SPATIO-TEMPORAL OSCILLATION FOR A SINGULAR PREDATOR-PREY MODEL

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Abstract. We study an initial boundary value problem for a reaction-diffusion system arising in the study of a singular predator-prey system. Under an assumption on the growth rates, we first prove that the unique co-existence state is a center for the kinetic system. Then we prove that solutions of the diffusion system with equal diffusivity become spatially homogeneous and are subject to the kinetic part asymptotically.

1. Introduction

In [2], a three-component model of ordinary differential equations is proposed for the control of rabbits to protect birds from cat predation in insular environments. By setting the rabbit population to be zero, the model is reduced to the following two-component ordinary differential system:

$$\begin{cases} B' = r_b \left(1 - \frac{B}{K}\right) B - \mu C, \\ C' = r_c \left(1 - \mu \frac{C}{B}\right) C, \end{cases} \tag{1.1}$$

where B (resp. C) denotes the population density of birds (resp. cats). Furthermore, r_b (resp. r_c) is the growth rate of birds (resp. cats), K is the carrying capacity of birds, μ is the annual intake of birds per individual predator (cat), and the carrying capacity of cats is assumed to be proportional to the birds. See also [3] for the model derivation of (1.1) in the framework of non-spatially structured populations.

By introducing the function $P := C/B$, system (1.1) is reduced to the following system of ordinary differential equations

$$\begin{cases} B' = \left[r_b(1 - \frac{B}{K}) - \mu P\right] B, \\ P' = \left[r_c - r_b + r_b \frac{B}{K} - \mu(r_c - 1)P\right] P. \end{cases} \tag{1.2}$$

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There are always two nontrivial constant equilibria, namely, $(K,0)$ and $(0,P^{**})$, $P^{**}:=\frac{r_c-r_b}{\mu(r_c-1)}$, when $r_c\neq 1$. Also, when $r_b>1$, there is the unique co-existence state $(B^*,P^*):=(K(1-1/r_b),1/\mu)$. The dynamical behaviors for solutions to system (1.2) has been studied completely in [8]. We refer the reader to [8] for the details.

However, some results in [8] are given only based on numerical simulations. For example, along the segment $\{r_b+r_c=2, r_b>1, r_c>0\}$, it is conjectured in [8] that the state (B^*,P^*) is a *center* by numerical simulations. On the other hand, both sets

$$\mathcal{P}^{\pm}:=\left\{(B,P)\mid B>0,P>0,\pm\left(\frac{B}{K}+\frac{P}{P^{**}}-1\right)>0\right\},$$

are positively (forward) invariant for system (1.2). By numerical simulations, it is also conjectured in [8] that solutions of (1.2) starting with initial data $(B_0,P_0)\in\mathcal{P}^+$ blow up in finite time. One of the main purposes of this work is to verify the above two numerical observations rigorously.

When taking into account the spatial dependence, the analysis of this predator-prey model becomes much more complex. For this aspect, we consider the following singular predator-prey model posed on a bounded smooth domain $\Omega\subset\mathbb{R}^N$:

$$\begin{cases} B_t=d_b\Delta B+r_b\left(1-\frac{B}{K}\right)B-\mu C, & x\in\Omega,\ t>0,\\ C_t=d_c\Delta C+r_c\left(1-\mu\frac{C}{B}\right)C, & x\in\Omega,\ t>0,\\ \frac{\partial B}{\partial\nu}=\frac{\partial C}{\partial\nu}=0, & x\in\partial\Omega,\ t>0,\\ B(\cdot,0)=B_0>0,\ C(\cdot,0)=C_0\geq 0, & x\in\Omega, \end{cases} \tag{1.3}$$

where d_b,d_c,r_b,r_c,K,μ are positive constants and ν denotes the outer normal on $\partial\Omega$.

The dynamical behaviors of problem (1.3) were studied in [8] when $d_b=d_c$. This includes the global existence and quenching (i.e., B reaches zero in finite time), depending on the range of parameters r_b and r_c. A simplified model in the whole space was studied in [5] in which a non-self-similar quenching was found. On the other hand, the case when $d_b\neq d_c$ is analyzed in a recent work [6] along with the associated shadow system.

In this paper, we are interested in the case when

$$r_b+r_c=2,\ r_b>1,\ r_c>0. \tag{1.4}$$

In fact, it is suggested by numerical simulations in [8] that solutions of (1.3) with $d_b=d_c$ may exhibit spatio-temporal oscillations under assumption (1.4). Hereafter, we shall always assume the condition (1.4) being enforced. It is one of our purposes to verify this interesting dynamical behavior rigorously.

This spatio-temporal oscillation phenomena (or, eventually spatially-homogeneous and time-periodic behavior) were also discussed in [11] for a predator-prey system and in [10] for the Gierer-Meinhardt system. Note that the nonlinearity considered in the works [11, 10] can be formulated as a Hamiltonian system. Unfortunately, our nonlinearity here does not follow in this category. However, by applying the method of Darboux (cf. [4, 12, 1, 7]), we are able to show that our nonlinearity is Darboux integrable and a generalization of Hamiltonian-like system (see the formulation (3.2) below) can be deduced.

The rest of this paper is organized as follows. In §2, we study the kinetic system (1.2) under the assumption (1.4). In particular, we prove that (B^*, P^*) is a center and solutions starting from $\mathcal{P}^+$ blow up in finite time. Then the reaction diffusion system is studied in §3. Using the results obtained in §2, we first reduce the reaction diffusion system into a Hamiltonian-like system so that a Lyapunov functional for problem (1.3) can be formulated. Finally, a theorem on the spatial-temporal oscillations for solutions of problem (1.3) is proved.

2. The kinetic system

This section is devoted to the study of the kinetic system (1.2). Under assumption (1.4), system (1.2) is reduced to

$$\begin{cases} B' = [r_b(1-\frac{B}{K}) - \mu P]B, \\ P' = [2 - 2r_b + r_b\frac{B}{K} + \mu(r_b-1)P]P. \end{cases} \tag{2.1}$$

Recall the unique positive co-existence state $(B^*, P^*) = (K(1-1/r_b), 1/\mu)$, $r_b \in (1,2)$. Then the corresponding system for $(\bar{B}, \bar{P})$ with $\bar{B} := B - B^*$ and $\bar{P} := P - P^*$ can be represented as

$$\frac{d}{dt}\begin{bmatrix} \bar{B} \\ \bar{P} \end{bmatrix} = A\begin{bmatrix} \bar{B} \\ \bar{P} \end{bmatrix} + \begin{bmatrix} -\frac{r_b}{K}\bar{B}^2 - \mu\bar{B}\bar{P} \\ \frac{r_b}{K}\bar{B}\bar{P} + \mu(r_b-1)\bar{P}^2 \end{bmatrix}. \tag{2.2}$$

Here A is the Jacobian matrix for (2.1) evaluated at the stationary state (B^*, P^*) and it reads

$$A = \begin{bmatrix} -\frac{r_b}{K}B^* & -\mu B^* \\ \frac{r_b}{K}P^* & \mu(r_b-1)P^* \end{bmatrix} = \begin{bmatrix} -(r_b-1) & -\mu B^* \\ \frac{r_b-1}{\mu B^*} & r_b - 1 \end{bmatrix}.$$

Therefore, the trace of A is 0 and its determinant is $(r_b-1)(2-r_b) > 0$. Thus the eigenvalues of the matrix A are $\pm i\omega$, where

$$\omega := \sqrt{(r_b-1)(2-r_b)}, \tag{2.3}$$

and whose corresponding eigenvectors are $\mathbf{p} \pm i\mathbf{q}$ respectively, where

$$\mathbf{p} := \begin{bmatrix} -\mu B^* \\ r_b - 1 \end{bmatrix} \quad \text{and} \quad \mathbf{q} := \begin{bmatrix} 0 \\ \omega \end{bmatrix}.$$

It is easy to check that these eigenvectors satisfy

$$A\mathbf{p} = -\omega\mathbf{q}, \qquad A\mathbf{q} = \omega\mathbf{p}. \tag{2.4}$$

Also, we have

$$\mathbf{e_1} := \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{-1}{\mu B^*}\Big\{\mathbf{p} - \frac{r_b - 1}{\omega}\mathbf{q}\Big\}, \quad \mathbf{e_2} := \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\omega}\mathbf{q} \tag{2.5}$$

2.1. **A simple reduction.** Now, let us define the variable $(\bar{X}, \bar{Y})$ through the relation

$$\begin{bmatrix} \bar{B} \\ \bar{P} \end{bmatrix} = \bar{X}\mathbf{p} + \bar{Y}\mathbf{q}. \tag{2.6}$$

Then

$$\bar{B} = -\mu B^* \bar{X}, \quad \bar{P} = (r_b - 1)\bar{X} + \omega\bar{Y} \tag{2.7}$$

By substituting (2.6) into (2.2) and applying (2.4), we get

$$\begin{aligned} \bar{X}'\mathbf{p} + \bar{Y}'\mathbf{q} &= \frac{d}{dt}\begin{bmatrix} \bar{B} \\ \bar{P} \end{bmatrix} \\ &= \omega\bar{Y}\mathbf{p} - \omega\bar{X}\mathbf{q} + \Big(-\frac{r_b}{K}\bar{B}^2 - \mu\bar{B}\bar{P}\Big)\mathbf{e}_1 + \Big(\frac{r_b}{K}\bar{B}\bar{P} + \mu(r_b - 1)\bar{P}^2\Big)\mathbf{e}_2 \end{aligned} \tag{2.8}$$

From (2.7), (2.3) and a simple calculation,

$$\begin{aligned} -\frac{r_b}{K}\bar{B}^2 - \mu\bar{B}\bar{P} &= \mu^2 B^* \omega \bar{X}\bar{Y}, \\ \frac{r_b}{K}\bar{B}\bar{P} + \mu(r_b - 1)\bar{P}^2 &= \mu(r_b - 1)\{\omega^2\bar{Y}^2 - \omega^2\bar{X}^2 + \omega(2r_b - 3)\bar{X}\bar{Y}\}. \end{aligned}$$

Substituting these equalities into (2.8), using (2.5), and equating the coefficients of $\mathbf{p}$ and $\mathbf{q}$, we get

$$\begin{cases} \bar{X}' &= \omega\bar{Y} - \mu\omega\bar{X}\bar{Y}, \\ \bar{Y}' &= -\omega\bar{X} + \mu(r_b - 1)\{\omega\bar{Y}^2 - \omega\bar{X}^2 + 2(r_b - 1)\bar{X}\bar{Y}\}. \end{cases}$$

By a time scaling and setting $(X, Y) = (\mu\bar{X}, \mu\bar{Y})$, we can reduce system (2.1) to

$$\begin{cases} X' &= Y - XY, \\ Y' &= -X + \alpha\{Y^2 - X^2 + 2\alpha\omega^{-1}XY\}, \end{cases} \tag{2.9}$$

hereafter, for notational convenience, we set $\alpha := r_b - 1 \in (0, 1)$ so that $\omega^2 = \alpha(1 - \alpha)$.

2.2. **Method of Darboux.** We first recall the method of Darboux (cf. [4, 12, 1, 7]) for a planar system

$$X' = p(X,Y), \quad Y' = q(X,Y), \tag{2.10}$$

where p, q are polynomials of degree r. We call a polynomial f an *invariant polynomial* of (2.10), if f is a solution to the following first order partial differential equation

$$\langle (p,q), \nabla f \rangle = p(X,Y)\partial_X f(X,Y) + q(X,Y)\partial_Y f(X,Y) = g(X,Y)f(X,Y)$$

for some polynomial g, which is called the *polynomial cofactor* associated with the invariant polynomial f. It is easily seen that the degree of g must be less than or equal to $r-1$. We also call the curve $\{f(X,Y)=0\}$ an *invariant algebraic curve* of system (2.10).

By the Darboux theory, if we can find invariant polynomials $\{f_i\}_{i=1}^p$ with cofactors $\{g_i\}_{i=1}^p$ such that the condition

$$\sum_{i=1}^{p} \lambda_i g_i = 0 \tag{2.11}$$

holds for some complex numbers $\{\lambda_i\}_{i=1}^p$ which are not all zeroes, then it is easy to check that the function

$$\Psi(X,Y) := \prod_{i=1}^{p} f_i^{\lambda_i}(X,Y)$$

is a *first integral* of the system (2.10), i.e., Ψ satisfies

$$p(X,Y)\Psi_X(X,Y) + q(X,Y)\Psi_Y(X,Y) = 0. \tag{2.12}$$

In particular, (2.11) holds, if the number of different irreducible invariant curves is strictly larger than $r(r+1)/2$. The reason is that the dimension of the vector space of polynomials in two variables whose degree is less than or equal to $r-1$ is $r(r+1)/2$, so that those associated cofactors must be linearly dependent.

Now, for system (2.9), the invariant polynomials are determined by solving the partial differential equation

$$[(1-X)Y]\partial_X f + [-X + \alpha(Y^2 - X^2 + 2\alpha\omega^{-1}XY)]\partial_Y f = gf \tag{2.13}$$

with some appropriate cofactor polynomial g. It is clear that equation (2.13) has a polynomial solution

$$f_1(X,Y) = 1 - X$$

with cofactor $g_1(X,Y) = -Y$.

Next, we look for an invariant polynomial f which is linear in Y with coefficients being unknown polynomials of X, that is, $f(X,Y) = k_0(X) + k_1(X)Y$, where k_0 and k_1 are to be determined from a cofactor polynomial g of degree 1. With these assumptions, we find the following two invariant straight lines

$$f_\pm(X,Y) = 1 + \alpha X + \frac{\alpha}{\omega}(-\alpha \pm 1)Y$$

with cofactors

$$g_\pm(X,Y) = \alpha Y + \frac{\alpha}{\omega}(\alpha \mp 1)X,$$

respectively. By setting

$$\lambda_1 = -1, \quad \lambda_\pm = \frac{1}{2\alpha}(\mp\alpha - 1),$$

we get $\lambda_1 g_1 + \lambda_+ g_+ + \lambda_- g_- = 0$. Thus $\Psi(X,Y) := f_1^{-1} f_+^{\lambda_+} f_-^{\lambda_-}$ is a first integral of (2.9) by the Darboux integrability theory.

Since $\alpha \in (0,1)$, we see that $\lambda_\pm < 0$, and this first integral Ψ is positive in the invariant domain (triangular region):

$$\mathcal{T} := \{(X,Y) \mid f_1(X,Y) > 0, f_+(X,Y) > 0, f_-(X,Y) > 0\} \ni (0,0).$$

Note also that Ψ diverges on the boundary of $\mathcal{T}$.

2.3. **The state (B^*, P^*) is a center.** We first claim that the function $\Psi = f_1^{-1} f_+^{\lambda_+} f_-^{\lambda_-}$ is a strictly convex function in $\mathcal{T}$. For this, we write

$$\log \Psi = \lambda_1 \log f_1 + \lambda_+ \log f_+ + \lambda_- \log f_-.$$

Note that $f_1, f_\pm$ are affine functions of X and Y. We recall that the composition of a (strictly) convex function with an affine mapping is (strictly) convex and that the minus logarithmic function $-\log : (0,\infty) \to \mathbb{R}$ is strictly convex. Also, any linear combination with positive coefficients of (strictly) convex functions is (strictly) convex. Moreover, the composition of a (strictly) convex function with the exponential function $\exp : \mathbb{R} \to \mathbb{R}$ is still (strictly) convex. Hence the function $\Psi : \mathcal{T} \to (0,\infty)$ is strictly convex.

Applying this convex property, we can prove

Theorem 2.1. *The constant state (B^*, P^*) is a center for system* (1.2).

Proof. It is easy to check that $(0,0)$ is a critical point of Ψ. Hence it follows from the strict convexity of Ψ that $(0,0)$ is the unique minimal point of Ψ in $\mathcal{T}$. This also implies that $(0,0)$ is an isolated critical point in $\mathcal{T}$. Note that (2.12) implies that

Ψ is *constant* along any solution trajectory of system (2.9). Hence Ψ is a non-trivial conserved quantity and it follows that all trajectories in $\mathcal{T}\setminus\{(0,0)\}$ are periodic orbits. Hence (B^*, P^*) is a center for system (1.2), by returning to the original variable (B, P). This proves the theorem. □

2.4. **Blow-up occurs in $\mathcal{P}^+$.** As before, we always assume that condition (1.4) holds. Then it is easy to check that $P^{**} = 2/\mu$ and so

$$\frac{B^*}{K} + \frac{P^*}{P^{**}} = 1 - \frac{1}{r_b} + \frac{1}{2} < 1,$$

using $r_b \in (1, 2)$ and $P^* = 1/\mu$. Hence (B^*, P^*) is outside of the region $\mathcal{P}^+$. By a simple phase plane analysis, it is easy to check that the boundary of $\mathcal{P}^+$ is an invariant set of (2.1). Indeed, on the boundary of $\mathcal{P}^+$ the flow is pointed leftward on the part on B-axis, upward on the part on P-axis, and to the same direction as the vector from $(K, 0)$ to $(0, P^{**})$ on the line segment connecting $(K, 0)$ and $(0, P^{**})$. We conclude that the region $\mathcal{P}^+$ is an invariant region which contains no critical points of (2.1).

On the other hand, it is easy to see that $B' < 0$ in the region $\{B \geq K, P > 0\}$. Since $B' < 0$ and $P' > 0$ on the line segment $\{P = P^{**} = 2/\mu, B \in (0, K)\}$, we see that $P(t_0) \geq P^{**}$ for some finite $t_0 > 0$ for any trajectory (B, P) of (2.1). Since $r_b > 1$, a simple comparison principle (using the P-equation in (2.1)) as that in [8] yields that the component P of any solution of (2.1) blows up in a finite time for any initial data (B_0, P_0) lying in $\mathcal{P}^+$.

3. The reaction diffusion system

We study in this section the initial boundary value problem (1.3) for the reaction diffusion system under the condition (1.4).

First, it follows from (2.7) that

$$B = B^*(1 - X), \quad P = (1 + \alpha X + \omega Y)/\mu. \tag{3.1}$$

and so we have

$$B = B^*(1 - X), \quad C = BP = B^*(1 - X)(1 + \alpha X + \omega Y)/\mu.$$

Since the Jacobian matrix $\partial(B, C)/\partial(X, Y)$ is nonsingular in $\mathcal{T}$, the inverse function $(X(B, C), Y(B, C))$ of (B, C) is well-defined in the image $(B, C)(\mathcal{T}) := \mathcal{U}$.

Now, we set $\Phi(B,C) := \Psi(X(B,C),Y(B,C))$. Since $|\nabla\Psi| \neq 0$ in $\mathcal{T}\setminus\{(0,0)\}$, we also have $|\nabla\Phi| \neq 0$ in the punctured domain $\mathcal{U}\setminus\{(B^*,C^*)\}$ in (B,C)-plane. On the other hand, along any trajectory $\{(B(t),C(t)) : t\in[0,T)\}$ of (1.1),

$$0 = \frac{d}{dt}\Phi(B(t),C(t)) = \nabla\Phi(B(t),C(t))\cdot(B'(t),C'(t)),\ t\in[0,T),$$

which implies $(B'(t),C'(t))$ is parallel to $(\Phi_C(B(t),C(t)),-\Phi_B(B(t),C(t)))$ for each $t\in[0,T)$. As a consequence, there is a function $\tilde{S}(t)$ such that

$$\begin{bmatrix} r_b\left(1-\frac{B(t)}{K}\right)B(t)-\mu C(t) \\ r_c\left(1-\mu\frac{C(t)}{B(t)}\right)C(t) \end{bmatrix} = \tilde{S}(t)\begin{bmatrix} \Phi_C(B(t),C(t)) \\ -\Phi_B(B(t),C(t)) \end{bmatrix}$$

for each $t\in[0,T)$. Since there is a unique trajectory passing through each point (B,C) in $\mathcal{U}\setminus\{(B^*,C^*)\}$, the function $S(B,C) := \tilde{S}(t)$ (for a certain t) is well-defined in $\mathcal{U}$ such that

$$\begin{bmatrix} r_b\left(1-\frac{B}{K}\right)B-\mu C \\ r_c\left(1-\mu\frac{C}{B}\right)C \end{bmatrix} = S(B,C)\begin{bmatrix} \Phi_C(B,C) \\ -\Phi_B(B,C) \end{bmatrix}.$$

Therefore, the first two equations in problem (1.3) can be re-formulated as

$$B_t = d_b\Delta B + S\Phi_C, \quad C_t = d_b\Delta C - S\Phi_B. \tag{3.2}$$

With this formulation, we are ready to prove the spatio-temporal oscillations of solutions to problem (1.3) as follows.

Theorem 3.1. *Assume $d_b = d_c$ and (1.4). Let (B,C) be a solution of problem (1.3) such that*

$$I_0 := \{(B_0(x),C_0(x)) \mid x\in\Omega\} \ \textit{is a compact subset of } \mathcal{U}. \tag{3.3}$$

Then there exists an orbit $\hat{\mathcal{O}}$ of (1.1) such that

$$\lim_{t\to\infty} dist_{C^2}((B(\cdot,t),C(\cdot,t));\hat{\mathcal{O}}) = 0. \tag{3.4}$$

Moreover, if this $\hat{\mathcal{O}}$ is not the center (B^*,C^*), $C^* := B^*P^*$, then

$$\lim_{t\to\infty} \|(B(\cdot,t+l),C(\cdot,t+l)) - (B(\cdot,t),C(\cdot,t))\|_{C^2(\Omega)} = 0 \tag{3.5}$$

for some constant $l>0$.

Here (in Theorem 3.1) the notation $dist_{C^2}$ is defined (as that in [11, 10]) by

$$dist_{C^2}((B(\cdot,t);C(\cdot,t)),\hat{\mathcal{O}}) := \inf_{(\hat{B},\hat{C})\in\hat{\mathcal{O}}} \|(B(\cdot,t),C(\cdot,t)) - (\hat{B},\hat{C})\|_{C^2},$$

where $\|\cdot\|_{C^2}$ is the standard C^2 norm.

Proof of Theorem 3.1. First, we show that $\mathcal{U}$ is convex. Indeed, using $\omega^2 = \alpha(1-\alpha)$ and (3.1), it is easy to check that $\{f_1(X,Y)=0\}$ is mapped to $\{B=0\}$, $\{f_+(X,Y)=0\}$ is sent to $\{P=0\}$, and $\{f_-(X,Y)=0\}$ is mapped to $\{B/K+P/P^{**}=1\}$, where $P^{**}=2/\mu$. Hence the image of $\mathcal{T}$ by the mapping (B,P) is the triangular region $\mathcal{P}^-$ with vertices $\{(0,0),(K,0),(0,P^{**})\}$. Furthermore, it is easy to check that the mapping $(B,C)(B,P)=(B,BP)$ sends $\mathcal{P}^-$ to the set $\mathcal{U}$ with

$$\mathcal{U}=\{(B,C)\mid B>0, C>0, B^2/K+C/P^{**}<B\}.$$

Therefore, the set $\mathcal{U}$ is convex.

Using the convexity of $\Psi(X,Y)$ and (3.1), it follows that the function $\hat{\Psi}(B,P):=\Psi(X(B,P),Y(B,P))$ is strictly convex. Moreover, since $(B,C)\to C/B$ is convex in the set $\{B>0,C>0\}$, it follows that $\Phi(B,C)$ is strictly convex on the convex set $\mathcal{U}$. Hence the Hessian of Φ, $\nabla^2\Phi(B,C)$, is positive definite in $\mathcal{U}$.

Next, by Theorem 2.1, the set $\mathcal{U}$ is consisted of all closed orbits of (1.1) lying in $\mathcal{U}$. These closed orbits are simply the (convex) level curves of Φ. Now, by assumption (3.3), there is an orbit $\mathcal{O}_0$ of (1.1) such that the set I_0 is contained in the interior of the region $\mathcal{U}_0$ which is enclosed by $\mathcal{O}_0$. Since the region $\mathcal{U}_0$ is convex, by the standard invariance theory of [13], the solution (B,C) stays in $\mathcal{U}_0$ for all $t\geq 0$. In particular, (B,C) exists globally in time, is uniformly bounded in $\Omega\times(0,\infty)$ and $B\geq\delta$ for some constant $\delta>0$. Then, by the standard parabolic regularity theory, the orbit $O:=\{(B(\cdot,t),C(\cdot,t))\}_{t\geq 0}$ is compact in $C^2(\bar{\Omega})\times C^2(\bar{\Omega})$. Hence the theory of dynamical system (cf. [9]) implies that the ω-limit set of the orbit O, $\omega(B_0,C_0)$, is a nonempty compact, connected and invariant set in $C^2(\bar{\Omega})\times C^2(\bar{\Omega})$.

Now, without loss of generality, we may assume that $d_b=d_c=1$. By a simple calculation and using (3.2), we have

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \Phi(B(x,t),C(x,t))\,dx=\int_\omega\{\Phi_B B_t+\Phi_C C_t\}\,dx\\
&=\int_\omega\{\Phi_B(\Delta B+S\Phi_c)+\Phi_C(\Delta C-S\Phi_B)\}\,dx \qquad (3.6)\\
&=-\int_\Omega\{\Phi_{BB}|\nabla B|^2+2\Phi_{BC}\nabla B\cdot\nabla C+\Phi_{CC}|\nabla C|^2\}\,dx.
\end{aligned}$$

Therefore, the right-hand side of (3.6) is non-positive and so the functional

$$E[B,C](t):=\int_\Omega \Phi(B(x,t),C(x,t))\,dx$$

is a Lyapunov functional for problem (1.3). Moreover, $E'[B,C](t)=0$ for some $t>0$ if and only if $|\nabla B(\cdot,t)|^2+|\nabla C(\cdot,t)|^2\equiv 0$ in Ω.

Recall from the definitions of f_1, $f_\pm$, and $\mathcal{T}$ that $\Phi(B,C)\geq 1$ for $(B,C)\in\mathcal{U}$. Since $E'[B,C](t)\leq 0$ for all $t\geq 0$, the value $E_\infty:=\lim_{t\to\infty}E[B,C](t)$ exists and $E_\infty\in[1,\infty)$. Let $(B_*,C_*)\in\omega(B_0,C_0)$, i.e., there exists a sequence $\{t_j\}$ tending to ∞ such that

$$\lim_{j\to\infty}\|(B(\cdot,t_j),C(\cdot,t_j))-(B_*(\cdot),C_*(\cdot))\|_{C^2}=0.$$

Then, up to extracting a subsequence of $\{t_j\}$, $E'[B,C](t_j)\to 0$ as $j\to\infty$. It follows that $|\nabla B_*|^2+|\nabla C_*|^2\equiv 0$ in Ω, i.e., (B_*,C_*) is a constant independent of the space variable x. Moreover, $\Phi(B_*,C_*)=E_\infty/|\Omega|$, where $|\Omega|$ is the Lebesgue measure of Ω. Since this is true for any (B_*,C_*) in $\omega(B_0,C_0)$, this determines a unique orbit $\hat{\mathcal{O}}$ of (1.1) such that (3.4) holds.

Finally, we prove (3.5) when the orbit $\hat{\mathcal{O}}$ is not the single point (B^*,C^*). Let l be the period of $\hat{\mathcal{O}}$. Then we have $l>0$. Given any sequence $\{t_j\}$ such that $t_j\to\infty$ as $j\to\infty$. By the parabolic regularity and Ascoli-Arzela theorem, the same argument as above implies that there exists a subsequence $\{t_{j_k}\}$ of $\{t_j\}$ such that

$$\lim_{k\to\infty}\sup_{s\in[-2l,2l]}\|(B(\cdot,s+t_{j_k}),C(\cdot,s+t_{j_k}))-(\hat{B}(s),\hat{C}(s))\|_{C^2}=0,$$

where $(\hat{B}(s),\hat{C}(s))\in\hat{\mathcal{O}}$. It follows from $(\hat{B}(s+l),\hat{C}(s+l))=(\hat{B}(s),\hat{C}(s))$ that

$$\begin{aligned}
&\limsup_{k\to\infty}\|(B(\cdot,s+t_{j_k}+l),B(\cdot,s+t_{j_k}+l))-(B(\cdot,s+t_{j_k}),C(\cdot,s+t_{j_k})\|_{C^2}\\
\leq\ &\lim_{k\to\infty}\|(B(\cdot,s+t_{j_k}+l),C(\cdot,s+t_{j_k}+l))-(\hat{B}(s+l),\hat{C}(s+l))\|_{C^2}\\
&\quad+\lim_{k\to\infty}\|(B(\cdot,s+t_{j_k}),C(\cdot,s+t_{j_k}))-(\hat{B}(s),\hat{C}(s))\|_{C^2}=0.
\end{aligned}$$

Hence we obtain

$$\lim_{k\to\infty}\|(B(\cdot,s+t_{j_k}+l),B(\cdot,s+t_{j_k}+l))-(B(\cdot,s+t_{j_k}),C(\cdot,s+t_{j_k})\|_{C^2}=0.$$

Since $\{t_j\}$ is arbitrary, we conclude that

$$\lim_{t\to\infty}\|(B(\cdot,s+t+l),C(\cdot,s+t+l))-(B(\cdot,s+t),C(\cdot,s+t)\|_{C^2}=0.$$

Then (3.5) follows by setting $s=0$. This completes the proof of Theorem 3.1. □

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