Traveling wave solutions for a continuous and discrete diffusive predator-prey model

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Abstract

We study a diffusive predator-prey model of Lotka-Volterra type functional response in which both species obey the logistic growth such that the carrying capacity of the predator is proportional to the prey population and the one for prey is a constant. Both continuous and discrete diffusion are addressed. Our aim is to see whether both species can survive eventually, if an alien invading predator is introduced to the habitat of an existing prey. The answer to this question is positive under certain restriction on the parameter. Applying Schauder's fixed point theory with the help of suitable upper and lower solutions, the existence of traveling wave solutions for this model is proven. Furthermore, by deriving the non-existence of traveling wave solutions, we also determine the minimal speed of traveling waves for this model. This provides an estimation of the invasion speed.

Keywords: predator-prey model, Lotka-Volterra type, traveling wave solution, minimal speed

1. Introduction

In this paper, we consider the following diffusive predator-prey model

$$\begin{cases} u_t &= u_{xx} + ru(1-u) - rkuv, \\ v_t &= dv_{xx} + sv\left(1 - \dfrac{v}{u}\right), \end{cases} \tag{1.1}$$

where the unknown functions u, v of (x,t), $x, t \in \mathbb{R}$, stand for the population densities of prey and predator species at position x and time t, respectively, d, r, s, k are positive constants such that $1, d$ are diffusion coefficients and r, s are intrinsic growth rates of species u, v, respectively. The functional response of predator to prey is given by rku, which is of *Lotka-Volterra* type. The prey obeys the logistic growth and its carrying capacity is normalized to be 1. However, the density of predator follows a logistic dynamics with a varying carrying capacity proportional to the density of prey.

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In fact, the model (1.1) is a special case of the following Holling-Tanner type predator-prey model (cf. [24, 25]):

$$\begin{cases} u_t &= u_{xx} + ru(1-u) - \dfrac{rku}{a+bu}v, \\ v_t &= dv_{xx} + sv\left(1 - \dfrac{v}{u}\right), \end{cases}$$

when $a = 1, b = 0$. For the case when $a = 0, b = 1$, it is possible that the density of prey may vanish so that quenching or extinction phenomenon may occur. For this singular behavior, we refer the reader to [4, 5, 7, 8, 10] and the references cited therein.

It is easy to see that (1.1) has two constant steady states $(1, 0)$ and $(1/(1+k), 1/(1+k))$. In [6], they studied the model (1.1) in a bounded domain with zero Neumann boundary condition. Among other things, by constructing a delicate Lyapunov function, they show that the unique positive constant state $(1/(1+k), 1/(1+k))$ is globally stable under certain restrictions on k. In other words, this constant state attracts every positive solution of (1.1) for the Neumann initial boundary value problem in a bounded domain. Since the predator will extinct if the prey vanish, the possibility of co-existence is very important from the ecological point view. For the case $a = 1, b > 0$, we refer the reader to [12, 13, 14].

In this paper, we consider the case when the habitat is the whole real line. We are interested in the question whether both species can survive eventually, if an alien predator is introduced into the habitat where a prey has been living there. In fact, this question is equivalent to whether the solution of (1.1) tend to the unique positive constant steady state as the time approaches infinity. Therefore, we study the so-called traveling wave solution defined as follows.

A solution of (1.1) is called a traveling wave with speed c if there exist positive functions ϕ_1 and ϕ_2 defined on $\mathbb{R}$ such that $u(x,t) = \phi_1(x+ct)$ and $v(x,t) = \phi_2(x+ct)$. Here ϕ_1 and ϕ_2 are the wave profiles. Set $z := x + ct$ and substitute $(u, v)(x, t) = (\phi_1, \phi_2)(z)$ into (1.1). Then the wave profile (ϕ_1, ϕ_2) satisfies the following system of equations:

$$\begin{cases} \phi_1''(z) - c\phi_1'(z) + r\phi_1(z)[1 - \phi_1(z) - k\phi_2(z)] = 0, \ z \in \mathbb{R}, \\ d\phi_2''(z) - c\phi_2'(z) + s\phi_2(z)\left[1 - \dfrac{\phi_2(z)}{\phi_1(z)}\right] = 0, \ z \in \mathbb{R}. \end{cases} \tag{1.2}$$

Here the prime denotes the derivative with respect to z. As described above, we are interested in the traveling wave solutions of (1.1) connecting $(1, 0)$ and $(1/(1+k), 1/(1+k))$. This implies that (ϕ_1, ϕ_2) satisfies the following asymptotic boundary conditions

$$\lim_{z\to-\infty}(\phi_1, \phi_2)(z) = (1, 0), \quad \lim_{z\to+\infty}(\phi_1, \phi_2)(z) = \left(\frac{1}{1+k}, \frac{1}{1+k}\right). \tag{1.3}$$

Note that the existence of such traveling wave solutions (with $c > 0$) means the successful invasion of the predator.

Biologically, it is also interesting to study the invasion speed. A constant c^* is called the minimal speed of traveling waves, if there is a traveling wave of speed c for any $c \geq c^*$

and no wave of speed c exists for $c < c^*$. The minimal speed of traveling waves plays an important role in the estimation of the invasion speed. We prove that the minimal speed of traveling wave solutions of (1.1) is given by $c^* := 2\sqrt{ds}$. Notice that this minimal speed is independent of the parameters r and k.

In this paper, we also consider the following lattice dynamical system (LDS)

$$\begin{cases} \dfrac{du_i}{dt} &= (u_{i+1} + u_{i-1} - 2u_i) + ru_i(1 - u_i - kv_i),\ i \in \mathbb{Z}, \\ \dfrac{dv_i}{dt} &= d(v_{i+1} + v_{i-1} - 2v_i) + sv_i\left(1 - \dfrac{v_i}{u_i}\right),\ i \in \mathbb{Z}, \end{cases} \tag{1.4}$$

where $u_i = u_i(t)$, $v_i = v_i(t)$, $t \in \mathbb{R}$. Here u_i, v_i (as functions of time t) stand for the population densities of prey and predator at niches i. In fact, when we divide the habitat into countable niches and replace the Laplace operator of (1.1) by a finite difference operator, we end up with the system (1.4) (with different scale of diffusion coefficients). For the aggregated dispersion, the discrete model (1.4) is more suitable than the continuous model (1.1) to describe the phenomenon of invasion. Indeed, lattice dynamic systems have been extensively used to model biological problems, see, for example, [25, 26]. Therefore, we also study the LDS model (1.4) in this paper.

A solution of (1.4) is called a traveling wave with speed c if there exist positive functions U, V (the wave profiles) defied on $\mathbb{R}$ such that $u_i(t) = U(i + ct)$ and $v_i(t) = V(i + ct)$ for $i \in \mathbb{Z}$, $t \in \mathbb{R}$. Set $\xi = i + ct$ and substitute $(u_i, v_i)(t) = (U, V)(\xi)$ into (1.4). Then (U, V) satisfies the following system of equations

$$\begin{cases} -cU'(\xi) + D[U](\xi) + rU(\xi)[1 - U(\xi) - kV(\xi)] = 0, \quad \xi \in \mathbb{R}, \\ -cV'(\xi) + dD[V](\xi) + sV(\xi)\left[1 - \dfrac{V(\xi)}{U(\xi)}\right] = 0, \quad \xi \in \mathbb{R}, \end{cases} \tag{1.5}$$

where $D[\phi](\xi) := \phi(\xi + 1) + \phi(\xi - 1) - 2\phi(\xi)$. Since we are interested in the traveling wave solution connecting $(1, 0)$ and $(1/(1 + k), 1/(1 + k))$, we also impose the following boundary conditions

$$\lim_{\xi \to -\infty} (U, V)(\xi) = (1, 0), \quad \lim_{\xi \to +\infty} (U, V)(\xi) = (1/(1 + k), 1/(1 + k)). \tag{1.6}$$

Notice that system (1.5) is a system of functional differential equations which is of infinite dimensional nature. We also derive the existence of traveling waves of (1.4) and prove that the minimal speed of traveling wave solutions of (1.4) is given by

$$c_* := \inf_{\lambda > 0} \frac{d(e^{\lambda} + e^{-\lambda} - 2) + s}{\lambda}. \tag{1.7}$$

Here the notion of minimal speed is defined as for the continuous case.

Note that the nonlinearity in the above systems ((1.1), (1.2), (1.4), (1.5)) does not enjoy the monotone property in which the standard comparison principle can be applied. For the existence of traveling wave solutions of non-monotone systems, the application of Schauder's fixed point theorem with the help of (generalized) upper and lower solutions has been proved to be quite successful. For this aspect, we refer the reader to the works [23, 17, 18, 21, 15, 19, 22] for the continuous case and [16, 20] for the discrete case. See also references cited therein. Although this method (by now) is very standard, the existence of suitable upper and lower solutions is not trivial. In fact, the construction of upper and lower solutions relies on a delicate formulation of appropriate functions with careful choices of suitable parameters (or constants). Due to a technical difficulty, we shall always assume that $k \in (0, 1)$ in this paper. Indeed, if one can construct a suitable pair of upper and lower solutions for $k \geq 1$, then all results of this paper can be readily derived. We leave it as an open problem.

Different from the existing works, there is a negative power nonlinearity in the predator equation of our model. In particular, we need a positive (everywhere) lower solution for the prey density. To overcome this singularity, a new formulation for the upper-lower-solutions is found for the prey component. Furthermore, in the case $c = c^*$, due to the feature of a double root λ_2 for the characteristic equation of ϕ_2 at $z = -\infty$, the asymptotic behavior of ϕ_2 is expected to be a combination of $e^{\lambda_2 z}$ and $ze^{\lambda_2 z}$. In fact, the leading term is only $ze^{\lambda_2 z}$. However, due to the special nonlinearity of our model, we take the perturbation term for the lower solution of ϕ_2 to be $\sqrt{-z}e^{\lambda_2 z}$. This is a new formulation and it works well for our model. A precise and detailed construction of upper and lower solutions is done in §2.1 for the continuous case and §3.1 for the discrete case.

As for the asymptotic boundary conditions, the one at $z = -\infty$ (or, $\xi = -\infty$) can be verified without any costs due to the construction of upper and lower solutions. For the conditions at $z = +\infty$ and $\xi = +\infty$, we borrow an idea from [15] in which a sequence of shrinking intervals is introduced (see the proofs of Theorems 2.5 and 3.6 for details).

The rest of this paper is organized as follows. In §2, we study the system (1.2) with the boundary conditions (1.3). First, we construct a pair of upper and lower solutions of (1.2) for any speed $c \geq c^* = 2\sqrt{ds}$. Next, we obtain the existence of a positive solution of system (1.2) by applying Schauder's fixed point theorem. Finally, we prove this solution satisfies the boundary conditions (1.3). For $c < c^*$, the non-existence of solution of (1.2)-(1.3) can be shown by using a contradiction argument with the help of the spreading phenomenon of the Cauchy problem for Fisher's equation. This implies that $c = c^*$ is the minimal wave speed for the continuous case. Then the discrete case is treated in §3. Although the construction of upper and lower solutions is by no means trivial, once the formula of upper and lower solutions are found, it is not very difficult to verify it. For reader's convenience, we give the details of the verifications of all constructed upper and lower solutions in §4.

2. Traveling wave solutions of (1.1)

2.1. Upper and lower solutions

First, we give the definition of upper and lower solutions of (1.2) as follows.

Definition 2.1. The functions $(\overline{\phi_1}, \overline{\phi_2})$ and $(\underline{\phi_1}, \underline{\phi_2})$ are called a pair of upper and lower solutions of (1.2), if $\overline{\phi_i}'$, $\underline{\phi_i}'$, $\overline{\phi_i}''$, $\underline{\phi_i}''$, $i = 1, 2$ are bounded and the inequalities

$$\begin{cases} \overline{\phi_1}''(z) - c\overline{\phi_1}'(z) + r\overline{\phi_1}(z)[1 - \overline{\phi_1}(z) - k\underline{\phi_2}(z)] \le 0, \\ \underline{\phi_1}''(z) - c\underline{\phi_1}'(z) + r\underline{\phi_1}(z)[1 - \underline{\phi_1}(z) - k\overline{\phi_2}(z)] \ge 0, \\ d\overline{\phi_2}''(z) - c\overline{\phi_2}'(z) + s\overline{\phi_2}(z)\left[1 - \overline{\phi_2}(z)/\overline{\phi_1}(z)\right] \le 0, \\ d\underline{\phi_2}''(z) - c\underline{\phi_2}'(z) + s\underline{\phi_2}(z)\left[1 - \underline{\phi_2}(z)/\underline{\phi_1}(z)\right] \ge 0 \end{cases} \tag{2.1}$$

hold for $z \in \mathbb{R}\backslash D$ with some finite set $D = \{z_1, z_2, \dots, z_m\}$.

To find upper and lower solutions of (1.2), we divide it into two cases: $c > c^*$ and $c = c^*$.

2.1.1. The case $c > c^*$.

For a given $c > c^* = 2\sqrt{ds}$, we define the following positive constants

$$\lambda_1 = \frac{c + \sqrt{c^2 + 4r}}{2}, \quad \lambda_2 = \frac{c - \sqrt{c^2 - 4ds}}{2d}, \quad \lambda_3 = \frac{c + \sqrt{c^2 - 4ds}}{2d}.$$

In fact, we have

$$\lambda_1^2 - c\lambda_1 - r = 0 \quad \text{and} \quad d\lambda_i^2 - c\lambda_i + s = 0, \ i = 2, 3.$$

First, for given constants $\mu, q > 1$, it is easy to check that the function

$$f(z) = e^{\lambda_2 z} - qe^{\mu\lambda_2 z}$$

has a unique zero point at $z_0 = -\ln q/[(\mu - 1)\lambda_2]$ and a unique maximum point at $z_M = -\ln(q\mu)/[(\mu - 1)\lambda_2] < z_0$. Since f is continuous on $\mathbb{R}$ and positive on $(-\infty, z_0)$, there exist δ and $z_2 \in (z_M, z_0)$ such that

$$0 < \delta < 1 - k \quad \text{and} \quad f(z_2) = \delta. \tag{2.2}$$

Note that $f'(z_2) < 0$.

Next, we choose the constants μ, ν, η, p, q and ϵ satisfying the following assumptions (A1)-(A3) *in sequence.*

(A1) $\mu \in (1, \min\{\lambda_3/\lambda_2, 2\})$, $\nu > \max\{1, \lambda_2/\lambda_1\}$ and $\eta > 0$ is small enough such that $\lambda_2 > \eta\lambda_1$ and $(\eta\lambda_1)^2 - c(\eta\lambda_1) - r + rk < 0$. Here we used the assumption $k < 1$.

(A2) $p > \dfrac{r(1+k)}{-[(\eta\lambda_1)^2 - c(\eta\lambda_1) - r + rk]}$ and $q > \max\left\{1, \dfrac{s}{-[d(\mu\lambda_2)^2 - c(\mu\lambda_2) + s](1-k)}\right\}$.

(A3) $0 < \epsilon < \min\left\{k, \dfrac{rk\delta}{[(\nu\lambda_1)^2 - c(\nu\lambda_1) - r] + rk\delta}, \dfrac{rk(1 - qe^{(\mu-1)\lambda_2 z_2})}{(\nu\lambda_1)^2 - c(\nu\lambda_1) - r + rk}\right\}$, where δ and z_2 satisfy (2.2).

Now we introduce the functions $\overline{\phi_1}(z), \underline{\phi_1}(z), \overline{\phi_2}(z), \underline{\phi_2}(z)$ as follows.

$$\overline{\phi_1}(z) = \begin{cases} 1, & z \geq 0, \\ 1 - \epsilon(e^{\lambda_1 z} - e^{\nu\lambda_1 z}), & z \leq 0, \end{cases} \tag{2.3}$$

$$\underline{\phi_1}(z) = \begin{cases} 1 - k, & z \geq z_1, \\ 1 - k(e^{\lambda_1 z} + pe^{\eta\lambda_1 z}), & z \leq z_1, \end{cases} \tag{2.4}$$

$$\overline{\phi_2}(z) = \begin{cases} 1, & z \geq 0, \\ e^{\lambda_2 z}, & z \leq 0, \end{cases} \tag{2.5}$$

$$\underline{\phi_2}(z) = \begin{cases} \delta, & z \geq z_2, \\ e^{\lambda_2 z} - qe^{\mu\lambda_2 z}, & z \leq z_2, \end{cases} \tag{2.6}$$

where $z_1 < 0$ is defined by $e^{\lambda_1 z_1} + pe^{\eta\lambda_1 z_1} = 1$. Then the following lemma holds.

Lemma 2.1. *Assume that $c > c^*$. Then the functions $(\overline{\phi_1}, \overline{\phi_2})$ and $(\underline{\phi_1}, \underline{\phi_2})$ defined by* (2.3)-(2.6) *are a pair of upper and lower solutions of* (1.2).

2.1.2. The case $c = c^*$.

In this subsection, we always assume that $c = c^* = 2\sqrt{ds}$. In this case, we have $\lambda_2 = \lambda_3 = c/(2d)$.

For given positive constants h and q, we consider the function

$$g(z) := [-hz - q(-z)^{1/2}]e^{\lambda_2 z}, \quad z \leq 0.$$

We claim that g has exactly two critical points in $(-\infty, 0)$. For this, we compute

$$g'(z) = [-h\lambda_2 z - q\lambda_2(-z)^{1/2} - h + q(-z)^{-1/2}/2]e^{\lambda_2 z}.$$

Note that $g(0) = 0$, $g'(0) = \infty$, $g(-\infty) = 0$ and $g'(z) > 0$ for $-z \gg 1$. Hence g has at least two critical points in $(-\infty, 0)$. On the other hand, we set

$$G(w) := h\lambda_2 w^3 - q\lambda_2 w^2 - hw + q/2, \quad w := \sqrt{-z}.$$

Then $G'(w) = 3h\lambda_2 w^2 - 2q\lambda_2 w - h$ and so G has at most one critical point in $(0, \infty)$. However, since $G(0) = q/2 > 0$, $G'(0) = -h < 0$ and $G(\infty) = \infty$, G has at least one critical point in $(0, \infty)$. Hence G has exactly one critical point in $(0, \infty)$ which is the (unique) minimum point so that G has at most two zeros in $(0, \infty)$. Therefore, g has exactly two critical points in $(-\infty, 0)$.

Set $z_0 = z_0(h, q) := -(q/h)^2$. Then z_0 is the unique zero of g in $(-\infty, 0)$. Moreover, $g > 0$ in $(-\infty, z_0)$ and g has a unique maximum point $\tilde{z}$ in $(-\infty, z_0)$. Note also that the function $-hz - q(-z)^{1/2}$ is positive and strictly decreasing in $(-\infty, z_0)$.

Now, we choose the appropriate constants *in sequence* as follows.

First, we take η with $0 < \eta \ll 1$ such that

$$(\eta\lambda_1)^2 - c(\eta\lambda_1) - r + rk < 0, \quad \lambda_2 > 2\eta\lambda_1. \tag{2.7}$$

Secondly, set $f(z) := -hze^{\lambda_2 z}$ with $h = \lambda_2 e^2/2$. Then f is strictly increasing on $(-\infty, -2/\lambda_2]$ and $f(-2/\lambda_2) = 1$. Let p be a constant with

$$p > \max\left\{e, \frac{r[k + h/(\eta\lambda_1 e)]}{-[(\eta\lambda_1)^2 - c(\eta\lambda_1) - r + rk]}\right\}. \tag{2.8}$$

Then there exists a z_1 such that $e^{\lambda_1 z_1} + pe^{\eta\lambda_1 z_1} = 1$ and $z_1 < -2/\lambda_2$, since $e^{\lambda_1 z} + pe^{\eta\lambda_1 z}$ is increasing in z and

$$e^{-2\lambda_1/\lambda_2} + pe^{-2\eta\lambda_1/\lambda_2} > e^{-2\lambda_1/\lambda_2} + pe^{-1} > 1.$$

Thirdly, we choose $\delta > 0$ small enough such that $\delta < 1 - k$. Let z_2 be the unique $z \in (\tilde{z}, z_0)$ such that $g(z) = \delta$. Note that $g' < 0$ in $(\tilde{z}, z_0)$. Then we choose $q > 1$ sufficiently large such that

$$(q/h)^2 > 2/\lambda_2, \quad q > \frac{4sh^2}{d(1-k)}\left(\frac{7}{2e\lambda_2}\right)^{7/2}. \tag{2.9}$$

The first inequality in (2.9) shows us that $z_2 < z_0 < -2/\lambda_2$.

Finally, we take $\nu > \max\{1, \lambda_2/\lambda_1\}$ and $\epsilon > 0$ small enough such that

$$\epsilon < \min\left\{k, \frac{rk\delta}{[(\nu\lambda_1)^2 - c(\nu\lambda_1) - r] + rk\delta}, \frac{rk(1-k)(-hz_2 - q(-z_2)^{1/2})}{[(\nu\lambda_1)^2 - c(\nu\lambda_1) - r]}\right\}. \tag{2.10}$$

Then we define $\overline{\phi_1}(z), \underline{\phi_1}(z), \overline{\phi_2}(z), \underline{\phi_2}(z)$ as follows:

$$\overline{\phi_1}(z) = \begin{cases} 1, & z \geq 0, \\ 1 - \epsilon(e^{\lambda_1 z} - e^{\nu\lambda_1 z}), & z \leq 0, \end{cases} \tag{2.11}$$

$$\underline{\phi_1}(z) = \begin{cases} 1-k, & z \geq z_1, \\ 1 - k(e^{\lambda_1 z} + pe^{\eta\lambda_1 z}), & z \leq z_1, \end{cases} \tag{2.12}$$

$$\overline{\phi_2}(z) = \begin{cases} 1, & z \geq -2/\lambda_2, \\ -hze^{\lambda_2 z}, & z \leq -2/\lambda_2, \end{cases} \tag{2.13}$$

$$\underline{\phi_2}(z) = \begin{cases} \delta, & z \geq z_2, \\ [-hz - q(-z)^{1/2}]e^{\lambda_2 z}, & z \leq z_2. \end{cases} \tag{2.14}$$

Lemma 2.2. *For $c = c^*$, the functions $(\overline{\phi_1}, \overline{\phi_2})$ and $(\underline{\phi_1}, \underline{\phi_2})$ defined by* (2.11)-(2.14) *are a pair of upper and lower solutions of* (1.2).

2.2. Existence of traveling wave solutions

With upper and lower solutions at hand, we shall apply Schauder's fixed point theorem to derive the existence of solution to (1.2).

First, we introduce the following function spaces

$$\begin{aligned} X &= \{\Phi = (\phi_1, \phi_2) \mid \Phi \text{ is a continuous function from } \mathbb{R} \text{ to } \mathbb{R}^2\}, \\ X_k &= \{(\phi_1, \phi_2) \in X \mid 1-k \le \phi_1(z) \le 1 \text{ and } 0 \le \phi_2(z) \le 1 \text{ for all } z \in \mathbb{R}\}. \end{aligned}$$

Define the functions

$$\begin{aligned} F_1(y_1, y_2) &:= \beta y_1 + r y_1 (1 - y_1 - k y_2), \\ F_2(y_1, y_2) &:= \beta y_2 + s y_2 \left(1 - \frac{y_2}{y_1}\right) \end{aligned}$$

for some constant β. By taking $\beta > \max\{r(1+k), s(1+k)/(1-k)\}$, we see that F_1 is nondecreasing in y_1 and is decreasing in y_2 for $1-k \le y_1 \le 1$ and $0 \le y_2 \le 1$. Also, F_2 is nondecreasing with respect to y_1 and y_2 for $1-k \le y_1 \le 1$ and $0 \le y_2 \le 1$.

For notational convenience, we set $d_1 = 1$ and $d_2 = d$. Then (1.2) can be re-written as

$$d_i \phi_i''(z) - c\phi_i'(z) - \beta\phi_i(z) + F_i(\phi_1, \phi_2)(z) = 0, \quad i = 1, 2. \tag{2.15}$$

Now we define

$$\lambda_{i1}(c) = \frac{c - \sqrt{c^2 + 4\beta d_i}}{2d_i}, \quad \lambda_{i2}(c) = \frac{c + \sqrt{c^2 + 4\beta d_i}}{2d_i}, \quad i = 1, 2.$$

Without ambiguity, we sometime omit the dependence of c and denote $\lambda_{i1} = \lambda_{i1}(c)$ and $\lambda_{i2} = \lambda_{i2}(c)$. It is easy to see that $\lambda_{i1} < 0 < \lambda_{i2}$ and

$$d_i \lambda_{i1}^2 - c\lambda_{i1} - \beta = 0, \quad d_i \lambda_{i2}^2 - c\lambda_{i2} - \beta = 0, \quad i = 1, 2.$$

For $(\phi_1, \phi_2) \in X_k$, we consider the operator $P = (P_1, P_2) : X_k \to X$ defined as follows

$$P_i(\phi_1, \phi_2)(z) = \frac{1}{d_i(\lambda_{i2} - \lambda_{i1})} \left[\int_{-\infty}^{z} e^{\lambda_{i1}(z-s)} + \int_{z}^{\infty} e^{\lambda_{i2}(z-s)} \right] F_i(\phi_1, \phi_2)(s) ds,$$

for $i = 1, 2$, $z \in \mathbb{R}$. It is easy to check that the operator P satisfies

$$d_i (P_i(\phi_1, \phi_2))''(z) - c(P_i(\phi_1, \phi_2))'(z) - \beta P_i(\phi_1, \phi_2)(z) + F_i(\phi_1, \phi_2)(z) = 0,$$

for $i = 1, 2$, $z \in \mathbb{R}$.

Although the proof of the following lemma is very standard (cf. [23, 17, 18]), for reader's convenience we provide some details here.

Lemma 2.3. *Let $c > 0$. Suppose that* (1.2) *has a pair of upper and lower solutions* $(\overline{\phi_1}, \overline{\phi_2})$ *and* $(\underline{\phi_1}, \underline{\phi_2})$ *in* X_k *satisfying*

(1) $\overline{\phi_i}(z) \geq \underline{\phi_i}(z)$, $z \in \mathbb{R}$, $i = 1, 2$;

(2) $\overline{\phi_i}'(z-) \geq \overline{\phi_i}'(z+)$, $\underline{\phi_i}'(z-) \leq \underline{\phi_i}'(z+)$, $z \in D$, $i = 1, 2$, *where*

$$\overline{\phi_i}'(z\pm) := \lim_{\xi \to z\pm} \overline{\phi_i}'(\xi), \quad \underline{\phi_i}'(z\pm) := \lim_{\xi \to z\pm} \underline{\phi_i}'(\xi).$$

Then (1.2) *has a positive solution* (ϕ_1, ϕ_2) *such that* $\overline{\phi_i}(z) \geq \phi_i(z) \geq \underline{\phi_i}(z)$ *for all* $z \in \mathbb{R}$ *for* $i = 1, 2$.

Proof. Choose a constant $\alpha \in (0, \min\{-\lambda_{11}, -\lambda_{21}\})$ and denote $\|\cdot\|$ the supremum norm in $\mathbb{R}^2$. Define

$$B_\alpha(\mathbb{R}, \mathbb{R}^2) := \left\{ \Phi \in X_k \,\middle|\, \sup_{z \in \mathbb{R}} \|\Phi(z)\| e^{-\alpha|z|} < \infty \right\}, \quad |\Phi|_\alpha := \sup_{z \in \mathbb{R}} \|\Phi(z)\| e^{-\alpha|z|}.$$

Then $(B_\alpha(\mathbb{R}, \mathbb{R}^2), |\cdot|_\alpha)$ is a Banach space. Also, we let

$$\Gamma := \{(\phi_1, \phi_2) \in X_k \mid \underline{\phi_i}(z) \leq \phi_i(z) \leq \overline{\phi_i}(z) \text{ for all } z \in \mathbb{R},\ i = 1, 2\}.$$

Then Γ is a nonempty convex bounded closed set with respect to the weighted norm $|\cdot|_\alpha$.

First, we show that P maps Γ into itself. For $(\phi_1, \phi_2)(z) \in \Gamma$ and each fixed $z \in \mathbb{R}$, we have

$$\begin{cases} P_1(\underline{\phi_1}, \overline{\phi_2})(z) \leq P_1(\phi_1, \phi_2)(z) \leq P_1(\overline{\phi_1}, \underline{\phi_2})(z), \\ P_2(\underline{\phi_1}, \underline{\phi_2})(z) \leq P_1(\phi_1, \phi_2)(z) \leq P_2(\overline{\phi_1}, \overline{\phi_2})(z), \end{cases}$$

by the choice of β. Thus we only need to show that

$$\begin{cases} \underline{\phi_1}(z) \leq P_1(\underline{\phi_1}, \overline{\phi_2})(z) \leq P_1(\overline{\phi_1}, \underline{\phi_2})(z) \leq \overline{\phi_1}(z), \ z \in \mathbb{R}, \\ \underline{\phi_2}(z) \leq P_2(\underline{\phi_1}, \underline{\phi_2})(z) \leq P_2(\overline{\phi_1}, \overline{\phi_2})(z) \leq \overline{\phi_2}(z), \ z \in \mathbb{R}. \end{cases} \tag{2.16}$$

Without loss of generality, we may assume that $z_1 > z_2 > ... > z_m$ and set $z_0 = \infty$, $z_{m+1} = -\infty$. For $z \in \mathbb{R}\backslash D$, there exists a $k \in \{0, 1, \cdots, m\}$ such that $z \in (z_{k+1}, z_k)$. By the definition of upper and lower solutions, if $z \in \mathbb{R}\backslash D$, then

$$\begin{aligned} P_1(\underline{\phi_1}, \overline{\phi_2})(z) &= \frac{1}{\lambda_{12} - \lambda_{11}} \left[\int_{-\infty}^{z} e^{\lambda_{11}(z-s)} + \int_{z}^{\infty} e^{\lambda_{12}(z-s)} \right] F_1(\underline{\phi_1}, \overline{\phi_2})(s) ds \\ &\geq \frac{1}{\lambda_{12} - \lambda_{11}} \left[\int_{-\infty}^{z} e^{\lambda_{11}(z-s)} + \int_{z}^{\infty} e^{\lambda_{12}(z-s)} \right] \left[-\underline{\phi_1}''(s) + c\underline{\phi_1}'(s) + \beta \underline{\phi_1}(s) \right] ds \\ &= \underline{\phi_1}(z) + \frac{1}{\lambda_{12} - \lambda_{11}} \sum_{j=1}^{k} e^{\lambda_{12}(z-z_j)} [\underline{\phi_1}'(z_j+) - \underline{\phi_1}'(z_j-)] \\ &\quad + \frac{1}{\lambda_{12} - \lambda_{11}} \sum_{j=k+1}^{m} e^{\lambda_{11}(z-z_j)} [\underline{\phi_1}'(z_j+) - \underline{\phi_1}'(z_j-)] \\ &\geq \underline{\phi_1}(z). \end{aligned}$$

The continuity of $P_1(\underline{\phi_1}, \overline{\phi_2})(z)$ and $\underline{\phi_1}(z)$ for $z \in \mathbb{R}$ implies that $P_1(\underline{\phi_1}, \overline{\phi_2})(z) \geq \underline{\phi_1}(z)$ for all $z \in \mathbb{R}$. The other inequalities in (2.16) can be shown similarly.

Next, that $P : \Gamma \to \Gamma$ is completely continuous in the sense of the weighted norm $|\cdot|_\alpha$ can be shown by a similar argument as in [23]. We shall not repeat it here. Finally, we apply Schauder's fixed point theorem to obtain the conclusion of the lemma. □

By Lemmas 2.1- 2.3, we have the following theorem.

Theorem 2.4. *For each $c \geq c^*$, there exists a positive solution (ϕ_1, ϕ_2) of (1.2) such that*

$$\lim_{z\to-\infty}(\phi_1, \phi_2)(z) = (1, 0) \quad \text{and} \quad \underline{\phi}_i(z) \leq \phi_i(z) \leq \overline{\phi}_i(z),\ z \in \mathbb{R},\ i = 1, 2.$$

Proof. It suffices to verify the conditions (1) and (2) in Lemma 2.3.

Now we treat the case $c > c^*$. First we show that $\overline{\phi_1}(z) \geq \underline{\phi_1}(z)$ for $z \in \mathbb{R}$. Recall that $\epsilon < k$. For $z \geq 0$, we have

$$\overline{\phi_1}(z) - \underline{\phi_1}(z) = k>0.$$

When $z_1 \leq z < 0$, we have

$$\overline{\phi_1}(z) - \underline{\phi_1}(z) = k - \epsilon(e^{\lambda_1 z} - e^{\nu\lambda_1 z})> k - \epsilon > 0.$$

For $z < z_1$, we know that

$$\overline{\phi_1}(z) - \underline{\phi_1}(z) = (k - \epsilon)e^{\lambda_1 z} + \epsilon e^{\nu\lambda_1 z} + kpe^{\eta\lambda_1 z}>0.$$

So we obtain the conclusion. Similarly, we can show that $\overline{\phi_2}(z) \geq \underline{\phi_2}(z)$ for $z \in \mathbb{R}$. Thus, condition (1) of Lemma 2.3 holds.

For condition (2), we have

$$\begin{aligned}
&\overline{\phi_1}'(0+) = 0 < \epsilon(\nu - 1)\lambda_1 = \overline{\phi_1}'(0-),\\
&\underline{\phi_1}'(z_1+) = 0 > -k(\lambda_1 e^{\lambda_1 z_1} + p\eta\lambda_1 e^{\eta\lambda_1 z_1}) = \underline{\phi_1}'(z_1-),\\
&\overline{\phi_2}'(0+) = 0 < \lambda_2 = \overline{\phi_2}'(0-),\\
&\underline{\phi_2}'(z_2+) = 0 > \underline{\phi_2}'(z_2-).
\end{aligned}$$

When $c = c^*$, by our construction, it is easy to verify the conditions (1) and (2) in Lemma 2.3.

That $\lim_{z\to-\infty}(\phi_1, \phi_2)(z) = (1, 0)$ is trivial. This completes the proof of the theorem. □

Now we study the tail behavior of the traveling wave solution obtained in Theorem 2.4 at $z = +\infty$. In order to do this, we define the functions $m(\theta)$ and $M(\theta)$ for $\theta \in [0, 1]$ as follows.

$$m(\theta) = \theta\frac{1}{1 + k}, \quad M(\theta) = \theta\frac{1}{1 + k} + (1 - \theta)(1 + \varepsilon).$$

Here, ε is small enough such that $k(1 + \varepsilon) < 1$. For $0 < \theta_1 < \theta_2 < 1$, it is easy to see that

$$0 = m(0) < m(\theta_1) < m(\theta_2) < m(1) = \frac{1}{1 + k} = M(1) < M(\theta_2) < M(\theta_1) < M(0) = 1 + \varepsilon.$$

We are ready to show the tail behavior of traveling wave solution at ∞ as follows.

Theorem 2.5. *Let (ϕ_1, ϕ_2) be a positive solution obtained in Theorem 2.4. Then*

$$\lim_{z\to+\infty}(\phi_1,\phi_2)(z) = \left(\frac{1}{1+k}, \frac{1}{1+k}\right). \tag{2.17}$$

Proof. By the facts $1-k = \underline{\phi_1}(z) \le \phi_1(z) \le \overline{\phi_1}(z) = 1$ and $\delta = \underline{\phi_2}(z) \le \phi_2(z) \le \overline{\phi_2}(z) = 1$ for all $z > 0$, we obtain that

$$\begin{aligned}
&\limsup_{z\to+\infty}\phi_1(z) \le 1, \quad \limsup_{z\to+\infty}\phi_2(z) \le 1,\\
&\liminf_{z\to+\infty}\phi_1(z) \ge 1-k > 0, \quad \liminf_{z\to+\infty}\phi_2(z) \ge \delta > 0.
\end{aligned}$$

Now we denote

$$\phi_i^- = \liminf_{z\to+\infty}\phi_i(z), \quad \phi_i^+ = \limsup_{z\to+\infty}\phi_i(z), \quad i = 1,2.$$

Obviously, we have

$$m(0) = 0 < \phi_i^- \le \phi_i^+ < 1+\varepsilon = M(0),\ i = 1{,}2.$$

Note that (2.17) holds if we can show that

$$m(\theta) < \phi_i^- \le \phi_i^+ < M(\theta),\ i = 1,2, \tag{2.18}$$

for all $\theta \in [0,1)$.

Set $\theta_0 := \sup\{\theta \in [0,1) \,|\, (2.18) \text{ holds}\}$. Then θ_0 is well-defined and it suffices to claim that $\theta_0 = 1$. For contradiction, we suppose that $\theta_0 < 1$. Then, by passing to the limit, we have

$$m(\theta_0) \le \phi_i^- \le \phi_i^+ \le M(\theta_0),\ i = 1,2.$$

Moreover, by the definition of θ_0, (2.18) does not hold for $\theta = \theta_0$, i.e., at least one of the following conditions holds:

$$\phi_1^- = m(\theta_0),\ \phi_1^+ = M(\theta_0),\ \phi_2^- = m(\theta_0),\ \phi_2^+ = M(\theta_0).$$

First, we assume that $\phi_1^- = m(\theta_0)$. If ϕ_1 is eventually monotone, we have $\phi_1(\infty)$ exists by ϕ_1 is bounded on $\mathbb{R}$. Since $\int_0^\infty \phi_1'(s)ds = \phi_1(\infty) - \phi_1(0)$ is finite, either $\liminf_{s\to+\infty}\phi_1'(s) = 0$ if $\phi_1'(s) \ge 0$ for $s \gg 1$ or $\limsup_{s\to+\infty}\phi_1'(s) = 0$ if $\phi_1'(s) \le 0$ for $s \gg 1$. Then we can find a sequence $\{z_n\}$ with $z_n \to +\infty$ as $n \to +\infty$ such that $\lim_{n\to+\infty}\phi_1(z_n) = m(\theta_0)$ and $\lim_{n\to+\infty}\phi_1'(z_n) = 0$. Also, we know that $\limsup_{n\to+\infty}\phi_2(z_n) \le M(\theta_0)$. So we have

$$\begin{aligned}
\liminf_{n\to+\infty}\{[1-\phi_1(z_n) - k\phi_2(z_n)]\} &\ge 1 - \frac{\theta_0}{1+k} - k\left[\frac{\theta_0}{1+k} + (1-\theta_0)(1+\varepsilon)\right]\\
&= (1-\theta_0)(1-k(1+\varepsilon)) > 0.
\end{aligned}$$

Integrating the first equation of the system (1.2) from 0 to z_n, we obtain that

$$\phi_1'(z_n) - \phi_1'(0) - c[\phi_1(z_n) - \phi_1(0)] = -r \int_0^{z_n} \phi_1(s)[1 - \phi_1(s) - k\phi_2(s)]ds. \tag{2.19}$$

Letting $n \to +\infty$, we get a contradiction, since the left-hand side of (2.19) remains bounded and the right-hand side of (2.19) tends to $-\infty$.

If ϕ_1 is oscillatory at ∞, then we can choose a sequence $\{z_n\}$ of minimal points of ϕ_1 with $z_n \to +\infty$ as $n \to +\infty$ such that $\lim_{n\to+\infty} \phi_1(z_n) = m(\theta_0)$. Note that $\phi_1''(z_n) - c\phi_1'(z_n) \geq 0$ for all n. Also, we have $\limsup_{n\to+\infty} \phi_2(z_n) \leq M(\theta_0)$ and

$$\begin{aligned} \liminf_{n\to+\infty}[1 - \phi_1(z_n) - k\phi_2(z_n)] &\geq 1 - \frac{\theta_0}{1+k} - k\left[\frac{\theta_0}{1+k} + (1-\theta_0)(1+\varepsilon)\right] \\ &= (1-\theta_0)(1 - k(1+\varepsilon)) > 0. \end{aligned}$$

This implies that

$$\liminf_{n\to+\infty}\{\phi_1''(z_n) - c\phi_1'(z_n) + r\phi_1(z_n)[1 - \phi_1(z_n) - k\phi_2(z_n)]\} > 0,$$

a contradiction. Hence $\phi_1^- = m(\theta_0)$ is impossible.

The case for $\phi_1^+ = M(\theta_0)$ can be treated similarly.

Next, we deal with the case $\phi_2^- = m(\theta_0)$. In this case, without loss of generality, we may assume that $m(\theta_0) < \phi_1^- \leq \phi_1^+ < M(\theta_0)$. If ϕ_2 is eventually monotone, we have $\phi_2(\infty)$ exists due to ϕ_2 is bounded on $\mathbb{R}$. By the similar argument as the case $\phi_1(\infty)$ exists, we can find a sequence $\{\xi_n\}$ with $\xi_n \to +\infty$ as $n \to +\infty$ such that $\lim_{n\to+\infty} \phi_2(\xi_n) = m(\theta_0)$ and $\lim_{n\to+\infty} \phi_2'(\xi_n) = 0$. Moreover, we have

$$\liminf_{n\to+\infty}\left[1 - \frac{\phi_2(\xi_n)}{\phi_1(\xi_n)}\right] > 1 - \frac{m(\theta_0)}{m(\theta_0)} = 0.$$

Integrating the second equation of the system (1.2) from 0 to ξ_n, we obtain that

$$d[\phi_2'(\xi_n) - \phi_2'(0)] - c[\phi_2(\xi_n) - \phi_2(0)] = -s \int_0^{\xi_n} \phi_2(s)\left[1 - \frac{\phi_2(s)}{\phi_1(s)}\right] ds. \tag{2.20}$$

Letting $n \to +\infty$, we get a contradiction. On the other hand, if ϕ_2 is oscillatory at ∞, then we can choose a sequence $\{\xi_n\}$ of minimal points of ϕ_2 with $\xi_n \to +\infty$ as $n \to +\infty$ such that $\lim_{n\to+\infty} \phi_2(\xi_n) = m(\theta_0)$. Note that $d\phi_2''(\xi_n) - c\phi_2'(\xi_n) \geq 0$ for all n. Also, we have

$$\liminf_{n\to+\infty}\left[1 - \frac{\phi_2(z_n)}{\phi_1(z_n)}\right] > \left[1 - \frac{m(\theta_0)}{m(\theta_0)}\right] = 0,$$

by using $\phi_1^- > m(\theta_0)$. Hence we obtain that

$$\liminf_{n\to+\infty}\left\{d\phi_2''(z_n) - c\phi_2'(z_n) + s\phi_2(z_n)\left[1 - \frac{\phi_2(z_n)}{\phi_1(z_n)}\right]\right\} > 0,$$

a contradiction. Similarly, we can deal with the case $\phi_2^+ = M(\theta_0)$ to reach a contradiction. Consequently, we must have $\theta_0 = 1$ and (2.18) follows. □

2.3. Determination of the minimal speed

In this section, we would like to show that $c = c^*$ is the minimal wave speed. This is equivalent to show that there is no positive solution of (1.2)-(1.3) for $c < c^*$, due to Theorems 2.4 and 2.5.

First, we recall the following spreading phenomenon (cf. [1]), namely, if $z_0 > 0$ and $c \in (0, 2\sqrt{dr})$, then

$$\liminf_{t\to+\infty} \inf_{|x|<ct} z(x,t) = \limsup_{t\to+\infty} \sup_{|x|<ct} z(x,t) = \frac{1}{a} \tag{2.21}$$

for the solution z of the following Cauchy problem for Fisher's equation (see, e.g., [9, 27])

$$\begin{cases} z_t(x,t) = dz_{xx}(x,t) + rz(x,t)[1 - az(x,t)], \quad x \in \mathbb{R},\ t > 0, \\ z(x,0) = z_0(x), \quad x \in \mathbb{R}, \end{cases} \tag{2.22}$$

with d, r, a are positive constants and $z_0(x)$ is a positive bounded continuous function. Then we have

Theorem 2.6. *For $c < c^*$, there is no positive solution of* (1.2)-(1.3).

Proof. For contradiction, we suppose that there exists a positive solution (ϕ_1, ϕ_2) of (1.2)-(1.3) for some $\tilde{c} < c^*$. Since $\tilde{c} < c^* = 2\sqrt{ds}$, there exists a $\theta \in (0,1)$ with $0 < 1 - \theta \ll 1$ such that $2\sqrt{ds\theta} > \tilde{c}$. By (1.3) and the positivity of ϕ_2, there exists a positive constant K such that $\alpha(x,t) := \phi_2(x + \tilde{c}t)$ satisfies

$$\begin{cases} \alpha_t(x,t) \geq d\alpha_{xx}(x,t) + s\alpha(x,t)\left[1 - K\alpha(x,t)\right], \\ \alpha(x,0) = \phi_2(x). \end{cases}$$

Now we consider that $y(t) = -(2\sqrt{ds\theta} + \tilde{c})t/2$. Note that $|y(t)| < 2\sqrt{ds\theta}|t|$. Then, by (2.21), we obtain

$$\liminf_{t\to+\infty} \alpha(y(t),t) \geq \frac{1}{K} > 0.$$

On the other hand, $y(t) + \tilde{c}t = (\tilde{c} - 2\sqrt{ds\theta})t/2 \to -\infty$ as $t \to +\infty$. Hence we obtain that

$$\limsup_{t\to+\infty} \alpha(y(t),t) = \limsup_{t\to+\infty} \phi_2(y(t) + \tilde{c}t) = \lim_{z\to-\infty} \phi_2(z) = 0,$$

a contradiction. Therefore, the proof of this theorem is done. □

3. Traveling wave solutions of (1.4)

3.1. Upper and lower solutions

For the system (1.5), the upper and lower solutions are defined as follows.

Definition 3.1. *The functions* $(\overline{U}, \overline{V})$ *and* $(\underline{U}, \underline{V})$ *are called a pair of upper and lower solutions of* (1.5)*, if* $\overline{U}'$, $\overline{V}'$, $\underline{U}'$, $\underline{V}'$ *exist and the inequalities*

$$[\overline{U}(\xi+1)+\overline{U}(\xi-1)-2\overline{U}(\xi)]-c\overline{U}'(\xi)+r\overline{U}(\xi)[1-\overline{U}(\xi)-k\underline{V}(\xi)] \le 0, \tag{3.1}$$

$$[\underline{U}(\xi+1)+\underline{U}(\xi-1)-2\underline{U}(\xi)]-c\underline{U}'(\xi)+r\underline{U}(\xi)[1-\underline{U}(\xi)-k\overline{V}(\xi)] \ge 0, \tag{3.2}$$

$$d[\overline{V}(\xi+1)+\overline{V}(\xi-1)-2\overline{V}(\xi)]-c\overline{V}'(\xi)+s\overline{V}(\xi)\left(1-\frac{\overline{V}(\xi)}{\overline{U}(\xi)}\right) \le 0, \tag{3.3}$$

$$d[\underline{V}(\xi+1)+\underline{V}(\xi-1)-2\underline{V}(\xi)]-c\underline{V}'(\xi)+s\underline{V}(\xi)\left(1-\frac{\underline{V}(\xi)}{\underline{U}(\xi)}\right) \ge 0 \tag{3.4}$$

hold for $\xi \in \mathbb{R} \setminus D$ *with some finite set* $D = \{\xi_1, \xi_2, \ldots, \xi_m\}$.

Now we construct a suitable pair of upper and lower solutions of (1.5) for each $c \ge c_*$.

3.1.1. The case $c > c_*$.

For $c > c_*$, there are positive constants λ_1, λ_2, λ_3 with $\lambda_2 < \lambda_3$ such that

$$e^{\lambda_1}+e^{-\lambda_1}-2-c\lambda_1-r=0 \quad \text{and} \quad d(e^{\lambda_i}+e^{-\lambda_i}-2)-c\lambda_i+s=0,\ i=2,3.$$

For constants $\mu, q > 1$, we consider the function

$$f(\xi) := e^{\lambda_2 \xi} - q e^{\mu \lambda_2 \xi}.$$

Then we have the following lemma.

Lemma 3.1. *For any given* $\mu, q > 1$*, there exist a sufficiently small* δ *with* $0 < \delta < 1-k$ *and a* $\xi_2 < 0$ *such that* $f(\xi_2) = \delta$ *and* $f(\xi) > \delta$ *for all* $\xi \in [\xi_2 - 1, \xi_2)$.

Proof. Obviously, there exists a unique $\xi_0 := -\frac{1}{\lambda_2(\mu-1)} \ln q < 0$ such that $f(\xi_0) = 0$ and $f(\xi) > 0$ for all $\xi < \xi_0$. So we have $f(\xi_0 - 1) - f(\xi_0) > 0$. Also, the function f has a unique maximal point at $\xi_M = -\frac{1}{\lambda_2(\mu-1)} \ln(q\mu)$. Since f is a continuous function, we can choose a ξ_2 with $0 < \xi_0 - \xi_2 \ll 1$ and a sufficiently small δ with $0 < \delta < 1-k$ such that $f(\xi_2) = \delta$ and $f(\xi_2 - 1) - f(\xi_2) > 0$. If $\xi_2 - 1 \ge \xi_M$, then we get the conclusion by the fact f is decreasing on $[\xi_2 - 1, \xi_2]$. Otherwise, for the case $\xi_2 - 1 < \xi_M$ since we have f is increasing in $[\xi_2 - 1, \xi_M]$ and is decreasing in $[\xi_M, \xi_2]$, the lemma follows by using $f(\xi_2 - 1) > f(\xi_2) = \delta$. □

Now, we define the functions $\overline{U}(\xi), \underline{U}(\xi), \overline{V}(\xi), \underline{V}(\xi)$ as follows:

$$\overline{U}(\xi) = \begin{cases} 1, & \xi \geq 0, \\ 1 - \epsilon(e^{\lambda_1 \xi} - e^{\nu\lambda_1 \xi}), & \xi \leq 0, \end{cases} \tag{3.5}$$

$$\underline{U}(\xi) = \begin{cases} 1 - k, & \xi \geq \xi_1, \\ 1 - k(e^{\lambda_1 \xi} + pe^{\eta\lambda_1 \xi}), & \xi \leq \xi_1, \end{cases} \tag{3.6}$$

$$\overline{V}(\xi) = \begin{cases} 1, & \xi \geq 0, \\ e^{\lambda_2 \xi}, & \xi \leq 0, \end{cases} \tag{3.7}$$

$$\underline{V}(\xi) = \begin{cases} \delta, & \xi \geq \xi_2, \\ e^{\lambda_2 \xi} - qe^{\mu\lambda_2 \xi}, & \xi \leq \xi_2, \end{cases} \tag{3.8}$$

where the constants η, μ, ν, q, p and ϵ are chosen *in sequence* such that

(C1) $\eta > 0$ is small enough such that $(e^{\eta\lambda_1} + e^{-\eta\lambda_1} - 2) - c(\eta\lambda_1) - r + rk < 0$ and $\lambda_2 > \eta\lambda_1$, $\mu \in \left(1, \min\left\{\frac{\lambda_3}{\lambda_2}, 2\right\}\right)$, $\nu > \max\{1, \lambda_2/\lambda_1\}$,

(C2) $q > \max\left\{1, \frac{s}{-[d(e^{\mu\lambda_2} + e^{-\mu\lambda_2} - 2) - c(\mu\lambda_2) + s](1-k)}\right\}$,
$p > \frac{r(1+k)}{-[(e^{\eta\lambda_1} + e^{-\eta\lambda_1} - 2) - c(\eta\lambda_1) - r + rk]}$,

(C3) $0 < \varepsilon < \min\{k, k_1, k_2\}$ with k_1, k_2 defined by

$$k_1 := \frac{rk\delta}{e^{\nu\lambda_1} + e^{-\nu\lambda_1} - 2 - c(\nu\lambda_1) - r + rk\delta},$$

$$k_2 := \frac{rk(1 - qe^{(\mu-1)\lambda_2\xi_2})}{e^{\nu\lambda_1} + e^{-\nu\lambda_1} - 2 - c(\nu\lambda_1) - r + rk},$$

where ξ_2, δ are defined as in Lemma 3.1,

and $\xi_1 < 0$ is uniquely defined by $e^{\lambda_1\xi_1} + pe^{\eta\lambda_1\xi_1} = 1$.
Then we have the following lemma.

Lemma 3.2. *Assume that $c > c_*$. Then the functions $(\overline{U}, \overline{V})$ and $(\underline{U}, \underline{V})$ defined by* (3.5)-(3.8) *are a pair of upper and lower solutions of* (1.5).

3.1.2. The case $c = c_*$.

For $c = c_*$, the equation

$$c_*\lambda = d(e^{\lambda} + e^{-\lambda} - 2) + s \tag{3.9}$$

has a unique positive root λ_2. This also implies that

$$c_* = d(e^{\lambda_2} - e^{-\lambda_2}). \tag{3.10}$$

Also, let λ_1 be the unique positive root of $c_*\lambda = (e^{\lambda} + e^{-\lambda} - 2) - r$.

For the later purpose, we consider

$$h_1(\xi) := \xi^2 + \xi\sqrt{\xi^2 + \xi} + \frac{1}{2}\xi, \quad h_2(\xi) := \xi^2 + \xi\sqrt{\xi^2 - \xi} - \frac{1}{2}\xi, \quad \xi \le -1.$$

Then $h_1(\xi)$ and $h_2(\xi)$ are positive on $(-\infty, -1]$. We compute by using l'Hôpital's rule that

$$\begin{aligned} \lim_{\xi\to-\infty} h_1(\xi) &= \lim_{\xi\to-\infty} \frac{1 - \left(1+\frac{1}{\xi}\right)^{1/2} + \frac{1}{2\xi}}{\frac{1}{\xi^2}} = \lim_{\xi\to-\infty} \frac{\frac{1}{2\xi^2}\left(1+\frac{1}{\xi}\right)^{-1/2} - \frac{1}{2\xi^2}}{\frac{-2}{\xi^3}} \\ &= \lim_{\xi\to-\infty} \frac{-\left(1+\frac{1}{\xi}\right)^{-1/2} + 1}{\frac{4}{\xi}} = \lim_{\xi\to-\infty} \frac{\left(1+\frac{1}{\xi}\right)^{-3/2}\frac{-1}{2\xi^2}}{\frac{-4}{\xi^2}} \\ &= \lim_{\xi\to-\infty} \frac{\left(1+\frac{1}{\xi}\right)^{-3/2}}{8} = \frac{1}{8}. \end{aligned}$$

Similarly, we have

$$\lim_{\xi\to-\infty} h_2(\xi) = \frac{1}{8}.$$

Thus the constants

$$l_1 := \inf_{\xi<-1} h_1(\xi) > 0 \quad \text{and} \quad l_2 := \inf_{\xi<-1} h_2(\xi) > 0 \tag{3.11}$$

are well-defined.

Now, we define the functions $\overline{U}(\xi), \underline{U}(\xi), \overline{V}(\xi), \underline{V}(\xi)$ as follows:

$$\overline{U}(\xi) = \begin{cases} 1, & \xi \ge 0, \\ 1 - \epsilon(e^{\lambda_1\xi} - e^{\nu\lambda_1\xi}), & \xi \le 0, \end{cases} \tag{3.12}$$

$$\underline{U}(\xi) = \begin{cases} 1 - k, & \xi \ge \xi_1, \\ 1 - k(e^{\lambda_1\xi} + pe^{\eta\lambda_1\xi}), & \xi \le \xi_1, \end{cases} \tag{3.13}$$

$$\overline{V}(\xi) = \begin{cases} 1, & \xi \ge -1/\lambda_2 - 1, \\ -h\xi e^{\lambda_2\xi}, & \xi \le -1/\lambda_2 - 1, \end{cases} \tag{3.14}$$

$$\underline{V}(\xi) = \begin{cases} \delta, & \xi \ge \xi_2, \\ [-h\xi - q(-\xi)^{1/2}]e^{\lambda_2\xi}, & \xi \le \xi_2, \end{cases} \tag{3.15}$$

where the constants $\eta, h, p, \xi_1, q, \delta, \xi_2, \nu, \epsilon$ are chosen *in sequence* as follows.

First, we take η with $0 < \eta \ll 1$ such that

$$e^{\eta\lambda_1} + e^{-\eta\lambda_1} - 2 - c(\eta\lambda_1) - r + rk < 0, \quad \frac{\lambda_2}{\lambda_2 + 1} > \eta\lambda_1. \tag{3.16}$$

Secondly, we set $h = \frac{\lambda_2}{\lambda_2+1} e^{\lambda_2+1}$ and consider $w(\xi) := -h\xi e^{\lambda_2\xi}$. Then w is strictly increasing on $(-\infty, -1/\lambda_2]$ and $w(-1/\lambda_2 - 1) = 1$. Let p be a constant with

$$p > \max\left\{ e, \frac{r[k + h(\lambda_2 + 1)/(\lambda_2^2 e)]}{-[e^{\eta\lambda_1} + e^{\eta\lambda_1} - 2 - c(\eta\lambda_1) - r + rk]} \right\}. \tag{3.17}$$

Then there exists a ξ_1 such that $e^{\lambda_1\xi_1} + pe^{\eta\lambda_1\xi_1} = 1$ and $\xi_1 < -1/\lambda_2 - 1$, since $e^{\lambda_1\xi} + pe^{\eta\lambda_1\xi}$ is increasing in ξ and

$$e^{-\lambda_1(1/\lambda_2+1)} + pe^{-\eta\lambda_1(1/\lambda_2+1)} > e^{-\lambda_1(1/\lambda_2+1)} + pe^{-1} > 1.$$

Thirdly, we choose $q > 1$ sufficiently large such that

$$(q/h)^2 > \frac{1}{\lambda_2} + 1, \quad q > \frac{sh^2}{d(l_1 e^{\lambda_2} + l_2 e^{-\lambda_2})(1-k)} \left(\frac{7}{2e\lambda_2}\right)^{7/2}. \tag{3.18}$$

For the chosen positive constants h and q, we consider the function

$$g(\xi) := [-h\xi - q(-\xi)^{1/2}]e^{\lambda_2\xi}, \quad \xi \le 0.$$

Then $g(\xi)$ has the unique zero $\xi_0 = \xi_0(h, q) := -(q/h)^2$ and a unique maximum point $\tilde{\xi}$ in $(-\infty, \xi_0)$. By using a similar argument as in Lemma 3.1, there exist a sufficiently small δ with $0 < \delta < 1 - k$ and a $\xi_2 < \xi_0$ such that $g(\xi_2) = \delta$ and $g(\xi) > \delta$ for all $\xi \in [\xi_2 - 1, \xi_2)$. Also, we have $\xi_2 < \xi_0 < -1/\lambda_2 - 1$ by the first inequality in (3.18).

Finally, we take $\nu > \max\{1, \lambda_2/\lambda_1\}$ and $\epsilon \in (0, k)$ small enough such that

$$\epsilon < \min\left\{ \frac{rk\delta}{[e^{\nu\lambda_1} + e^{-\nu\lambda_1} - 2 - c(\nu\lambda_1) - r] + rk\delta}, \frac{rk(1-k)(-h\xi_2 - q\sqrt{-\xi_2})\}}{[e^{\nu\lambda_1} + e^{-\nu\lambda_1} - 2 - c(\nu\lambda_1) - r]} \right\}. \tag{3.19}$$

Then the following lemma holds.

Lemma 3.3. *For $c = c_*$, the functions $(\overline{U}, \overline{V})$ and $(\underline{U}, \underline{V})$ defined by* (3.12)-(3.15) *are a pair of upper and lower solutions of* (1.5).

3.2. Existence of traveling wave solutions

As in Section 2.2, we would like to apply Schauder's fixed point theory to derive the existence of solutions of (1.5). Recall that X is the space of continuous functions from $\mathbb{R}$ to $\mathbb{R}^2$ with the usual sup-norm $\|\cdot\|$. Set

$$K = \{(U, V) \in X \mid (U(\xi), V(\xi)) \in [1-k, 1] \times [0, 1] \text{ for all } \xi \in \mathbb{R}\}.$$

Also, we introduce the functions

$$F_1(y_1, y_2) := (\beta - 2)y_1 + ry_1(1 - y_1 - ky_2),$$
$$F_2(y_1, y_2) := (\beta - 2d)y_2 + sy_2 \left(1 - \frac{y_2}{y_1}\right)$$

for a constant $\beta > \max\{2 + r(1 + k), 2d + s(1 + k)/(1 - k)\}$. Then, for $(U_i, V_i) \in K, i = 1, 2$, with $U_1 \leq U_2$ and $V_1 \leq V_2$, we have

$$F_1(U_1, V_2) \leq F_1(U_1, V_1) \leq F_1(U_2, V_1), \quad F_2(U_1, V_1) \leq F_2(U_2, V_2). \tag{3.20}$$

Let $c > 0$. For $(U, V) \in K$, we define the operator $G = (G_1, G_2) : K \to X$ by

$$G_i(U, V)(\xi) = \frac{1}{c} e^{-\frac{\beta}{c}\xi} \int_{-\infty}^{\xi} e^{\frac{\beta}{c}s} H_i(U, V)(s)ds, \ i = 1, 2, \ \xi \in \mathbb{R},$$

where

$$H_1(U, V)(\xi) := U(\xi + 1) + U(\xi - 1) + F_1(U, V)(\xi),$$
$$H_2(U, V)(\xi) := d[V(\xi + 1) + V(\xi - 1)] + F_2(U, V)(\xi).$$

It easy to see that (G_1, G_2) satisfies the system (1.5). Moreover, by (3.20), we have

$$G_1(U_1, V_2) \leq G_1(U_1, V_1) \leq G_1(U_2, V_1), \quad G_2(U_1, V_1) \leq G_2(U_2, V_2). \tag{3.21}$$

for $(U_i, V_i) \in K, i = 1, 2$, with $U_1 \leq U_2$ and $V_1 \leq V_2$.

Let

$$\mathcal{S} := \{(U, V) \in K \mid \underline{U}(\xi) \leq U(\xi) \leq \overline{U}(\xi), \ \underline{V}(\xi) \leq V(\xi) \leq \overline{V}(\xi) \text{ for all } \xi \in \mathbb{R}\}.$$

Also, we choose a $\alpha \in (0, \beta/c)$ and define the function space $B_\alpha(\mathbb{R}, \mathbb{R}^2)$ and the weighted norm $|\cdot|_\alpha$ of X by

$$B_\alpha(\mathbb{R}, \mathbb{R}^2) := \{\Phi \in X \,|\, |\Phi|_\alpha < \infty\}, \quad |\Phi|_\alpha := \sup_{\xi \in \mathbb{R}} \|\Phi(\xi)\| e^{-\alpha|\xi|}, \quad \Phi \in X.$$

Then $(B_\alpha(\mathbb{R}, \mathbb{R}^2), |\cdot|_\alpha)$ is a Banach space and $\mathcal{S}$ is a nonempty bounded closed convex set with respect to the weighted norm $|\cdot|_\alpha$.

The following lemma gives the existence of a positive solution of (1.5) if a pair of upper and lower solutions of (1.5) exists.

Lemma 3.4. *Let $c > 0$. Suppose that there exists a pair of upper and lower solutions $(\overline{U}, \overline{V})$ and $(\underline{U}, \underline{V})$ of (1.5) in K such that $\underline{U}(\xi) \leq \overline{U}(\xi)$ and $\underline{V}(\xi) \leq \overline{V}(\xi)$ for all $\xi \in \mathbb{R}$. Then G has a fixed point $(U, V) \in \mathcal{S}$ which is a solution of (1.5).*

Proof. First, we show that $G(\mathcal{S}) \subset \mathcal{S}$. Given any $(U, V) \in \mathcal{S}$, by (3.21), we know that

$$\begin{cases} G_1(\underline{U}, \overline{V})(\xi) \leq G_1(U, V)(\xi) \leq G_1(\overline{U}, \underline{V})(\xi), \ \xi \in \mathbb{R}, \\ G_2(\underline{U}, \underline{V})(\xi) \leq G_2(U, V)(\xi) \leq G_2(\overline{U}, \overline{V})(\xi), \ \xi \in \mathbb{R}. \end{cases}$$

Thus, we only need to show that

$$\begin{cases} \underline{U}(\xi) \le G_1(\underline{U}, \overline{V})(\xi) \le G_1(\overline{U}, \underline{V})(\xi) \le \overline{U}(\xi), \ \xi \in \mathbb{R}, \\ \underline{V}(\xi) \le G_2(\underline{U}, \underline{V})(\xi) \le G_2(\overline{U}, \overline{V})(\xi) \le \overline{V}(\xi), \ \xi \in \mathbb{R}. \end{cases}$$

By the definition of the upper and lower solutions, we obtain that

$$\begin{aligned} G_1(\underline{U}, \overline{V})(\xi) &= \frac{1}{c} e^{-\frac{\beta}{c}\xi} \int_{-\infty}^{\xi} e^{\frac{\beta}{c}s} H_1(\underline{U}, \overline{V})(s) ds \\ &\ge \frac{1}{c} e^{-\frac{\beta}{c}\xi} \int_{-\infty}^{\xi} e^{\frac{\beta}{c}s} (c\underline{U}' + \beta \underline{U})(s) ds \\ &= e^{-\frac{\beta}{c}\xi} \int_{-\infty}^{\xi} \left[e^{\frac{\beta}{c}s} \underline{U}'(s) + \frac{\beta}{c} e^{\frac{\beta}{c}s} \underline{U}(s) \right] ds \\ &= e^{-\frac{\beta}{c}\xi} \cdot e^{\frac{\beta}{c}\xi} \underline{U}(\xi) = \underline{U}(\xi). \end{aligned}$$

Similarly, the other inequalities hold. Hence we have $G(\mathcal{S}) \subset \mathcal{S}$.

By a similar argument as in [16], the operator $G : \mathcal{S} \to \mathcal{S}$ is completely continuous with respect to the weighted norm $|\cdot|_\alpha$. Then the lemma can be proved by using Schauder's fixed point theorem. □

By Lemmas 3.2-3.4, we have the following theorem.

Theorem 3.5. *For each $c \ge c_*$, there exists a positive solution (U, V) of (1.5) such that*

$$\underline{U}(\xi) \le U(\xi) \le \overline{U}(\xi), \quad \underline{V}(\xi) \le V(\xi) \le \overline{V}(\xi), \quad \forall \xi \in \mathbb{R},$$

where $(\overline{U}, \overline{V})$ and $(\underline{U}, \underline{V})$ are defined by (3.5)-(3.8) for $c > c_*$ and by (3.12)-(3.15) for $c = c_*$. Moreover, we have $(U, V)(-\infty) = (1, 0)$.

Note that, for $c = c_*$, we have $\underline{V}(\xi) \le \overline{V}(\xi)$ for all $\xi \in \mathbb{R}$, since $\xi_2 < -1/\lambda_2 - 1$, by (3.18). The other cases can be easily checked and hence Lemma 3.4 can be applied.

It remains to derive the tail behavior of the solution obtained in Theorem 3.5 at $\xi = \infty$. To do this, we recall that $[m(\theta), M(\theta)] \times [m(\theta), M(\theta)]$, $\theta \in [0, 1]$, where

$$m(\theta) := \theta \frac{1}{1+k}, \quad M(\theta) := \theta \frac{1}{1+k} + (1-\theta)(1+\varepsilon)$$

and ε is a small positive constant such that $k(1+\varepsilon) < 1$.

Theorem 3.6. *Let (U, V) be a positive solution obtained in Theorem 3.5. Then*

$$\lim_{\xi \to \infty} (U, V)(\xi) = \left(\frac{1}{1+k}, \frac{1}{1+k} \right). \tag{3.22}$$

Proof. The proof is similar to the one given in Theorem 2.5, we present the proof for the completeness.

Set

$$U^- := \liminf_{\xi\to\infty} U(\xi), \quad U^+ := \limsup_{\xi\to\infty} U(\xi),$$
$$V^- := \liminf_{\xi\to\infty} V(\xi), \quad V^+ := \limsup_{\xi\to\infty} V(\xi).$$

Since

$$\lim_{\xi\to\infty} \overline{U}(\xi) = 1, \ \lim_{\xi\to\infty} \underline{U}(\xi) = 1-k, \ \lim_{\xi\to\infty} \overline{V}(\xi) = 1, \ \lim_{\xi\to\infty} \underline{V}(\xi) = \delta,$$

we have

$$U^+ \le 1, \ U^- \ge 1-k, \quad V^+ \le 1, \ V^- \ge \delta.$$

Hence we obtain

$$m(0) = 0 < U^- \le U^+ < 1+\epsilon = M(0), \ m(0) = 0 < V^- \le V^+ < 1+\epsilon = M(0). \tag{3.23}$$

We claim that

$$m(\theta) < U^- \le U^+ < M(\theta), \quad m(\theta) < V^- \le V^+ < M(\theta), \tag{3.24}$$

for all $\theta \in [0,1)$. First we set $\theta_0 := \sup\{\theta \in [0,1) \,|\, (3.24) \text{ holds}\}$. Since (3.24) holds for $0 < \theta \ll 1$ due to (3.23), θ_0 is well-defined. By a contradiction argument, we assume that $\theta_0 < 1$. Then, by passing to the limit, we have

$$m(\theta_0) \le U^- \le U^+ \le M(\theta_0), \quad m(\theta_0) \le V^- \le V^+ \le M(\theta_0). \tag{3.25}$$

Hence at least one of the following equalities holds:

$$U^- = m(\theta_0), \ U^+ = M(\theta_0), \ V^- = m(\theta_0), \ V^+ = M(\theta_0).$$

Assume that $U^- = m(\theta_0)$. Then we can find a sequence $\{\xi_n\}$ of U with $\xi_n \to +\infty$ as $n \to +\infty$ such that $\lim_{n\to+\infty} U(\xi_n) = m(\theta_0)$ and $\lim_{n\to+\infty} U'(\xi_n) = 0$. Indeed, if U is eventually monotone, we know that $U(\infty)$ exists by the boundedness of U. Since $\int_0^\infty U'(s)ds = U(\infty) - U(0)$ is finite, either $\liminf_{s\to+\infty} U'(s) = 0$ if $U'(s) > 0$ for $s \gg 1$ or $\limsup_{s\to+\infty} U'(s) = 0$ if $U'(s) < 0$ for $s \gg 1$. This implies that the sequence $\{\xi_n\}$ can be found. If U is oscillatory at ∞, then the sequence $\{\xi_n\}$ can be chosen as the minimal points of U. By the definition of U^-, we have

$$\liminf_{n\to+\infty} U(\xi_n + 1) \ge U^-, \quad \liminf_{n\to+\infty} U(\xi_n - 1) \ge U^-.$$

Also, using (3.25) and $\limsup_{n\to+\infty} V(\xi_n) \le M(\theta_0)$, we have

$$\begin{aligned}\liminf_{n\to\infty}[1-U(\xi_n)-kV(\xi_n)] &\ge 1-\frac{\theta_0}{1+k}-k\left[\frac{\theta_0}{1+k}+(1-\theta_0)(1+\epsilon)\right]\\ &= (1-\theta_0)[1-k(1+\epsilon)]>0.\end{aligned}$$

This implies that

$$\begin{aligned}0 &= \liminf_{n\to\infty}\{[U(\xi_n+1)+U(\xi_n-1)-2U(\xi_n)]-cU'(\xi_n)+rU(\xi_n)[1-U(\xi_n)-kV(\xi_n)]\}\\ &\ge (U^-+U^--2U^-)-c\cdot 0+rU^-\cdot\liminf_{n\to\infty}[1-U(\xi_n)-kV(\xi_n)]>0,\end{aligned}$$

a contradiction. Similarly, we can show that $U^+ = M(\theta_0)$ does not hold.

For the other cases $V^- = m(\theta_0)$ and $V^+ = M(\theta_0)$, we may assume that $m(\theta_0) < U^- \le U^+ < M(\theta_0)$. Then they can be treated by the similar argument as previous cases. Therefore, (3.24) holds for all $\theta < 1$ and (3.22) follows by taking $\theta \to 1$. Hence the theorem is proved. □

3.3. Determination of the minimal speed

First, we have the following lemma.

Lemma 3.7. *If (c, U, V) is a solution of (1.5)-(1.6) such that $U, V \ge 0$, then U, V are positive on $\mathbb{R}$ and $c > 0$.*

Proof. Although the proof is similar to the one given in [11], we provide the details here for the completeness. First, we claim that $U > 0$ in $\mathbb{R}$. Otherwise, due to $U(+\infty) = 1/(1+k)$, we can find ξ_0 such that $U(\xi_0) = 0$ and $U(\xi) > 0$ for all $\xi > \xi_0$. On the other hand, $U'(\xi_0) = 0$ due to $U \ge 0$. By the first equation of (1.5), we obtain that $U(\xi_0+1) = U(\xi_0-1) = 0$ and get a contradiction. So $U(\xi) > 0$ for all $\xi \in \mathbb{R}$. Similarly, we also have $V(\xi) > 0$ for all $\xi \in \mathbb{R}$.

Now we show that $c > 0$. Since $\lim_{\xi\to-\infty}(U, V)(\xi) = (1, 0)$, there exists a sufficiently large N such that for any $\xi \le -N$

$$1-\frac{V(\xi)}{U(\xi)} \ge \frac{1}{2}.$$

Then we integrate the second equation of (1.5) from $-\infty$ to $\xi < -N$ and derive that

$$cV(\xi) = d\left[\int_{\xi}^{\xi+1} V(\theta)d\theta - \int_{\xi-1}^{\xi} V(\theta)d\theta\right] + \int_{-\infty}^{\xi} sV(\theta)\left[1-\frac{V(\theta)}{U(\theta)}\right]d\theta. \tag{3.26}$$

Thus, we obtain that

$$\begin{aligned}|c|+2d &\ge cV(\xi)-d\left[\int_{\xi}^{\xi+1}V(\theta)d\theta-\int_{\xi-1}^{\xi}V(\theta)d\theta\right]\\ &= \int_{-\infty}^{\xi} sV(\theta)\left[1-\frac{V(\theta)}{U(\theta)}\right]d\theta \ge \frac{s}{2}\int_{-\infty}^{\xi}V(\theta)d\theta.\end{aligned}$$

This shows us that $I(\xi) := \int_{-\infty}^{\xi} V(\theta)d\theta$ is well-defined for all $\xi < -N$ and $I(\xi)$ is increasing in $(-\infty, -N)$. Now, we integrate (3.26) from $-\infty$ to $\eta < -N$ and obtain that

$$cI(\eta) = d\left[\int_{\eta}^{\eta+1} I(\xi)d\xi - \int_{\eta-1}^{\eta} I(\xi)d\xi\right] + \int_{-\infty}^{\eta}\int_{-\infty}^{\xi} sV(\theta)\left[1 - \frac{V(\theta)}{U(\theta)}\right]d\theta d\xi > 0.$$

This implies that $c > 0$. Hence the lemma follows. □

The next theorem provides a lower bound for the admissible wave speeds.

Theorem 3.8. *Let* (c, U, V) *be a solution of* (1.5)-(1.6) *such that* $U, V \geq 0$. *Then* $c \geq c_*$, *where* c_* *is given in* (1.7).

Proof. Suppose that (c, U, V) is a positive solution of (1.5)-(1.6). Set

$$z(\xi) = V'(\xi)/V(\xi), \quad B(\xi) = s[1 - V(\xi)/U(\xi)] - 2d.$$

Then we can derive from the second equation of (1.5) that

$$cz(\xi) = d\left[e^{\int_{\xi}^{\xi+1} z(s)ds} + e^{-\int_{\xi-1}^{\xi} z(s)ds}\right] + B(\xi).$$

It follows from [3, Theorem 4] (see also [2]) that the limit $\omega := \lim_{\xi\to-\infty} z(\xi)$ exists and it satisfies

$$c\omega = d(e^{\omega} + e^{-\omega} - 2) + s.$$

Hence $c \geq c_*$, by the definition of c_*, and the theorem is proved. □

From the results shown in Theorems 3.5, 3.6 and 3.8, $c = c_*$ is the minimal speed of the traveling wave solution of (1.5)-(1.6).

4. Verifications of upper and lower solutions

In this section, we provide the details of verifications of all upper and lower solutions constructed in the previous two sections.

Proof of Lemma 2.1. First, we claim that

$$\overline{\phi_1}''(z) - c\overline{\phi_1}'(z) + r\overline{\phi_1}(z)[1 - \overline{\phi_1}(z) - k\underline{\phi_2}(z)] \leq 0$$

holds for $z \in \mathbb{R}\backslash\{0\}$. For $z > 0$, we have $\overline{\phi_1}(z) = 1$ and

$$\overline{\phi_1}''(z) - c\overline{\phi_1}'(z) + r\overline{\phi_1}(z)[1 - \overline{\phi_1}(z) - k\underline{\phi_2}(z)] = -rk\underline{\phi_2}(z) \leq 0.$$

When $z_2 \leq z < 0$, we have $\overline{\phi_1}(z) = 1 - \epsilon(e^{\lambda_1 z} - e^{\nu\lambda_1 z})$ and so

$$\begin{aligned}
&\overline{\phi_1}''(z) - c\overline{\phi_1}'(z) + r\overline{\phi_1}(z)[1 - \overline{\phi_1}(z) - k\underline{\phi_2}(z)] \\
= \ & \epsilon[(\nu\lambda_1)^2 - c(\nu\lambda_1) - r]e^{\nu\lambda_1 z} - r\epsilon^2(e^{\lambda_1 z} - e^{\nu\lambda_1 z})^2 - rk[1 - \epsilon(e^{\lambda_1 z} - e^{\nu\lambda_1 z})]\delta \\
\leq \ & \epsilon[(\nu\lambda_1)^2 - c(\nu\lambda_1) - r]e^{\nu\lambda_1 z} + \epsilon rke^{\lambda_1 z}\delta - rk\delta \\
\leq \ & \epsilon\left\{[(\nu\lambda_1)^2 - c(\nu\lambda_1) - r] + rk\delta\right\} - rk\delta \leq 0,
\end{aligned}$$

by the choices of ν and ϵ. If $z < z_2$, we have

$$\overline{\phi_1}(z) = 1 - \epsilon(e^{\lambda_1 z} - e^{\nu\lambda_1 z}), \quad \underline{\phi_2}(z) = e^{\lambda_2 z} - q e^{\mu\lambda_2 z}$$

and so

$$\begin{aligned}
& \overline{\phi_1}''(z) - c\overline{\phi_1}'(z) + r\overline{\phi_1}(z)[1 - \overline{\phi_1}(z) - k\underline{\phi_2}(z)] \\
= \ & \epsilon[(\nu\lambda_1)^2 - c(\nu\lambda_1) - r]e^{\nu\lambda_1 z} - r\epsilon^2(e^{\lambda_1 z} - e^{\nu\lambda_1 z})^2 \\
& -rk[1 - \epsilon(e^{\lambda_1 z} - e^{\nu\lambda_1 z})](e^{\lambda_2 z} - q e^{\mu\lambda_2 z}) \\
\leq \ & e^{\lambda_2 z}\{\epsilon[(\nu\lambda_1)^2 - c(\nu\lambda_1) - r]e^{(\nu\lambda_1 - \lambda_2)z} \\
& +rk\epsilon(e^{\lambda_1 z} - e^{\nu\lambda_1 z})(1 - q e^{(\mu-1)\lambda_2 z}) - rk(1 - q e^{(\mu-1)\lambda_2 z})\} \\
\leq \ & e^{\lambda_2 z}\{\epsilon[(\nu\lambda_1)^2 - c(\nu\lambda_1) - r + rk] - rk(1 - q e^{(\mu-1)\lambda_2 z_2})\} \leq 0,
\end{aligned}$$

by the assumptions (A1) and (A3).

Next, for $z \neq z_1$, we would like to show that

$$\underline{\phi_1}''(z) - c\underline{\phi_1}'(z) + r\underline{\phi_1}(z)[1 - \underline{\phi_1}(z) - k\overline{\phi_2}(z)] \geq 0.$$

When $z > z_1$, we have $\underline{\phi_1}(z) = 1 - k$ and

$$\underline{\phi_1}''(z) - c\underline{\phi_1}'(z) + r\underline{\phi_1}(z)[1 - \underline{\phi_1}(z) - k\overline{\phi_2}(z)] = r(1-k)k(1 - \overline{\phi_2}(z)) \geq 0$$

by $\overline{\phi_2}(z) \leq 1$. Otherwise, for $z < z_1$, we have $\underline{\phi_1}(z) = 1 - k(e^{\lambda_1 z} + p e^{\eta\lambda_1 z})$, $\overline{\phi_2}(z) = e^{\lambda_2 z}$ and

$$\begin{aligned}
& \underline{\phi_1}''(z) - c\underline{\phi_1}'(z) + r\underline{\phi_1}(z)[1 - \underline{\phi_1}(z) - k\overline{\phi_2}(z)] \\
= \ & -k[\lambda_1^2 e^{\lambda_1 z} + p(\eta\lambda_1)^2 e^{\eta\lambda_1 z}] + ck[\lambda_1 e^{\lambda_1 z} + p(\eta\lambda_1)e^{\eta\lambda_1 z}] \\
& +r[1 - k(e^{\lambda_1 z} + p e^{\eta\lambda_1 z})][k(e^{\lambda_1 z} + p e^{\eta\lambda_1 z}) - k e^{\lambda_2 z}] \\
= \ & k\{-(\lambda_1^2 - c\lambda_1 - r)e^{\lambda_1 z} - p[(\eta\lambda_1)^2 - c(\eta\lambda_1) - r]e^{\eta\lambda_1 z}\} \\
& -rk^2(e^{\lambda_1 z} + p e^{\eta\lambda_1 z})^2 - rk e^{\lambda_2 z} + rk^2(e^{\lambda_1 z} + p e^{\eta\lambda_1 z})e^{\lambda_2 z} \\
\geq \ & k\{-p[(\eta\lambda_1)^2 - c(\eta\lambda_1) - r]e^{\eta\lambda_1 z}\} - rk^2(e^{\lambda_1 z} + p e^{\eta\lambda_1 z}) - rk e^{\lambda_2 z} \\
\geq \ & k e^{\eta\lambda_1 z}\{-p[(\eta\lambda_1)^2 - c(\eta\lambda_1) - r + rk] - rk e^{(1-\eta)\lambda_1 z} - r e^{(\lambda_2 - \eta\lambda_1)z}\} \\
\geq \ & k e^{\eta\lambda_1 z}\{-p[(\eta\lambda_1)^2 - c(\eta\lambda_1) - r + rk] - r(1+k)\} \geq 0,
\end{aligned}$$

by the choices of p and η.

Now we show that

$$d\overline{\phi_2}''(z) - c\overline{\phi_2}'(z) + s\overline{\phi_2}(z)\left[1 - \frac{\overline{\phi_2}(z)}{\overline{\phi_1}(z)}\right] \leq 0$$

for $z \neq 0$. In the case $z > 0$, we have $\overline{\phi_1}(z) = 1$ and $\overline{\phi_2}(z) = 1$ and

$$d\overline{\phi_2}''(z) - c\overline{\phi_2}'(z) + s\overline{\phi_2}(z)\left[1 - \frac{\overline{\phi_2}(z)}{\overline{\phi_1}(z)}\right] = 0.$$

For $z<0$, we have $\overline{\phi_2}(z)=e^{\lambda_2 z}$ and

$$d\overline{\phi_2}''(z)-c\overline{\phi_2}'(z)+s\overline{\phi_2}(z)\left[1-\frac{\overline{\phi_2}(z)}{\underline{\phi_1}(z)}\right]=-s\frac{\overline{\phi_2}^2(z)}{\underline{\phi_1}(z)}\leq 0.$$

Finally, we prove that

$$d\underline{\phi_2}''(z)-c\underline{\phi_2}'(z)+s\underline{\phi_2}(z)\left[1-\frac{\underline{\phi_2}(z)}{\underline{\phi_1}(z)}\right]\geq 0$$

holds for $z\neq z_2$. If $z>z_2$, we know that $\underline{\phi_2}(z)=\delta$. Then we obtain that

$$d\underline{\phi_2}''(z)-c\underline{\phi_2}'(z)+s\underline{\phi_2}(z)\left[1-\frac{\underline{\phi_2}(z)}{\underline{\phi_1}(z)}\right]\geq s\delta\left[1-\frac{\delta}{(1-k)}\right]>0$$

by $\delta<1-k$. On the other hand, if $z<z_2$, we have $\underline{\phi_2}(z)=e^{\lambda_2 z}-qe^{\mu\lambda_2 z}$ and so

$$\begin{aligned}
& d\underline{\phi_2}''(z)-c\underline{\phi_2}'(z)+s\underline{\phi_2}(z)\left[1-\frac{\underline{\phi_2}(z)}{\underline{\phi_1}(z)}\right]\\
\geq\ & d\underline{\phi_2}''(z)-c\underline{\phi_2}'(z)+s\underline{\phi_2}(z)\left[1-\frac{\underline{\phi_2}(z)}{1-k}\right]\\
=\ & d[\lambda_2^2e^{\lambda_2 z}-q(\mu\lambda_2)^2e^{\mu\lambda_2 z}]-c[\lambda_2e^{\lambda_2 z}-q(\mu\lambda_2)e^{\mu\lambda_2 z}]\\
& +s(e^{\lambda_2 z}-qe^{\mu\lambda_2 z})-\frac{s}{1-k}(e^{\lambda_2 z}-qe^{\mu\lambda_2 z})^2\\
\geq\ & e^{\mu\lambda_2 z}\left\{-q[d(\mu\lambda_2)^2-c(\mu\lambda_2)+s]-\frac{s}{1-k}e^{(2-\mu)\lambda_2 z}\right\}\\
\geq\ & e^{\mu\lambda_2 z}\left\{-q[d(\mu\lambda_2)^2-c(\mu\lambda_2)+s]-\frac{s}{1-k}\right\}\geq 0,
\end{aligned}$$

by the fact $\underline{\phi_1}(z)\geq 1-k$ and the choices of q and μ. Therefore, the proof of this lemma has been completed. □

Proof of Lemma 2.2. First, we claim that

$$\overline{\phi_1}''(z)-c\overline{\phi_1}'(z)+r\overline{\phi_1}(z)[1-\overline{\phi_1}(z)-k\underline{\phi_2}(z)]\leq 0 \tag{4.1}$$

holds for $z\in\mathbb{R}\backslash\{0\}$. For $z>0$ and $z_2\leq z<0$, the inequality (4.1) holds by a similar argument as in the case $c>c^*$, using

$$\nu>1,\quad \epsilon<\frac{rk\delta}{[(\nu\lambda_1)^2-c(\nu\lambda_1)-r]+rk\delta}.$$

For $z<z_2$, we have $\overline{\phi_1}(z)=1-\epsilon(e^{\lambda_1 z}-e^{\nu\lambda_1 z})$ and $\underline{\phi_2}(z)=[-hz-q(-z)^{1/2}]e^{\lambda_2 z}$. Then

$$\begin{aligned}
& \overline{\phi_1}''(z)-c\overline{\phi_1}'(z)+r\overline{\phi_1}(z)[1-\overline{\phi_1}(z)-k\underline{\phi_2}(z)]\\
=\ & \epsilon[(\nu\lambda_1)^2-c(\nu\lambda_1)-r]e^{\nu\lambda_1 z}-r\epsilon^2(e^{\lambda_1 z}-e^{\nu\lambda_1 z})^2\\
& -rk[1-\epsilon(e^{\lambda_1 z}-e^{\nu\lambda_1 z})][-hz-q(-z)^{1/2}]e^{\lambda_2 z}\\
\leq\ & e^{\lambda_2 z}\{\epsilon[(\nu\lambda_1)^2-c(\nu\lambda_1)-r]e^{(\nu\lambda_1-\lambda_2)z}-rk[1-\epsilon(e^{\lambda_1 z}-e^{\nu\lambda_1 z})][-hz-q(-z)^{1/2}]\}\\
\leq\ & e^{\lambda_2 z}\left\{\epsilon[(\nu\lambda_1)^2-c(\nu\lambda_1)-r]-rk(1-k)[-hz-q(-z)^{1/2}]\right\}\\
\leq\ & e^{\lambda_2 z}\left\{\epsilon[(\nu\lambda_1)^2-c(\nu\lambda_1)-r]-rk(1-k)[-hz_2-q(-z_2)^{1/2}]\right\}\leq 0
\end{aligned}$$

for all $z < z_2$, by using (2.10). Hence (4.1) holds for all $z \in \mathbb{R}\backslash\{0\}$.

Next, for $z \neq z_1$, we would like to show that

$$\underline{\phi_1}''(z) - c\underline{\phi_1}'(z) + r\underline{\phi_1}(z)[1 - \underline{\phi_1}(z) - k\overline{\phi_2}(z)] \geq 0.$$

When $z > z_1$, we have $\underline{\phi_1}(z) = 1 - k$ and so

$$\underline{\phi_1}''(z) - c\underline{\phi_1}'(z) + r\underline{\phi_1}(z)[1 - \underline{\phi_1}(z) - k\overline{\phi_2}(z)] = r(1-k)k(1 - \overline{\phi_2}(z)) \geq 0,$$

by $\overline{\phi_2}(z) \leq 1$. For $z < z_1 < -2/\lambda_2$, we have $\underline{\phi_1}(z) = 1 - k(e^{\lambda_1 z} + pe^{\eta\lambda_1 z})$ and $\overline{\phi_2}(z) = -hze^{\lambda_2 z}$. Hence we obtain

$$\begin{aligned}
& \underline{\phi_1}''(z) - c\underline{\phi_1}'(z) + r\underline{\phi_1}(z)[1 - \underline{\phi_1}(z) - k\overline{\phi_2}(z)] \\
= \;& -k[\lambda_1^2 e^{\lambda_1 z} + p(\eta\lambda_1)^2 e^{\eta\lambda_1 z}] + ck[\lambda_1 e^{\lambda_1 z} + p(\eta\lambda_1)e^{\eta\lambda_1 z}] \\
& \quad + r[1 - k(e^{\lambda_1 z} + pe^{\eta\lambda_1 z})][k(e^{\lambda_1 z} + pe^{\eta\lambda_1 z}) - k(-hze^{\lambda_2 z})] \\
\geq \;& k\{-(\lambda_1^2 - c\lambda_1 - r)e^{\lambda_1 z} - p[(\eta\lambda_1)^2 - c(\eta\lambda_1) - r]e^{\eta\lambda_1 z}\} \\
& \quad - rk^2(e^{\lambda_1 z} + pe^{\eta\lambda_1 z})^2 - rk(-hze^{\lambda_2 z}) \\
\geq \;& k\{-p[(\eta\lambda_1)^2 - c(\eta\lambda_1) - r]e^{\eta\lambda_1 z}\} - rk^2(e^{\lambda_1 z} + pe^{\eta\lambda_1 z}) - rk(-hze^{\lambda_2 z}) \\
= \;& ke^{\eta\lambda_1 z}\{-p[(\eta\lambda_1)^2 - c(\eta\lambda_1) - r + rk] - rke^{(1-\eta)\lambda_1 z} - r(-hz)e^{(\lambda_2 - \eta\lambda_1)z}\} \\
\geq \;& ke^{\eta\lambda_1 z}\{-p[(\eta\lambda_1)^2 - c(\eta\lambda_1) - r + rk] - r[k + h/(\eta\lambda_1 e)]\} \geq 0
\end{aligned}$$

for all $z < z_1$, by using (2.7) and (2.8). Here we have also used the fact that

$$-hze^{(\lambda_2 - \eta\lambda_1)z} \leq -hze^{\eta\lambda_1 z} \leq h/(\eta\lambda_1 e) \quad \text{for all } z \leq 0.$$

Now we show that

$$d\overline{\phi_2}''(z) - c\overline{\phi_2}'(z) + s\overline{\phi_2}(z)\left[1 - \frac{\overline{\phi_2}(z)}{\overline{\phi_1}(z)}\right] \leq 0$$

for $z \neq -2/\lambda_2$. In the case $z > -2/\lambda_2$, we have $\overline{\phi_2}(z) = 1$ and

$$d\overline{\phi_2}''(z) - c\overline{\phi_2}'(z) + s\overline{\phi_2}(z)\left[1 - \frac{\overline{\phi_2}(z)}{\overline{\phi_1}(z)}\right] = s\left[1 - \frac{1}{\overline{\phi_1}(z)}\right] \leq 0$$

from $\overline{\phi_1}(z) \leq 1$. For $z < -2/\lambda_2$, we have $\overline{\phi_2}(z) = -hze^{\lambda_2 z}$ and

$$d\overline{\phi_2}''(z) - c\overline{\phi_2}'(z) + s\overline{\phi_2}(z)\left[1 - \frac{\overline{\phi_2}(z)}{\overline{\phi_1}(z)}\right] = -s\frac{\overline{\phi_2}^2(z)}{\overline{\phi_1}(z)} \leq 0.$$

Finally, we prove that

$$d\underline{\phi_2}''(z) - c\underline{\phi_2}'(z) + s\underline{\phi_2}(z)\left[1 - \frac{\underline{\phi_2}(z)}{\underline{\phi_1}(z)}\right] \geq 0 \tag{4.2}$$

for all $z \neq z_2$. For $z > z_2$, (4.2) holds by a similar argument as in the case $c > c^*$. When $z < z_2$, we have $\underline{\phi_2}(z) = [-hz - q(-z)^{1/2}]e^{\lambda_2 z}$ and

$$\begin{aligned}
& d\underline{\phi_2}''(z) - c\underline{\phi_2}'(z) + s\underline{\phi_2}(z)\left[1 - \frac{\underline{\phi_2}(z)}{\underline{\phi_1}(z)}\right] \\
\geq\ & d\underline{\phi_2}''(z) - c\underline{\phi_2}'(z) + s\underline{\phi_2}(z)\left[1 - \frac{\underline{\phi_2}(z)}{1-k}\right] \\
=\ & \frac{dq}{4}(-z)^{-3/2}e^{\lambda_2 z} - \frac{s}{1-k}[-hz - q(-z)^{1/2}]^2 e^{2\lambda_2 z} \\
=\ & (-z)^{-3/2}e^{\lambda_2 z}\left\{\frac{dq}{4} - \frac{s(-z)^{3/2}[-hz - q(-z)^{1/2}]^2 e^{\lambda_2 z}}{1-k}\right\} \\
\geq\ & (-z)^{-3/2}e^{\lambda_2 z}\left\{\frac{dq}{4} - \frac{sh^2}{1-k}(-z)^{7/2}e^{\lambda_2 z}\right\} \\
\geq\ & (-z)^{-3/2}e^{\lambda_2 z}\left\{\frac{dq}{4} - \frac{sh^2}{1-k}\left(\frac{7}{2e\lambda_2}\right)^{7/2}\right\} \geq 0,
\end{aligned}$$

by the facts $\underline{\phi_1}(z) \geq 1-k$, $\lambda_2 = c/(2d)$ and the choice of q in (2.9). Here we have used the fact that

$$(-z)^{7/2}e^{\lambda_2 z} \leq \left(\frac{7}{2e\lambda_2}\right)^{7/2} \quad \text{for all } z \leq 0.$$

Therefore, the proof of the lemma is completed. □

Proof Lemma 3.2. To prove the lemma, we note that

$$\begin{aligned}
&\overline{U}(\xi) \leq 1, \quad \overline{U}(\xi) \leq 1 - \epsilon(e^{\lambda_1 \xi} - e^{\nu\lambda_1 \xi}), \\
&\underline{U}(\xi) \geq 1-k, \quad \underline{U}(\xi) \geq 1 - k(e^{\lambda_1 \xi} + pe^{\eta\lambda_1 \xi}), \\
&\overline{V}(\xi) \leq 1, \quad \overline{V}(\xi) \leq e^{\lambda_2 \xi}, \quad \underline{V}(\xi) \geq e^{\lambda_2 \xi} - qe^{\mu\lambda_2 \xi}
\end{aligned}$$

for all $\xi \in \mathbb{R}$, and $\underline{V}(\xi) \geq \delta$ for $\xi \geq \xi_2 - 1$ due to Lemma 3.1.

First, we show that (3.1) holds for all $\xi \neq 0$. It is trivial that (3.1) holds for $\xi > 0$, since $\overline{U}(\xi+1) = 1$, $\overline{U}(\xi-1) \leq 1$ and $\overline{U}(\xi) = 1$. For $\xi_2 \leq \xi < 0$, we have

$$\begin{aligned}
& [\overline{U}(\xi+1) + \overline{U}(\xi-1) - 2\overline{U}(\xi)] - c\overline{U}'(\xi) + r\overline{U}(\xi)[1 - \overline{U}(\xi) - k\underline{V}(\xi)] \\
\leq\ & 1 - \epsilon(e^{\lambda_1(\xi+1)} - e^{\nu\lambda_1(\xi+1)}) + 1 - \epsilon(e^{\lambda_1(\xi-1)} - e^{\nu\lambda_1(\xi-1)}) - 2(1 - \epsilon(e^{\lambda_1\xi} - e^{\nu\lambda_1\xi})) \\
& -c(-\epsilon\lambda_1 e^{\lambda_1\xi} + \epsilon\nu\lambda_1 e^{\nu\lambda_1\xi}) + r(\epsilon e^{\lambda_1\xi} - \epsilon e^{\nu\lambda_1\xi}) \\
& -r\epsilon^2(e^{\lambda_1\xi} - e^{\nu\lambda_1\xi})^2 - rk\delta[1 - \epsilon(e^{\lambda_1\xi} - e^{\nu\lambda_1\xi})] \\
\leq\ & \epsilon[e^{\nu\lambda_1} + e^{-\nu\lambda_1} - 2 - c(\nu\lambda_1) - r]e^{\nu\lambda_1\xi} + \epsilon rk\delta e^{\lambda_1\xi} - rk\delta \\
\leq\ & \epsilon[e^{\nu\lambda_1} + e^{-\nu\lambda_1} - 2 - c(\nu\lambda_1) - r + rk\delta] - rk\delta \leq 0,
\end{aligned}$$

by the choices of ϵ and ν. If $\xi < \xi_2$, we have

$$\begin{aligned}
& [\overline{U}(\xi+1)+\overline{U}(\xi-1)-2\overline{U}(\xi)]-c\overline{U'}(\xi)+r\overline{U}(\xi)[1-\overline{U}(\xi)-k\underline{V}(\xi)] \\
\leq\ & \epsilon[e^{\nu\lambda_1}+e^{-\nu\lambda_1}-2-c(\nu\lambda_1)-r]e^{\nu\lambda_1\xi}-rk[1-\epsilon(e^{\lambda_1\xi}-e^{\nu\lambda_1\xi})][e^{\lambda_2\xi}-qe^{\mu\lambda_2\xi}] \\
\leq\ & e^{\lambda_2\xi}\{\epsilon[e^{\nu\lambda_1}+e^{-\nu\lambda_1}-2-c(\nu\lambda_1)-r]e^{(\nu\lambda_1-\lambda_2)\xi} \\
& +rk\epsilon(e^{\lambda_1\xi}-e^{\nu\lambda_1\xi})(1-qe^{(\mu-1)\lambda_2\xi})-rk(1-qe^{(\mu-1)\lambda_2\xi})\} \\
\leq\ & e^{\lambda_2\xi}\{\epsilon[e^{\nu\lambda_1}+e^{-\nu\lambda_1}-2-c(\nu\lambda_1)-r+rk]-rk(1-qe^{(\mu-1)\lambda_2\xi_2})\}\leq 0,
\end{aligned}$$

by the assumptions (C1) and (C3). Hence (3.1) holds for all $\xi \neq 0$.

Secondly, we claim that (3.2) holds for $\xi \neq \xi_1$. The case when $\xi > \xi_1$ is trivial. For $\xi < \xi_1$, we compute

$$\begin{aligned}
& [\underline{U}(\xi+1)+\underline{U}(\xi-1)-2\underline{U}(\xi)]-c\underline{U}'(\xi)+r\underline{U}(\xi)[1-\underline{U}(\xi)-k\overline{V}(\xi)] \\
\geq\ & 1-k(e^{\lambda_1(\xi+1)}+pe^{\eta\lambda_1(\xi+1)})+1-k(e^{\lambda_1(\xi-1)}+pe^{\eta\lambda_1(\xi-1)}) \\
& -2[1-k(e^{\lambda_1\xi}+pe^{\eta\lambda_1\xi})]-c[-k(\lambda_1e^{\lambda_1\xi}+p(\eta\lambda_1)e^{\eta\lambda_1\xi})] \\
& +rk[1-k(e^{\lambda_1\xi}+pe^{\eta\lambda_1\xi})][e^{\lambda_1\xi}+pe^{\eta\lambda_1\xi}-e^{\lambda_2\xi}] \\
=\ & k\{-p[e^{\eta\lambda_1}+e^{-\eta\lambda_1}-2-c(\eta\lambda_1)-r]e^{\eta\lambda_1\xi}-rk(e^{\lambda_1\xi}+pe^{\eta\lambda_1\xi})^2 \\
& -re^{\lambda_2\xi}+rk(e^{\lambda_1\xi}+pe^{\eta\lambda_1\xi})e^{\lambda_2\xi}\} \\
\geq\ & ke^{\eta\lambda_1\xi}\{-p[e^{\eta\lambda_1}+e^{-\eta\lambda_1}-2-c(\eta\lambda_1)-r+rk]-rke^{(1-\eta)\lambda_1\xi} \\
& -re^{(\lambda_2-\eta\lambda_1)\xi}\} \\
\geq\ & ke^{\eta\lambda_1\xi}\{-p[e^{\eta\lambda_1}+e^{-\eta\lambda_1}-2-c(\eta\lambda_1)-r+rk]-r(1+k)\}\geq 0,
\end{aligned}$$

by the choices of p and η. Hence (3.2) holds for $\xi \neq \xi_1$.

Thirdly, since it is trivial that (3.3) holds for all $\xi \neq 0$, we omit the details.

Finally, we prove that (3.4) holds for all $\xi \neq \xi_2$. For $\xi > \xi_2$, we have

$$\begin{aligned}
& d[\underline{V}(\xi+1)+\underline{V}(\xi-1)-2\underline{V}(\xi)]-c\underline{V}'(\xi)+s\underline{V}(\xi)\left(1-\frac{\underline{V}(\xi)}{\underline{U}(\xi)}\right) \\
\geq\ & s\delta\left[1-\frac{\delta}{1-k}\right]>0,
\end{aligned}$$

since $\underline{V}(\xi+1)=\underline{V}(\xi)=\delta$, $\underline{V}(\xi-1)\geq\delta$, and $\delta<1-k$. For $\xi<\xi_2$, we have

$$\begin{aligned}
& d[\underline{V}(\xi+1)+\underline{V}(\xi-1)-2\underline{V}(\xi)]-c\underline{V}'(\xi)+s\underline{V}(\xi)\left(1-\frac{\underline{V}(\xi)}{\underline{U}(\xi)}\right) \\
\geq\ & d[e^{\lambda_2(\xi+1)}-qe^{\mu\lambda_2(\xi+1)}+e^{\lambda_2(\xi-1)}-qe^{\mu\lambda_2(\xi-1)}-2e^{\lambda_2\xi}+2qe^{\mu\lambda_2\xi}] \\
& -c[\lambda_2e^{\lambda_2\xi}-q(\mu\lambda_2)e^{\mu\lambda_2\xi}]+s[e^{\lambda_2\xi}-qe^{\mu\lambda_2\xi}]\left[1-\frac{1}{1-k}(e^{\lambda_2\xi}-qe^{\mu\lambda_2\xi})\right] \\
\geq\ & e^{\mu\lambda_2\xi}\left\{-q[d(e^{\mu\lambda_2}+e^{-\mu\lambda_2}-2)-c(\mu\lambda_2)+s]-\frac{s}{1-k}e^{(2-\mu)\lambda_2\xi}\right\} \\
\geq\ & e^{\mu\lambda_2\xi}\left\{-q[d(e^{\mu\lambda_2}+e^{-\mu\lambda_2}-2)-c(\mu\lambda_2)+s]-\frac{s}{1-k}\right\}\geq 0
\end{aligned}$$

by the fact $\underline{U}(\xi) \geq 1-k$ and the choice of q. Hence (3.4) holds for all $\xi \neq \xi_2$ and the lemma is proved. □

Proof of Lemma 3.3. Later, we will use the following facts.

$$\begin{aligned}
&\overline{U}(\xi) \leq 1, \quad \overline{U}(\xi) \leq 1-\epsilon(e^{\lambda_1\xi}-e^{\nu\lambda_1\xi}) \quad \text{for} \ \ \xi \in \mathbb{R},\\
&\underline{U}(\xi) \geq 1-k, \quad \underline{U}(\xi) \geq 1-k(e^{\lambda_1\xi}+pe^{\eta\lambda_1\xi}) \quad \text{for} \ \ \xi \in \mathbb{R},\\
&\overline{V}(\xi) \leq 1 \quad \text{for} \ \ \xi \in \mathbb{R}, \quad \overline{V}(\xi) \leq -h\xi e^{\lambda_2\xi} \quad \text{for} \ \ \xi \leq -1/\lambda_2,\\
&\underline{V}(\xi) \geq \delta \quad \text{for} \ \ \xi \geq \xi_2-1, \quad \underline{V}(\xi) \geq [-h\xi-q(-\xi)^{1/2}]e^{\lambda_2\xi} \quad \text{for} \ \ \xi \leq \xi_2+1.
\end{aligned}$$

First, we claim that (3.1) holds for $\xi \in \mathbb{R}\setminus\{0\}$. For $\xi > 0$ and $\xi_2 \leq \xi < 0$, the inequality (3.1) holds by a similar argument as in the case $c > c_*$, using

$$\nu > 1, \quad \epsilon < \frac{rk\delta}{[e^{\nu\lambda_1}+e^{-\nu\lambda_1}-2-c(\nu\lambda_1)-r]+rk\delta}.$$

For $\xi < \xi_2$, we have $\overline{U}(\xi) = 1-\epsilon(e^{\lambda_1\xi}-e^{\nu\lambda_1\xi})$ and $\underline{V}(\xi) = [-h\xi-q(-\xi)^{1/2}]e^{\lambda_2\xi}$. Then

$$\begin{aligned}
&\overline{U}(\xi+1)+\overline{U}(\xi-1)-2\overline{U}(\xi)-c\overline{U}'(\xi)+r\overline{U}(\xi)(1-\overline{U}(\xi)-k\underline{V}(\xi))\\
\leq\ & \epsilon[e^{\nu\lambda_1}+e^{-\nu\lambda_1}-2-c(\nu\lambda_1)-r]e^{\nu\lambda_1\xi}-r\epsilon^2(e^{\lambda_1\xi}-e^{\nu\lambda_1\xi})^2\\
&-rk[1-\epsilon(e^{\lambda_1\xi}-e^{\nu\lambda_1\xi})][-h\xi-q(-\xi)^{1/2}]e^{\lambda_2\xi}\\
\leq\ & e^{\lambda_2\xi}\{\epsilon[e^{\nu\lambda_1}+e^{-\nu\lambda_1}-2-c(\nu\lambda_1)-r]e^{(\nu\lambda_1-\lambda_2)\xi}\\
&-rk[1-\epsilon(e^{\lambda_1\xi}-e^{\nu\lambda_1\xi})][-h\xi-q(-\xi)^{1/2}]\}\\
\leq\ & e^{\lambda_2\xi}\left\{\epsilon[e^{\nu\lambda_1}+e^{-\nu\lambda_1}-2-c(\nu\lambda_1)-r]-rk(1-k)[-h\xi-q(-\xi)^{1/2}]\right\}\\
\leq\ & e^{\lambda_2\xi}\left\{\epsilon[e^{\nu\lambda_1}+e^{-\nu\lambda_1}-2-c(\nu\lambda_1)-r]-rk(1-k)[-h\xi_2-q(-\xi_2)^{1/2}]\right\} \leq 0
\end{aligned}$$

for all $\xi < \xi_2$, by using (3.19). Hence (3.1) holds for all $\xi \in \mathbb{R}\setminus\{0\}$.

Next, we check (3.2) for $\xi \neq \xi_1$. The case when $\xi > \xi_1$ is trivial. For $\xi < \xi_1 < -1/\lambda_2-1$, we have

$$\underline{U}(\xi) = 1-k(e^{\lambda_1\xi}+pe^{\eta\lambda_1\xi}), \quad \overline{V}(\xi) = -h\xi e^{\lambda_2\xi}.$$

Hence we obtain

$$\begin{aligned}
&\underline{U}(\xi+1)+\underline{U}(\xi-1)-2\underline{U}(\xi)-c\underline{U}'(\xi)+r\underline{U}(\xi)(1-\underline{U}(\xi)-k\overline{V}(\xi))\\
\geq\ & k\{-p[e^{\eta\lambda_1}+e^{-\eta\lambda_1}-2-c(\eta\lambda_1)-r]e^{\eta\lambda_1\xi}-rk(e^{\lambda_1\xi}+pe^{\eta\lambda_1\xi})^2\\
&-r(-h\xi)e^{\lambda_2\xi}+rk(e^{\lambda_1\xi}+pe^{\eta\lambda_1\xi})(-h\xi)e^{\lambda_2\xi}\}\\
\geq\ & ke^{\eta\lambda_1\xi}\{-p[e^{\eta\lambda_1}+e^{-\eta\lambda_1}-2-c(\eta\lambda_1)-r+rk]-rke^{(1-\eta)\lambda_1\xi}-r(-h\xi)e^{(\lambda_2-\eta\lambda_1)\xi}\}\\
\geq\ & ke^{\eta\lambda_1\xi}\{-p[e^{\eta\lambda_1}+e^{-\eta\lambda_1}-2-c(\eta\lambda_1)-r+rk]-r[k+h(\lambda_2+1)/(\lambda_2^2e)]\} \geq 0
\end{aligned}$$

for all $\xi < \xi_1$, by using (3.16) and (3.17). Here we have also used the fact that

$$-h\xi e^{(\lambda_2-\eta\lambda_1)\xi} \leq -h\xi e^{\frac{\lambda_2^2}{\lambda_2+1}\xi} \leq \frac{h(\lambda_2+1)}{\lambda_2^2 e} \quad \text{for all } \xi \leq 0.$$

Now we show that (3.3) holds for $\xi \neq -1/\lambda_2 - 1$. In the case $\xi > -1/\lambda_2 - 1$, we have $\overline{V}(\xi) = 1$ and so

$$d[\overline{V}(\xi+1) + \overline{V}(\xi-1) - 2\overline{V}(\xi)] - c\overline{V}'(\xi) + s\overline{V}(\xi)\left(1 - \frac{\overline{V}(\xi)}{\overline{U}(\xi)}\right) \leq s\left[1 - \frac{1}{\overline{U}(\xi)}\right] \leq 0,$$

from $\overline{U}(\xi) \leq 1$. For $\xi < -1/\lambda_2 - 1$, we have $\overline{V}(\xi) = -h\xi e^{\lambda_2\xi}$ and so

$$\begin{aligned}
& d[\overline{V}(\xi+1) + \overline{V}(\xi-1) - 2\overline{V}(\xi)] - c\overline{V}'(\xi) + s\overline{V}(\xi)\left(1 - \frac{\overline{V}(\xi)}{\overline{U}(\xi)}\right) \\
\leq\ & d[-h(\xi+1)e^{\lambda_2(\xi+1)} - h(\xi-1)e^{\lambda_2(\xi-1)} + 2h\xi e^{\lambda_2\xi}] \\
& -c(-\lambda_2 h\xi e^{\lambda_2\xi} - he^{\lambda_2\xi}) - sh\xi e^{\lambda_2\xi} - s\frac{\overline{V}^2(\xi)}{\overline{U}(\xi)} \\
=\ & -[d(e^{\lambda_2} + e^{-\lambda_2} - 2) - c\lambda_2 + s]h\xi e^{\lambda_2\xi} - [d(e^{\lambda_2} - e^{-\lambda_2}) - c]he^{\lambda_2\xi} - s\frac{\overline{V}^2(\xi)}{\overline{U}(\xi)} \\
=\ & -s\frac{\overline{V}^2(\xi)}{\overline{U}(\xi)} \leq 0.
\end{aligned}$$

Hence (3.3) is verified.

Finally, we prove that (3.4) holds for all $\xi \neq \xi_2$. The case for $\xi > \xi_2$ is trivial. When $\xi < \xi_2 < -1$, we have

$$\begin{aligned}
& d[\underline{V}(\xi+1) + \underline{V}(\xi-1) - 2\underline{V}(\xi)] - c\underline{V}'(\xi) + s\underline{V}(\xi)\left(1 - \frac{\underline{V}(\xi)}{\underline{U}(\xi)}\right) \\
\geq\ & d[-q\sqrt{-\xi-1}e^{\lambda_2(\xi+1)} - q\sqrt{-\xi+1}e^{\lambda_2(\xi-1)} + 2q\sqrt{-\xi}e^{\lambda_2\xi}] \\
& -c\left(-q\lambda_2\sqrt{-\xi} + q\frac{1}{2\sqrt{-\xi}}\right)e^{\lambda_2\xi} - sq\sqrt{-\xi}e^{\lambda_2\xi} - \frac{s}{1-k}(-h\xi - q\sqrt{-\xi})^2 e^{2\lambda_2\xi} \\
=\ & dq\left[\left(\xi^2 + \xi\sqrt{\xi^2+\xi} + \frac{1}{2}\xi\right)e^{\lambda_2} + \left(\xi^2 + \xi\sqrt{\xi^2-\xi} - \frac{1}{2}\xi\right)e^{-\lambda_2}\right](-\xi)^{-3/2}e^{\lambda_2\xi} \\
& -\frac{s}{1-k}(-h\xi - q\sqrt{-\xi})^2 e^{2\lambda_2\xi} \\
\geq\ & [dq(l_1 e^{\lambda_2} + l_2 e^{-\lambda_2})](-\xi)^{-3/2}e^{\lambda_2\xi} - \frac{s}{1-k}(-h\xi - q\sqrt{-\xi})^2 e^{2\lambda_2\xi} \\
=\ & (-\xi)^{-3/2}e^{\lambda_2\xi}\left\{[dq(l_1 e^{\lambda_2} + l_2 e^{-\lambda_2})] - \frac{s(-\xi)^{3/2}[-h\xi - q(-\xi)^{1/2}]^2 e^{\lambda_2\xi}}{1-k}\right\} \\
\geq\ & (-\xi)^{-3/2}e^{\lambda_2\xi}\left\{[dq(l_1 e^{\lambda_2} + l_2 e^{-\lambda_2})] - \frac{sh^2}{1-k}(-\xi)^{7/2}e^{\lambda_2\xi}\right\} \\
\geq\ & (-\xi)^{-3/2}e^{\lambda_2\xi}\left\{[dq(l_1 e^{\lambda_2} + l_2 e^{-\lambda_2})] - \frac{sh^2}{1-k}\left(\frac{7}{2e\lambda_2}\right)^{7/2}\right\} \geq 0,
\end{aligned}$$

by the facts $\underline{U}(\xi) \geq 1-k$, (3.9)-(3.11) and the choice of q in (3.18). Here we have also used the fact that

$$(-\xi)^{7/2}e^{\lambda_2\xi} \leq \left(\frac{7}{2e\lambda_2}\right)^{7/2} \quad \text{for all } \xi \leq 0.$$

Therefore, the proof of the lemma is completed. □

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