EXISTENCE AND UNIQUENESS OF TRAVELING WAVES FOR A MONOSTABLE 2-D LATTICE DYNAMICAL SYSTEM

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Abstract. We study traveling waves for a two-dimensional lattice dynamical system with monostable nonlinearity. We prove that there is a minimal speed such that a traveling wave exists if and only if its speed is above this minimal speed. Then we show the uniqueness (up to translations) of wave profile for each given speed. Moreover, any wave profile is strictly monotone.

1. Introduction

In this paper, we study the existence and uniqueness of traveling waves to the following two-dimensional (2-D) lattice dynamical system:

$$\dot{u}_{i,j} = u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} + f(u_{i,j}), \quad i,j \in \mathbb{Z}, \tag{1.1}$$

where f is monostable: $f(0) = f(1) = 0 < f(u)$, $\forall u \in (0,1)$. The equation (1.1) is a spatial discrete version of the following reaction-diffusion equation

$$u_t = \Delta u + f(u), \quad x \in \mathbb{R}^N,\ t \in \mathbb{R}, \tag{1.2}$$

for $N = 2$. When $f(u) = u(1-u)$, the equation (1.2) is called Fisher's equation [9] or KPP equation [11] which arises in the study of gene development or population dynamics.

A solution $\{u_{i,j}\}_{i,j\in\mathbb{Z}}$ is called a traveling wave with speed c, if there exists a $\theta \in [0, 2\pi)$ and a differentiable function $U : \mathbb{R} \to [0,1]$ such that $U(-\infty) = 1$, $U(+\infty) = 0$, and $u_{i,j}(t) = U(ip + jq - ct)$ for all $i,j \in \mathbb{Z}$, $t \in \mathbb{R}$, where $p := \cos\theta$ and $q := \sin\theta$. The parameter θ represents the direction of movement of wave and U is called the wave profile. Set $\xi := ip + jq - ct$. Then it is easy to see that (1.1) has a traveling wave with speed c if and only if the equation

$$c\,U'(\xi) + D_2[U](\xi) + f(U(\xi)) = 0,\ \xi \in \mathbb{R}, \tag{1.3}$$

has a solution U defined on $\mathbb{R}$ with $0 \le U \le 1$, $U(-\infty) = 1$, and $U(+\infty) = 0$, where

$$D_2[U](\xi) := U(\xi + q) + U(\xi + p) + U(\xi - q) + U(\xi - p) - 4U(\xi).$$

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In particular, if $\theta = 0$, then the problem (1.1) is reduced to a one-dimensional (1-D) lattice dynamical system on $\mathbb{Z}$.

The study of traveling wave for lattice dynamical systems has attracted a lot attentions for past years. The main concerns are the existence, uniqueness, and stability of traveling waves for the lattice dynamical system. For the 1-D lattice dynamical system, we refer the readers to, e.g., [3]-[7], [10, 12, 13], [15]-[18] and the references cited therein. The nonlinearity f under consideration in the above references is either monostable or bistable. Here f is called a bistable nonlinearity, if there is $a \in (0,1)$ such that $f(0) = f(a) = f(1) = 0$, $f'(0) < 0$, and $f'(1) < 0$.

On the other hand, Cahn, Mallet-Paret, and van Vleck [1] studied a two-dimensional (2-D) lattice dynamical system with bistable nonlinearity. They obtained the existence and non-existence (so-called propagation failure) of traveling waves for the studied lattice dynamical system. The purpose of this paper is to study a 2-D lattice dynamical system with monostable nonlinearity.

We shall make the following assumptions.

(A) $f \in C^1([0,1])$, $f(0) = f(1) < f(u)$, $\forall u \in (0,1)$ and $f'(0) > 0$.

(B) There exists $M_0 = M_0(f) > 0$ and $\alpha \in (0,1]$ such that

$$f'(0)u - M_0 u^{1+\alpha} \le f(u) \le f'(0)u, \ \forall u \in [0,1]. \tag{1.4}$$

(C) $f'(1) < 0$ and $f(u) - f'(1)(u-1) = O(|u-1|^{1+\alpha})$ as $u \to 1^-$.

By the symmetry of $D_2[U]$, we may only consider $\theta \in [0, \pi/2)$. Since we are dealing with a 2-D problem, we shall always assume that $\theta \in (0, \pi/2)$. Therefore, for a given $\theta \in (0, \pi/2)$, our problem is to find $(c, U) \in \mathbb{R} \times C^1(\mathbb{R})$ such that

$$\begin{cases} cU'(\xi) + D_2[U](\xi) + f(U(\xi)) = 0, \ \xi \in \mathbb{R}, \\ U(+\infty) = 0, \quad U(-\infty) = 1, \\ 0 \le U(\xi) \le 1 \ \forall \xi \in \mathbb{R}. \end{cases} \tag{1.5}$$

Note that, by integrating (1.3) from $-\infty$ to $+\infty$, we have

$$c = \int_{-\infty}^{\infty} f(U(\xi)) d\xi \tag{1.6}$$

for any solution (c, U) of (1.5). Hence $c > 0$ for any solution (c, U) of (1.5).

We now state the main results of this paper as follows.

Theorem 1. *Assume* **(A)** and **(B)**. *Then the following holds:*
(i) *The problem* (1.5) *admits a solution if and only if* $c \ge c_*$, *where*

$$c_* := \min_{\lambda > 0} \{ \frac{e^{\lambda q} + e^{\lambda p} + e^{-\lambda q} + e^{-\lambda p} - 4 + f'(0)}{\lambda} \}.$$

(ii) *Every solution* (c, U) *of* (1.5) *satisfies* $0 < U(\xi) < 1, \forall \xi \in \mathbb{R}$.
(iii) *For each* $c \ge c_*$, (1.5) *admits a solution* (c, U) *with* $U' < 0$ *on* $\mathbb{R}$.

Theorem 2. *Assume* **(A)**, **(B)**, *and* **(C)**. *Then, for each* $c \geq c_*$, *wave profiles of* (1.5) *are unique up to translations.*

To prove this uniqueness theorem, we need the following result on the monotonicity of wave profiles.

Theorem 3. *Assume* **(A)**, **(B)**, *and* **(C)**. *Then all wave profiles of* (1.5) *are strictly decreasing.*

To prove the existence of traveling waves, we use the monotone iteration method developed by Wu and Zou [15] (see also [4, 10]) with the help of a pair of super-sub-solutions. We shall define the notion of super-sub-solutions and prove a key lemma for the existence of traveling wave in §2. Then, in §3, we prove Theorem 1.

To derive the uniqueness of wave profiles, we shall first apply Ikehara's Theorem (cf. [14, 8]) to study the asymptotic behavior of wave profiles. This idea is originated from Carr and Chmaj [2] in studying the uniqueness of waves for a nonlocal monostable equation. To derive the asymptotic behavior of wave profiles, another method can be found in [5, 6] for 1-D case. Here we use a different method which can be easily applied to any higher dimensional case. With this information on the asymptotic behaviors of wave tails, we then apply a method developed in [5] to prove Theorem 2 in §4.

Finally, we remark that the existence and uniqueness results presented in this paper for 2-D case can be extended to general higher dimensional case. But, the stability of these traveling waves in the multi-dimensional case is much more complicated. We leave here as an open problem for the future study.

2. Preliminaries

First, we define the notion of super-sub-solutions. Given a positive constant c. A non-increasing continuous function U^+ is called a *super-solution* of (1.5), if $U^+(+\infty) = 0$ and U^+ is differentiable a.e. in $\mathbb{R}$ such that

$$-c(U^+)' - D_2[U^+] - f(U^+) \geq 0 \ \text{ a.e. in } \mathbb{R}.$$

A continuous function U^- is called a *sub-solution* of (1.5), if $U^-(+\infty) = 0$, $U^- \not\equiv 0$, and U^- is differentiable a.e. in $\mathbb{R}$ such that

$$-c(U^-)' - D_2[U^-] - f(U^-) \leq 0 \ \text{ a.e. in } \mathbb{R}.$$

Next, we introduce the operator $H_\mu : C(\mathbb{R}) \to C(\mathbb{R})$ by

$$H_\mu(U)(\xi) = \mu U(\xi) + \frac{1}{c} D_2[U](\xi) + \frac{1}{c} f(U(\xi))$$

for any constant $\mu > (4 + \max_{0\le u\le 1} |f'(u)|)/c$. It is easy to see that U satisfies (1.3) and $U(+\infty) = 0$ if and only if U satisfies

$$U(\xi) = e^{\mu\xi} \int_{\xi}^{\infty} e^{-\mu s} H_\mu(U)(s)ds = \int_{\xi}^{\infty} e^{-\mu(s-\xi)} H_\mu(U)(s)ds,\ \xi \in \mathbb{R}. \tag{2.1}$$

Here, by choosing $\mu > (4 + \max_{0\le u\le 1} |f'(u)|)/c$, we see that (2.1) is well-defined and the following property holds:

$$H_\mu(U)(\xi) \le H_\mu(V)(\xi),\ \forall \xi \in \mathbb{R}, \ \text{ if } 0 \le U \le V \le 1 \text{ in } \mathbb{R}. \tag{2.2}$$

Lemma 2.1. *Assume* **(A)**. *Then* (1.5) *has a solution U satisfying $U' \le 0$, if there exists a super-solution U^+ and a sub-solution U^- of* (1.5) *such that $0 \le U^- \le U^+ \le 1$ in $\mathbb{R}$.*

Proof. Assume that there exist a super-solution U^+ and a sub-solution U^- of (1.5) such that $0 \le U^- \le U^+ \le 1$ in $\mathbb{R}$. Define

$$U_1(\xi) = e^{\mu\xi} \int_{\xi}^{\infty} e^{-\mu s} H_\mu(U^+)(s)ds,\ \xi \in \mathbb{R}.$$

Then U_1 is a well-defined C^1 function. Form the definition of super-solution, we have

$$U^+(\xi) \ge e^{\mu\xi} \int_{\xi}^{\infty} e^{-\mu s} H_\mu(U^+)(s)ds = U_1(\xi),\ \forall \xi \in \mathbb{R}.$$

Also, by the definition of sub-solution and the property (2.2) of H_μ, we get

$$U^-(\xi) \le e^{\mu\xi} \int_{\xi}^{\infty} e^{-\mu s} H_\mu(U^-)(s)ds \le e^{\mu\xi} \int_{\xi}^{\infty} e^{-\mu s} H_\mu(U^+)(s)ds = U_1(\xi),\ \forall \xi \in \mathbb{R}.$$

Hence $U^-(\xi) \le U_1(\xi) \le U^+(\xi)$ for all $\xi \in \mathbb{R}$. Moreover, we have

$$U_1'(\xi) = \mu e^{\mu\xi} \int_{\xi}^{\infty} e^{-\mu s} \{H_\mu(U^+)(s) - H_\mu(U^+)(\xi)\} ds \le 0,$$

since $H_\mu(U^+)(s) \le H_\mu(U^+)(\xi)$ for all $s \ge \xi$, by using the fact that U^+ is non-increasing and $\mu > (4 + \max_{0\le u\le 1} |f'(u)|)/c$.

Now, we define

$$U_{n+1}(\xi) = e^{\mu\xi} \int_{\xi}^{\infty} e^{-\mu s} H_\mu(U_n)(s)ds,\ n = 1, 2, \cdots .$$

By induction, it is easy to see that $0 \le U^- \le U_{n+1} \le U_n \le U^+ \le 1$ and $U_{n+1}' \le 0$ in $\mathbb{R}$ for all $n \ge 1$. Then the limit $U(\xi) := \lim_{n\to+\infty} U_n(\xi)$ exists for all $\xi \in \mathbb{R}$ and $U(\xi)$ is non-increasing in $\mathbb{R}$. By Lebesgue's Dominated Convergence Theorem, U satisfies (2.1). Hence U satisfies (1.3).

Finally, we claim that $U(+\infty) = 0$ and $U(-\infty) = 1$. Since U is non-increasing and bounded, both $U(+\infty)$ and $U(-\infty)$ exist. From $0 \le U(\xi) \le U^+(\xi)$ and $U^+(+\infty) = 0$, it follows that $U(+\infty) = 0$. By L'Hospital's rule, we have

$$\lim_{\xi\to-\infty} U(\xi) = \lim_{\xi\to-\infty} \frac{e^{-\mu\xi} H_\mu(U)(\xi)}{\mu e^{-\mu\xi}} = \lim_{\xi\to-\infty} \frac{\mu U(\xi) + D_2[U(\xi)]/c + f(U(\xi))/c}{\mu}.$$

This implies that $f(U(-\infty)) = 0$. Hence $U(-\infty) \in \{0,1\}$. Since $U^-(\xi_0) > 0$ for some $\xi_0 \in \mathbb{R}$, we have $U(-\infty) \geq U(\xi_0) \geq U^-(\xi_0) > 0$. Thus $U(-\infty) = 1$. The lemma follows. □

Recall that $p := \cos\theta$ and $q := \sin\theta$ for a given $\theta \in (0, \pi/2)$.

Lemma 2.2. *Assume $f'(0) > 0$. Set*

$$C(\lambda) := \frac{e^{\lambda q} + e^{\lambda p} + e^{-\lambda q} + e^{-\lambda p} - 4 + f'(0)}{\lambda},$$
$$\Psi(c,\lambda) := c\lambda - [e^{\lambda q} + e^{\lambda p} + e^{-\lambda q} + e^{-\lambda p} - 4 + f'(0)].$$

Then there exists a unique $\lambda_* > 0$ such that $C(\lambda_*) = \min_{\lambda>0} C(\lambda) := c_*$. Moreover, if $c < c_*$, then $\Psi(c,\lambda) < 0$, $\forall \lambda \in \mathbb{R}$; if $c > c_*$, then there exist $\lambda_2(c) > \lambda_1(c) > 0$ such that $\Psi(c, \lambda_i(c)) = 0$, $i = 1, 2$, $\Psi(c,\cdot) > 0$ in $(\lambda_1(c), \lambda_2(c))$, and $\Psi(c,\cdot) < 0$ in $\mathbb{R} \setminus [\lambda_1(c), \lambda_2(c)]$; if $c = c_*$, then there exists a unique $\lambda_1(c) > 0$ such that $\lambda_1(c)$ is a double root of $\Psi(c,\cdot) = 0$ and $\Psi(c,\lambda) < 0$ for all $\lambda \neq \lambda_1(c)$.

Proof. The lemma follows by noting that $C(\lambda)$ is convex and $C(0^+) = C(+\infty) = +\infty$. □

3. Existence

In this section, we shall establish the existence of traveling waves by constructing a suitable pair of super-sub-solutions.

First, we derive two properties of solutions of (1.5).

Lemma 3.1. (i) *Every solution (c, U) of* (1.5) *satisfies* $0 < U(\xi) < 1, \forall \xi \in \mathbb{R}$.
(ii) *Every solution (c, U) of* (1.5) *satisfying $U' \leq 0$ in $\mathbb{R}$ satisfies $U' < 0$ in $\mathbb{R}$.*

Proof. Let (c, U) be a solution of (1.5).

Suppose that there exists $\xi_0 \in \mathbb{R}$ such that $U(\xi_0) = 0$. Without loss of generality, we may assume ξ_0 is the left-most point such that $U(\xi_0) = 0$, since $U(-\infty) = 1$. By (1.3), using $U \geq 0$ and $U'(\xi_0) = 0$ we have $U(\xi_0 \pm p) = U(\xi_0 \pm q) = U(\xi_0) = 0$. This contradicts the definition of ξ_0. Hence $U > 0$ in $\mathbb{R}$. Similarly, $U < 1$ in $\mathbb{R}$. Thus (i) is proved.

To prove (ii), for a contradiction, we suppose that there exists ξ_1 such that $U'(\xi_1) = 0$. By differentiating (2.1) with respect to ξ, we obtain

$$0 = \mu e^{\mu\xi_1} \int_{\xi_1}^{\infty} e^{-\mu s} [\, H_\mu(U)(s) - H_\mu(U)(\xi_1)\,] ds \leq 0,$$

since $U' \leq 0$. Hence we have $H_\mu(U)(s) = H_\mu(U)(\xi_1)$, $\forall s \geq \xi_1$. Letting $s \to +\infty$, we obtain that $H_\mu(U)(\xi_1) = 0$. Then, from (1.3) and using $U'(\xi_1) = 0$, it follows that $\mu U(\xi_1) = 0$, a contradiction to (i). Hence the lemma is proved. □

Hence Theorem 1(ii) is proved.

We now construct a pair of super-sub-solution for $c > c_*$ as follows.

Lemma 3.2. *Assume* **(A)** and **(B)**. *For each $c > c_*$, let $0 < r < \min\{\lambda_1\alpha, \lambda_2 - \lambda_1\}$, where $\lambda_i = \lambda_i(c)$, $i = 1, 2$, are defined in Lemma 2.2. Then $U^-(\xi) := \max\{0, (1 - Me^{-r\xi})e^{-\lambda_1\xi}\}$ is a sub-solution of (1.5), provided $M \ge [M_0/\Psi(c, \lambda_1 + r)]^{r/(\lambda_1\alpha)}$.*

Proof. For $\xi < \ln M/r$, we have $U^-(\xi) = 0$ and so

$$\{-c(U^-)' - D_2[U^-] - f(U^-)\}(\xi) = -[U^-(\xi + p) + U^-(\xi + q)] \le 0.$$

For $\xi > \ln M/r$, we have $(U^-)'(\xi) = [\,(r + \lambda_1)Me^{-r\xi} - \lambda_1\,]e^{-\lambda_1\xi}$. Then, using (1.4), we compute that, for $\xi > \ln M/r$,

$$\begin{aligned}
& \{-c(U^-)' - D_2[U^-] - f(U^-)\}(\xi) \\
\le\ & \{-c(U^-)' - D_2[U^-] - f'(0)U^- + M_0(U^-)^{1+\alpha}\}(\xi) \\
\le\ & \Psi(c, \lambda_1)e^{-\lambda_1\xi} - M\Psi(c, \lambda_1 + r)e^{-(\lambda_1+r)\xi} + M_0e^{-\lambda_1(1+\alpha)\xi} \\
=\ & -M\Psi(c, \lambda_1 + r)e^{-(\lambda_1+r)\xi} + M_0e^{-\lambda_1(1+\alpha)\xi} \\
\le\ & 0
\end{aligned}$$

as long as $M \ge [M_0/\Psi(c, \lambda_1 + r)]^{r/(\lambda_1\alpha)}$. Also, note that $U^- \not\equiv 0$ and $U^-(+\infty) = 0$. Hence U^- is a sub-solution of (1.5) and the lemma follows. □

Lemma 3.3. *Assume that* **(A)** and **(B)**. *Then, for each $c > c_*$, the function $U^+(\xi) := \min\{1, e^{-\lambda_1(c)\xi}\}$ is a super-solution of (1.5).*

Proof. For $\xi < 0$, we have $U^+(\xi) = 1$ and so

$$\begin{aligned}
& \{-c(U^+)' - D_2[U^+] - f(U^+)\}(\xi) \\
=\ & -U^+(\xi + p) - U^+(\xi + q) + 2 \\
\ge\ & 0.
\end{aligned}$$

For $\xi > 0$, we have $U^+(\xi) = e^{-\lambda_1(c)\xi}$ and hence

$$\begin{aligned}
& \{-c(U^+)' - D_2[U^+] - f(U^+)\}(\xi) \\
\ge\ & c\lambda_1(c)e^{-\lambda_1(c)\xi} - [e^{-\lambda_1(c)(\xi+p)} + e^{-\lambda_1(c)(\xi-p)} \\
& +e^{-\lambda_1(c)(\xi+q)} + e^{-\lambda_1(c)(\xi-q)} - 4e^{-\lambda_1(c)\xi}] - f'(0)e^{-\lambda_1(c)\xi} \\
=\ & \Psi(c, \lambda_1(c))e^{-\lambda_1(c)\xi} \\
=\ & 0.
\end{aligned}$$

Since U^+ is non-increasing and $U^+(+\infty) = 0$, U^+ is a super-solution of (1.5) and the lemma is proved. □

Therefore, by applying Lemma 2.1, it follows from Lemmas 3.2 and 3.3 that (1.5) admits a solution (c, U) with $U' \le 0$ for any $c > c^*$.

Next, we prove that (1.5) has a solution (c, U) with $U' \le 0$ for $c = c_*$.

Lemma 3.4. *Assume that* **(A)** and **(B)**. *Then* (1.5) *admits a solution* (c, U) *with* $U' \leq 0$ *for* $c = c_*$.

Proof. Let $\{c_i, U_i\}_{i=1}^{\infty}$ be a sequence of solutions of (1.5) such that $c_i \downarrow c_*$ as $i \to \infty$ and $U_i' \leq 0$ for all i. By appropriate translations, we may assume $U_i(0) = 1/2$ for all i. From $0 \leq U_i(\cdot) \leq 1$ in $\mathbb{R}$ for all i and (1.3), we know that $\{U_i'\}$ is uniformly bounded in $\mathbb{R}$. It then follows that $\{U_i\}$ is equicontinuous on $\mathbb{R}$. By Arzela-Ascoli Theorem, there exists a subsequence $\{U_{i_k}\}$ of $\{U_i\}$ such that $U_{i_k} \to U_*$ on $\mathbb{R}$ as $k \to \infty$, uniformly on any compact subset of $\mathbb{R}$, for some $U_* \in C(\mathbb{R} \to [0, 1])$. Moreover, since U_{i_k} satisfies (2.1), by taking $k \to +\infty$, we have

$$U_*(\xi) = e^{\mu\xi} \int_{\xi}^{\infty} e^{-\mu s} H_\mu(U_*)(s) ds, \ \forall \xi \in \mathbb{R}.$$

Thus U_* satisfies (1.3) and $U_* \in C^1(\mathbb{R})$.

Next, we claim $U_*(+\infty) = 0$ and $U_*(-\infty) = 1$. Note that $U_*' \leq 0$. Since U_* is bounded, we know $U_*(\pm\infty)$ exists and $0 \leq U(\pm\infty) \leq 1$. Recall from (1.6) that

$$c_{i_k} = \int_{-\infty}^{+\infty} f(U_{i_k}(s)) ds, \ \forall k.$$

Then, by applying Fatou's Lemma, we obtain

$$\int_{-\infty}^{+\infty} f(U_*(s)) ds = \int_{-\infty}^{+\infty} \liminf_{k\to\infty} f(U_{i_k}(s)) ds \leq \liminf_{k\to\infty} \int_{-\infty}^{+\infty} f(U_{i_k}(s)) ds = c_*.$$

It follows that $f(U_*(\pm\infty)) = 0$. Hence $U_*(\pm\infty) \in \{0, 1\}$. On the other hand, since U_* satisfies (1.3) and $U_*(0) = 1/2$, we have

$$c_*[U_*(-\infty) - U_*(+\infty)] = \int_{-\infty}^{+\infty} f(U_*(s)) ds > 0.$$

It follows that $U_*(+\infty) = 0$ and $U_*(-\infty) = 1$. The lemma is proved. □

Hence we have proved the necessary condition in Theorem 1(i) and Theorem 1(iii).

To prove the sufficient condition in Theorem 1(i), we need the following lemma.

Lemma 3.5. *Assume* **(A)**. *Suppose that* (c, U) *is a solution of* (1.5). *Then*

(i) $U(\xi + s)/U(\xi)$ *is uniformly bounded for* $\xi \in \mathbb{R}, s \in [-1, 1]$,

(ii) $U'(\xi)/U(\xi)$ *is bounded and uniformly continuous in* $\mathbb{R}$.

Proof. Since $\mu > (4 + \max_{0\leq u\leq 1} |f'(u)|)/c$, $U'(\xi) - \mu U(\xi) \leq 0, \forall \xi \in \mathbb{R}$. By an integration from ξ to $\xi + s$, $s > 0$, we have $U(\xi + s) \leq U(\xi)e^{\mu s}$ for $\xi \in \mathbb{R}$, $s > 0$. In particular, for any $s > 0$, we have

$$U(\xi - s/2) = U(y - s + \xi + s/2 - y) \leq e^{\mu(\xi + s/2 - y)} U(y - s) \leq e^{\mu s/2} U(y - s)$$

for all $y \in [\xi, \xi + s/2]$; and $U(y) \leq e^{\mu s} U(\xi)$ for all $y \in [\xi, \xi + s]$.

Integrating (1.3) from ξ to $+\infty$ gives

$$\begin{aligned} cU(\xi) &= \int_\xi^\infty D_2[U](y)dy + \int_\xi^\infty f(U(y))dy \\ &\geq \int_\xi^{\xi+q} U(y-q)dy - \int_\xi^{\xi+q} U(y)dy + \int_\xi^{\xi+p} U(y-p)dy - \int_\xi^{\xi+p} U(y)dy \\ &\geq \int_\xi^{\xi+q/2} U(y-q)dy - \int_\xi^{\xi+q} U(y)dy - \int_\xi^{\xi+p} U(y)dy \\ &\geq e^{-\mu q/2}U(\xi-q/2)q/2 - U(\xi)(qe^{\mu q}+pe^{\mu p}). \end{aligned}$$

It follows that

$$\frac{U(\xi-q/2)}{U(\xi)} \leq \frac{2(c+qe^{\mu q}+pe^{\mu p})e^{\mu q/2}}{q}, \ \forall \xi \in \mathbb{R}.$$

Hence, by a finite number of iterations, we can easily show that $U(\xi+s)/U(\xi)$ is uniformly bounded for $\xi \in \mathbb{R}$ for any $s \in [-1,0]$. Hence (i) follows. Moreover, (ii) follows from (1.3) by applying (i). The lemma is proved. □

Now, we are ready to prove the sufficient condition in Theorem 1(i).

Lemma 3.6. *Assume* **(A)**. *If* $(c\,,U)$ *is a solution of* (1.5), *then* $c \geq c_*$.

Proof. Let (c,U) be a solution of (1.5) and $\varepsilon > 0$ be given. Since

$$\lim_{\xi\to\infty}\{f(U(\xi))/U(\xi)\} = f'(0),$$

we can choose $x = x(\varepsilon)$ such that

$$\frac{f(U(\xi))}{U(\xi)} \geq f'(0) - \varepsilon, \quad \forall \xi > x.$$

Set

$$R(\xi) := \frac{U(\xi+q)}{U(\xi)} + \frac{U(\xi+p)}{U(\xi)} + \frac{U(\xi-q)}{U(\xi)} + \frac{U(\xi-p)}{U(\xi)}.$$

Dividing (1.3) by U and integrating it over $[x,y]$, $y > x$, we have

$$\begin{aligned} c\,[\ln U(x) - \ln U(y)] &= \int_x^y \{R(\xi) - 4 + \frac{f(U(\xi))}{U(\xi)}\}d\xi \\ &\geq \int_x^y R(\xi)d\xi + (f'(0) - 4 - \varepsilon)(y-x). \end{aligned}$$

Hence

$$c\lambda(x,y) \geq \frac{1}{y-x}\int_x^y R(\xi)d\xi + (f'(0) - 4 - \varepsilon),$$

where

$$\lambda(x,y) := \frac{\ln U(x) - \ln U(y)}{y-x} = \frac{\ln[U(x)/U(y)]}{y-x}.$$

We can write

$$
\begin{aligned}
R(\xi) &= \exp\left\{\ln\frac{U(\xi+q)}{U(\xi)}\right\} + \exp\left\{\ln\frac{U(\xi+p)}{U(\xi)}\right\} \\
&\quad + \exp\left\{\ln\frac{U(\xi-q)}{U(\xi)}\right\} + \exp\left\{\ln\frac{U(\xi-p)}{U(\xi)}\right\}.
\end{aligned}
$$

Then, by Jensen's Inequality, we obtain

$$
\begin{aligned}
&\frac{1}{y-x}\int_x^y R(\xi)d\xi \\
\geq\ & \exp\left\{\frac{1}{y-x}\int_x^y \ln\frac{U(\xi+q)}{U(\xi)}d\xi\right\} + \exp\left\{\frac{1}{y-x}\int_x^y \ln\frac{U(\xi+p)}{U(\xi)}d\xi\right\} \\
&\quad + \exp\left\{\frac{1}{y-x}\int_x^y \ln\frac{U(\xi-q)}{U(\xi)}d\xi\right\} + \exp\left\{\frac{1}{y-x}\int_x^y \ln\frac{U(\xi-p)}{U(\xi)}d\xi\right\} \\
=\ & e^{-\lambda(x,y)q+\triangle_1} + e^{-\lambda(x,y)p+\triangle_2} + e^{\lambda(x,y)q+\triangle_3} + e^{\lambda(x,y)p+\triangle_4},
\end{aligned}
$$

where

$$
\begin{aligned}
\triangle_1 = \triangle_1(x,y) &:= \frac{1}{y-x}\left\{\int_y^{y+q} \ln\frac{U(\xi)}{U(y)}d\xi - \int_x^{x+q} \ln\frac{U(\xi)}{U(x)}d\xi\right\}, \\
\triangle_2 = \triangle_2(x,y) &:= \frac{1}{y-x}\left\{\int_y^{y+p} \ln\frac{U(\xi)}{U(y)}d\xi - \int_x^{x+p} \ln\frac{U(\xi)}{U(x)}d\xi\right\}, \\
\triangle_3 = \triangle_3(x,y) &:= \frac{1}{y-x}\left\{\int_{x-q}^{x} \ln\frac{U(\xi)}{U(x)}d\xi - \int_{y-q}^{y} \ln\frac{U(\xi)}{U(y)}d\xi\right\}, \\
\triangle_4 = \triangle_4(x,y) &:= \frac{1}{y-x}\left\{\int_{x-p}^{x} \ln\frac{U(\xi)}{U(x)}d\xi - \int_{y-p}^{y} \ln\frac{U(\xi)}{U(y)}d\xi\right\}.
\end{aligned}
$$

Hence we get

$$
c\lambda(x,y) \geq e^{-\lambda(x,y)q+\triangle_1} + e^{-\lambda(x,y)p+\triangle_2} + e^{\lambda(x,y)q+\triangle_3} + e^{\lambda(x,y)p+\triangle_4} + (f'(0)-4-\varepsilon). \tag{3.1}
$$

Also, from Lemma 3.5 it follows that $(y-x)\triangle_i(x,y)$ is bounded in y for each i. Hence there exists $z > x$ large enough such that $|\triangle_i(x,y)| < \varepsilon, \forall y \geq z$, $i = 1,2,3,4$. Now taking y large enough so that $\lambda(x,y) > 0$ and $y \geq z$. Then it follows from (3.1) that

$$
\begin{aligned}
c &\geq \frac{e^{-\lambda(x,y)q-\varepsilon} + e^{-\lambda(x,y)p-\varepsilon} + e^{\lambda(x,y)q-\varepsilon} + e^{\lambda(x,y)p-\varepsilon} + (f'(0)-4-\varepsilon)}{\lambda(x,y)} \\
&\geq \inf_{\lambda>0} \frac{e^{-\lambda q-\varepsilon} + e^{-\lambda p-\varepsilon} + e^{\lambda q-\varepsilon} + e^{\lambda p-\varepsilon} + (f'(0)-4-\varepsilon)}{\lambda}.
\end{aligned}
$$

Letting $\varepsilon \to 0$, we obtain that

$$
c \geq \inf_{\lambda>0} \frac{e^{\lambda q} + e^{\lambda p} + e^{-\lambda q} + e^{-\lambda p} + f'(0) - 4}{\lambda} = c_*.
$$

Hence the lemma follows. □

Therefore, the proof of Theorem 1 is completed.

4. Uniqueness

In this section, we always assume that **(A)**, **(B)**, and **(C)** hold. Let (c, U) be a solution of (1.5). We shall follow a method of Carr and Chmaj [2] to prove that for each (c, U) there exists $\eta = \eta(U) \in \mathbb{R}$ such that

$$\lim_{\xi\to\infty} \frac{U(\xi+\eta)}{e^{-\lambda_1(c)\xi}} = 1 \text{ for } c > c_*; \quad \lim_{\xi\to\infty} \frac{U(\xi+\eta)}{\xi e^{-\lambda_1(c)\xi}} = 1 \text{ for } c = c_*, \tag{4.1}$$

where $\lambda_1(c)$ is the smaller root of $\Psi(c, \lambda) = 0$. Hereafter we shall always assume that $c \geq c_*$.

Lemma 4.1. *Let $(c\,, U)$ be a solution of (1.5). Then $U(\xi) = O(e^{-\lambda_0 \xi})$ as $\xi \to \infty$ for some $\lambda_0 > 0$.*

Proof. Given $s \in \mathbb{R}$. Integrating (1.3) over $[s, y]$, $y > s$, we obtain

$$c[U(s) - U(y)] = \int_s^y D_2[U](\xi)d\xi + \int_s^y f(U(\xi))d\xi. \tag{4.2}$$

Introduce

$$a(s) := \inf_{\xi\geq s} \frac{f(U(\xi))}{U(\xi)} = \inf_{u\in[0,\delta(s)]} \frac{f(u)}{u} > 0, \quad \delta(s) := \sup_{\xi\geq s} U(\xi) \in (0, 1).$$

Then (4.2) implies that

$$c[U(s) - U(y)] \geq \int_s^y D_2[U](\xi)d\xi + a(s) \int_s^y U(\xi)d\xi, \tag{4.3}$$

Set

$$W(x) := \int_{x-q}^x U(\xi)d\xi + \int_{x-p}^x U(\xi)d\xi - \int_x^{x+q} U(\xi)d\xi - \int_x^{x+p} U(\xi)d\xi.$$

Then

$$\int_s^y D_2[U](\xi)d\xi = W(s) - W(y).$$

Since $U(\xi) \to 0$ as $\xi \to \infty$, $W(y) \to 0$ as $y \to \infty$. Letting $y \to \infty$ in (4.3), we see that

$$cU(s) \geq \int_s^\infty D_2[U](\xi)d\xi + a(s) \int_s^\infty U(\xi)d\xi \tag{4.4}$$

and so $U \in L((s, +\infty))$ for all $s \in \mathbb{R}$. Moreover, by (4.2),

$$cU(s) = \int_s^\infty D_2[U](\xi)d\xi + \int_s^\infty f(U(\xi))d\xi. \tag{4.5}$$

Set $V(s) := \int_s^\infty U(\xi)d\xi$. Then $0 < V < +\infty$ and V is decreasing. Note also that $a(s)$ is non-decreasing in s and $a(+\infty) = f'(0)$. Set $a_0 := a(0)$. Integrating (4.4) over $[x, \infty)$ for $x \geq 0$, we obtain

$$cV(x) \geq \int_x^\infty D_2[V](s)ds + a_0 \int_x^\infty V(s)ds. \tag{4.6}$$

Note that

$$\int_x^\infty D_2[V](s)ds = \int_{x-q}^x V(s)ds + \int_{x-p}^x V(s)ds - \int_x^{x+q} V(s)ds - \int_x^{x+p} V(s)ds \geq 0,$$

since V is decreasing. Then form (4.6) it follows that

$$cV(x) \geq a_0 \int_x^{x+z} V(s)ds \geq a_0 z V(x+z)$$

for all $z > 0$ and $x \geq 0$. This implies that

$$\frac{c}{a_0 z} V(x) \geq V(x+z),\ \forall z > 0,\ x \geq 0.$$

Choose $z > 0$ such that $c < a_0 z$. Then there exists $\lambda_0 > 0$ such that $e^{-\lambda_0 z} = c/(a_0 z)$ and so

$$e^{\lambda_0(x+z)} V(x+z) \leq e^{\lambda_0 x} V(x),\ \forall x \geq 0.$$

Set $K := \max\{e^{\lambda_0 x} V(x) \mid x \in [0, z]\}$. Then $K \in (0, \infty)$ and $e^{\lambda_0 y} V(y) \leq K$ for all $y \geq 0$. Hence $V(x) = O(e^{-\lambda_0 x})$ as $x \to \infty$. From (4.5) and noting that

$$\int_s^\infty D_2[U](\xi)d\xi = V(s+q) + V(s+p) + V(s-q) + V(s-p) - 4V(s),$$

$$\int_s^\infty f(U(\xi))d\xi \leq f'(0) \int_s^\infty U(\xi)d\xi = f'(0)V(s),$$

the lemma follows. □

To derive the asymptotic behavior of wave profile U, we first recall the following theorem.

Ikehara's Theorem. *For a positive non-increasing function U, we define*

$$F(\Lambda) := \int_0^{+\infty} e^{-\Lambda\xi} U(\xi)d\xi.$$

If F can be written as $F(\Lambda) = H(\Lambda)/(\Lambda+\gamma)^{k+1}$ for some constants $k > -1, \gamma > 0$, and some analytic function H in the strip $-\gamma \leq \mathrm{Re}\Lambda < 0$, then

$$\lim_{\xi\to+\infty} \frac{U(\xi)}{\xi^k e^{-\gamma\xi}} = \frac{H(-\gamma)}{\Gamma(\gamma+1)}.$$

Here we only need the case when $k = 0$ and $k = 1$. The proof of Ikehara's Theorem can be found in, e.g., [14, 8].

Applying Lemma 4.1 and choosing $\Lambda \in \mathbb{C}$ such that $-\lambda_0 < \mathrm{Re}\Lambda < 0$, we can define the bilateral Laplace transform of U by

$$\mathcal{L}(\Lambda) := \int_{-\infty}^{+\infty} e^{-\Lambda\xi} U(\xi)d\xi.$$

Note that

$$\int_{-\infty}^{+\infty} e^{-\Lambda\xi} D_2[U](\xi)d\xi = [e^{\Lambda q} + e^{\Lambda p} + e^{-\Lambda q} + e^{-\Lambda p} - 4]\mathcal{L}(\Lambda).$$

Rewrite (1.3) as $cU' + D_2[U] + f'(0)U = f'(0)U - f(U)$, we deduce that

$$\begin{aligned} & c\int_{-\infty}^{+\infty} e^{-\Lambda\xi}U'(\xi)d\xi + [e^{\Lambda q} + e^{\Lambda p} + e^{-\Lambda q} + e^{-\Lambda p} - 4 + f'(0)]\mathcal{L}(\Lambda) \\ = & \int_{-\infty}^{+\infty} e^{-\Lambda\xi}[f'(0)U(\xi) - f(U(\xi))]d\xi. \end{aligned}$$

An integration by parts gives

$$c\int_{-\infty}^{+\infty} e^{-\Lambda\xi}U'(\xi)d\xi = -c(-\Lambda)\mathcal{L}(\Lambda),$$

so we have

$$-\Psi(c,-\Lambda)\mathcal{L}(\Lambda) = \int_{-\infty}^{+\infty} e^{-\Lambda\xi}[f'(0)U(\xi) - f(U(\xi))]d\xi. \tag{4.7}$$

It follows from (4.7) that

$$\int_0^{+\infty} e^{-\Lambda\xi}U(\xi)d\xi = -\frac{\int_{-\infty}^{+\infty} e^{-\Lambda\xi}[f'(0)U(\xi) - f(U(\xi))]d\xi}{\Psi(c,-\Lambda)} - \int_{-\infty}^{0} e^{-\Lambda\xi}U(\xi)d\xi \tag{4.8}$$

whenever $\mathcal{L}$ is well-defined.

In order to apply Ikehara's Theorem, we define

$$H(\Lambda) := -\frac{\int_{-\infty}^{+\infty} e^{-\Lambda\xi}[f'(0)U(\xi) - f(U(\xi))]d\xi}{\Psi(c,-\Lambda)/[\Lambda+\lambda_1(c)]^{k+1}} - \left(\int_{-\infty}^{0} e^{-\Lambda\xi}U(\xi)d\xi\right)[\Lambda+\lambda_1(c)]^{k+1},$$

where $k = 0$ if $c > c_*$; $k = 1$ if $c = c_*$. We claim that H is analytic in the strip

$$S := \{\Lambda \in \mathbb{C} \mid -\lambda_1(c) \le \mathrm{Re}\Lambda < 0\}.$$

It is trivial that

$$\left(\int_{-\infty}^{0} e^{-\Lambda\xi}U(\xi)d\xi\right)[\Lambda+\lambda_1(c)]^{k+1}$$

is analytic in $\{\mathrm{Re}\Lambda < 0\}$. Thus it suffices to show that the function

$$\mathcal{H}(\Lambda) := -\frac{\int_{-\infty}^{+\infty} e^{-\Lambda\xi}[f'(0)U(\xi) - f(U(\xi))]d\xi}{[\Psi(c,-\Lambda)/(\Lambda+\lambda_1(c))^{k+1}]}$$

is analytic in S. First, we show that $\mathcal{L}(\Lambda)$ is well-defined for $-\lambda_1(c) < \mathrm{Re}\Lambda < 0$. Since $U(+\infty) = 0$ and, by the assumption **(B)**, $f'(0)U - f(U) = O(U^{1+\alpha})$ as $\xi \to +\infty$, the right-hand side of (4.7) is well-defined for $-(1+\alpha)\lambda_0 < \mathrm{Re}\Lambda < 0$. Hence $\mathcal{L}(\Lambda)$ is well-defined until Λ is a zero of $\Psi(c,-\Lambda)$. Recall a property of Laplace transform (cf. Theorem 5b in p.58 of [14]): *if* $\mathcal{L}(\Lambda)$ *is well-defined (convergent) in* $\{\mathrm{Re}\Lambda > -s\}$ *and diverges in* $\{\mathrm{Re}\Lambda < -s\}$, *then necessarily* $\Lambda = -s$ *is a singularity of* $\mathcal{L}(\Lambda)$. It follows from Lemma 2.2 that $\mathcal{L}(\Lambda)$ is well-defined for $-\lambda_1(c) < \mathrm{Re}\Lambda < 0$. Since $\mathcal{H}(\Lambda) = \mathcal{L}(\Lambda)[\Lambda+\lambda_1(c)]^{k+1}$ in $\{-\lambda_1(c) < \mathrm{Re}\Lambda < 0\}$, it follows that $\mathcal{H}(\Lambda)$ is analytic in $\{-\lambda_1(c) < \mathrm{Re}\Lambda < 0\}$.

Next, we claim that $\mathcal{H}(\Lambda)$ is analytic on $\mathrm{Re}\Lambda = -\lambda_1(c)$. For this, we first claim that the only zero of $\Psi(c,-\Lambda)$ on $\mathrm{Re}\Lambda = -\lambda_1(c)$ is $\Lambda = -\lambda_1(c)$. Indeed, if $\Psi(c,-\Lambda) = 0$ with

$\mathrm{Re}(-\Lambda)=\lambda_1(c)$ and $\mathrm{Im}(-\Lambda)=\beta$ for some $\beta\in\mathbb{R}$, then we have $\beta p=2m\pi$ and $\beta q=2k\pi$ for some integers m and k, by using $\Psi(c,\lambda_1(c))=0$ and $\mathrm{Re}\{\Psi(c,-\Lambda)\}=0$. Then, by considering the imaginary part of the equation $\Psi(c,-\Lambda)=0$, we conclude that $\beta=0$. Therefore, the only zero of $\Psi(c,-\Lambda)$ on $\mathrm{Re}\Lambda=-\lambda_1(c)$ is $\Lambda=-\lambda_1(c)$. Hence $\mathcal{H}(\Lambda)$ is analytic on $\mathrm{Re}\Lambda=-\lambda_1(c)$, since the zeroes of $\Psi(c,-\Lambda)$ are isolated. We conclude that $H(\Lambda)$ is analytic in S.

Now, we are ready to derive the asymptotic behavior of wave profile U as follows.

Lemma 4.2. *Let (c,U) be a solution of* (1.5)*. Then* (4.1) *holds for some $\eta=\eta(U)\in\mathbb{R}$. Moreover,*

$$\lim_{\xi\to\infty}\frac{U'(\xi)}{U(\xi)}=-\lambda_1(c) \tag{4.9}$$

for all $c\geq c_*$.

Proof. Recall (4.8). If U is non-increasing, then, by applying Ikehara's Theorem with a suitable translation, we can easily deduce (4.1).

In general, by (1.3), we have

$$cU'(\xi)=-D_2[U](\xi)-f(U(\xi))\leq 4U(\xi).$$

Hence the function $\overline{U}(\xi):=U(\xi)e^{-4\xi/c}$ is non-increasing in $\mathbb{R}$. Now, we define the bilateral Laplace transform of $\overline{U}$ by

$$\overline{\mathcal{L}}(\Lambda):=\int_{-\infty}^{+\infty}e^{-\Lambda\xi}\,\overline{U}(\xi)d\xi.$$

Note that $\overline{\mathcal{L}}(\Lambda)=\mathcal{L}(\Lambda+4/c)$. Then, by Ikehara's Theorem again, we have

$$\lim_{\xi\to\infty}\frac{\overline{U}(\xi+\bar{\eta})}{e^{-(\lambda_1(c)+4/c)\xi}}=1 \text{ for } c>c_*;\ \lim_{\xi\to\infty}\frac{\overline{U}(\xi+\bar{\eta})}{\xi e^{-(\lambda_1(c)+4/c)\xi}}=1 \text{ for } c=c_*$$

for some $\bar{\eta}=\bar{\eta}(\overline{U})\in\mathbb{R}$. Hence (4.1) follows for some $\eta=\eta(U)\in\mathbb{R}$.

Finally, (4.9) follows from (1.3) and (4.1). This proves the lemma. □

Next, for each $c\geq c_*$, we let $\nu=\nu(c)$ be the unique positive root of

$$c\nu+e^{\nu p}+e^{\nu q}+e^{-\nu p}+e^{-\nu q}-4+f'(1)=0. \tag{4.10}$$

Set $V=1-U$ and $F(s)=f(1-s)$. Then by a similar argument as above we can derive the following lemma. Since the proof is very similar to that of deriving (4.1), we omit its details here (see also Theorem 4.5 in [5]).

Lemma 4.3. *Any solution (c,U) of* (1.5) *satisfies*

$$\lim_{\xi\to-\infty}\frac{U'(\xi)}{U(\xi)-1}=\nu(c), \tag{4.11}$$

where $\nu(c)$ is the unique positive root of (4.10)*.*

In order to prove the monotonicity result, we shall need the following strong comparison principle.

Lemma 4.4. *Let (c, U_1) and (c, U_2) are solutions of (1.5) with $U_1 \geq U_2$ on $\mathbb{R}$. Then either $U_1 \equiv U_2$ or $U_1 > U_2$ in $\mathbb{R}$.*

Proof. Suppose that there exists ξ_0 such that $U_1(\xi_0) = U_2(\xi_0)$. Then

$$0 = U_1(\xi_0) - U_2(\xi_0) = e^{\mu\xi_0} \int_{\xi_0}^{\infty} e^{-\mu s}[H_\mu(U_1)(s) - H_\mu(U_2)(s)]ds.$$

It follows that $H_\mu(U_1)(s) = H_\mu(U_2)(s)$ for all $s \geq \xi_0$, since $U_1 \geq U_2$ in $\mathbb{R}$. By the definitions of H_μ and D_2, we have

$$\begin{aligned}
0 \ &\leq \ [U_1(s+q) - U_2(s+q)] + [U_1(s-q) - U_2(s-q)] \\
&\qquad +[U_1(s+p) - U_2(s+p)] + [U_1(s-p) - U_2(s-p)] \\
&= \ (-c\mu + 4)[U_1(s) - U_2(s)] - [f(U_1(s)) - f(U_2(s)] \\
&\leq \ -(c\mu - 4 - \max_{0\leq u\leq 1} |f'(u)|)[U_1(s) - U_2(s)] \\
&\leq \ 0
\end{aligned}$$

for all $s \geq \xi_0$. Hence $U_1(\xi) = U_2(\xi)$ for all $\xi \in [\xi_0 - r, \infty)$, where r can be either $p = \cos\theta$ or $q = \sin\theta$. Note that r is a positive constant. Repeating the above argument with ξ_0 replaced by $\xi_0 - r$ (infinitely many times), we conclude that $U_1 \equiv U_2$ in $\mathbb{R}$. □

Proof of Theorem 3. Let (c, U) be a solution of (1.5). Then it follows from (4.9) and (4.11) that there exists $x_1 > 0$ and $x_2 > 0$ such that $U'(\xi) < 0$ for all $\xi \geq x_1$ and $\xi \leq -x_2$.

Now, since $0 < U < 1$ and $U(-\infty) = 1$, we can define

$$\eta^* := \inf\{\ \eta > 0 \mid U(\xi + s) \leq U(\xi)\ \forall \xi \in \mathbb{R},\ s \geq \eta\}.$$

In particular, $U(\xi + \eta^*) \leq U(\xi)$ for all $\xi \in \mathbb{R}$. We claim that $\eta^* = 0$. Otherwise, $\eta^* > 0$. By Lemma 4.4, we have $U(\xi + \eta^*) < U(\xi)$ for all $\xi \in \mathbb{R}$. Also, by the continuity of U, there exists $\eta_0 \in (0, \eta^*)$ such that $U(\xi + \eta_0) < U(\xi)$ for all $\xi \in [-x_2 - 2\eta_0, x_1 + 2\eta_0]$. Since $U' < 0$ on $\xi \in \mathbb{R} \setminus [-x_2, x_1]$, we have $U(\xi + \eta_0) \leq U(\xi)$ for all $\xi \in \mathbb{R} \setminus [-x_2 - \eta_0, x_1 + \eta_0]$. Hence $U(\cdot + \eta_0) \leq U(\cdot)$ in $\mathbb{R}$. But, $\eta_0 < \eta^*$, a contradiction to the definition of η^*. This implies that $\eta^* = 0$. Therefore, $U' \leq 0$ in $\mathbb{R}$. By Lemma 3.1(ii), $U' < 0$ in $\mathbb{R}$. Hence the theorem follows. □

With this monotonicity result, we now apply a method developed in [5] to derive the uniqueness of wave profiles.

Hereafter we shall always assume that $c \geq c_*$.

Lemma 4.5. *Let (c, U) be a solution of* (1.5). *Then there exists $\rho_0 = \rho_0(c, f) \in (0, 1)$ such that for any $\rho \in (0, \rho_0]$,*

$$f((1+\rho)U(\xi)) - (1+\rho)f(U(\xi)) < 0$$

on $\{\ \xi \mid 1-\rho_0 < U(\xi) \le 1/1+\rho\}$.

Proof. Note that $\{f((1+\rho)u)-(1+\rho)f(u)\}|_{\rho=0} = 0$. Since $f'(1) < 0$, we may choose $\rho_0 > 0$ small enough such that $uf'(u) - f(u) < 0$ for $u \in (1-\rho_0, 1]$. Also,

$$\frac{d}{d\rho}\{f((1+\rho)u) - (1+\rho)f(u)\}|_{\rho=0} = uf'(u) - f(u) < 0$$

for $u \in (1-\rho_0, 1]$. Then the lemma follows by choosing $\rho_0 > 0$ smaller (if necessary). □

For a given solution (c, U) of (1.5), we define

$$\kappa = \kappa(U) := \sup\{\ \frac{U(\xi)}{|U'(\xi)|} \mid U(\xi) \le 1-\rho_0\}.$$

Note that $0 < \kappa < +\infty$, since $\lim_{\xi\to\infty} U'(\xi)/U(\xi) = -\lambda_1$ and $U' < 0$ in $\mathbb{R}$.

Lemma 4.6. *Let (c, U_1) and (c, U_2) are two solutions of* (1.5) *and there exists $\rho \in (0, \rho_0]$ such that $(1+\rho)U_1(\cdot + \kappa\rho) \ge U_2(\cdot)$ in $\mathbb{R}$, where $\kappa = \kappa(U_1)$. Then $U_1(\cdot) \ge U_2(\cdot)$ in $\mathbb{R}$.*

Proof. First, we define $W(\rho, \xi) := (1+\rho)U_1(\xi + \kappa\rho) - U_2(\xi)$ and

$$\rho^* := \inf\{\ \rho > 0 \mid W(\rho, \xi) \ge 0,\ \forall\xi \in \mathbb{R}\}.$$

Then, by the continuity of W, $W(\rho^*, \xi) \ge 0$ for all $\xi \in \mathbb{R}$.

Now, we claim $\rho^* = 0$. For a contradiction, we suppose that $\rho^* \in (0, \rho_0]$. Then, by the definition of κ,

$$\frac{d}{d\rho}W(\rho, \xi) = U_1(\xi + \kappa\rho) + \kappa(1+\rho)U_1'(\xi + \kappa\rho) < 0$$

on $\{\xi \mid U_1(\xi + \kappa\rho) \le 1-\rho_0\}$. Also note that $W(\rho^*, -\infty) = \rho^* > 0$. Hence there exists ξ_0 with $U_1(\xi_0 + \kappa\rho^*) > 1-\rho_0$ such that $0 = W(\rho^*, \xi_0) = W_\xi(\rho^*, \xi_0)$, $W(\rho^*, \xi_0 \pm p) \ge 0$, and $W(\rho^*, \xi_0 \pm q) \ge 0$. Then

$$(1+\rho^*)U_1(P_0) = U_2(\xi_0), \quad (1+\rho^*)U_1'(P_0) = U_2'(\xi_0),$$
$$(1+\rho^*)U_1(P_0 \pm p) \ge U_2(\xi_0 \pm p), \quad (1+\rho^*)U_1(P_0 \pm q) \ge U_2(\xi_0 \pm q),$$

where $P_0 := \xi_0 + \kappa\rho^*$. So we have

$$\begin{aligned} 0 &= cU_2'(\xi_0) + D_2[U_2](\xi_0) + f(U_2(\xi_0)) \\ &\le c(1+\rho^*)U_1'(P_0) + D_2[(1+\rho^*)U_1](P_0) + f((1+\rho^*)U_1(P_0)) \\ &= -(1+\rho^*)D_2[U_1](P_0) - (1+\rho^*)f(U_1(P_0)) + D_2[(1+\rho^*)U_1](P_0) + f((1+\rho^*)U_1(P_0)) \\ &= f((1+\rho^*)U_1(P_0)) - (1+\rho^*)f(U_1(P_0)). \end{aligned}$$

But, by Lemma 4.5, the last quantity is negative, a contradiction. Hence we must have $\rho^* = 0$ and the lemma follows. □

Proof of Theorem 2. Let (c, U_1) and (c, U_2) be two solutions of (1.5). By translation, we may assume $U_1(0) = U_2(0) = 1/2$. From (4.1), we have $\lim_{\xi\to\infty}[U_2(\xi)/U_1(\xi)] = e^{\lambda_1\eta}$ for some $\eta \in \mathbb{R}$. Hence we may assume that $\lim_{\xi\to\infty}[U_2(\xi)/U_1(\xi)] \le 1$, by exchanging U_1 and U_2 if necessary. Then $\lim_{\xi\to\infty}[U_2(\xi+z)/U_1(\xi)] < 1$ for all $z > 0$.

Fix $z > 0$, then there exists $M > 0$ such that $U_1(\xi) > U_2(\xi + z)$ for all $\xi \ge M$. Since $U_1(-\infty) = 1$, we can find $z_0 > 0$ large enough such that $(1 + \rho_0)U_1(\xi + \kappa\rho_0) \ge U_2(\xi + z_0)$ for all $\xi \in \mathbb{R}$. Applying Lemma 4.6, we have $U_1(\xi) \ge U_2(\xi + z_0)$ for all $\xi \in \mathbb{R}$. Hence we can define

$$z^* := \inf\{\, z > 0 \mid U_1(\xi) \ge U_2(\xi + z)\ \forall \xi \in \mathbb{R}\}.$$

We clam that $z^* = 0$.

For a contradiction, we assume that $z^* > 0$. From

$$\lim_{\xi\to\infty} \frac{U_2(\xi + z^*)}{U_1(\xi + z^*/2)} < 1,$$

it follows that there exists $M_1 > 0$ such that

$$U_1(\cdot + z^*/2) \ge U_2(\cdot + z^*) \quad \text{on} \quad [M_1, \infty). \tag{4.12}$$

Next, since $U_1(-\infty) = 1$ and $U_1'(-\infty) = 0$, there exists $M_2 > 0$ large enough such that

$$\frac{d}{d\rho}\{(1 + \rho)U_1(\xi + 2\kappa\rho)\} = U_1(\xi + 2\kappa\rho) + 2\kappa(1 + \rho)U_1'(\xi + 2\kappa\rho) > 0$$

for all $\rho \in [0, 1]$ and $\xi \in (-\infty, -M_2]$. So we have

$$(1 + \rho)U_1(\xi + 2\kappa\rho) \ge U_1(\xi) \ge U_2(\xi + z^*) \tag{4.13}$$

for all $\rho \in [0, 1]$ and $\xi \in (-\infty, -M_2]$.

Now, since $U_1(\cdot) \ge U_2(\cdot + z^*)$ in $\mathbb{R}$, by Lemma 4.4, $U_1(\cdot) > U_2(\cdot + z^*)$ in $\mathbb{R}$. Also, U_1 is uniformly continuous on $\mathbb{R}$, we can choose $0 < \varepsilon < \min\{z^*/(4\kappa), \rho_0\}$ small enough such that

$$U_1(\xi + 2\kappa\varepsilon) \ge U_2(\xi + z^*) \tag{4.14}$$

for all $\xi \in [-M_2, M_1]$.

Combining (4.12), (4.13), and (4.14), we have $(1 + \varepsilon)U_1(\cdot + 2\kappa\varepsilon) \ge U_2(\cdot + z^*)$ in $\mathbb{R}$. By Lemma 4.6, we have

$$U_1(\xi + \kappa\varepsilon) \ge U_2(\xi + z^*) \quad \forall\, \xi \in \mathbb{R}.$$

This contradicts the definition of z^*. Hence $z^* = 0$, i.e., $U_1(\cdot) \ge U_2(\cdot)$ in $\mathbb{R}$. Since $U_1(0) = U_2(0) = 1/2$, by Lemma 4.4, the theorem follows. □

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