Differential Operators versus Vector Operators

Any vector operator $\circledast$ possessing *distributive property* (eg. $+, -, \bullet, \times$) will satisfy the following:

$d(\mathbf{A} \circledast \mathbf{B}) = (d\mathbf{A}) \circledast \mathbf{B} + \mathbf{A} \circledast (d\mathbf{B})$;

$\frac{d}{dt}(\mathbf{A} \circledast \mathbf{B}) = \frac{d\mathbf{A}}{dt} \circledast \mathbf{B} + \mathbf{A} \circledast \frac{d\mathbf{B}}{dt}$, if t is possibly the only variable of $\mathbf{A}$ and $\mathbf{B}$;

$\frac{\partial}{\partial t}(\mathbf{A} \circledast \mathbf{B}) = \frac{\partial \mathbf{A}}{\partial t} \circledast \mathbf{B} + \mathbf{A} \circledast \frac{\partial \mathbf{B}}{\partial t}$, if t is possibly a variable of $\mathbf{A}$ and $\mathbf{B}$.

Curvature (more intuitive approaches)

Let $\mathbf{r}(t)$ be a parametric curve in $\mathbb{R}^n$, and s denote *arc length* from some fixed point on the curve. Define $\mathbf{T}(t) \stackrel{\text{def}}{=} \dfrac{\dot{\mathbf{r}}(t)}{\|\dot{\mathbf{r}}(t)\|}$ a *unit tangent* at t, then $\mathbf{T} = \dfrac{\dot{\mathbf{r}}}{\|\dot{\mathbf{r}}\|} = \dfrac{\frac{d\mathbf{r}}{dt}}{\frac{ds}{dt}} = \dfrac{d\mathbf{r}}{ds}$.

A particle moves in a very short period (from t to $t + \Delta t$, $\Delta t \approx 0$) and assume that this locus is an arc of a circle of radius R sweeping through angle $\theta\,(\approx 0)$. The arc length $R\theta \approx \|\mathbf{r}(t + \Delta t) - \mathbf{r}(t)\|$. Since $\mathbf{T}$ is a *unit tangent*, $\therefore \|\mathbf{T}(t + \Delta t) - \mathbf{T}(t)\| \approx \theta$. Hence, as $\Delta t \rightsquigarrow 0$, the instantaneous radius

$$R \approx \frac{\|\mathbf{r}(t + \Delta t) - \mathbf{r}(t)\|}{\|\mathbf{T}(t + \Delta t) - \mathbf{T}(t)\|} = \frac{\left\|\frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t}\right\|}{\left\|\frac{\mathbf{T}(t+\Delta t) - \mathbf{T}(t)}{\Delta t}\right\|} \xrightarrow{\Delta t \rightsquigarrow 0} \frac{\|\dot{\mathbf{r}}(t)\|}{\left\|\dot{\mathbf{T}}(t)\right\|}$$

Define the *curvature* $\kappa \stackrel{\text{def}}{=} \dfrac{1}{R}$, and R is hereby called the *radius of curvature*, then

$$\kappa = \frac{\left\|\dot{\mathbf{T}}(t)\right\|}{\|\dot{\mathbf{r}}(t)\|} = \frac{\left\|\frac{d\mathbf{T}}{dt}\right\|}{\left\|\frac{d\mathbf{r}}{dt}\right\|} = \left\|\frac{d\mathbf{T}}{ds}\right\|.$$

Similarly, since $R\theta \approx \|\mathbf{r}(t + \Delta t) - \mathbf{r}(t)\|$ and $\mathbf{T}(t + \Delta t) - \mathbf{T}(t) \approx \theta\mathbf{N}$ (where $\mathbf{N}$ is a unit vector $\parallel \mathbf{T}(t + \Delta t) - \mathbf{T}(t)$, roughly $\perp \mathbf{T}$),

$$\frac{\mathbf{T}(t + \Delta t) - \mathbf{T}(t)}{\|\mathbf{r}(t + \Delta t) - \mathbf{r}(t)\|} \approx \frac{1}{R}\mathbf{N}, \overset{\Delta t \rightsquigarrow 0}{\Longrightarrow} \frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}.$$

Clearly, $\mathbf{N}$ must be a unit vector perpendicular to $\mathbf{T}$($\because \|\mathbf{T}\|^2 = 1 \overset{\frac{d}{ds}}{\Longrightarrow} 2\mathbf{T} \bullet \frac{d\mathbf{T}}{ds} = 0$) and we call it the *principal normal* to the curve $\mathbf{N} \stackrel{\text{def}}{=} \frac{1}{\kappa}\frac{d\mathbf{T}}{ds}$.

Now, define $\mathbf{B} \stackrel{\text{def}}{=} \mathbf{T} \times \mathbf{N}$, called *bi-normal* to the curve.
We have shown in class that since $\frac{d\mathbf{B}}{ds} \perp \mathbf{B}$ and $\mathbf{N} \Longrightarrow \frac{d\mathbf{B}}{ds} \parallel \mathbf{N}$. Moreover, $\frac{d\mathbf{B}}{ds}$ is opposite to $\mathbf{N}$. Therefore $\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$ for some $\tau > 0$ called *torsion*.

Now we can easily derive the *Frenet-Serret formulae* by the orthonormal basis $\mathbf{T}, \mathbf{N}, \mathbf{B}$:

$$\left.\begin{array}{lll} \text{unit tangent} & \mathbf{T} & \stackrel{\text{def}}{=} \frac{d\mathbf{r}}{ds}, \\ \text{curvature} & \kappa & \stackrel{\text{def}}{=} \left\|\frac{d\mathbf{T}}{ds}\right\|, \\ \text{unit normal} & \mathbf{N} & \stackrel{\text{def}}{=} \frac{1}{\kappa}\frac{d\mathbf{T}}{ds}, \\ \text{binormal} & \mathbf{B} & \stackrel{\text{def}}{=} \mathbf{T} \times \mathbf{N}, \\ \text{torsion} & \tau & \text{is a constant such that} \\ & & \frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}, \end{array}\right\} \text{ then, } \frac{d\mathbf{N}}{ds} = \tau\mathbf{B} - \kappa\mathbf{T}.$$