$ on $\mathbf{S_1}, \mathbf{S_2}, \mathbf{S_3}$:

$$(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3})\Delta x_2 \Delta x_3 = \operatorname{curl}\mathbf{f} \bullet (\Delta x_2 \Delta x_3 \hat{i}) = \operatorname{curl}\mathbf{f} \bullet \mathbf{S_1},$$

$$(\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1})\Delta x_3 \Delta x_1 = \operatorname{curl}\mathbf{f} \bullet (\Delta x_3 \Delta x_1 \hat{j}) = \operatorname{curl}\mathbf{f} \bullet \mathbf{S_2},$$

$$(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2})\Delta x_1 \Delta x_2 = \operatorname{curl}\mathbf{f} \bullet (\Delta x_1 \Delta x_2 \hat{k}) = \operatorname{curl}\mathbf{f} \bullet \mathbf{S_3}.$$

From the deriviations of divergence and curl, we can directly come up with the conclusions:

Divergence Theorem V is the region enclosed by closed surface S. Then

$$\oint\!\!\!\oint_S \mathbf{f} \bullet d\mathbf{S} = \iiint_V \operatorname{div}\mathbf{f}\, dV$$

Stokes' Theorem S is a surface with simple closed boundary C. Then

$$\iint_S \operatorname{curl}\mathbf{f} \bullet d\mathbf{S} = \oint_C \mathbf{f} \bullet d\mathbf{r}$$

Green's Theorem A special case of *Stokes' Theorem*:

Let $\mathbf{f}(x,y,z) = (M(x,y), N(x,y), 0)$ and a flat surface $S = R$ is lying on the x-y plane with boundary C, then the normal of S is $(0,0,1)$

so that $\operatorname{curl}\mathbf{f} \bullet d\mathbf{S} = (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})\, dx\, dy$ and $\mathbf{f} \bullet d\mathbf{r} = M\, dx + N\, dy$.

i.e. $$\iint_R (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})\, dx\, dy = \oint_C M\, dx + N\, dy.$$