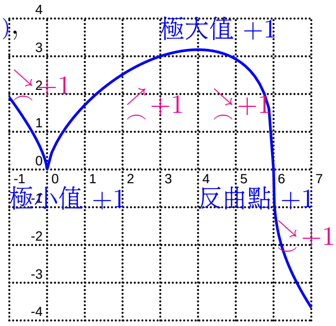
微積分 之『求極值, 積分』 班級_____ 學號________________ 座號___ 姓名 ***Grading Key*** 分數 **41**

1.**(5)** Show that $\sqrt{1+x} < 1 + \frac{1}{2}x$ for any $x > 0$.

令 $f(x) = \sqrt{1+x}$ (+1), $f'(x) = \frac{\frac{1}{2}}{\sqrt{1+x}}$ (+1) 。依均值定理, 存在 $c \in (0,x)$ 使得 $\frac{f(x)-f(0)}{x-0} = f'(c)$ (+1),

即 $\frac{\sqrt{1+x}-1}{x} = \frac{\frac{1}{2}}{\sqrt{1+c}} < \frac{1}{2} \overset{\cdot x(>0)}{\Longrightarrow} \sqrt{1+x} - 1 < \frac{1}{2}x$ (+1) $\iff \sqrt{1+x} < 1 + \frac{1}{2}x$ (+1)

2.**(21)** $f(x) = x^{\frac{2}{3}}(6-x)^{\frac{1}{3}}$ 。若 f 有奇點、極值、反曲點、漸近線, 則圖形要反映出來, 沒計算過程不給分。



$f'(x) = \frac{2}{3}x^{\frac{-1}{3}}(6-x)^{\frac{1}{3}} + x^{\frac{2}{3}}\frac{1}{3}(6-x)^{\frac{-2}{3}}(-1)$ (1+1) $= \frac{\frac{2}{3}(6-x)-\frac{1}{3}x}{x^{\frac{1}{3}}(6-x)^{\frac{2}{3}}}$ (+1) $= \frac{4-x}{x^{\frac{1}{3}}(6-x)^{\frac{2}{3}}}$ (+1),

$f''(x) = (-1)x^{\frac{-1}{3}}(6-x)^{\frac{-2}{3}} + (4-x)\frac{-1}{3}x^{\frac{-4}{3}}(6-x)^{\frac{-2}{3}} + (4-x)x^{\frac{-1}{3}}\frac{-2}{3}(6-x)^{\frac{-5}{3}}(-1)$ (1+1+1)

$= \frac{-x(6-x)-(4-x)\cdot\frac{1}{3}(6-x)+(4-x)x\cdot\frac{2}{3}}{x^{\frac{4}{3}}(6-x)^{\frac{5}{3}}}$ (+1) $= \frac{-8}{x^{\frac{4}{3}}(6-x)^{\frac{5}{3}}}$ (+1)

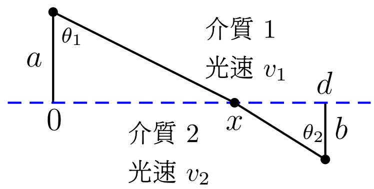
$f'(4) = 0$, $f''(4) < 0$, $\therefore f(4) = 2^{5/3}$ 是 local max (+1) 。

明顯地, f'、f'' 在 $0, 6$ 皆不存在, $\begin{array}{l} f'(0^-) = -\infty \\ f'(0^+) = +\infty \end{array}$ (+1), $\begin{array}{l} f'(6^-) = -\infty \\ f'(6^+) = -\infty \end{array}$ (+1)

	0		4		6	
f'	−		+	−		−
f''	−		−	−		+

$\therefore f(0) = 0$ 是 local min. (+1), 6 是反曲點 (+1), $f(6) = 0$

3.**(11)** 光速 在介質 1 為 v_1、在介質 2 為 v_2, 根據「費馬定律」— 光總是走最速路線、並且依此圖設定, 推導出「斯涅爾定律」: $\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}$ 。



根據圖, 時間必須設定為 $t(x) = \frac{\sqrt{a^2+x^2}}{v_1} + \frac{\sqrt{b^2+(d-x)^2}}{v_2}$ (+1) 。

$t'(x) = \frac{x}{v_1\sqrt{a^2+x^2}} - \frac{d-x}{v_2\sqrt{b^2+(d-x)^2}}$ (1+1) $= \frac{xv_2\sqrt{b^2+(d-x)^2}-(d-x)v_1\sqrt{a^2+x^2}}{v_1\sqrt{a^2+x^2}\cdot v_2\sqrt{b^2+(d-x)^2}}$ (+1) ,

$t'(x) = 0 \Leftrightarrow xv_2\sqrt{b^2+(d-x)^2} = (d-x)v_1\sqrt{a^2+x^2}$, $\Leftrightarrow \frac{x}{\sqrt{a^2+x^2}}v_2 = \frac{d-x}{\sqrt{b^2+(d-x)^2}}v_1$ (+1) — $(*)$,

即: $\sin\theta_1\ v_2 = \sin\theta_2\ v_1$ (+1), 但不表示此 x 會使 t 達極小值。

$t''(x) = \left(\frac{x}{v_1\sqrt{a^2+x^2}}\right)' + \left(\frac{x-d}{v_2\sqrt{b^2+(d-x)^2}}\right)'$

$= \frac{1}{v_1}\left(\frac{1}{(a^2+x^2)^{1/2}} + \frac{x(\frac{-1}{2})(2x)}{(a^2+x^2)^{3/2}}\right) + \frac{1}{v_2}\left(\frac{1}{(b^2+(d-x)^2)^{1/2}} + \frac{(x-d)(\frac{-1}{2})2(d-x)(-1)}{(b^2+(d-x)^2)^{3/2}}\right)$ (1+1)

$= \frac{a^2}{v_1(a^2+x^2)^{3/2}} + \frac{b^2}{v_2(b^2+(d-x)^2)^{3/2}}$ (1+1), 總是 > 0 (+1), 即: 滿足 $(*)$ 的 x、即 $\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}$, 使時間達到最小值。

4.**(4)** 若 $f(x) = \int_0^{\sin x}\sqrt{1+t^2}\,dt$ 且 $g(y) = \int_0^y f(x)\,dx$, 求 $g''(\pi/6)$ 。

$g''(y) = \frac{d}{dy}\left[\frac{dg}{dy}\right] = \frac{d}{dy}[f(y) - 0]$ (+1) $= \sqrt{1+\sin^2 y}\cdot\cos y - 0$ (+1) , $\therefore g''(\pi/6) = \sqrt{1+(\frac{1}{2})^2}\cdot\frac{\sqrt{3}}{2}$ (+1) $= \frac{\sqrt{15}}{4}$ (+1)

過程、算式 寫於背面 或 此線以下 (草稿區) 皆不記分