EXISTENCE OF A ROTATING WAVE PATTERN IN A DISK FOR A WAVE FRONT INTERACTION MODEL

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Abstract. We study the rotating wave patterns in an excitable medium in a disk. This wave pattern is rotating along the given disk boundary with a constant angular speed. To study this pattern we use the wave front interaction model proposed by Zykov in 2007. This model is derived from the FitzHugh-Nagumo equation and it can be described by two systems of ordinary differential equations for wave front and wave back respectively. Using a delicate shooting argument with the help of the comparison principle, we derive the existence and uniqueness of rotating wave patterns for any admissible angular speed with convex front in a given disk.

Keywords: rotating wave pattern, front, back, angular speed

1. Introduction

Wave propagation in excitable medium has been studied widely due to its various applications in physical model, chemical reaction, and biological system. In particular, the spiral wave has been recognized as a fascinating and important spatio-temporal pattern (cf., e.g., [18, 8, 19, 14]). We also refer the reader to the survey papers on spirals by Tyson and Keener [17], Meron [9], Mikhailov [12], Fiedler and Scheel [1], etc.

Until recently, most studies of spiral patterns were for unbounded media; though cannot be applied to describe spiral waves rotating within a disk. In [22], by using the free-boundary approach, two types of rigidly rotating patterns within a disk are studied, namely, *spots* moving along the disk boundary and *spiral waves* rotating around the disk center. In particular, rotating spots are intrinsically unstable and can be observed in excitable media only under a stabilizing feedback as in the wave segments ([10, 11, 21]).

The study in [22] indicates a selection mechanism that uniquely determines the shape and angular velocity of these two patterns as a function of the medium excitability and the disk radius. A wave pattern corresponds to the domain of excitation with sharp transition layer. In the free boundary approach proposed in [22], this sharp transition layer is taken to be a planar curve to enclose the excited region. We define the *tip* (or, phase change point)

Date: October 30, 2011.

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This work was supported in part by the National Science Council of the Republic of China under the grant NSC 99-2115-M-032-006-MY3, by Challenging Exploratory Research (No. 23654042), Japan Society for the Promotion of Science, and by the National Science Council of the Republic of China under the grant NSC 99-2115-M-005-001. We would like to thank Vladimir Zykov for some valuable discussions. Also, we would like to express our gratitude to the anonymous referee for the valuable comments.

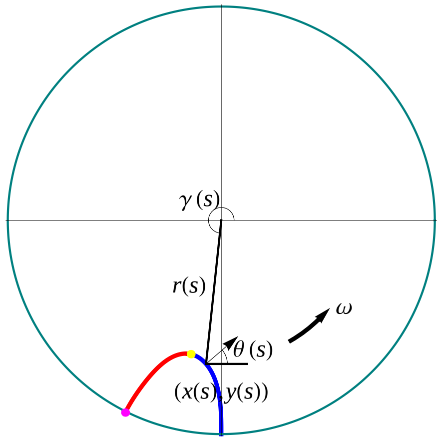


Figure 1. The profile of a rotating wave

to be the unique point on the boundary of the excited region with zero normal velocity. We shall call the wave boundary before this point the *front* and the one after this point the *back*. In fact, the mathematical difference between a rotating spot and a spiral wave is the curvature of the touching point of the front on the domain disk boundary. The curvature of this touching point is positive for a spot, while it is negative for a spiral wave. Here the sign of curvature is defined to be positive if the curve is winding in the counter-clock direction. It follows from the result of [3] that the curvature of the front is always positive for a spot, while the curvature of the front changes sign exactly once for a spiral wave. Hence the front of a spot is convex. See Figure 1.

The purpose of this paper is to provide a proof of the existence of a rotating wave pattern in an excitable medium in a disk. Our approach is based on the approximation by the interface equations derived by Zykov [22] from the FitzHugh-Nagumo equation when the excited region rotates with a constant angular speed. This system is called a *wave front interaction model*, which consists of two systems of ordinary differential equations for the front and the back, respectively. We shall describe these systems in details in the next section.

The outline of the paper is as follows. In section 2, we provide a description of the system of equations for the rotating wave pattern in a disk and state our main result. Then, in section 3, we study the existence of the front part for any given domain disk and angular speed. This is carried out by using a useful transformation and a phase plane analysis as done in [3]. Finally, we study the back part to derive the existence and uniqueness of the rotating patterns in section 4. The proof is by a delicate shooting argument with the help of the comparison principle and the continuous dependence on the parameter b related to the excitability of the medium. Due to some technical difficulties, we can only derive rigorously the existence of rotating spots. The existence of rotating spiral wave patterns is open.

2. Preliminary and the main theorem

To explain the wave front interaction model, we introduce several notation for a planar rotating curve.

A planar curve can be described by (x, y, θ) with (x, y) the Euclidean coordinate of a point on this curve and θ the angle of the normal vector (right-hand to the tangent) measured from the positive x-axis. Then we have the relations

$$\frac{dx}{ds} = -\sin\theta, \quad \frac{dy}{ds} = \cos\theta, \quad \frac{d\theta}{ds} = \kappa,$$

where κ is the (signed) curvature and s is the arc length parameter. Using polar coordinates (r, γ) instead of (x, y) with $(x, y) = (r\cos\gamma, r\sin\gamma)$, we have

$$\frac{dr}{ds} = \sin(\gamma - \theta), \quad \frac{d\gamma}{ds} = \frac{1}{r}\cos(\gamma - \theta), \quad \frac{d\theta}{ds} = \kappa. \tag{2.1}$$

In this paper, we consider a rotating wave pattern (the excited region) which lies inside a disk and is rotating counter-clockwise along the disk boundary with a positive angular speed ω. Let the center of this disk be the origin. Then the function $(\hat{r}, \hat{\gamma}, \hat{\theta})$ describing the boundary of this excited region in the disk satisfies

$$\hat{r}(s,t) = r(s), \quad \hat{\gamma}(s,t) = \gamma(s) + \omega t, \quad \hat{\theta}(s,t) = \theta(s) + \omega t \tag{2.2}$$

for some function (r, γ, θ) depending only on the arc length s.

The functions that describe the front curve and the back curve are denoted by

$$(x_+, y_+, r_+, \gamma_+, \theta_+), \quad (x_-, y_-, r_-, \gamma_-, \theta_-),$$

respectively. Since the normal velocity V can be computed as

$$V = \frac{dx}{dt}\cos\theta + \frac{dy}{dt}\sin\theta = \frac{dr}{dt}\cos(\gamma - \theta) - r\frac{d\gamma}{dt}\sin(\gamma - \theta),$$

it follows from (2.2) that

$$V_\pm = -\omega r_\pm \sin\varphi_\pm, \tag{2.3}$$

where $\varphi_\pm = \gamma_\pm - \theta_\pm$ and V_+ (resp. V_-) is the normal velocity of the front (resp. the back).

Recall from [22] that the normalized interface equation for the front is given by

$$V_+ = 1 - \kappa_+.$$

Using (2.1) and (2.3), we obtain

$$\begin{cases} \dfrac{dr_+}{ds} &= \sin\varphi_+, \\ \dfrac{d\varphi_+}{ds} &= \dfrac{\cos\varphi_+}{r_+} - 1 - \omega r_+ \sin\varphi_+. \end{cases} \tag{2.4}$$

Here we measure the arc length starting from the touching point of the front on the disk boundary. Also, we set this touching point to be $(r_+, \gamma_+, \theta_+)|_{s=0} = (R_D, 3\pi/2, 0)$, where R_D is the radius of the domain disk. Therefore, (2.4) is equipped with the initial condition

$$r_+|_{s=0} = R_D, \quad \varphi_+|_{s=0} = \frac{3}{2}\pi. \tag{2.5}$$

We look for solutions such that the radius function is monotone in s. Hence the tip corresponds to the point when φ_+ reaches the value π for some $s_1 > 0$.

After solving (r_+, φ_+), we can obtain V_+ using (2.3). Then by the relation $\kappa_+ = 1 - V_+$ and the definition of curvature, we can solve the function θ_+ and then γ_+. Define

$$(r_*, \theta_*, \gamma_*) = (r_+(s_1;\omega), \theta_+(s_1;\omega), \gamma_+(s_1;\omega)).$$

Also, we introduce $\Gamma_+(r) = \gamma_+(s(r);\omega)$ for $r \in [r_*, R_D]$. The existence of Γ_+ is to be explained in the next section.

The back of a rotating wave is influenced by the front through the inhibitor of the excitable medium. Actually the normalized interface equation for the back derived by Zykov [22] is given by

$$V_- = 1 - \kappa_- - b(\Gamma_+(r_-) - \gamma_-),$$

where b is a nonnegative constant to be determined. Recall from (2.1) that the back of rotating wave is controlled by the system:

$$\frac{dr_-}{ds} = \sin(\gamma_- - \theta_-), \quad \frac{d\gamma_-}{ds} = \frac{\cos(\gamma_- - \theta_-)}{r_-}, \quad \frac{d\theta_-}{ds} = \kappa_-. \tag{2.6}$$

For the simplicity of the notation, we shall drop the subscript minus sign. Then, using the identity

$$V = -\omega r \sin\varphi,$$

we can rewrite (2.6) as

$$\begin{cases} \dfrac{dr}{ds} &= \sin\varphi, \\ \dfrac{d\gamma}{ds} &= \dfrac{\cos\varphi}{r}, \\ \dfrac{d\varphi}{ds} &= \dfrac{\cos\varphi}{r} - 1 - \omega r \sin\varphi + b(\Gamma_+(r) - \gamma). \end{cases} \tag{2.7}$$

Note that the function $\Gamma_+(r(s))$ is well-defined as long as $r(s) \in [r_*, R_D]$. Here we measure the arc length starting from the tip of the rotating wave. Then the initial condition is given by

$$r|_{s=0} = r_*, \quad \gamma|_{s=0} = \gamma_*, \quad \varphi|_{s=0} = \pi. \tag{2.8}$$

Therefore, for a given $R_D > 0$ and a given $\omega > 0$, the existence of a rotating wave pattern is equivalent to looking for a solution (r, γ, φ) of (2.7)-(2.8) such that

$$r(s_2) = R_D, \quad \varphi(s) \in (0, \pi) \text{ for } s \in (0, s_2), \quad \varphi(s_2) = \frac{\pi}{2}, \tag{2.9}$$

for some positive arc length s_2 and a certain constant $b \geq 0$.

Now, we are ready to state our main theorem as follows.

Theorem 2.1. *For a given $R_D > 0$ and a given $\omega > 0$ with $\omega R_D \leq 1$, there exists a unique $b > 0$ such that a unique solution (r, γ, φ) of* (2.7)*,* (2.8) *and* (2.9) *exists.*

3. Existence of the front

In this section, we study the existence of the front for a rotating wave in a disk. We denote the solution of (2.4) with (2.5) by $(r_+(s;\omega), \varphi_+(s;\omega))$, or simply $(r_+(s), \varphi_+(s))$.

For the special case when $\omega = 0$, the wave is stationary. Then we have $V_+ \equiv 0$ and so $\kappa_+ \equiv 1$. Hence the front is a part of the unit circle with center at $(x, y) = (-1, -R_D)$. It starts at $(r_+, \gamma_+) = (R_D, 3\pi/2)$, runs counter clockwise and ends at $(r_+, \gamma_+) = (\sqrt{1 + R_D^2} - 1, \pi + \alpha)$ with $\alpha := \arcsin(R_D/\sqrt{1 + R_D^2})$.

To show the existence of the front for $\omega > 0$, we introduce new variables

$$X(\tau) := \omega r_+(s) \cos\varphi_+(s), \quad Y(\tau) := 1 + \omega r_+(s) \sin\varphi_+(s), \quad \tau = -s. \tag{3.1}$$

Then, by (2.4), we have

$$\begin{cases} \dfrac{dX}{d\tau} &= Y(1 - Y), \\ \dfrac{dY}{d\tau} &= -\omega + XY \end{cases} \tag{3.2}$$

with the terminal condition:

$$X(0) = 0, \quad Y(0) = 1 - \omega R_D := a.$$

Recall that the system (3.2) has been studied completely in [3]. In our case, we always have

$$a = 1 - \omega R_D < 1.$$

Note that Y is the curvature function.

Since we look for solutions such that the radius function is monotone in s and

$$\varphi_+(0;\omega) = \frac{3\pi}{2},$$

we see from (2.4) that $\varphi_+ \in [\pi, 2\pi]$ for the front. This gives us the condition that $Y < 1$ for the front, except when $\varphi_+ = \pi$ or 2π. On the other hand, at $\varphi_+ = \pi$, we have

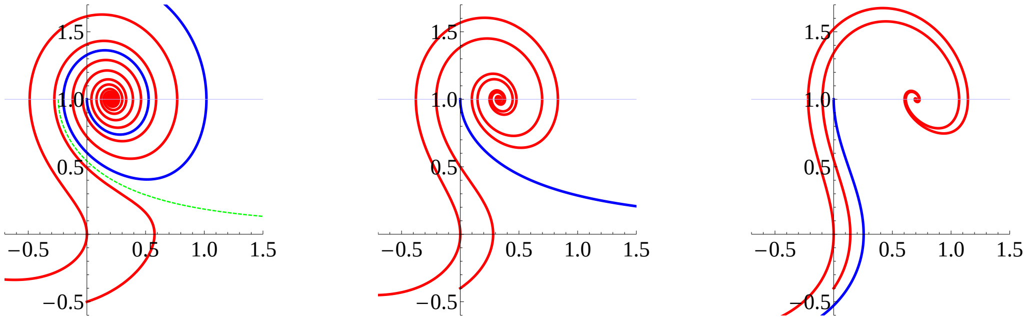
$$\frac{d\varphi_+}{ds}\Big|_{\varphi_+=\pi} = -\frac{1}{r_+} - 1 < 0,$$

so that

$$\frac{d^2 r_+}{ds^2}\Big|_{\varphi_+=\pi} = \left(\cos\varphi_+ \frac{d\varphi_+}{ds}\right)\Big|_{\varphi_+=\pi} = \frac{1 + r_+}{r_+} > 0.$$

Hence the tip of the rotating wave corresponds to the point where φ_+ reaches the value π. This also gives the initial condition

$$X(-s_1) = -\omega r_+(s_1), \quad Y(-s_1) = 1$$



(a) $\omega < \omega^*$ (b) $\omega = \omega^*$ (c) $\omega > \omega^*$

FIGURE 2. Dynamics of (3.2).

for some $s_1 > 0$. Note that $X > 0$ (resp. $X < 0$) if and only if $\varphi_+ \in (3\pi/2, 2\pi)$ (resp. $(\pi, 3\pi/2)$).

We can prove the following proposition for the existence of the front in the same manner as in [3, Lemma 2.1 and Theorem 1].

Proposition 3.1. *There exists a positive constant ω_* such that the followings hold.*

(i) *Let $\omega \in (0, \omega_*]$. For each $R_D > 0$, there exists a unique solution (r_+, φ_+) of (2.4)–(2.5) defined on $[0, s_1]$ for some $s_1 = s_1(R_D, \omega) > 0$ such that $\varphi_+(s;\omega) \in (\pi, 2\pi)$ for all $s \in [0, s_1)$ and $\varphi_+(s_1;\omega) = \pi$.*
(ii) *Let $\omega > \omega_*$. Then there exists a positive constant $a^* = a^*(\omega)$ such that a unique solution (r_+, φ_+) of (2.4)–(2.5) defined on $[0, s_1]$ for some $s_1 = s_1(R_D, \omega) > 0$ with $\varphi_+(s;\omega) \in (\pi, 2\pi)$ for all $s \in [0, s_1)$ and $\varphi_+(s_1;\omega) = \pi$ exists if and only if $R_D \le [1 + a^*(\omega)]/\omega$. Moreover, $a^*(\omega) \to \infty$ as $\omega \downarrow \omega_*$.*
(iii) *For each $\omega > 0$, there holds $\varphi_+(s;\omega) \in (\pi, 3\pi/2)$ for all $s \in (0, s_1(R_D, \omega))$ if and only if $R_D \in (0, 1/\omega]$.*

Proof. First, by [3, Lemma 2.1], for each $\omega > 0$ there exists a unique trajectory in the phase plane of the system (3.2) (see Figure 2) such that it connects the point $(0,0)$ and the point $(-X^*(\omega), 1)$ for some positive constant $X^* = X^*(\omega)$. This trajectory can be continued forward, stays in the third quadrant after $(0,0)$ and tends to $(-\infty, 0)$ as $s \to \infty$.

Next, by [3, Theorem 1], there exists a positive constant ω_* such that, in the phase plane of the system (3.2), a trajectory with $(X,Y)(s) \to (\infty, 0)$ as $s \to \infty$ exists if and only if $\omega \in (0, \omega_*]$. This trajectory is unique whenever it exists. For $\omega = \omega_*$, this trajectory goes backward to the point $(0,1)$. For $\omega < \omega_*$, it reaches a point $(-X_1(\omega), 1)$ for some constant $X_1 = X_1(\omega) > 0$. Moreover, for $\omega > \omega_*$, the trajectory starting at $(0,1)$ crosses the positive X-axis first, travels in the fourth quadrant, reaches the negative Y-axis at a point $(0, -a^*(\omega))$ for some positive constant $a^* = a^*(\omega)$ and then stays in the third quadrant. Moreover, the function $a^*(\omega)$ is decreasing in ω for $\omega > \omega_*$ with $a^*(\omega) \to \infty$ as $\omega \downarrow \omega_*$.

Finally, the assertions follow from the above phase plane analysis by defining $a^*(\omega) = \infty$ for $\omega \in (0, \omega_*]$. □

Remark 3.2. Proposition 3.1 implies that, for each $R_D > 0$, there is a $\omega^*(R_D) > \omega_*$ such that a front exists if and only if $0 \le \omega \le \omega^*(R_D)$.

Note that $r_* > 0$ for $\omega < \omega^*(R_D)$ and $r_* = 0$ when $\omega = \omega^*(R_D)$. Also, we have

$$\frac{dr_+}{ds} = \sin\varphi_+ < 0 \qquad \text{for } s \in (0, s_1).$$

Thus we can define the inverse function $s = s(r)$ on (r_*, R_D) of $r = r_+(s)$ and

$$\Gamma_+(r) := \gamma_+(s(r);\omega), \quad \Phi_+(r) := \varphi_+(s(r);\omega).$$

Note that $\Gamma_+(r)$ is increasing in r whenever $\varphi_+ \in (\pi, 3\pi/2)$. In particular, for $\omega = 0$ we have

$$\Gamma_+(r) = 3\pi/2 + \alpha - \arcsin\left\{\frac{r^2 + R_D^2}{2r\sqrt{1+R_D^2}}\right\}, \qquad \sqrt{1+R_D^2} - 1 \le r \le R_D.$$

In the case when $\omega R_D \le 1$, it follows from Proposition 3.1 that $\pi < \varphi_+(s) < 3\pi/2$ for $s \in (0, s_1)$, which means that Γ_+ is increasing in r. On the other hand, when $\omega R_D > 1$, then $a < 0$ and X is positive near $s = 0$. Thus Γ_+ is decreasing in r close to r_*. Hence we shall say that the front is *convex* if $\omega R_D \le 1$ and *nonconvex* if $\omega R_D > 1$. Moreover, in the case $\omega R_D \le 1$, we have

$$\frac{d\varphi_+}{dr} > 0 \qquad \text{for } \ r \in (r_*, R_D).$$

Indeed, since $X \le 0$ and $0 < Y \le 1$ when $\omega R_D \le 1$, the last statement follows from

$$\frac{d(-\cos\varphi_+)}{dr} = \frac{\cos\varphi_+}{r} - 1 - \omega r \sin\varphi_+ = \frac{X}{\omega r^2} - Y < 0.$$

From now on we fix R_D to be a positive constant and let $\omega \in [0, \omega^*(R_D))$. Using

$$\frac{d\gamma_+/ds}{dr/ds}\Big|_{\varphi_+ = 3\pi/2} = 0,$$

we can extend the function Γ_+ to $[R_D, R_D + 1]$ as the constant function. Also, we have

Lemma 3.3. *The following expansions hold:*

$$\begin{aligned} s(r) &= s_1 - \sqrt{\frac{2r_*(r - r_*)}{1 + r_*}} + O(|r - r_*|), \\ \Gamma_+(r) &= \gamma_* + \sqrt{\frac{2(r - r_*)}{r_*(1 + r_*)}} + O(|r - r_*|) \end{aligned}$$

for $0 < r - r_* \ll 1$.

Proof. We have

$$\frac{dr_+}{ds}\Big|_{\varphi_+=\pi} = 0, \quad \frac{d^2 r_+}{ds^2}\Big|_{\varphi_+=\pi} = \frac{1 + r_*}{r_*}, \quad \frac{d\gamma_+}{ds}\Big|_{\varphi_+=\pi} = -\frac{1}{r_*} < 0, \quad \frac{d^2\gamma_+}{ds^2}\Big|_{\varphi_+=\pi} = 0.$$

The lemma follows. ∎

4. Existence of the back

In this section, we study the back part of the rotating wave pattern in a disk. We denote the solution of (2.7) with (2.8) by $(r(s;\omega,b),\gamma(s;\omega,b),\varphi(s;\omega,b))$.

Note that r_* and γ_* are independent of b. But, r_* and γ_* are functions of ω.

Since Γ_+ is not Lipschitz continuous at $r=r_*$ (see Lemma 3.3), we cannot apply the standard uniqueness theorem of ordinary differential equations to (2.7) with the initial condition (2.8). Nevertheless, the local existence of the solutions can be shown in the same manner as in [4, Lemma 3.1].

Lemma 4.1. *If $R_D>0$ and $\omega\in[0,\omega^*(R_D))$, then there exists a unique solution (r,γ,φ) of* (2.7) *and* (2.8) *such that*

$$\begin{aligned} r(s;\omega,b) &= r_*+\frac{r_*+1}{r_*}s^2+O(s^3),\\ \gamma(s;\omega,b) &= \gamma_*-\frac{1}{r_*}s+O(s^3),\\ \varphi(s;\omega,b) &= \pi-\frac{r_*+1}{r_*}s+\left[\frac{b}{r_*}-\omega(r_*+1)\right]s^2+O(s^3)\end{aligned}$$

for small $s\geq 0$.

Proof. The existence and uniqueness of local solutions can be proved by a fixed point argument as in [4, Lemma 3.1]. Using (2.7), the expansions near $s=0$ are obtained using the following observations:

$$\begin{aligned}&\frac{dr}{ds}\Big|_{s=0}=0,\ \frac{d\gamma}{ds}\Big|_{s=0}=-\frac{1}{r_*},\ \frac{d\varphi}{ds}\Big|_{s=0}=-\frac{r_*+1}{r_*},\\ &\frac{d^2r}{ds^2}\Big|_{s=0}=\frac{r_*+1}{r_*},\ \frac{d^2\gamma}{ds^2}\Big|_{s=0}=0,\ \frac{d^2\varphi}{ds^2}\Big|_{s=0}=-\omega(r_*+1)+\frac{b}{r_*}.\end{aligned}$$

Hence the lemma is proved. ☐

Using Lemma 4.1, we have a local solution of (2.7) and (2.8) while r belongs to the interval $[r_*,R_D]$. For a given $R_D>0$ and a given $\omega\in[0,\omega^*(R_D))$, we are looking for a solution (r,γ,φ) of (2.7)–(2.8) such that

$$r(s_2;\omega,b)=R_D,\quad \varphi(s;\omega,b)\in(0,\pi)\ \text{ for } s\in(0,s_2),\quad \varphi(s_2;\omega,b)=\frac{\pi}{2},$$

for some positive arc length s_2 and a certain constant $b\geq 0$.

In fact, when $\omega=0$, we can choose $b=0$ and the back is the arc of the unit circle with center at $(-1,-R_D)$ from the point $(\sqrt{1+R_D^2}-1,\pi+\alpha)$ running counter clockwise till the intersection point of the unit circle with the domain disk boundary.

From now on, we fix a $R_D>0$ and $\omega\in(0,\omega^*(R_D))$. Hereafter, we shall sometimes omit the dependence of ω and/or b, if there is no ambiguity.

The following lemma provides a useful property for the component φ of a solution of the system (2.7).

Lemma 4.2. *Let $b > 0$. Then the following statements hold:*

(i) *If $\varphi(s_*) = 0$ for some $s_* > 0$ and $0 < \varphi(s) < \pi$ for $0 < s < s_*$, then $\varphi(s) < 0$ for $s > s_*$ with $s - s_*$ small.*
(ii) *If $\varphi(s_*) = \pi$ for some $s_* > 0$ and $0 < \varphi(s) < \pi$ for $0 < s < s_*$, then $\varphi(s) > \pi$ for $s > s_*$ with $s - s_*$ small.*

Proof. Assume that there is a orbit (r, γ, φ) satisfying the assumption in (i). Then we have $d\varphi/ds(s_*) \le 0$. If $d\varphi/ds(s_*) < 0$, then we are done. Otherwise, let $d\varphi/ds(s_*) = 0$. We have

$$\frac{d^2\varphi}{ds^2} = -\frac{\sin\varphi}{r}\frac{d\varphi}{ds} - \frac{\cos\varphi}{r^2}\frac{dr}{ds} - \omega\frac{dr}{ds}\sin\varphi - \omega r\cos\varphi\frac{d\varphi}{ds} + b\frac{d\Gamma_+}{dr}\frac{dr}{ds} - b\frac{d\gamma}{ds},$$

Using $\varphi(s^*) = 0$ and $dr/ds(s_*) = 0$, we have

$$\frac{d^2\varphi}{ds^2} = -b\frac{d\gamma}{ds} = -\frac{b\cos\varphi(s^*)}{r(s_*)} < 0,$$

which gives the conclusion.

Similarly, in the case (ii), we can also obtain

$$\frac{d^2\varphi}{ds^2}(s_*) = -b\frac{d\gamma}{ds} = -\frac{b\cos\pi}{r(s_*)} > 0.$$

This proves the lemma. $\square$

Now, we consider the following open strip domain

$$Q := (r_*, R_D) \times \mathbb{R} \times (0, \pi)$$

for a given $R_D > 0$ and $\omega \in (0, \omega^*(R_D))$. Note that $r_* = r_*(\omega)$. Lemma 4.1 implies that the orbit of the solution (r, γ, φ) stays in Q for small $s \ge 0$. Moreover, we have

Lemma 4.3. *Given $b > 0$, there exists a finite length $S = S(b) > 0$ such that $(r, \gamma, \varphi)(s) \in Q$ for $s \in (0, S)$ and that either $r(S) \in \{r_*, R_D\}$ or $\varphi(S) \in \{0, \pi\}$. Moreover, there exists a positive constant $\gamma_{**}(b)$ depending on b such that*

$$-\gamma_{**}(b) \le \gamma(s) \le \gamma_{**}(b)$$

for $0 \le s \le S$.

Proof. We show this lemma by contradiction. Suppose that $\varphi(s) \in (0, \pi)$ and $r(s) \in (r_*, R_D)$ for all $s > 0$. It follows from (2.7) that

$$-\frac{1}{r_*} \le \frac{d\gamma}{ds}(s) \le \frac{1}{r_*} \quad \text{for all } s \ge 0. \tag{4.1}$$

We claim that

$$\gamma(s) > -\frac{1}{b}\left(\frac{1}{r_*} + 3 + \omega R_D\right) - \frac{\pi}{r_*} \quad \text{for all } s > 0. \tag{4.2}$$

Otherwise, assume that

$$\gamma(\tau) \le -\frac{1}{b}\left(\frac{1}{r_*} + 3 + \omega R_D\right) - \frac{\pi}{r_*}$$

for some $\tau > 0$. Then (4.1) implies that

$$\gamma(s) \le -\frac{1}{b}\left(\frac{1}{r_*} + 3 + \omega R_D\right) \quad \text{for } \tau \le s \le \tau + \pi.$$

Hence we have

$$\frac{d\varphi}{ds}(s) \ge -\frac{1}{r_*} - 1 - \omega R_D - b\gamma > 1 \quad \text{for } \tau \le s \le \tau + \pi.$$

This implies that φ has to attain the value π at some $s \le \tau + \pi$, which contradicts the assumption that $\varphi(s) \in (0, \pi)$. Thus (4.2) is proved.

Similarly, we can show that

$$\gamma(s) < \frac{1}{b}\left(\frac{1}{r_*} + \omega R_D + \max_{r_* \le r \le R_D} \Gamma_+(r)\right) + \frac{\pi}{r_*} \quad \text{for all } s > 0.$$

Define

$$\gamma_{**}(b) := \frac{1}{b}\left(\frac{1}{r_*} + \omega R_D\right) + \max\left\{3, \max_{r_* \le r \le R_D} \Gamma_+(r)\right\} + \frac{\pi}{r_*}.$$

Then we have

$$-\gamma_{**} < \gamma(s) < \gamma_{**} \quad \text{for all } s > 0. \tag{4.3}$$

Since $r(s)$ is increasing in s while $0 < \varphi(s) < \pi$, the limit $r_\infty = \lim_{s\to\infty} r(s)$ exists. If $\pi/2 < \varphi(s) < \pi$ for large s, then γ is monotone decreasing for large s and the finite limit $\gamma_\infty := \lim_{s\to\infty} \gamma(s)$ exists. We can take a sequence $\{\tau_j\}$ with $\tau_j \to \infty$ as $j \to \infty$ such that $(r, \gamma)'(\tau_j) \to (0, 0)$ as $j \to \infty$. Taking $\tau_j \to \infty$ in (2.7), we obtain that

$$\lim_{j\to\infty} \sin\varphi(\tau_j) = 0 = \lim_{j\to\infty} \frac{\cos\varphi(\tau_j)}{r(\tau_j)},$$

which is a contradiction. Similarly, we can treat the case where $0 < \varphi(s) < \pi/2$ for large s.

Now, suppose that there is a sequence $\{s_j\}$ with $s_j \to \infty$ such that $\varphi(s_j) = \pi/2$. We also have a sequence $\{t_j\}$ with $t_j \to \infty$ such that either $\varphi(t_j) \to 0$ or $\varphi(t_j) \to \pi$ as $j \to \infty$ due to the existence of the limit of r. Suppose that the former case happens. Relabeling the indices (if necessary), we may assume that $t_j \in (s_{2j}, s_{2j+1})$ for all $j \ge 1$. Choose $\bar{s}_j \in (s_{2j}, t_j)$ such that $\varphi(\bar{s}_j) = \pi/4$ and $\varphi(s) \in (\pi/4, \pi/2)$ for all $s \in (s_{2j}, \bar{s}_j)$ for all $j \ge j_0$ for a certain $j_0 \gg 1$. Hence we have $r'(s) = \sin\varphi(s) \ge 1/\sqrt{2}$ for all $s \in (s_{2j}, \bar{s}_j)$. For any $i > j_0$, we have

$$R_D - r_* \ge r(s_{2i+1}) - r(0) = \int_0^{s_{2i+1}} r'(s)ds \ge \sum_{j=j_0}^{j=i} \int_{s_{2j}}^{\bar{s}_j} r'(s)ds \ge \frac{1}{\sqrt{2}} \sum_{j=j_0}^{j=i} (\bar{s}_j - s_{2j}).$$

Hence

$$\sum_{j=j_0}^{\infty} (\bar{s}_j - s_{2j}) < \infty$$

and so $\lim_{j\to\infty}(\bar{s}_j - s_{2j}) = 0$. This implies that φ' is unbounded. From the equation of φ it follows that γ is unbounded, which contradicts (4.3). Therefore, $S(b)$ is finite and the same argument implies (4.3) as long as $\varphi \in (0, \pi)$ and $r \in (r_*, R_D)$. $\square$

Let us define the *exit-length* $S = S(b)$ and the *exit-point* $(r_e, \gamma_e, \varphi_e)(b)$ as follows:

(i) if there is a positive number $\hat{s}$ such that the orbit stays in Q for $0 < s < \hat{s}$ and $r(\hat{s}) = R_D$, then $S = S(b) = \hat{s}$ and $(r_e, \gamma_e, \varphi_e)(b) = (R_D, \gamma(S), \varphi(S))$;
(ii) if there is a positive number $\overline{s}$ such that the orbit stays in Q for $0 < s < \overline{s}$, $r(\overline{s}) < R_D$ and $\varphi(\tau) > \pi$ for some $\tau > \overline{s}$ and close to $\overline{s}$, then $S = S(b) = \overline{s}$ and $(r_e, \gamma_e, \varphi_e)(b) = (r(S), \gamma(S), \pi)$;
(iii) if there is a positive number $\underline{s}$ such that the orbit stays in Q for $0 < s < \underline{s}$, $r(\underline{s}) < R_D$ and $\varphi(\tau) < 0$ for some $\tau > \underline{s}$ and close to $\underline{s}$, then $S = S(b) = \underline{s}$ and $(r_e, \gamma_e, \varphi_e)(b) = (r(S), \gamma(S), 0)$.

We remark that since r is increasing in s while the orbit stays in Q, the orbit never touches the plane $r = r_*$. Moreover, by Lemmas 4.2 and 4.3, the definitions of exit-length and exit-point are well-defined and exclusive.

Next, for a given $R_D > 0$ and $\omega \in (0, \omega^*(R_D))$, we study the set

$$J := \{(r_e, \gamma_e, \varphi_e)(b) \mid b \geq 0\}.$$

To show the continuity of the exit-point in b we prepare new functions and the information of the exit-point when $b = 0$.

Since $(dr/ds)(s) > 0$ for all $s \in (0, S)$, the functions $\Gamma := \Gamma(r)$ and $\Phi := \Phi(r)$ are well-defined for $r \in (r_*, r(S))$. Moreover, (Γ, Φ) satisfies the system

$$\frac{d\Gamma}{dr} = f(r, \Gamma, \Phi), \quad \frac{d\Phi}{dr} = g(r, \Gamma, \Phi), \tag{4.4}$$

where $\Phi \in (0, \pi)$ and

$$f(r, \Gamma, \Phi) := \frac{\cos\Phi}{r\sin\Phi},$$
$$g(r, \Gamma, \Phi) := \frac{(\cos\Phi/r) - 1 - \omega r\sin\Phi + b(\Gamma_+(r) - \Gamma)}{\sin\Phi}.$$

Here the function $\Gamma_+(r)$ is well-defined for $r \in (r_*, r(S)) \subset (r_*, R_D)$.

For $b = 0$, the system (2.7) is the same as the one for a front. Hence the same argument as in section 2 for Proposition 3.1 can be applied. Here the terminal condition is $(X, Y)(0) = (-\omega r_*, 1)$. Note that $Y = 1$ if and only if $\varphi \in \{0, \pi\}$. So we can just trace backward the trajectory passing through the terminal point in the (X, Y) phase plane. Since any trajectory reaches the line $Y = 1$ in a finite length, we see that $S(0) < \infty$.

Lemma 4.4. *When $b = 0$, one of the following alternatives holds:*

(i) *$S(0) = \hat{s}$ and $\varphi(S(0)) \in (0, \pi/2)$, or*
(ii) *$S(0) = \underline{s}$ and $r(S(0)) < R_D$.*

Proof. Recall that $\varphi_+ \in [\pi, 2\pi]$ and $\varphi \in [0, \pi]$. Hence $\sin\varphi_+ \le 0$ and $\sin\varphi \ge 0$. From (2.4) and (4.4) with $b = 0$, we have

$$\begin{aligned}\frac{d\cos\Phi_+}{dr} &= -\frac{\cos\Phi_+}{r} + 1 - \omega r\sqrt{1-\cos^2\Phi_+}, \quad \cos\Phi_+(r_*) = -1,\\ \frac{d\cos\Phi}{dr} &= -\frac{\cos\Phi}{r} + 1 + \omega r\sqrt{1-\cos^2\Phi}, \quad \cos\Phi(r_*) = -1.\end{aligned}$$

This implies that

$$\frac{d(\cos\Phi - \cos\Phi_+)}{dr} = -\frac{\cos\Phi - \cos\Phi_+}{r} + \omega r\left(\sqrt{1-\cos^2\Phi} + \sqrt{1-\cos^2\Phi_+}\right). \tag{4.5}$$

Since $\cos\Phi(r), \cos\Phi_+(r) > -1$ for $r > r_*$ with $r - r_*$ small, by an integration of (4.5) it follows that $(\cos\Phi - \cos\Phi_+)(r) > 0$ for $r > r_*$. Therefore, if the exit-length $S = \hat{s}$, then $r(S) = R_D$ and $\Phi_+(R_D) = 3\pi/2$. Hence $\varphi(S(0)) \in (0, \pi/2)$.

Suppose that $S(0) = \bar{s}$. Then $r(S) < R_D$ and $\Phi(r(S)) = \pi$. This is impossible, since $\cos\Phi > \cos\Phi_+$. This completes the proof of the lemma. ☐

By extending $\Gamma_+(r)$ to be constant over $[R_D, R_D + 1]$, we can apply the theory of continuous dependence to (r, γ, φ) in the extended domain

$$Q_\sigma := [r_*, R_D + 1] \times \mathbb{R} \times (-\sigma, \pi + \sigma)$$

for a small positive constant σ. We shall show the continuity of the exit-time and the exit-point with respect to b.

Lemma 4.5. *The exit-length $S(b)$ is continuous in b for $b \ge 0$.*

Proof. Suppose that $S(b)$ is not continuous at some $b = b_0$. Then there exists a positive constant ε such that $|S(b_n) - S(b_0)| \ge \varepsilon$ for a sequence $\{b_n\}$ such that $b_n \to b_0$ as $n \to \infty$.

Consider any subsequence $\{b_{n_j}\}$ of $\{b_n\}$ with $S(b_{n_j}) \ge S(b_0) + \varepsilon$ for all j. Then we have $(r, \gamma, \varphi)(s; b_{n_j}) \in Q$ for all $s \le S(b_0) + \varepsilon/2$. The theory of continuous dependence implies that

$$(r, \gamma, \varphi)(s; b_{n_j}) \to (r, \gamma, \varphi)(s; b_0)$$

for all $s \in [0, S(b_0) + \varepsilon/2]$. Notice that $(r, \gamma, \varphi)(s; b_0) \in \bar{Q}$ for all $s \in [S(b_0), S(b_0) + \varepsilon/2]$. In particular, $\varphi(s) \in [0, \pi]$ and $r(s) \le R_D$ for all $s \in [S(b_0), S(b_0) + \varepsilon/2]$.

First consider the case where $b_0 > 0$. If $\varphi_e(b_0) = 0$ or π, then φ attains its minimum or maximum at $s = S(0)$ over $(0, S(0) + \varepsilon/2)$. This contradicts Lemma 4.2. If $\varphi_e(b_0) \in (0, \pi)$, then $r_e(b_0) = R_D$ and we have $\varphi(s; b_0) \in (0, \pi)$ for $s \in [0, S(b_0) + \delta]$ for some small $\delta \in (0, \varepsilon/2)$. Since $dr/ds = \sin\varphi > 0$ for $\varphi \in (0, \pi)$, we would have $r(S(b_0) + \delta; b_0) > R_D$, a contradiction.

When $b_0 = 0$, from Lemma 4.4, we have either (i) $r(S(0)) = R_D$ and $\varphi(S(0)) \in (0, \pi/2)$ or (ii) $\varphi(S(0)) = 0$ and $r(S(0)) < R_D$. For the former case, the derivative $dr/ds = \sin\varphi > 0$ at $s = S(0)$ implies that $r(S(0) + \delta; 0) > R_D$ for some positive constant $\delta \in (0, \varepsilon/2)$, a contradiction. For the latter case, since there are no difference between the front equation and the back one when $b_0 = 0$, we can use the system (3.2) for the the transformed function

(X, Y) defined by (3.1) replaced r_+, γ_+, φ_+ by r, γ, φ respectively. Namely the orbit (X, Y) satisfies (3.2) and

$$X(0) = -\omega r_* < 0, \quad Y(0) = 1, \quad 0 < X(-S(0)) < \omega R_D, \quad Y(-S(0)) = 1.$$

Note that $Y(\tau) > 1$ for all $\tau \in (-S(0), 0)$, since $\varphi(s) \in (0, \pi)$ for all $s \in (0, S(0))$. By the phase plane analysis, we have $Y(\tau) < 1$ for $0 < -S(0) - \tau \ll 1$. Since $Y(\tau) = 1 + \omega r(-\tau) \sin\varphi(-\tau)$ and $\varphi(S(0)) = 0$, we see that $\varphi(s) < 0$ for $0 < s - S(0) \ll 1$. This contradicts $(r, \gamma, \varphi)(s; b_0) \in \bar{Q}$ for all $s \in [S(0), S(0) + \varepsilon/2]$.

On the other hand, if $S(b_{n_j}) \leq S(b_0) - \varepsilon$ for a subsequence $\{b_{n_j}\}$ of $\{b_n\}$, then a subsequence of $(r, \gamma, \varphi)(S(b_{n_j}); b_{n_j})$ converges to $(r, \gamma, \varphi)(s^*; b_0)$ for some $s^* \leq S(b_0) - \varepsilon$. Since $(r, \gamma, \varphi)(S(b_n); b_n) \in \partial Q$ for each n and ∂Q is a closed set, we also have $(r, \gamma, \varphi)(s^*; b_0) \in \partial Q$, a contradiction to the definition of $S(b_0)$ for any $b_0 \geq 0$.

Therefore, $S(b)$ is continuous in b for $b \geq 0$ and the proof of the lemma is completed. □

The following lemma gives the continuous dependence of the exit point on b.

Lemma 4.6. *The exit-point $(r_e, \gamma_e, \varphi_e)(b)$ is continuous in b for $b \geq 0$.*

Proof. By Lemma 4.1, the solution of (2.7) depends on b and s continuously for small positive s and so for all positive s as long as the solution exists. This lemma immediately follows from Lemma 4.5. □

We note that (4.4) does not enjoy the usual comparison principle. By introducing the following ordering in $\mathbb{R}^2$:

$$(\Gamma_1, \Phi_1) \leq (\Gamma_2, \Phi_2) \qquad \Leftrightarrow \qquad \Gamma_1 \leq \Gamma_2,\ \Phi_1 \geq \Phi_2$$

and by applying a comparison theorem from [5, p.28] to $(\Gamma, -\Phi)$, we have the following comparison principle for solutions of (4.4) in this order.

Lemma 4.7. *Let (Γ, Φ) and $(\widetilde{\Gamma}, \widetilde{\Phi})$ be two solutions of (4.4) with $b > 0$ on the interval $[R_0, R_1]$ where $r_* < R_0 < R_1 \leq R_D$. If $(\Gamma(R_0), \Phi(R_0)) \leq (\widetilde{\Gamma}(R_0), \widetilde{\Phi}(R_0))$ and $\Phi(r), \widetilde{\Phi}(r) \in (0, \pi)$ for all $r \in [R_0, R_1]$, then $(\Gamma(r), \Phi(r)) \leq (\widetilde{\Gamma}(r), \widetilde{\Phi}(r))$ for $R_0 \leq r \leq R_1$.*

Proof. We have

$$\frac{\partial f}{\partial \Phi}(r, \Gamma, \Phi) = -\frac{1}{r \sin^2 \Phi} < 0, \quad \frac{\partial g}{\partial \Gamma}(r, \Gamma, \Phi) = -\frac{b}{\sin \Phi} < 0,$$

as long as $0 < \Phi < \pi$. This implies the lemma. □

Similarly, we have

Lemma 4.8. *Let $(\Gamma_j(r), \Phi_j(r)) := (\Gamma(r; b_j), \Phi(r; b_j))$ be the solution of (4.4) with $\Gamma_j(r_*) = \gamma_*$ and $\Phi_j(r_*) = \pi$ defined on $[r_*, r(S(b_j))]$, $j = 1, 2$. If $0 < b_1 < b_2$, then*

$$\Gamma_1(r) > \Gamma_2(r), \quad \Phi_1(r) < \Phi_2(r) \tag{4.6}$$

on $(r_*, \min\{r(S(b_1)), r(S(b_2)), R_D\})$ as long as $\Gamma_+(r) > \Gamma_1(r)$.

Proof. Since there is a singularity at $r = r_*$, we cannot apply the comparison principle near $r = r_*$. However, Lemma 4.1 implies that (4.6) holds for $0 < r - r_* \ll 1$. From this, the lemma follows from a comparison principle from [5, p.28] by the same argument as in the proof of Lemma 4.7 as long as $\Gamma_+(r) > \Gamma_1(r)$. □

We now deal with the case of a convex front, i.e., $\omega R_D \le 1$.

Lemma 4.9. *Suppose that $0 < \omega R_D \le 1$. Then there is a positive constant $b^* = b^*(\omega, R_D)$ such that $S(b) = \overline{s}$ for all $b \ge b^*$.*

Proof. First, we take constants $r_0 \in (r_*, R_D)$ and $b_0 > 0$ satisfying $\varphi_0 := \Phi(r_0; b_0) \in (3\pi/4, \pi)$ and $\Phi(r; b_0) \in [\pi/2, \pi)$ for all $r \in (r_*, r_0]$. Note that we have $\Gamma(r; b_0) \le \gamma_*$ for all $r \in [r_*, r_0]$, since $\Gamma(r)$ is non-increasing if $\Phi \in [\pi/2, \pi)$. By the assumption $\omega R_D \le 1$ and Proposition 3.1, we have $\Phi_+ \in (\pi, 3\pi/2)$ so that $\Gamma_+(r)$ is increasing in r. Hence we have $\Gamma_+(r) > \Gamma_+(r_0)$ for all $r \in (r_0, R_D]$ and $\Gamma_+(r) > \gamma_*$ for all $r \in (r_*, R_D]$. Then Lemma 4.8 implies that

$$\Gamma(r; b_0) \ge \Gamma(r; b), \quad \Phi(r; b_0) \le \Phi(r; b) \tag{4.7}$$

for $b \ge b_0$ and $r \in [r_*, \min\{r_0, r(S(b))\}]$.

Suppose $b \ge b^*$, where

$$b^* := \max\left\{ b_0, \frac{\dfrac{2(\pi - \varphi_0)}{R_D - r_0} + \dfrac{1}{r_*} + 1 + \omega R_D}{\Gamma_+(r_0) - \gamma_0} \right\}, \quad \gamma_0 := \Gamma(r_0; b_0).$$

Then $r(S(b)) \ge r_0$ by (4.7). Note that for $r \ge r_0$ we have

$$\Gamma(r; b) \le \Gamma(r_0; b) \le \Gamma(r_0; b_0) = \gamma_0$$

whenever $\Phi(r; b) \in [\pi/2, \pi)$. The definition of b^* and the equation of φ imply

$$\frac{d\Phi}{dr}(r; b) \ge \frac{2(\pi - \varphi_0)}{R_D - r_0} > 0$$

for $b \ge b^*$ and $r \ge r_0$. Moreover, $\Phi(r; b) (= \varphi(s; b))$ reaches to π for some $r \in (r_0, (r_0 + R_D)/2)$. Thus $S(b) = \overline{s}$ and the lemma is proved. □

Now, we are ready to prove the main existence and uniqueness theorem for the case of a convex front.

Theorem 4.10. *For a given $R_D > 0$ and $\omega \in (0, \omega^*(R_D))$ with $\omega R_D \le 1$, there is a unique constant $b^\sharp := b^\sharp(\omega)$ such that a solution (Γ, Φ) of (4.4) with*

$$(\Gamma(r_*), \Phi(r_*)) = (\gamma_*, \pi)$$

exists for $r \in (r_*, R_D]$ and satisfies

$$\Phi(R_D; b^\sharp) = \frac{\pi}{2}, \quad 0 < \Phi(r; b^\sharp) < \pi \ \text{ for } r_* < r < R_D.$$

Proof. Consider the set $J = \{(r_e, \gamma_e, \varphi_e)(b) \mid b \geq 0\}$. Lemmas 4.6, 4.4 and 4.9 imply the existence of a positive b such that

$$r_e(b) = R_D, \quad \varphi_e(b) = \pi/2, \tag{4.8}$$

which is the required constant of this lemma.

For the uniqueness of b, we note that

$$\begin{aligned}
\frac{d(\Gamma_+ - \Gamma)}{dr} &= \frac{\sin(\Phi - \Phi_+)}{r \sin\Phi_+ \sin\Phi}, \\
\frac{d(\Phi - \Phi_+)}{dr} &= \frac{-\frac{1}{r}\sin(\Phi - \Phi_+) + \sin\Phi - \sin\Phi_+ + b(\Gamma_+ - \Gamma)\sin\Phi_+}{\sin\Phi \sin\Phi_+}, \\
\frac{d^2(\Gamma_+ - \Gamma)}{dr^2} &= \frac{\cos(\Phi - \Phi_+)}{r \sin\Phi_+ \sin\Phi} \frac{d(\Phi - \Phi_+)}{dr} \\
&\quad - \frac{\sin(\Phi - \Phi_+)(\sin\Phi_+ \sin\Phi + r\cos\Phi_+ \frac{d\Phi_+}{dr}\sin\Phi + r\sin\Phi_+ \cos\Phi \frac{d\Phi}{dr})}{r^2 \sin^2\Phi_+ \sin^2\Phi}.
\end{aligned}$$

Let b be a positive constant such that (4.8) holds. We will show $\Gamma_+(r) > \Gamma(r; w, b)$ for all $r \in (r_*, R_D)$ by contradiction. If not, then there is a $r_1 \in (r_*, R_D)$ such that $\Gamma_+(r) > \Gamma(r)$ for $r \in (r_*, r_1)$ and $\Gamma_+(r_1) = \Gamma(r_1)$. Then

$$\frac{d}{dr}(\Gamma_+ - \Gamma)(r_1) \leq 0.$$

If

$$\frac{d}{dr}(\Gamma_+ - \Gamma)(r_1) = 0,$$

then $(\Phi - \Phi_+)(r_1) = -\pi$, since $\Phi \in (0, \pi)$ and $\Phi_+ \in (\pi, 3\pi/2)$. Moreover, we have

$$\begin{aligned}
\frac{d(\Phi - \Phi_+)}{dr}\Big|_{r=r_1} &= -\frac{2}{\sin\Phi}\Big|_{r=r_1} < 0, \\
\frac{d^2(\Gamma_+ - \Gamma)}{dr^2}\Big|_{r=r_1} &= \frac{1}{r\sin^2\Phi} \frac{d(\Phi - \Phi_+)}{dr}\Big|_{r=r_1} < 0,
\end{aligned}$$

which implies a contradiction. So we must have

$$\frac{d}{dr}(\Gamma_+ - \Gamma)(r_1) < 0.$$

Suppose that $r_2 \in (r_1, R_D]$ is the point closest to the point r_1 with

$$\frac{d}{dr}(\Gamma_+ - \Gamma)(r_2) = 0.$$

Then we have $\Gamma_+(r_2) \leq \Gamma(r_2)$ and $(\Phi - \Phi_+)(r_2) = -\pi$, since $\Phi_+ \in (\pi, 3\pi/2)$ and $\Phi \in (0, \pi)$. Hence

$$\begin{aligned}
\frac{d(\Phi - \Phi_+)}{dr}\Big|_{r=r_2} &= \frac{2\sin\Phi - b(\Gamma_+ - \Gamma)\sin\Phi}{-\sin^2\Phi}\Big|_{r=r_2} < 0, \\
\frac{d^2(\Gamma_+ - \Gamma)}{dr^2}\Big|_{r=r_2} &= \frac{1}{r\sin^2\Phi} \frac{d(\Phi - \Phi_+)}{dr}\Big|_{r=r_2} < 0.
\end{aligned}$$

This is impossible. Thus

$$\frac{d}{dr}(\Gamma_+ - \Gamma)(r) < 0 \quad \text{for all } r > r_1$$

holds. However, since $(\Phi - \Phi_+)(R_D) = \pi/2 - 3\pi/2 = -\pi$, we have

$$\frac{d}{dr}(\Gamma_+ - \Gamma)(R_D) = 0$$

which is a contradiction. Hence $\Gamma_+(r) > \Gamma(r; w, b)$ for all $r \in (r_*, R_D)$. Therefore, Lemma 4.8 is applicable and the uniqueness of $b^\sharp$ follows. This completes the proof of the theorem. □

From this, Theorem 2.1 follows.

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