微積分 之『求極值, 積分』卷一　班級_____　學號________________　座號___　姓名 ***Grading Key***　　分數 **38**

1.**(4)** Show that $\sqrt{1+x} < 1 + \frac{1}{2}x$ for any $x > 0$.

依均值定理, 存在 $c \in (0, x)$ 使得 $\frac{\sqrt{1+x}-1}{x} = \frac{1}{2}(1+c)^{\frac{-1}{2}}$ (+1) $< \frac{1}{2}$ (+1), 即 $\frac{\sqrt{1+x}-1}{x} < \frac{1}{2}$

$\overset{\cdot x(>0)}{\Longrightarrow} \sqrt{1+x} - 1 < \frac{1}{2}x$ (+1) $\iff \sqrt{1+x} < 1 + \frac{1}{2}x$ (+1)

2.**(17)** $f(x) = x \cdot (\ln x)^2$ 。若 f 有奇點、極值、反曲點、漸近線, 則圖形要反映出來, 沒計算過程不給分。

$f'(x) = (\ln x)^2 + \cancel{x} \cdot 2\ln x \cdot \frac{1}{\cancel{x}}$ (+1) $= \ln x \cdot (\ln x + 2)$, 奇點 $1, \frac{1}{e^2}$ (+1)

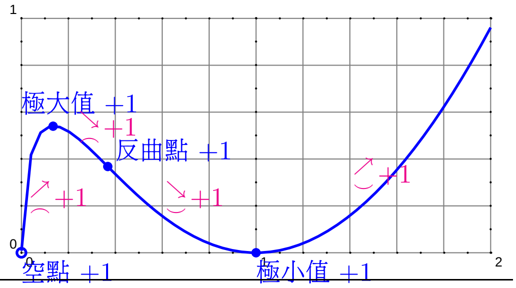
$f''(x) = \frac{1}{x}(\ln x + 2) + \ln x \cdot \frac{1}{x}$ (+1) $= \frac{2}{x}(\ln x + 1)$, 反曲點 $\frac{1}{e}$ (+1)

$f''(1) > 0,\ f(1) = 0$ 極小值 (+1) $f''(\frac{1}{e^2}) < 0,\ f(\frac{1}{e^2}) = \frac{4}{e^2}$ 極大值 (+1)

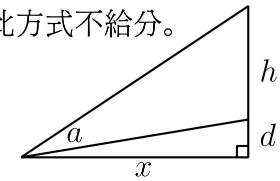
明顯地, 在 0^+ 有爭議。必須檢查:

$\lim\limits_{x\to0^+} f(x) = \lim\limits_{x\to0^+} \frac{(\ln x)^2}{\frac{1}{x}} \overset{\mathscr{L}}{=} \lim\limits_{x\to0^+} \frac{2\ln x \cdot \frac{1}{x}}{\frac{-1}{x^2}}$ (+1) $= \lim\limits_{x\to0^+} -2\frac{\ln x}{\frac{1}{x}}$

$\overset{\mathscr{L}}{=} -2 \lim\limits_{x\to0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}}$ (+1) $= 2 \lim\limits_{x\to0^+} x = 0^+$ (+1)



3.**(13)** h, d 長度固定, 如圖。將夾角 a 寫成 x 的函數後求最大值, 但不使用 $a''(x)$。不依此方式不給分。

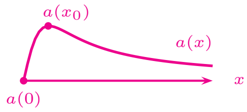


$a(x) = \arctan(\frac{x}{d}) - \arctan(\frac{x}{h+d})$ (1+1) $\left(或 = \operatorname{arccot}(\frac{x}{h+d}) - \operatorname{arccot}(\frac{x}{d})\right)$, $x \ge 0$

很明顯地 $0 = a(0) \le a(x)$, $\lim\limits_{x\to+\infty} a(x) = \frac{\pi}{2} - \frac{\pi}{2} = 0$

$a'(x) = \frac{1}{1+\frac{x^2}{d^2}}\frac{1}{d} - \frac{1}{1+\frac{x^2}{(h+d)^2}}\frac{1}{h+d}$ (1+1) $= \frac{d}{x^2+d^2} - \frac{h+d}{x^2+(h+d)^2}$ (+1)

$= \frac{d(x^2+(h+d)^2)-(h+d)(x^2+d^2)}{(x^2+d^2)(x^2+(h+d)^2)}$ (+1) $= \frac{dh^2+d^2h-hx^2}{(x^2+d^2)(x^2+(h+d)^2)}$ (+1)



$a'(x) = 0 \Leftrightarrow x^2 = d(d+h)$ (+1), 令 $x_0 = \sqrt{d(d+h)}$ 。$a(x_0) = \underbrace{\arctan\frac{\sqrt{d+h}}{\sqrt{d}}}_{\theta, > \frac{\pi}{4}}$ (+1) $- \underbrace{\arctan\frac{\sqrt{d}}{\sqrt{h+d}}}_{\frac{\pi}{2}-\theta, < \frac{\pi}{4}}$ (+1) > 0

因為 x_0 是 $[0, \infty)$ 裡唯一的奇點, 一定是開口向下的, 如圖, $\therefore a(x_0)$ 一定是最大值。(+3)

4.**(4)** 若 $f(x) = \int_0^{\sin x} \sqrt{1+t^2}\,dt$ 且 $g(y) = \int_0^y f(x)\,dx$, 求 $g''(\pi/6)$ 。

$g''(y) = \frac{d}{dy}\left[\frac{dg}{dy}\right] = \frac{d}{dy}[f(y) - 0]$ (+1) $= \sqrt{1+\sin^2 y} \cdot \cos y - 0$ (+1) , $\therefore g''(\pi/6) = \sqrt{1+(\frac{1}{2})^2} \cdot \frac{\sqrt{3}}{2}$ (+1) $= \frac{\sqrt{15}}{4}$ (+1)

過程、算式 寫於背面 或 此線以下 (草稿區) 皆不記分

禁止交談、傳遞物品、掀示考卷、四處張望 .... 做弊者、疑似做弊 警告後再犯者, 以零分計算。