ON A NONLOCAL PARABOLIC PROBLEM ARISING IN ELECTROSTATIC MEMS CONTROL

JONG-SHENQ GUO AND NIKOS I. KAVALLARIS

Abstract. We consider a nonlocal parabolic equation associated with Dirichlet boundary and initial conditions arising in MEMS control. First, we investigate the structure of the associated steady-state problem for a general star-shaped domain. Then we classify radially symmetric stationary solutions and their radial Morse indices. Finally, we study under which circumstances the solution of the time-dependent problem is global-in-time or *quenches* in finite time.

1. Introduction

In this paper, we study the following nonlocal parabolic problem (P):

$$u_t - \Delta u = \frac{\lambda}{(1-u)^2\left(1+\alpha\int_\Omega \frac{1}{1-u}dx\right)^2} \quad \text{in} \quad Q_T := \Omega\times(0,T), \tag{1.1}$$

$$u = 0 \quad \text{on} \quad \partial\Omega\times(0,T), \tag{1.2}$$

$$u(x,0) = u_0(x), \quad \text{for} \quad x\in\overline{\Omega}, \tag{1.3}$$

where λ,α are positive constants, $T>0$, $\Omega\subset\mathbb{R}^N$, $N\geq 2$, is a bounded domain with smooth boundary $\partial\Omega$, and u_0 is a continuous function in $\bar{\Omega}$ such that $0\leq u_0<1$.

Problem (1.1)-(1.3) models the dynamic deflection of an elastic membrane inside a micro-electro mechanical system (MEMS). This kind of systems combine electronics with various types of micro-size mechanical devices and could be found in accelerometers for airbag deployment in automobiles, in ink jet printer heads, in optical switches, in chemical sensors and so on, for more details see [26] and the references therein. Typically, this kind of MEMS consists of an electric membrane hanged above a rigid ground plate, connected in series with a fixed voltage source and a fixed capacitor. In the case the distance between the two plates is relative small compared to the length of the device (a realistic assumption for a typical MEMS), the original mathematical system describing the operation of the MEMS is reduced to the following single nonlocal equation in dimensionless variables

$$\varepsilon u_{tt} + u_t - \Delta u = \frac{\lambda f(x,t)}{(1-u)^2\left(1+\alpha\int_\Omega \frac{1}{1-u}dx\right)^2} \quad \text{in} \quad \Omega, \tag{1.4}$$

where u is the deflection of the membrane, see also [13].

In (1.4), ε is the ratio of the interaction due to the inertial and damping terms in the model, while

$$\lambda = \frac{V^2L^2\varepsilon_0}{2\mathcal{T}l^2},$$

1991 *Mathematics Subject Classification.* Primary 35K55, 35J60; Secondary 74H35, 74G55, 74K15.

Key words and phrases. Electrostatic MEMS, touchdown, quenching, Morse index.

The work of the first author is partially supported by the National Science Council of the Republic of China under the grants NSC 98-2115-M-003-001 and NSC 99-2115-M-032-006-MY3.

where V stands for the applied voltage, $\mathcal{T}$ is the tension in the membrane, L the characteristic length (diameter) of the domain Ω, l the characteristic width of the gap between the membrane and the fixed ground plate (electrode), and ε_0 the permittivity of the free space. The function $f(x,t)$ represents varying dielectric properties of the membrane and an applied alternating current. By the physics of the problem f is forced to be positive and for a typical MEMS is independent of time. Some physically suggested dielectric profiles are the power-law profile

$$f(x) = |x|^p, \ p > 0,$$

and the exponential-law profile

$$f(x) = e^{k(|x|^2 - c)}, \ k, c > 0,$$

see for example [9].

The integral in the equation (1.4) arises due to the fact that the device is embedded in a electrical circuit with a capacitor of fixed capacitance. The parameter α denotes the ratio of this fixed capacitance to a reference capacitance of the device. The limiting case $\alpha = 0$ corresponds to the case where there is no capacitor in the circuit. It is also assumed that the edges of the membrane are kept fixed leading to Dirichlet boundary conditions of the form (1.2), whereas it is usually considered that initially the elastic membrane is in rest corresponding to $u(x,0) \equiv 0$. Though in this work, we consider more general nonnegative initial conditions $u(x,0) = u_0(x) \geq 0$. For a more detailed analysis and derivation of (1.4) see [25, 26].

When $\alpha = 0$ and under the extra assumption that ε is very small, (1.4) is transformed to the local equation

$$u_t - \Delta u = \frac{\lambda f(x)}{(1-u)^2}, \tag{1.5}$$

which has been extensively studied in the papers [6, 9, 10, 11, 16]. Under different physical circumstances, we have that the u_t term in the left-hand side of (1.4) is much smaller than u_{tt} and Δu terms and then under proper scaling (1.4) is reduced to the following nonlocal hyperbolic equation

$$u_{tt} - \Delta u = \frac{\lambda f(x)}{(1-u)^2 \left(1 + \alpha \int_\Omega \frac{1}{1-u} dx\right)^2} \quad \text{in} \quad Q_T,$$

associated with the Dirichlet boundary condition

$$u(x,t) = 0, \quad \text{on} \quad \partial\Omega \times (0,T),$$

and the initial conditions

$$u(x,0) = u_0(x), \ \ u_t(x,0) = u_1(x), \quad \text{for} \quad x \in \overline{\Omega},$$

which for the one-dimensional case is studied in [17].

In the following, we focus on equation (1.1), though most of our results could be easily extended to the equation

$$u_t - \Delta u = \frac{\lambda f(x)}{(1-u)^2 \left(1 + \alpha \int_\Omega \frac{1}{1-u} dx\right)^2}, \tag{1.6}$$

where $f(x) \geq 0$. It will be indicated when the presented results could be carried out for (1.6) as well.

When the solution $u(x,t)$ of (1.5) achieves the value 1 at some point of Ω in finite time, we say that u *quenches in finite time* and this creates a singularity, since the source term in the right-hand side of (1.5) becomes infinite. This situation, physically, corresponds to the phenomenon of *touch-down*, i.e. when the elastic membrane touches the ground electrode. In applications, *touch-down* phenomenon is observed when the applied voltage V at the ends of the electrical circuit exceeds a fixed value. Analogously, *quenching* for the solution u to (1.5) occurs when $\lambda > \lambda^*$. Here, λ^* is the supremum of the spectrum of the corresponding steady-state problem

$$\Delta w + \frac{\lambda f(x)}{(1-w)^2} = 0, \quad x \in \Omega; \qquad w = 0, \quad x \in \partial\Omega. \tag{1.7}$$

It is well-known that the steady states of the parabolic problem play an important role in the study of its long-time behavior. See, for example, the classical work of Levine [19] for the homogeneous equation when $f(x) \equiv 1$. Furthermore, when $0 < \lambda < \lambda^*$, the solution u of (1.5) exists for all time and converges to the minimal steady state when the initial data remain below this minimal steady state. For $\lambda = \lambda^*$, the situation is more complicated since two alternative behaviors are permitted, either (i) global existence and convergence to the regular steady state, or (ii) quenching in infinite time, i.e. $\sup_\Omega u(\cdot,t) \to 1$ as $t \to \infty$, and convergence to a "singular" steady-state w^* with $||w^*||_\infty = 1$, depending on the dimension (cf. [6, 16]). The steady-state problem (1.7) was first studied by [7]. A complete study was done later in the works [1, 2, 3, 5, 12, 16].

Without loss of generality, we may assume that $\alpha = 1$ and $f(x) \equiv 1$, though some of the results in the following sections hold true for any positive function f. Under these assumptions, the nonlocal problem (P) in one-spatial-dimensional case was studied in [8]. This paper continues the study of [8] for problem (P) for higher spatial dimensions. Our purpose is to study the structure of steady states and global/non-global existence of solutions to (P). For the structure of steady states in radial symmetric case, we found that the structure of local problem is preserved (see Theorem 3.5 below). That is, the structures of steady states of local and nonlocal problems are the same. This result is expected, but the proof is highly nontrivial due to the nonlocal feature. Indeed, a more delicate analysis of the orbit $\mathcal{O}$ (defined in [14], see §3) is needed, especially, for the case $2 \le N < 7$.

This paper is organized as follows. In section 2, we first provide the local existence and uniqueness of classical solutions to problem (P). Then we study the structure of the classical solutions of the associated steady-state problem in section 3. Finally, some results on global existence and quenching of solutions to problem (P) are given in section 4.

2. Local existence and uniqueness

We shall study the local existence and uniqueness of classical solutions to problem (P) in this section. Note that since the integrand in the nonlocal term is an increasing function, the usual comparison principle for parabolic problems is not applicable for problem (P) (cf. [18]). Therefore, in the following we define the notion of lower-upper solution pairs which will be applied for comparison purposes.

Definition 2.1. *A pair of functions $0 \le v(x,t), z(x,t) < 1$ with $v, z \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ is called a lower-upper solution pair of (P), if $v(x,t) \le z(x,t)$ for $(x,t) \in Q_T$, $v(x,0) \le$*

$u_0(x) \le z(x,0)$ in $\overline{\Omega}$, $v(x,t) \le 0 \le z(x,t)$ for $(x,t) \in \partial\Omega \times [0,T]$, and

$$\begin{aligned} v_t &\le \Delta v + \frac{\lambda}{(1-v)^2\Big(1+\int_\Omega \frac{dx}{1-z}\Big)^2} \quad \text{in} \quad Q_T, \\ z_t &\ge \Delta z + \frac{\lambda}{(1-z)^2\Big(1+\int_\Omega \frac{dx}{1-v}\Big)^2} \quad \text{in} \quad Q_T. \end{aligned}$$

If the above inequalities are strict, then (v,z) is called a strict lower-upper solution pair.

Proposition 2.2. *Let (v,z) be a lower-upper solution pair to (P) in Q_T for some $T>0$. Then there exists a unique (classical) solution u to (P) such that $v \le u \le z$ in Q_T.*

Proof. First, we define the iteration scheme starting with $\overline{u}_0 = z, \underline{u}_0 = v$ and proceeding according to

$$\begin{aligned} \underline{u}_{nt} &= \Delta \underline{u}_n + \frac{\lambda}{(1-\underline{u}_{n-1})^2\Big(1+\int_\Omega \frac{dx}{1-\overline{u}_{n-1}}\Big)^2} \quad \text{in} \quad Q_T \\ \overline{u}_{nt} &= \Delta \overline{u}_n + \frac{\lambda}{(1-\overline{u}_{n-1})^2\Big(1+\int_\Omega \frac{dx}{1-\underline{u}_{n-1}}\Big)^2} \quad \text{in} \quad Q_T \\ \underline{u}_n(x,t) &= \overline{u}_n(x,t) = 0, \quad \text{on} \quad \partial\Omega \times (0,T) \\ \underline{u}_n(x,0) &= \overline{u}_n(x,0) = u_0(x), \quad \text{for} \quad x \in \overline{\Omega}, \end{aligned}$$

for $n = 1, 2, \cdots$. The above problems are local, therefore, using the standard comparison arguments for parabolic problems and Definition 2.1, we easily see that the sequences $\{\underline{u}_n\}_{n=1}^\infty, \{\overline{u}_n\}_{n=1}^\infty \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ are strictly positive and satisfy

$$v \le \underline{u}_{n-1} \le \underline{u}_n \le ... \le \overline{u}_n \le \overline{u}_{n-1} \le z.$$

Hence, by the parabolic regularity theory, $\{\underline{u}_n\}_{n=1}^\infty, \{\overline{u}_n\}_{n=1}^\infty$ converge as $n \to \infty$ to $u_1, u_2 \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$, respectively, such that $v \le u_1 \le u_2 \le z$.

We claim that $u_1 = u_2$. Indeed, u_1, u_2 satisfy

$$\begin{aligned} u_{1t} &= \Delta u_1 + \frac{\lambda}{(1-u_1)^2\Big(1+\int_\Omega \frac{dx}{1-u_2}\Big)^2} \quad \text{in} \quad Q_T \\ u_{2t} &= \Delta u_2 + \frac{\lambda}{(1-u_2)^2\Big(1+\int_\Omega \frac{dx}{1-u_1}\Big)^2} \quad \text{in} \quad Q_T \\ u_1(x,t) &= u_2(x,t) = 0, \quad \text{on} \quad \partial\Omega \times (0,T) \\ u_1(x,0) &= u_2(x,0) = u_0(x), \quad \text{for} \quad x \in \overline{\Omega}. \end{aligned}$$

Denote $\psi(x,t) = u_1(x,t) - u_2(x,t)$. Then $\psi(x,t)$ satisfies

$$\begin{aligned} &\psi_t = \Delta\psi + A(x,t)\psi + B(x,t)\int_\Omega \int_0^1 \frac{d\theta}{\Big[1-\theta u_1 - (1-\theta)u_2\Big]^2}\,\psi dx, \quad \text{in} \quad Q_T \\ &\psi(x,t) = 0, \quad \text{for} \quad (x,t) \in (\partial\Omega \times (0,T)) \cap (\overline{\Omega} \times \{0\}), \end{aligned}$$

where

$$A(x,t) := 2\lambda \frac{\displaystyle\int_0^1 \frac{d\theta}{[1-\theta u_1-(1-\theta)u_2]^3}}{\left(1+\int_\Omega \frac{dx}{1-u_2}\right)^2} > 0,$$

$$B(x,t) := \frac{\lambda}{(1-u_2)^2} \frac{2+\int_\Omega \frac{dx}{1-u_1}+\int_\Omega \frac{dx}{1-u_2}}{\left(1+\int_\Omega \frac{dx}{1-u_1}\right)^2\left(1+\int_\Omega \frac{dx}{1-u_2}\right)^2} > 0.$$

Applying Proposition 52.24 in [28](which is actually a maximum principle for nonlocal problems) we easily obtain that $\psi(x,t)=0$ in $\overline{Q}_T$. Hence $u_1=u_2:=u$ in $\overline{Q}_T$.

Finally, suppose that there is a second solution U satisfying $v \le U \le z$. Then by the preceding iterative scheme we derive that $\underline{u}_n \le U \le \overline{u}_n$ for every $n=1,2,\cdots$. By sending $n\to\infty$, we obtain that $U=u$. This proves the proposition. □

Remark 2.3. *By the above analysis we derive that the solution of problem (P) continues to exist as long as it remains less than or equal to b for some $b<1$. This argument implies that u ceases to exist only by "quenching", i.e., if there exists a sequence $(x_n,t_n)\to(x^*,t^*)$ as $n\to\infty$ with $t^*\le\infty$ such that $u(x_n,t_n)\to 1$ as $n\to\infty$.*

Let $z(x,t)$ be the unique solution of the following problem (Q):

$$z_t-\Delta z=\frac{\lambda}{(1-z)^2(1+|\Omega|)^2} \quad \text{in} \quad Q_T,$$
$$z=0 \quad \text{on} \quad \partial\Omega\times(0,T),$$
$$z|_{t=0}=z_0 \quad \text{for} \quad x\in\overline{\Omega}.$$

Then we have the following.

Corollary 2.4. *If $z_0(x)\ge u_0(x)$ for every $x\in\Omega$, then problem (P) has a unique (classical) solution u on $\Omega\times[0,T)$, where $[0,T)$ is the maximal existence time interval for the solution $z(x,t)$ of problem (Q), and $u(x,t)\le z(x,t)$ on $\Omega\times[0,T)$.*

Proof. Set $v(x,t)=0$, then we have

$$z_t-\Delta z=\frac{\lambda}{(1-z)^2(1+|\Omega|)^2}=\frac{\lambda}{(1-z)^2\left(1+\int_\Omega\frac{dx}{1-v}\right)^2} \quad \text{in} \quad Q_T,$$
$$z=0 \quad \text{on} \quad \partial\Omega\times(0,T),$$
$$z|_{t=0}=z_0 \quad \text{for} \quad x\in\overline{\Omega},$$

while $v(x,t)$ satisfies

$$v_t-\Delta v=0\le\frac{\lambda}{(1-v)^2\left(1+\int_\Omega\frac{dx}{1-z}\right)^2} \quad \text{in} \quad Q_T,$$
$$v=0 \quad \text{on} \quad \partial\Omega\times(0,T),$$
$$v|_{t=0}=0 \quad \text{for} \quad x\in\overline{\Omega}.$$

Therefore, (v,z) is a lower-upper solution pair for problem (P) and the result immediately follows by Proposition 2.2. □

Remark 2.5. *All the results of this section could be carried out without any change for equation (1.6) as well, when $f(x)>0$.*

3. Steady state problem

The corresponding steady-state problem to (P) has the form

$$\Delta w + \frac{\lambda}{K(1-w)^2} = 0, \quad x \in \Omega, \quad w = 0, \quad x \in \partial\Omega, \tag{3.1}$$

where

$$K = K(w) := \Big(1 + \int_\Omega \frac{dx}{1-w}\Big)^2. \tag{3.2}$$

Note that we always have $0 < w < 1$ in Ω for a (classical) solution of (3.1).

By setting

$$\mu = \frac{\lambda}{K} = \frac{\lambda}{\Big(1 + \int_\Omega \frac{dx}{1-w}\Big)^2}, \tag{3.3}$$

then (3.1) is transformed to

$$\Delta w + \mu(1-w)^{-2} = 0 \quad x \in \Omega, \quad w = 0 \quad x \in \partial\Omega. \tag{3.4}$$

Problems (3.1) and (3.4) are equivalent through (3.3), i.e. w is a solution of (3.1) corresponding to λ if and only if w satisfies (3.4) for $\mu = \lambda/K$. Thus, some features of the solution set

$$\mathcal{C} = \{(\lambda, w) | w = w_\lambda(x) \text{ is a classical solution to (3.1) for } \lambda > 0\}$$

resemble those of the solution set

$$\mathcal{S} = \{(\mu, w) | w = w_\mu(x) \text{ is a classical solution to (3.4) for } \mu > 0\}.$$

The structure of $\mathcal{S}$ is well known. Indeed, for any N, we recall from [7] that there exists a positive constant μ^* such that a solution of (3.4) exists, if $\mu < \mu^*$, and no solutions of (3.4) exist if $\mu > \mu^*$. Moreover, for each $\mu \in (0, \mu^*)$ the minimal solution of (3.4), denoted by w_μ, satisfies

$$0 < w_{\mu_1}(x) < w_{\mu_2}(x) < 1 \quad \text{ for } x \in \Omega, \quad \text{ if } 0 < \mu_1 < \mu_2 < \mu^*, \tag{3.5}$$

i.e. the minimal solutions are ordered with respect to the parameter μ.

When Ω is the unit ball of $\mathbb{R}^N$, $N \geq 2$, it is proved in [14] (see also the proof of Theorem 3.5 below) that $\mathcal{S}$ is homeomorphic to $\mathbb{R}$ and has end points $(0, 0)$ and $(\hat{\mu}, 1-|x|^{2/3})$ with $\hat{\mu} = [4 + 6(N-2)]/9$. Moreover, when $2 \leq N < 7$ the solution curve $(\mu(s), w(\cdot, s))$, $s \in \mathbb{R}$, of (3.4) bends infinitely many times around the singular point $(\hat{\mu}, 1-|x|^{2/3})$. Whereas for $N \geq 7$ the solution curve terminates at $(\hat{\mu}, 1-|x|^{2/3})$ and no bendings occur. For a detailed analysis on the structure of problem (3.4) when the term $\mu(1-w)^{-2}$ is also multiplied by the function $f(x) = |x|^p$, $p > 0$, see [1, 5, 16].

Recall that for $N = 1$ there is a $\lambda^* > 0$ such that a steady state of (P) exists if and only if $\lambda \leq \lambda^*$. In fact, by Theorem 2.1 of [8], there are exactly two solutions of (3.1) if $\lambda \in (0, \lambda^*)$, while (3.1) has a unique solution if $\lambda = \lambda^*$ and no solution if $\lambda > \lambda^*$. This indicates that the solution structure of the local problem (cf. [19]) is preserved by the nonlocal one when $N = 1$. However, it is not obvious that the aforementioned features of the local problem (3.4) are preserved by the nonlocal problem (3.1) for the case $N \geq 2$.

In the case where the equation (1.1) of problem (P) is substituted by (1.6), then the related stationary problem is given by

$$\Delta w + \frac{\lambda f(x)}{K(1-w)^2} = 0, \quad x \in \Omega, \quad w = 0, \quad x \in \partial\Omega, \tag{3.6}$$

where K is given by (3.2) and $f(x) \geq 0$.

In the following, we shall focus on the case $N \geq 2$. We first prove some results regarding the case of a general smooth domain Ω and then we shift to the radial symmetric case, i.e. when $\Omega = \{x \in \mathbb{R}^N : |x| < 1\} := B(0,1) = B$.

Using the monotonicity property (3.5) we can prove the following existence theorem.

Theorem 3.1. *There exists a (classical) steady-state solution of (P) if* $\lambda \in (0, (1+|\Omega|)^2\mu^*)$.

Proof. We look for $w = w_\mu$ for some μ. Note that, by (3.5),

$$(1+|\Omega|)^2 = K(0) < K(w_{\mu_1}) < K(w_{\mu_2}), \quad \text{if } 0 < \mu_1 < \mu_2 < \mu^*.$$

Hence there exists a unique $\mu \in (0, \mu^*)$ such that

$$\mu K(w_\mu) = \lambda, \tag{3.7}$$

if $\lambda \in (0, (1+|\Omega|)^2\mu^*)$. This proves the theorem, since problems (3.1) and (3.4) are equivalent through (3.3). □

Let Ω be a strictly star-shaped domain of $\mathbb{R}^N$, $N \geq 2$. Then there exists a constant $\beta > 0$ such that

$$s \cdot \nu \geq \beta \int_{\partial\Omega} ds \quad \text{for any} \quad s \in \partial\Omega,$$

where ν is the unit outward normal at s. Let v be a solution of the problem

$$\Delta v + \mu f(v) = 0 \ \ x \in \Omega, \ \ v = 0 \ \ x \in \partial\Omega.$$

Then we have the following Pohozaev's identity (see [27]):

$$\mu N \int_\Omega F(v)dx - \frac{\mu(N-2)}{2} \int_\Omega v f(v)dx = \frac{1}{2} \int_{\partial\Omega} (s\cdot\nu)\Big(\frac{\partial v}{\partial \nu}\Big)^2 ds, \tag{3.8}$$

where $F(v) := \int_0^v f(s)ds$.

Theorem 3.2. *Let Ω be a strictly star-shaped domain of $\mathbb{R}^N$, $N \geq 2$. Then problem (3.1) has no solution if*

$$\lambda > \frac{N(1+|\Omega|)^4}{2\beta|\Omega|^2},$$

where $|\Omega|$ is the N-dimensional Lebesgue measure of Ω. Moreover, for $N \geq 2$ there exists $\lambda_* > 0$ such that problem (3.1) has a unique solution for $0 < \lambda < \lambda_*$.

Proof. We first prove the second statement of the theorem. For $N \geq 3$, by Theorem 3.1 in [3] there exists a positive constant μ_* such that problem (3.4) has a unique solution for $0 < \mu < \mu_*$. A similar result holds true for $N = 2$, see Theorem 4.1 in [12]. On the other hand, any solution $w = w(x)$ to (3.1) solves (3.4) with

$$\mu = \frac{\lambda}{\big(1+\int_\Omega \frac{dx}{1-w}\big)^2} \leq \frac{\lambda}{(1+|\Omega|)^2},$$

due to the fact that $1/(1-w) \geq 1$, when w is a regular solution i.e. $||w||_\infty < 1$. We claim that problem (3.1) has a unique solution for $0 < \lambda < \lambda_* := \mu_*(1+|\Omega|)^2$. Indeed, let us assume that problem (3.1) has two distinct solutions w_1, w_2 for some $0 < \lambda < \lambda_*$. We claim that

$$\int_\Omega \frac{dx}{1-w_1} = \int_\Omega \frac{dx}{1-w_2}. \tag{3.9}$$

Otherwise, if, for example,

$$\int_\Omega \frac{dx}{1-w_1} < \int_\Omega \frac{dx}{1-w_2},$$

then we have

$$\frac{\lambda}{(1+|\Omega|)^2} \geq \mu_1 := \frac{\lambda}{\left(1+\int_\Omega \frac{dx}{1-w_1}\right)^2} > \mu_2 := \frac{\lambda}{\left(1+\int_\Omega \frac{dx}{1-w_2}\right)^2}. \tag{3.10}$$

Since $0 < \lambda < \lambda_* = \mu_*(1+|\Omega|)^2$, we obtain by (3.10) that $0 < \mu_2 < \mu_1 < \mu_*$, hence $w_1(x) = w(x;\mu_1)$ and $w_2(x) = w(x;\mu_2)$ are minimal solutions of the local problem (3.4). It follows by the monotonicity property (3.5) for the minimal solution branch of (3.4) that $w_1(x) > w_2(x)$ for $x \in \Omega$. This implies that

$$\int_\Omega \frac{dx}{1-w_1} > \int_\Omega \frac{dx}{1-w_2},$$

which is a contradiction. Using the same arguments we can exclude the possibility of

$$\int_\Omega \frac{dx}{1-w_1} > \int_\Omega \frac{dx}{1-w_2}.$$

Hence (3.9) is established and so local problem (3.4) has two distinct solutions w_1, w_2 corresponding to the same μ, which contradicts to [3, Theorem 3.1] and [12, Theorem 4.1]. Therefore, the second statement of the theorem is proved.

For the first statement, we suppose that problem (3.1) has a solution w for some $\lambda > 0$. Then applying Pohozaev's identity (3.8) to problem (3.1) with

$$f(w) = \frac{1}{K(1-w)^2}, \quad F(w) = \frac{w}{K(1-w)},$$

we obtain

$$\frac{\lambda N}{K}\int_\Omega \frac{w}{1-w}dx - \frac{\lambda(N-2)}{2K}\int_\Omega \frac{w}{(1-w)^2}dx = \frac{1}{2}\int_{\partial\Omega}(s\cdot\nu)\Big(\frac{\partial w}{\partial \nu}\Big)^2 ds. \tag{3.11}$$

By Hölder's inequality, we have

$$0 \leq -\int_{\partial\Omega}\frac{\partial w}{\partial\nu}ds \leq \Big(\int_{\partial\Omega}\Big(\frac{\partial w}{\partial\nu}\Big)^2 ds\Big)^{1/2}\Big(\int_{\partial\Omega} ds\Big)^{1/2},$$

hence due to the divergence theorem we derive

$$\begin{aligned}\frac{1}{2}\int_{\partial\Omega}(s\cdot\nu)\Big(\frac{\partial w}{\partial\nu}\Big)^2 ds \quad &\geq \frac{\beta}{2}\Big(\int_{\partial\Omega} ds\Big)\int_{\partial\Omega}\Big(\frac{\partial w}{\partial\nu}\Big)^2 ds \geq \frac{\beta}{2}\Big(\int_{\partial\Omega} -\frac{\partial w}{\partial\nu}ds\Big)^2 \\ &= \frac{\beta}{2}\Big(\int_\Omega -\Delta w dx\Big)^2 = \frac{\lambda^2\beta}{2K^2}\Big(\int_\Omega \frac{1}{(1-w)^2}dx\Big)^2. \end{aligned} \tag{3.12}$$

By dropping out the negative terms in (3.11) and using again Hölder's inequality, it follows from (3.11) and (3.12) that

$$\begin{aligned}
\frac{\lambda N \int_\Omega \frac{dx}{1-w}}{\left(1+\int_\Omega \frac{dx}{1-w}\right)^2} &\geq \frac{\lambda N}{K}\int_\Omega \frac{w}{1-w}dx \geq \frac{\lambda^2\beta}{2}\Big[\frac{\int_\Omega \frac{dx}{(1-w)^2}}{\left(1+\int_\Omega \frac{dx}{1-w}\right)^2}\Big]^2 \\
&\geq \frac{\lambda^2\beta}{2|\Omega|^2}\Big[\frac{\int_\Omega \frac{dx}{1-w}}{\left(1+\int_\Omega \frac{dx}{1-w}\right)}\Big]^4 \geq \frac{\lambda^2\beta}{2|\Omega|^2}\Big(\frac{|\Omega|}{1+|\Omega|}\Big)^4 \\
&= \frac{\lambda^2\beta}{2}\Big(\frac{|\Omega|^{1/2}}{1+|\Omega|}\Big)^4, \qquad (3.13)
\end{aligned}$$

where the last inequality in (3.13) comes out from the fact that

$$\int_\Omega \frac{dx}{1-w} \geq |\Omega|$$

and since the function $g(s) = s/(1+s)$ is increasing in $s > 0$. Note also that

$$s/(1+s)^2 \leq 1/4 \quad \text{for all } s \geq 0. \qquad (3.14)$$

Combining (3.13) and (3.14), we get

$$\frac{\lambda N}{4} \geq \frac{\lambda^2\beta}{2}\Big(\frac{|\Omega|^{1/2}}{1+|\Omega|}\Big)^4$$

or equivalently

$$\lambda \leq \frac{N(1+|\Omega|)^4}{2\beta|\Omega|^2}.$$

This completes the proof of the theorem. □

Remark 3.3. *The second statement of Theorem 3.2 is still true for a general domain Ω. Indeed, for $N=2$ Theorem 4.1 in* [12] *guarantees the existence of μ_* such that the local problem (3.4) has a unique solution for $0<\mu<\mu_*$, see also Theorem 3.3 in* [3]*. On the other hand in higher dimensions an analogous result can be found in section 5 of* [31]*. Furthermore, the second statement of Theorem 3.2 holds for problem (3.6) when $f(x)=|x|^p$, $p>0$, and Ω is a bounded domain in $\mathbb{R}^N$, $N\geq 3$, since in this case Theorem 3.2 in* [3] *guarantees the existence of $\mu_*>0$ such that the local problem (1.7) has only the minimal solution for $0<\mu<\mu_*$.*

Remark 3.4. *Let ω_N be the volume of the unit ball B in $\mathbb{R}^N$. Then Theorem 3.2 implies that no steady states exist when $\Omega = B$, if $\lambda > N^2(1+\omega_N)^4/(2\omega_N)$.*

When $\Omega = B$, then the solution of (3.1) is radial symmetric, [4], i.e., $w(x) = w(r)$, where $r = |x|$ and (3.1) is reduced to

$$(r^{N-1}w_r)_r + \frac{\lambda r^{N-1}}{K\,(1-w)^2} = 0 \quad \text{for } r \in (0,1), \qquad (3.15)$$

$$w(1) = 0, \qquad w_r(0) = 0, \qquad (3.16)$$

where now K has the form

$$K = \Big(1 + N\omega_N \int_0^1 \frac{r^{N-1}}{1-w}dr\Big)^2,$$

or equivalently to the local problem

$$(r^{N-1}w_r)_r + \frac{\mu r^{N-1}}{(1-w)^2} = 0 \quad \text{for } r \in (0,1), \tag{3.17}$$

$$w(1) = 0, \qquad w_r(0) = 0, \tag{3.18}$$

where $\mu = \lambda/K$.

We then write $\mathcal{S}_r, \mathcal{C}_r$ instead of $\mathcal{S}, \mathcal{C}$ respectively. Also, set

$$\mathcal{C}_r^\lambda = \left\{ w \in C^2(B) \cap C^0(\overline{B}) \mid w \text{ solves (3.15)-(3.16)} \right\},$$

the section of $\mathcal{C}_r$ cut by $\lambda > 0$.

Theorem 3.5. *If $N \geq 2$, then $\mathcal{C}_r$ is homeomorphic to $\mathbb{R}$ and has end points $(0,0)$ and $(\widehat{\lambda}, 1-|x|^{2/3})$, where*

$$\widehat{\lambda} = \Lambda\Big(1 + \frac{N\omega_N}{\gamma}\Big)^2, \quad \Lambda := \frac{2}{3}(N - 4/3) = \hat{\mu}, \quad \gamma = N - 2/3.$$

Moreover, if $2 \leq N < 7$, then $\mathcal{C}_r$ bends infinitely many times with respect to λ around $\widehat{\lambda}$ and there exist two positive constants λ^* and λ_* with $0 < \lambda_* < \widehat{\lambda} < \lambda^*$ such that problem (3.15)-(3.16) has

- *a unique solution for $0 < \lambda < \lambda_*$ and $\lambda = \lambda^*$,*
- *a finite number of solutions for $\lambda \in (\lambda_*, \lambda^*)$ and $\lambda \neq \widehat{\lambda}$,*
- *infinite number of solutions for $\lambda = \widehat{\lambda}$,*
- *no solutions for $\lambda > \lambda^*$,*

see Fig. 2.

Whereas no bending occurs in the case of $N \geq 7$, i.e., problem (3.15)-(3.16) has a unique solution for any $0 < \lambda < \widehat{\lambda} = \lambda^*$ and no solution for $\lambda \geq \widehat{\lambda} = \lambda^*$, see Fig. 3.

Proof. We proceed as in [14] using phase-plane analysis. First we note that every positive solution of problem (3.17)-(3.18) can be obtained as a solution of the following initial-value problem

$$(r^{N-1}w_r)_r + \frac{\mu r^{N-1}}{(1-w)^2} = 0, \quad r > 0, \tag{3.19}$$

$$w(0) = A, \qquad w_r(0) = 0, \tag{3.20}$$

with a certain positive constant $A \in (0,1)$. We always require that $0 < w(r) < 1$ for $r \in (0,1)$.

Putting

$$k = \sqrt{\Lambda(1-A)^3/\mu} = \sqrt{K\Lambda(1-A)^3/\lambda},$$

we apply the Emden transformation

$$w(r;t) = 1 - (1-A)\, e^{2t/3} z(t), \qquad r = ke^t, \tag{3.21}$$

for $\Lambda = [6(N-2)+4]/9$. Then problem (3.19)-(3.20) is transformed to

$$\ddot{z} + \gamma \dot{z} + \Lambda\left(z - \frac{1}{z^2}\right) = 0, \quad t > -\infty, \tag{3.22}$$

$$\lim_{t\to-\infty} e^{2t/3} z(t) = 1, \qquad \lim_{t\to-\infty} e^{2t/3} \dot{z}(t) = -\frac{2}{3}. \tag{3.23}$$

Here we require that $0 < z(t) < e^{-2t/3}/(1-A)$.

If we set

$$w(r;t) = 1 - (1-A)\, e^{-2t/3}\widetilde{z}(t), \qquad r = ke^{-t},$$

then $\widetilde{z}(t)$ satisfies

$$\ddot{\widetilde{z}} - \gamma\dot{\widetilde{z}} + \Lambda\left(\widetilde{z} - \frac{1}{\widetilde{z}^2}\right) = 0, \quad t > -\infty,$$

$$\lim_{t\to+\infty} e^{-2t/3}\widetilde{z}(t) = 1, \qquad \lim_{t\to+\infty} e^{-2t/3}\dot{\widetilde{z}}(t) = \frac{2}{3}.$$

As it is proved in [14], problem (3.22)-(3.23) has a global-in-time solution with the orbit $\mathcal{O} = \{(z(t), \dot{z}(t)),\ t\in\mathbb{R}\}$ starting at $t=-\infty$ from the point $(+\infty, -\infty)$ above its tangent line $\dot{z} + 2z/3 = 0$ and approaching the point $(1,0)$ as $t\to+\infty$, see Fig. 1. On the other hand the orbit $\widetilde{\mathcal{O}} = \{(\widetilde{z}(t), \dot{\widetilde{z}}(t)),\ t\in\mathbb{R}\}$ starts at $t=-\infty$ from the point $(+\infty,+\infty)$ below its tangent line $\dot{z} - 2z/3 = 0$ and approaches the point $(1,0)$ as $t\to+\infty$. Moreover, the orbits $\mathcal{O}$ and $\widetilde{\mathcal{O}}$ are symmetric with respect to z-axis in the phase plane, see [14], hence we obtain that

$$\left|\frac{\dot{z}(t)}{z(t)}\right| < \frac{2}{3} \tag{3.24}$$

for any $t\in\mathbb{R}$.

Through the transformation (3.21), the boundary condition $w(1) = 0$ in (3.16) is converted to

$$z(\tau) = [\Lambda/\mu(\tau)]^{1/3}, \quad \tau := -\frac{1}{2}\ln\frac{[6\,(N-2)+4]\,(1-A)^3}{9\mu} = -\ln k.$$

In other words, for any $\tau\in\mathbb{R}$, $(A(\tau), \mu(\tau), w(\cdot,\tau))$ defined by

$$w(r;\tau) = 1 - \frac{z(\tau+\ln r)}{z(\tau)} r^{2/3} \tag{3.25}$$

$$\mu(\tau) = \frac{\Lambda}{z^3(\tau)} \tag{3.26}$$

$$A(\tau) = 1 - \frac{1}{e^{2\tau/3} z(\tau)} \tag{3.27}$$

satisfies the initial-value problem (3.19)-(3.20) or equivalently the local problem (3.17)-(3.18). Let

$$K(\tau) := \left(1 + N\omega_N \int_0^1 \frac{r^{N-1}}{1-w(r;\tau)}dr\right)^2,$$

then $(\lambda(\tau), w(\cdot,\tau))$ with $\lambda(\tau) = \mu(\tau)K(\tau)$ satisfies the nonlocal steady-state problem (3.15)-(3.16). Conversely, every solution of (3.15)-(3.16) corresponds, through the Emden transformation (3.21) (where μ and A are given by (3.26) and (3.27) respectively) and $\lambda = \mu K$, to an element of the orbit $\mathcal{O}$ for a certain $\tau\in\mathbb{R}$. Therefore, there is an one-to-one and onto correspondence between the solutions to (3.15)-(3.16) and the elements of the orbit $\mathcal{O}$, hence $\mathcal{C}_r$ is homeomorphic to $\mathcal{O}$ and so to $\mathbb{R}$.

Now, taking the limit as $\tau\to-\infty$ in (3.26), we obtain that $\mu(\tau)\to 0$, which implies as well that $w(\cdot;\tau)\to 0$ in B as $\tau\to-\infty$, hence $\lambda(\tau)\to 0$ as $\tau\to-\infty$. On the other hand, taking the limit as $\tau\to+\infty$ we obtain that $\mu(\tau)\to\Lambda$ and $w(\cdot;\tau)\to w^*(\cdot)$ in B, where

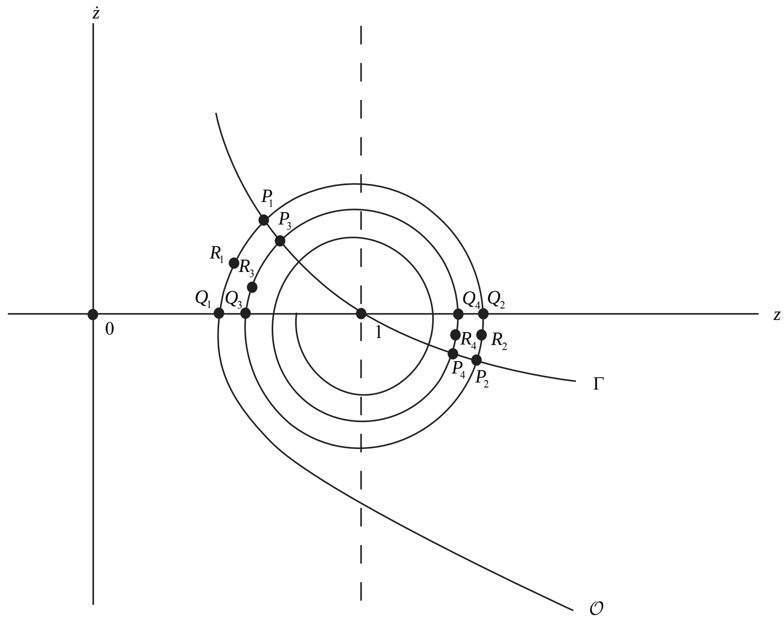

Figure 1. The phase plane

$w^*(r) := 1 - r^{2/3}$ is actually a singular, $||w^*||_\infty = 1$, solution to (3.15)-(3.16). This implies that

$$\lambda(\tau) \to \widehat{\lambda} := \Lambda\Big(1 + N\omega_N \int_0^1 r^{N-5/3} dr\Big)^2 = \Lambda\Big(1 + \frac{N\omega_N}{\gamma}\Big)^2 \quad \text{as } \tau \to +\infty. \tag{3.28}$$

Hence the terminating point of the solution curve $\mathcal{C}_r$ is $(\widehat{\lambda}, w^*)$. Besides, the singular points $(+\infty, -\infty)$ and $(0,1)$ of (3.22)-(3.23) correspond to the end points $(0,0)$ and $(\widehat{\lambda}, w^*)$ of the solution curve $\mathcal{C}_r$. This proves the first part of the theorem.

Moreover, by Lemma 6 in [14], for $2 \le N < 7$ we have that the orbit $\mathcal{O}$ of (3.22)-(3.23) starts from $(+\infty, -\infty)$ above the tangent line $\dot{z} + 2z/3 = 0$ and terminates to $(1,0)$ crossing clockwise infinitely many times the positive part of the z-axis, say for $t_1 < t_2 < \cdots < t_k < \cdots$, $k \in \mathbb{N}$, as well as the curve Γ defined by

$$\dot{z} = -\frac{\Lambda}{\gamma}\Big(z - \frac{1}{z^2}\Big) := f(z), \tag{3.29}$$

for $s_1 < s_2 < \cdots < s_k < \cdots$, $k \in \mathbb{N}$, without crossing itself. This follows from the fact that the equilibrium $(1,0)$ is a spiral point when $2 \le N < 7$. On the other hand, $(1,0)$ is a saddle point when $N \ge 7$.

Let $Q_k = (z(t_k), 0)$ and $P_k = (z(s_k), \dot{z}(s_k))$, $k \in \mathbb{N}$, be the intersection points of the orbit $\mathcal{O}$ with the z-axis and the curve Γ respectively, see Fig. 1. Then it is enough to show that $\dot{\lambda}(\tau)$ changes sign across the arcs $Q_{2k-1}P_{2k-1}$ and $Q_{2k}P_{2k}$, see also [21].

We first note that $\lambda(\tau)$ can be expressed as

$$\lambda(\tau) = \mu(\tau) G^2(\tau) = \Lambda \frac{G^2(\tau)}{z^3(\tau)}, \quad G(\tau) := K^{1/2}(\tau). \tag{3.30}$$

Also, by (3.25) it follows that

$$\begin{aligned} G(\tau) &= 1+N\omega_N z(\tau)\int_0^1 \frac{1}{z(\tau+\ln r)} r^{\gamma-1}dr \\ &= 1+N\omega_N z(\tau)e^{-\gamma\tau}\int_{-\infty}^{\tau} \frac{1}{z(s)} e^{\gamma s}ds, \end{aligned} \tag{3.31}$$

and hence

$$\dot{G}(\tau)=\frac{\dot{z}(\tau)}{z(\tau)}[G(\tau)-1]-\gamma[G(\tau)-1]+N\omega_N. \tag{3.32}$$

Then

$$\begin{aligned} \dot{\lambda}(\tau) &= 2\Lambda\frac{G(\tau)\dot{G}(\tau)}{z^3(\tau)}-3\Lambda\frac{G^2(\tau)}{z^4(\tau)}\dot{z}(\tau) \\ &= \frac{\Lambda G(\tau)}{z^4(\tau)}\Big[2\dot{G}(\tau)z(\tau)-3\dot{z}(\tau)G(\tau)\Big]. \end{aligned} \tag{3.33}$$

Using (3.32), relation (3.33) reads

$$\dot{\lambda}(\tau)=\frac{\Lambda G(\tau)}{z^4(\tau)}\Big\{-\dot{z}(\tau)[G(\tau)+2]+2\Big(N\omega_N-\gamma[G(\tau)-1]\Big)z(\tau)\Big\} \tag{3.34}$$

and so at points Q_{2k-1} and P_{2k-1} we have

$$\dot{\lambda}(t_{2k-1})=\frac{2\Lambda G(t_{2k-1})}{z^3(t_{2k-1})}\Big(\gamma+N\omega_N-\gamma G(t_{2k-1})\Big) \tag{3.35}$$

and

$$\begin{aligned} \dot{\lambda}(s_{2k-1}) &= \frac{\Lambda G(s_{2k-1})}{z^4(s_{2k-1})}\Big[-(G(s_{2k-1})+2)f(z(s_{2k-1})) \\ &+ 2\Big(\gamma+N\omega_N-\gamma G(s_{2k-1})\Big)z(s_{2k-1})\Big] \end{aligned} \tag{3.36}$$

respectively, since the points Q_{2k-1} are on the z-axis.

Now by using mean value theorem (3.31) is written in the form

$$G(\tau)=1+N\omega_N\frac{z(\tau)}{z(\tau+\ln r_0(\tau))}\int_0^1 r^{\gamma-1}dr=1+\frac{N\omega_N}{\gamma}\frac{z(\tau)}{z(\tau+\ln r_0(\tau))} \tag{3.37}$$

where $r_0(\tau)\in(0,1)$. In particular, we have

$$G(t_{2k-1})=1+\frac{N\omega_N}{\gamma}\frac{z(t_{2k-1})}{z(t_{2k-1}+\ln r_0(t_{2k-1}))} \tag{3.38}$$

for any k.

Moreover, due to (3.28) we have

$$G(\tau)\to 1+\frac{N\omega_N}{\gamma} \quad \text{as} \quad \tau\to\infty,$$

hence by (3.37) and the continuity of $z(\tau)$ we derive that

$$\ln r_0(\tau)\to 0 \quad \text{as} \quad \tau\to\infty.$$

This implies that at t_{2k-1} with k sufficiently large we have $|\ln r_0(t_{2k-1})| \ll 1$. But, at t_{2k-1} a local minimum for $z(t)$ occurs hence $z(t_{2k-1}) < z(t_{2k-1} + \ln r_0(t_{2k-1}))$ for k sufficiently large which finally yields

$$G(t_{2k-1}) < 1 + \frac{N\omega_N}{\gamma} \quad \text{for } k \text{ sufficiently large,} \tag{3.39}$$

by (3.38). Then (3.35) and (3.39) immediately imply that

$$\dot{\lambda}(t_{2k-1}) > 0 \tag{3.40}$$

when k is sufficiently large.

On the other hand, at the point P_{2k-1} we have $z(s_{2k-1}) < 1$ hence the first term in the bracket in the RHS of (3.36) is strictly negative. Besides, there holds

$$G(s_{2k-1}) = 1 + \frac{N\omega_N}{\gamma} \frac{z(s_{2k-1})}{z(s_{2k-1} + \ln r_0(s_{2k-1}))} \tag{3.41}$$

with $0 < -\ln r_0(s_{2k-1}) \ll 1$ whenever k is large enough. But, z is strictly increasing along the arc $Q_{2k-1}Q_{2k}$, hence (3.41) yields that

$$G(s_{2k-1}) > 1 + \frac{N\omega_N}{\gamma} \tag{3.42}$$

for k sufficiently large. Therefore, we conclude from (3.36) that

$$\dot{\lambda}(s_{2k-1}) < 0 \tag{3.43}$$

for k sufficiently large.

Now, (3.40) and (3.43) imply that whenever k is large enough there exists a point of the orbit $\mathcal{O}$, say $R_{2k-1} = (z(\tau_{2k-1}), \dot{z}(\tau_{2k-1}))$, with $t_{2k-1} < \tau_{2k-1} < s_{2k-1}$, in the arc $Q_{2k-1}P_{2k-1}$ such that $\dot{\lambda}(\tau_{2k-1}) = 0$.

In a similar manner, it is derived that $\dot{\lambda}(t_{2k}) < 0$ and $\dot{\lambda}(s_{2k}) > 0$, since z achieves a local maximum at t_{2k} as well as $z(\tau)$ is strictly decreasing along the arc $Q_{2k}Q_{2k+1}$. Therefore, we can again find, at least for k large enough, a point $R_{2k} = (z(\tau_{2k}), \dot{z}(\tau_{2k}))$, $t_{2k} < \tau_{2k} < s_{2k}$, lying in the arc $Q_{2k}P_{2k}$ such that $\dot{\lambda}(\tau_{2k}) = 0$. Also note that by (3.34) using the above analysis we obtain that $\dot{\lambda}(\tau) < 0$ for any $s_{2k-1} < \tau < t_{2k}$ while $\dot{\lambda}(\tau) > 0$ for any $s_{2k} < \tau < t_{2k+1}$, which means that only the points R_k correspond to the bendings of $\mathcal{C}_r$. For the distribution of points R_k across the orbit $\mathcal{O}$ see Fig. 1. Finally the above analysis implies that $\mathcal{C}_r$ bends infinitely many times around $\widehat{\lambda}$, see Fig. 2.

On the other hand, when $N \geq 7$ the orbit starts from $(+\infty, -\infty)$ and stays inside the area $\{(z, \dot{z}) \mid z > 0,\ 0 > \dot{z} > -2z/3\}$ terminating to $(1, 0)$ without crossing neither the z-axis nor itself, implying, through the homeomorphism between $\mathcal{S}_r$ and $\mathcal{O}$, that $\mathcal{S}_r$ presents no bending, hence the local problem (3.17)-(3.18) has a unique solution for $0 < \mu < \widehat{\mu} = \Lambda$. Now regarding the nonlocal problem (3.15)-(3.16) we obtain that it has a unique solution for any $0 < \lambda < \widehat{\lambda}$, by using the same arguments as in the first part of the proof of Theorem 3.2, see Fig. 3. This completes the proof of the theorem. □

Remark 3.6. *Using the same reasoning as above, an analogous result to Theorem 3.5 could be proven for problem (3.6), see also Theorem 1.2 in* [16].

Remark 3.7. *Theorem 3.5 implies that if* $2 \leq N < 7$ *then there is a strictly increasing sequence* $\{\tau_k\}_{k=1}^{\infty} \subset \mathbb{R}$ *such that the functions* $\tau \in [\tau_{2k-1}, \tau_{2k}] \longmapsto \lambda(\tau)$ *and* $\tau \in [\tau_{2k}, \tau_{2k+1}] \longmapsto \lambda(\tau)$ *are strictly decreasing and increasing respectively, and* $\lambda(\tau_2) < \lambda(\tau_4) < \cdots < \lambda(\tau_{2k}) <$

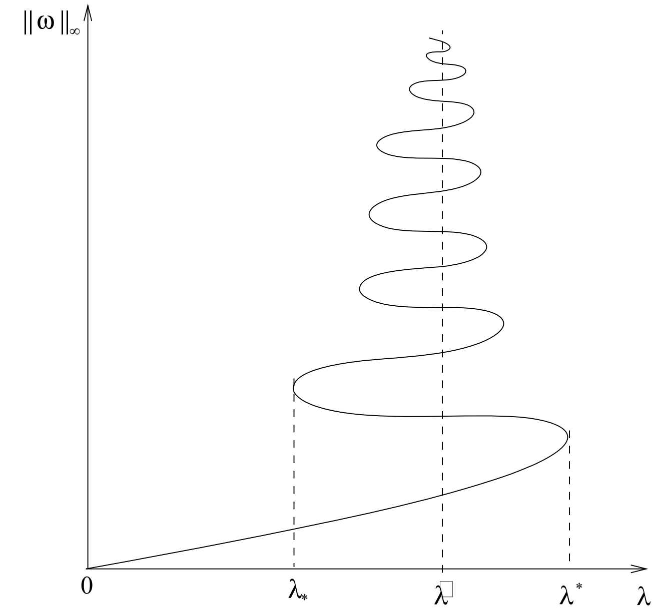

FIGURE 2. The bifurcation diagram of (3.15)-(3.16) when $2 \leq N < 7$.

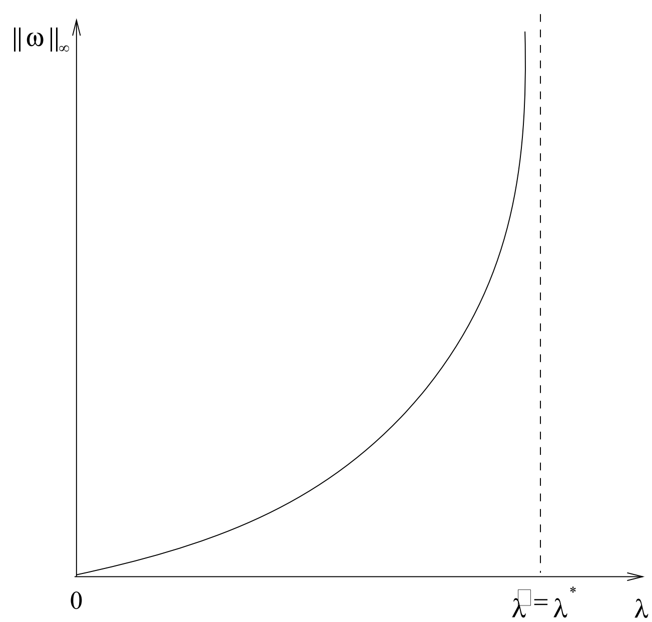

FIGURE 3. The bifurcation diagram of (3.15)-(3.16) when $N \geq 7$.

$\lambda(\tau_{2k+2}) < \cdots < \widehat{\lambda} < \cdots < \lambda(\tau_{2k+1}) < \lambda(\tau_{2k-1}) < \cdots < \lambda(\tau_3) < \lambda(\tau_1)$. *On the other hand, if* $N \geq 7$ *the function* $\tau \in (-\infty, \infty) \longmapsto \lambda(\tau)$ *is strictly increasing and for each* $\lambda \in (0, \widehat{\lambda})$ *the nonlocal steady-state problem (3.15)-(3.16) has a unique solution.*

Given $w \in \mathcal{C}^\lambda$, the linearized eigenvalue problem around (λ, w) is given as follows

$$-\Delta\phi - \frac{2\lambda\phi}{(1-w)^3\Big(1+\int_\Omega \frac{dx}{1-w}\Big)^2} + \frac{2\lambda\int_\Omega \frac{\phi dx}{(1-w)^2}}{(1-w)^2\Big(1+\int_\Omega \frac{dx}{1-w}\Big)^3} = \rho\phi,\ x \in \Omega, \tag{3.44}$$

$$\phi = 0, \quad x \in \partial\Omega. \tag{3.45}$$

The number of the negative eigenvalues ρ of problem (3.44)-(3.45), denoted by $i = i(\lambda, w)$, is called the Morse index at (λ, w). In the case $\Omega = B$, the number of the negative eigenvalues of the corresponding linearized eigenvalue problem around the radial symmetric solution $w(r) \in \mathcal{C}_r^\lambda$ is denoted by $i_R = i_R(\lambda, w)$ and is called the radial Morse index at (λ, w). Regarding the radial Morse index the following result holds. For an analogous result associated to the nonlocal Gelfand problem see [21, 22].

Theorem 3.8. *If $\Omega = B$, then $i(\lambda, w) = i_R(\lambda, w)$ and this index increases by one at each turning (bending) point of the solution curve $\mathcal{C}_r$.*

Proof. The first statement of the theorem is proven by using a similar argument to the proof of Proposition 3.3 in [20]. Indeed, when $\Omega = B$ any solution w to (3.1) is radially symmetric, i.e., $w(x) = w(r),\ r = |x|$. Hence, applying the separation of variables, any solution to (3.44)-(3.45) can be written in the form

$$\phi_k(x) = \psi_k(r)e_k(x/|x|),\ x \in B,\ k = 1, 2, \cdots,$$

where $\{e_k\}_{k=1}^\infty$ is the sequence of eigenfunctions of the Laplace operator on the unit sphere ∂B with corresponding eigenvalues $0 = \nu_1 < N-1 = \nu_2 \le \nu_3 \le \cdots$. Due to the radial symmetry of w and ψ_k we obtain

$$\int_B \frac{\phi_k dx}{(1-w)^2} = 0 \quad \text{for} \quad k \ge 2.$$

Then $\psi_k(r)$ satisfies the problem

$$-\psi_k'' - \frac{N-1}{r}\psi_k' + \frac{\nu_k}{r^2}\psi_k - \frac{2\lambda\psi_k}{(1-w)^3\Big(1+\int_B \frac{dx}{1-w}\Big)^2} = \rho\psi_k,\ 0 < r < 1, \tag{3.46}$$

$$\psi_k'(0) = \psi_k(1) = 0, \tag{3.47}$$

for any $k \ge 2$. Note that problem (3.46)-(3.47) has the same form with the problem obtained by (3.4) under linearization and separation of variables.

We claim that, under the assumption $\rho < 0$, $\psi_k = 0$ and so $\phi_k = 0$ for $k \ge 2$. Indeed, let assume that $\psi_k \ne 0$ and r_1 be the first positive zero of ψ_k such that $\psi_k > 0$ in $(0, r_1)$. Then using similar arguments as in [20] we derive for $0 < r < r_1$

$$\left[\frac{\partial\psi_k}{\partial r}\frac{\partial w}{\partial r}r^{N-1} - \psi_k\frac{\partial^2 w}{\partial r^2}r^{N-1}\right]_{r=0}^{r=r_1} + \int_0^{r_1} \frac{N-1-\nu_k}{r^2}\psi_k\frac{\partial w}{\partial r}r^{N-1}dr$$
$$= -\rho\int_0^{r_1} \psi_k\frac{\partial w}{\partial r}r^{N-1}dr. \tag{3.48}$$

Taking into account that $w(r)$ is radial decreasing as well as that $\nu_k \ge N-1$ for $k \ge 2$ we derive by (3.48) that $\rho > 0$, leading to a contradiction. Therefore under the hypothesis $\rho < 0$ we have that the solution of (3.44)-(3.45) could be written in the form $\phi(x) = c\psi(r)$, for some constant c with $\psi(r)$ satisfying (3.46)-(3.47), which implies the desired result $i(\lambda, w) = i_R(\lambda, w)$.

The second part of the theorem is proven by the method of [23]; note that each $(\lambda, w) \in \mathcal{C}_r$ corresponds to $(z(\tau), \dot{z}(\tau)) \in \mathcal{O}$ and therefore, it is parameterized by $\tau \in \mathbb{R}$: $(\lambda, w) = (\lambda(\tau), w(\tau))$. We denote by ρ_τ^ℓ, $\ell = 1, 2, \ldots$, the ℓ-th eigenvalue of the linearized problem (3.44)-(3.45) arround $(\lambda(\tau), w(\tau))$ which corresponds to a radially symmetric eigenfunction. Each ρ_τ^ℓ is simple. If $(\lambda(\tau), w(\tau))$ is on a turning point of $\mathcal{C}_r$, then, by the implicit function theorem, there is $\ell \geq 1$ such that $\rho_\tau^\ell = 0$. If $\rho_\tau^\ell = 0$ holds for some $\ell \geq 1$ with $(\lambda(\tau), w(\tau)) \in \mathcal{C}_r$ not on a turning point, then, by the bifurcation theory on the critical point of odd multiplicity, [29, 30], it is actually a bifurcation point of $\mathcal{C}_r$. But, this is impossible by the geometric features of $\mathcal{C}_r$ provided by Theorem 3.5, and therefore, $(\lambda(\tau), w(\tau))$ is on a turning point of $\mathcal{C}_r$ if and only if (3.44)- (3.45) has a zero eigenvalue $\rho_\tau^\ell = 0$ for some $\ell \geq 1$. Denoting the k-th turning point of $\mathcal{C}_r$ by $T_k = (\lambda(\tau_k), w(\cdot, \tau_k))$, $\tau_1 < \tau_2 < \cdots < \tau_k < \cdots$, then the assertion follows if we can prove that $\dot{\rho}_{\tau_k}^{\ell(k)} < 0$ for $k \geq 1$, where $\ell(k)$ is such that $\rho_{\tau_k}^{\ell(k)} = 0$.

By differentiating (3.1) with respect to τ, we deduce

$$\begin{aligned} \Delta \dot{w} + \frac{\dot{\lambda}(1-w)^{-2}}{\left(1+\int_B \frac{dx}{1-w}\right)^2} + \frac{2\lambda(1-w)^{-3}\dot{w}}{\left(1+\int_B \frac{dx}{1-w}\right)^2} & \\ - \frac{2\lambda(1-w)^{-2}\int_B (1-w)^{-2}\dot{w}dx}{\left(1+\int_B \frac{dx}{1-w}\right)^3} &= 0 \quad \text{in} \quad B, \qquad (3.49) \\ \dot{w} = 0 \quad \text{on} \quad \partial B, & \qquad (3.50) \end{aligned}$$

where $\dot{w} := \partial w / \partial \tau$ and $\dot{\lambda} := d\lambda / d\tau$. Hence $w_k := w(\cdot, \tau_k)$ satisfies

$$\begin{aligned} \Delta \dot{w}_k + \frac{2\lambda(1-w_k)^{-3}\dot{w}_k}{\left(1+\int_B \frac{dx}{1-w_k}\right)^2} & \\ - \frac{2\lambda(1-w_k)^{-2}\int_B (1-w_k)^{-2}\dot{w}_k dx}{\left(1+\int_B \frac{dx}{1-w_k}\right)^3} &= 0 \quad \text{in} \quad B, \\ \dot{w}_k = 0 \quad \text{on} \quad \partial B, & \end{aligned}$$

recalling that $\dot{\lambda}(\tau_k) = 0$.

We claim that

$$\dot{w}(r, \tau) = r^{2/3} \frac{z(\tau + \ln r)\dot{z}(\tau) - \dot{z}(\tau + \ln r) z(\tau)}{z^2(\tau)} \not\equiv 0, \ \forall \, \tau \in \mathbb{R}.$$

Indeed, if there is a $\tau \in \mathbb{R}$ such that $\dot{w}(r, \tau) \equiv 0$, then we have $\dot{z}/z$ is identically equal to a constant. This constant must be $-2/3$ by (3.23). Then (3.22) implies that $1/z^2 \equiv 0$, a contradiction. Hence the claim is proved and we obtain that $\dot{w}_k$ is an eigenfunction of the linearized problem (3.44)-(3.45) corresponding to $\rho = \rho_{\tau_k}^{\ell(k)} = 0$. Therefore, the standard perturbation theory ([15]) guarantees the existence of a function $\psi = \psi(\cdot, \tau)$ and $\rho = \rho(\tau)$ satisfying the linearized problem (3.44)-(3.45) such that $\psi(\cdot, \tau_k) = \dot{w}_k$ and $\rho(\tau_k) = \rho_{\tau_k}^{\ell(k)} = 0$.

Differentiating problems (3.49)-(3.50) and (3.44)-(3.45) with respect to τ and taking into account that $\dot\lambda(\tau_k) = \rho(\tau_k) = 0$, we obtain

$$\begin{aligned}
&\Delta \ddot w_k + \frac{\ddot\lambda(1-w_k)^{-2}}{\left(1+\int_B \frac{dx}{1-w_k}\right)^2} + \frac{6\lambda(1-w_k)^{-4}\dot w_k^2}{\left(1+\int_B \frac{dx}{1-w_k}\right)^2} + \frac{2\lambda(1-w_k)^{-3}\ddot w_k}{\left(1+\int_B \frac{dx}{1-w_k}\right)^2}\\
&-\frac{8\lambda(1-w_k)^{-3}\dot w_k\int_B(1-w_k)^{-2}\dot w_k dx}{\left(1+\int_B \frac{dx}{1-w_k}\right)^3} - \frac{4\lambda(1-w_k)^{-2}\int_B(1-w_k)^{-3}\dot w_k^2 dx}{\left(1+\int_B \frac{dx}{1-w_k}\right)^3}\\
&+\frac{6\lambda(1-w_k)^{-2}\Big(\int_B(1-w_k)^{-2}\dot w_k dx\Big)^2}{\left(1+\int_B \frac{dx}{1-w_k}\right)^4} - \frac{2\lambda(1-w_k)^{-2}\int_B(1-w_k)^{-2}\ddot w_k dx}{\left(1+\int_B \frac{dx}{1-w_k}\right)^3} = 0 \quad (3.51)
\end{aligned}$$

for $x \in B$ with $\ddot w_k = 0$ on ∂B, and

$$\begin{aligned}
&\Delta \dot\psi_k + \frac{6\lambda(1-w_k)^{-4}\psi_k\dot w_k}{\left(1+\int_B \frac{dx}{1-w_k}\right)^2} + \frac{2\lambda(1-w_k)^{-3}\dot\psi_k}{\left(1+\int_B \frac{dx}{1-w_k}\right)^2}\\
&-\frac{4\lambda(1-w_k)^{-3}\psi_k\int_B(1-w_k)^{-2}\dot w_k dx}{\left(1+\int_B \frac{dx}{1-w_k}\right)^3} - \frac{4\lambda(1-w_k)^{-3}\dot w_k\int_B(1-w_k)^{-2}\psi_k dx}{\left(1+\int_B \frac{dx}{1-w_k}\right)^3}\\
&-\frac{4\lambda(1-w_k)^{-2}\int_B(1-w_k)^{-3}\psi_k^2 dx}{\left(1+\int_B \frac{dx}{1-w_k}\right)^3} + \frac{6\lambda(1-w_k)^{-2}\Big(\int_B(1-w_k)^{-2}\psi_k dx\Big)^2}{\left(1+\int_B \frac{dx}{1-w_k}\right)^4}\\
&-\frac{2\lambda(1-w_k)^{-2}\int_B(1-w_k)^{-2}\dot\psi_k dx}{\left(1+\int_B \frac{dx}{1-w_k}\right)^3} = -\dot\rho\psi_k \qquad (3.52)
\end{aligned}$$

for $x \in B$ with $\dot\psi_k = 0$ on ∂B. Using $\psi_k = \dot w_k$, it follows from (3.51) and (3.52) that

$$\frac{\ddot\lambda(\tau_k)(1-w_k)^{-2}}{\left(1+\int_B \frac{dx}{1-w_k}\right)^2} = \dot\rho(\tau_k)\dot w_k. \qquad (3.53)$$

Multiplying (3.53) by $\dot w_k$ and integrating over B, we derive that

$$\frac{\ddot\lambda(\tau_k)\int_B(1-w_k)^{-2}\dot w_k dx}{\left(1+\int_B \frac{dx}{1-w_k}\right)^2} = \dot\rho(\tau_k)\int_B \dot w_k^2 dx. \qquad (3.54)$$

Thus, in order to obtain the desired result we need to prove that the left-hand side of (3.54) is negative.

Using that $\dot\lambda(\tau_k) = 0$ relation (3.33) implies

$$\dot G(\tau_k) = \frac{3}{2}\frac{\dot z(\tau_k)G(\tau_k)}{z(\tau_k)}, \qquad (3.55)$$

while (3.34) also reads

$$\dot z(\tau_k)\Big(G(\tau_k)+2\Big) = 2\Big(N\omega_N - \gamma(G(\tau_k)-1)\Big)z(\tau_k) \qquad (3.56)$$

for every $k = 1, 2, \ldots$.

Differentiating (3.34) with respect to τ, we obtain

$$\begin{aligned}\ddot{\lambda}(\tau) &= \Lambda\frac{\dot{G}(\tau)z^4(\tau)-4G(\tau)z^3(\tau)\dot{z}(\tau)}{z^8(\tau)}\\ &\times\Big[-\dot{z}(\tau)(G(\tau)+2)+2\Big(N\omega_N-\gamma(G(\tau)-1)\Big)z(\tau)\Big]\\ &+\frac{\Lambda G(\tau)}{z^4(\tau)}\Big[-\ddot{z}(\tau)(G(\tau)+2)-\dot{z}(\tau)\dot{G}(\tau)-2\gamma\dot{G}(\tau)z(\tau)\\ &+2\Big(N\omega_N-\gamma(G(\tau)-1)\Big)\dot{z}(\tau)\Big]\end{aligned}$$

and due to (3.56) yields

$$\begin{aligned}\ddot{\lambda}(\tau_k) &= \frac{\Lambda G(\tau_k)}{z^4(\tau_k)}\Big[-\ddot{z}(\tau_k)(G(\tau_k)+2)-\dot{z}(\tau_k)\dot{G}(\tau_k)-2\gamma\dot{G}(\tau_k)z(\tau_k)\\ &+2\Big(N\omega_N-\gamma(G(\tau_k)-1)\Big)\dot{z}(\tau_k)\Big]. \end{aligned}\tag{3.57}$$

Now (3.57), via (3.55), yields

$$\begin{aligned}\ddot{\lambda}(\tau_k) &= \frac{2}{3}\frac{\Lambda\dot{G}(\tau_k)}{z^3(\tau_k)}\Big[-\frac{\ddot{z}(\tau_k)}{\dot{z}(\tau_k)}(G(\tau_k)+2)-\dot{G}(\tau_k)-2\gamma\dot{G}(\tau_k)\frac{z(\tau_k)}{\dot{z}(\tau_k)}\\ &\quad+2\Big(N\omega_N-\gamma(G(\tau_k)-1)\Big)\Big].\end{aligned}$$

Using (3.32), (3.55) and (3.56), we finally end up with the following

$$\ddot{\lambda}(\tau_k)=\frac{2}{3}\frac{\Lambda\dot{G}(\tau_k)}{z^3(\tau_k)}\Big[-\frac{\ddot{z}(\tau_k)}{\dot{z}(\tau_k)}(G(\tau_k)+2)-\gamma(1+2G(\tau_k))+3\frac{\dot{z}(\tau_k)}{z(\tau_k)}-N\omega_N\Big].$$

Noting that

$$\dot{G}(\tau_k)=\int_B(1-w_k)^{-2}\dot{w}_k dx,$$

it follows by (3.54)

$$\frac{2\Lambda\dot{G}(\tau_k)^2}{3z^3(\tau_k)}\frac{C(\tau_k)}{\left(1+\int_B\frac{dx}{1-w_k}\right)^2}=\dot{\rho}(\tau_k)\int_B\dot{w}_k^2dx,$$

where

$$C(\tau_k):=-\frac{\ddot{z}(\tau_k)}{\dot{z}(\tau_k)}(G(\tau_k)+2)-\gamma(1+2G(\tau_k))+3\frac{\dot{z}(\tau_k)}{z(\tau_k)}-N\omega_N.$$

Therefore, it is enough to prove that $C(\tau_k)<0$ as well as that $\dot{G}(\tau_k)\neq 0$.

Note that for $\tau=\tau_{2k-1}$ there holds

$$\frac{\ddot{z}(\tau_{2k-1})}{\dot{z}(\tau_{2k-1})}>0,$$

since $\dot{z}(\tau_{2k-1})>0$ and $\ddot{z}(\tau_{2k-1})>0$. The latter actually holds because the point $R_{2k-1}=(z(\tau_{2k-1}),\dot{z}(\tau_{2k-1}))$ is below the curve Γ, i.e. $\dot{z}(\tau_{2k-1})<f(z(\tau_{2k-1}))$ and due to the orbit equation (3.22) the strictly positivity of $\ddot{z}(\tau_{2k-1})$ is obtained, see also Fig. 1. Analogously,

we have that $\dot{z}(\tau_{2k}) < 0$ and $\ddot{z}(\tau_{2k}) < 0$, since now $R_{2k} = (z(\tau_{2k}), \dot{z}(\tau_{2k}))$ is above the curve Γ. Hence at every turning point $R_k = (z(\tau_k), \dot{z}(\tau_k))$ we have

$$\frac{\ddot{z}(\tau_k)}{\dot{z}(\tau_k)} > 0. \tag{3.58}$$

Furthermore, by (3.24)

$$3\frac{\dot{z}(\tau_k)}{z(\tau_k)} - N\omega_N < 2 - N\omega_N < 0 \quad \text{for any} \quad N \geq 2. \tag{3.59}$$

Hence $C(\tau_k)$ is strictly negative as an immediate consequence of (3.58) and (3.59) since also $G(\tau) > 0$.

Moreover, differentiating (3.30) with respect to τ we obtain

$$\dot{\mu}(\tau_k) = -\frac{2\lambda(\tau_k)\dot{G}(\tau_k)}{G^3(\tau_k)}. \tag{3.60}$$

On the other hand, differentiating (3.26) with respect to τ we have

$$\dot{\mu}(\tau_k) = -3\Lambda z^{-4}(\tau_k)\dot{z}(\tau_k) \neq 0, \tag{3.61}$$

since at $\tau = \tau_k$ the orbit $\mathcal{O}$ does not meet the curve z-axis. Therefore by (3.60) and (3.61) we derive that $\dot{G}(\tau_k) \neq 0$ and we finally conclude that $\dot{\rho}(\tau_k) < 0$ which proves the desired assertion. This completes the proof of the theorem. □

Remark 3.9. *The above theorem characterizes the stage of stability of the steady-state solutions. In particular, for $2 \leq N < 7$ we have $i = i_R = k$ on the arc T_kT_{k+1} of the curve $\mathcal{C}_r$, where $T_k = (\lambda(\tau_k), w(\tau_k))$, $k = 0, 1, 2, \dots$ with $\tau_0 = -\infty$. That means that moving from the minimal to the maximal branch of the steady-state problem, for $0 < \lambda < \widehat{\lambda}$ the solutions become less stable. On the other hand, there always holds $i = i_R = 0$ for $N \geq 7$, i.e. the unique steady-state solution for $0 < \lambda < \widehat{\lambda}$ is asymptotically stable.*

Remark 3.10. *Following the same arguments as above, the second statement of Theorem 3.8 could be also proved for problem (3.6) when $f(x) = |x|^p$, $p > 0$, see also Theorem 1.3 in* [16]. *On the other hand, the first statement of Theorem 3.8 does not hold for problem (3.6), since general theorems, see* [4, 24], *of radial symmetry for stationary problems are not valid when $f_r > 0$, $r = |x|$ (which is the case for $f(x) = |x|^p$, $p > 0$).*

4. Global existence and quenching

Given $\lambda \in (0, (1+|\Omega|)^2)\mu^*)$, by the result of section 3, there exists a minimal steady-state solution of (P), denoted by w_λ, such that $w_\lambda = w_\mu$ with μ satisfying (3.7). Then we have the following theorem on the global existence for small values of λ.

Theorem 4.1. *For any $\lambda \in (0, (1+|\Omega|)^2\mu^*)$, the problem (P) with $u_0 \leq w_\lambda$ has a solution which exists globally in time and converges to the minimal steady state w_λ as $t \to \infty$.*

Proof. By Corollary 2.4, we obtain that $0 \leq u \leq z \leq w_\lambda$, where z is the unique solution of the local problem (Q) with $z_0 = u_0 \leq w_\lambda$. Therefore we derive that $0 \leq u \leq z \leq w_\mu$ for $0 < \mu = \lambda/(1+|\Omega|)^2 < \mu^*$. But it is known that the local problem (Q) has a global-in-time solution for $0 < \mu < \mu^*$ and $z_0 \leq w_\mu$ hence the nonlocal problem (P) does so.

The proof of convergence result can be carried out by a similar argument as that in [8] and so we omit it here. □

Next, we shall show that the solution of problem (P) quenches in finite time for certain initial data. It is easy to see that problem (P) admits an energy functional of the form

$$E[u](t) \equiv E(t) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{\lambda}{\left(1+\int_\Omega (1-u)^{-1}dx\right)}, \tag{4.1}$$

which decreases with respect to time across any solution of problem (P). More precisely there holds

$$\frac{dE}{dt} = -\int_\Omega u_t^2(x,t)dx < 0, \tag{4.2}$$

hence

$$0 \le \int_0^T \int_\Omega u_t^2(x,t)dxdt = E(0) - E(T) \le E(0) < \infty$$

for any $0 < T < T_{max}$, where T_{max} is the maximal existence time of problem (P). The following quenching result reveals the fact that quenching can also be controlled via initial conditions too.

Theorem 4.2. *Assume that $|\Omega| \le 1/2$, then for any fixed $\lambda > 0$, there exist initial data such that the solution of problem (P) quenches in finite time provided the corresponding initial energy is chosen sufficiently small.*

Proof. Suppose that the problem (P) has a global-in-time (classical) solution u. Set

$$A(t) = \int_\Omega u^2(x,t)dx.$$

Multiplying equation (1.1) by u and integrating by parts over Ω, we derive

$$\begin{aligned}\frac{1}{2}\frac{dA}{dt} &= \int_\Omega u\Big[\Delta u + \frac{\frac{\lambda}{(1-u)^2}}{\left(1+\int_\Omega (1-u)^{-1}dx\right)^2}\Big]dx \\ &= -\int_\Omega |\nabla u|^2 dx + \lambda\frac{\int_\Omega \frac{u}{(1-u)^2}dx}{\left(1+\int_\Omega (1-u)^{-1}dx\right)^2}.\end{aligned} \tag{4.3}$$

Using (4.1) and the energy dissipation formula (4.2), relation (4.3) reads

$$\begin{aligned}\frac{1}{2}\frac{dA}{dt} &= -2E(t) + \frac{2\lambda}{1+\int_\Omega (1-u)^{-1}dx} + \lambda\frac{\int_\Omega \frac{u}{(1-u)^2}dx}{\left(1+\int_\Omega (1-u)^{-1}dx\right)^2} \\ &\ge -2E(0) + \lambda\frac{2\left(1+\int_\Omega \frac{dx}{1-u}\right) + \int_\Omega \frac{u}{(1-u)^2}dx}{\left(1+\int_\Omega (1-u)^{-1}dx\right)^2} \\ &= -2E(0) + \lambda\frac{2+\int_\Omega \frac{2-u}{(1-u)^2}dx}{\left(1+\int_\Omega (1-u)^{-1}dx\right)^2}.\end{aligned} \tag{4.4}$$

On the other hand, using Hölder's and Young's inequalities, we can deduce that

$$\left(1+\int_\Omega \frac{dx}{1-u}\right)^2 \le 2 + 2|\Omega|\int_\Omega \frac{dx}{(1-u)^2}. \tag{4.5}$$

Combining (4.4) and (4.5) yields

$$\frac{1}{2}\frac{dA}{dt} \geq -2E(0) + \lambda \frac{2 + \int_\Omega \frac{2-u}{(1-u)^2}dx}{2 + 2|\Omega| \int_\Omega \frac{dx}{(1-u)^2}}.$$

Under the assumption that $|\Omega| \leq 1/2$ and using also the fact that

$$\int_\Omega \frac{2-u}{(1-u)^2}dx \geq |\Omega|$$

for a classical solution u (i.e. $||u||_\infty < 1$) we derive

$$\frac{1}{2}\frac{dA}{dt} \geq -2E(0) + \lambda.$$

The latter relation implies that

$$|\Omega| \geq A(t) \geq 2\Big(\lambda - 2E(0)\Big)t, \tag{4.6}$$

and hence by (4.6) we derive

$$A(t) \to |\Omega| \quad \text{as} \quad t \to T^* := \frac{1}{2}\Big(\lambda - 2E(0)\Big)^{-1}|\Omega|$$

which implies that $u(x,t)$ quenches in finite time $t_q \leq T^*$, i.e.

$$||u(\cdot,t)||_\infty \to 1 \quad \text{as} \quad t \to t_q \leq T^*$$

provided that

$$E(0) < \frac{\lambda}{2}. \tag{4.7}$$

But this contradicts the initial assumption that $T_{max} = \infty$. The later leads to the conclusion of the theorem. □

Remark 4.3. *Note that the condition* (4.7) *holds if, for example,* $N = 1$, $\Omega = (0, 1/2)$ *and we choose*

$$u_0(x) = \begin{cases} \dfrac{1}{1-\delta}x, & 0 \leq x < \frac{\delta}{2}(1-\delta), \\ \frac{\delta}{2}, & \frac{\delta}{2}(1-\delta) \leq x \leq \frac{1}{2} - \frac{\delta}{2}(1-\delta), \\ \frac{1}{1-\delta}(\frac{1}{2} - x), & \frac{1}{2} - \frac{\delta}{2}(1-\delta) \leq x \leq \frac{1}{2}, \end{cases}$$

where $0 < \delta < 1$ *and for* λ *sufficiently large.*

Indeed for such initial data we have

$$E_0 := \frac{1}{2}\int_0^{1/2}(u_0'(x))^2dx + \frac{\lambda}{1 + \int_0^{1/2}\frac{dx}{1-u_0(x)}} = \frac{\delta}{2(1-\delta)} + \frac{\lambda}{1 + \frac{1}{2-\delta} - 2(1-\delta)\ln(1-\frac{\delta}{2})}$$

and then condition (4.7) *is satisfied for any* $\lambda > 5.5$ *by choosing for example* $0.97 < \delta < 1$. *Such initial data as above, leading to quenching, could be constructed in higher dimensions as well.*

Acknowledgments

We would like to thank P. Esposito for sending us the manuscript [2]. N.I. Kavallaris would also like to thank T. Suzuki for stimulating discussions during the preparation of this manuscript. Part of this work was completed while the second author was visiting the Department of Mathematics, National Taiwan Normal University, in July, 2010, and he would like to express his thanks for the warm hospitality. We also thank the referee for some valuable comments.

References

[1] P. Esposito, N. Ghoussoub and Y. Guo, *Compactness along the first branch of unstable solutions for an elliptic problem with a singular nonlinearity*, Comm. Pure Appl. Math. **60** (2007), 1731–1768.

[2] P. Esposito, *Compactness of a nonlinear eigenvalue problem with a singular nonlinearity*, Comm. Contemp. Math. **10** (2008), 17–45.

[3] P. Esposito and N. Ghoussoub, *Uniqueness of solutions for an elliptic equation modeling MEMS*, Methods Appl. Anal. **15** (2008), 341–354.

[4] B. Gidas, W.-M. Ni, and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979) 209–243.

[5] N. Ghoussoub and Y. Guo, *On the partial differential equations of electrostatic MEMS devices: stationary case*, SIAM J. Math. Anal. **38** (2007), 1423–1449.

[6] N. Ghoussoub and Y. Guo, *On the partial differential equations of electrostatic MEMS devices II: dynamic case*, Nonlinear Diff. Eqns. Appl. **15** (2008), 115–145.

[7] J.-S. Guo, *Quenching problem in nonhomogeneous media*, Differential and Integral Equations **10** (1997), 1065-1074.

[8] J.-S. Guo, B. Hu and C.-J. Wang, *A nonlocal quenching problem arising in micro-electro mechanical systems*, Quarterly Appl. Math. **67** (2009), 725–734.

[9] Y. Guo, Z. Pan, and M.J. Ward, *Touchdown and pull-in voltage behavior of a MEMS device with varying dielectric properties*, SIAM J.Appl. Math **166** (2006), 309–338.

[10] Y. Guo, *On the partial differential equations of electrostatic MEMS devices III: refined touchdown behavior*, J. Diff. Eqns., **244** (2008), 2277–2309.

[11] Y. Guo, *Global solutions of singular parabolic equations arising from electrostatic MEMS*, J. Diff. Eqns., *245* (2008), 809–844.

[12] Z. Guo and J. Wei, *Asymptotic Behavior of touch-down solutions and global bifurcations for an elliptic problem with a singular nonlinearity*, Comm. Pure Appl. Anal. **7** (2008), 765–786.

[13] G. Flores, G. Mercado, J.A. Pelesko and N. Smyth, *Analysis of the dynamics and touchdown in a model of electrostatic MEMS*, SIAM J. Appl. Math. **67** (2007), 434–446.

[14] D.D. Joseph and T.S. Lundgren, *Quasilinear Dirichlet problems driven by positive sources*, Arch. Rational Mech. Anal. **49** (1973), 241–269.

[15] T. Kato, *Perturbation Theory for Linear Operators*, Springer, Berlin, 1966.

[16] N.I. Kavallaris, T. Miyasita, and T. Suzuki, *Touchdown and related problems in electrostatic MEMS device equation*, Nonlinear Diff. Eqns. Appl. **15** (2008), 363–385.

[17] N.I. Kavallaris, A.A. Lacey, C.V. Nikolopoulos and D.E. Tzanetis, *On the behaviour of a nonlocal hyperbolic problem arising in MEMS control*, to appear in Rocky Mountain J. Math.

[18] A. A. Lacey, *Thermal runaway in a non–local problem modelling Ohmic heating. Part I: Model derivation and some specail cases*, Euro. J. Appl. Math. **6** (1995), 127–144.

[19] H.A. Levine, *Quenching, nonquenching, and beyond quenching for solution of some parabolic equations*, Ann. Mat. Pura Appl. **155** (1989), 243–260.

[20] C.-S. Lin and W.-M. Ni, *A counterexample to the nodal domain conjecture and a related semilinear equation*, Proc. Amer. Math. Soc. **102** (1988), 271–277.

[21] T. Miyasita, *Non-local elliptic problem in higher dimension*, Osaka J. Math. **44** (2007), 159–172.

[22] T. Miyasita and T. Suzuki, *Non-local Gel'fand problem in higher dimension*, Nonlocal Elliptic and Parabolic Problems, Banach Center Publ. **66** (2004), 221–235.

[23] K. Nagasaki and T. Suzuki, *Spectral and related properties about the Emden-Fowler equation $-\Delta u = \lambda e^u$ on circular domains*, Math. Ann. **299** (1994), 1–15.

[24] Y. Naito and T. Suzuki, *Radial symmetry of positive solutions for semilinear elliptic equations on the unit ball in* $\mathbb{R}^n$, Funkcial Ekvac. **41** (1998) 215–234.
[25] J.A. Pelesko and A.A. Triolo, *Nonlocal problems in MEMS device control,* J. Engrg. Math., **41** (2001), 345–366.
[26] J.A. Pelesko and D.H. Bernstein, *Modeling MEMS and NEMS*, Chapman Hall and CRC Press, 2002.
[27] S. I. Pohozaev, *Eigenfunctions of the equation* $\Delta u + \lambda f(u) = 0$, Doklady **165** (1965), 1408–1411.
[28] P. Quittner and Ph. Souplet, *Superlinear parabolic problems. Blow-up, global existence and steady states*, Birkhäuser Advanced Texts, Birkhäuser Verlag, Basel, 2007.
[29] P.H. Rabinowitz, *Some global results for nonlinear eigenvalue problems*, J. Func. Anal. **7** (1971), 487–513.
[30] P.H. Rabinowitz, *Some aspects of nonlinear eigenvalue problems*, Rocky Mountain J. Math. **3** (1973), 161–202.
[31] R. Schaaf, *Uniqueness for semilinear elliptic problems supercritical growth and domain geometry*, Adv. Diff. Equations **5** (2000), 1201–1220.

Department of Mathematics, Tamkang University, Tamsui, New Taipei City 25137, Taiwan
E-mail address: jsguo@mail.tku.edu.tw

Department of Statistics and Actuarial-Financial Mathematics, University of Aegean, TGr-83200 Karlovassi, Samos, Greece
E-mail address: nkaval@aegean.gr