TRAVELING WAVE FRONT FOR A TWO-COMPONENT LATTICE DYNAMICAL SYSTEM ARISING IN COMPETITION MODELS

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Abstract. We study traveling front solutions for a two-component system on a one-dimensional lattice. This system arises in the study of the competition between two species with diffusion (or migration), if we divide the habitat into discrete regions or niches. We consider the case when the nonlinear source terms are of Lotka-Volterra type and of monostable case. We first show that there is a positive constant (the minimal wave speed) such that a traveling front exists if and only if its speed is above this minimal wave speed. Then we show that any wave profile is strictly monotone. Moreover, under some conditions, we show that the wave profile is unique (up to translations) for a given wave speed. Finally, we characterize the minimal wave speed by the parameters in the system.

1. Introduction

In this paper, we study the following two-component lattice dynamical system (LDS):

$$\begin{cases} \dfrac{du_j}{dt} = (u_{j+1} + u_{j-1} - 2u_j) + u_j(1 - u_j - kv_j), \\ \dfrac{dv_j}{dt} = d(v_{j+1} + v_{j-1} - 2v_j) + rv_j(1 - v_j - hu_j), \end{cases} \tag{1.1}$$

where $t \in \mathbb{R}$, $j \in \mathbb{Z}$, $d > 0$, $h > 0$, $k > 0$, and $r > 0$. This system arises in the study of the competition between two species with diffusion (or migration) when the habitat is of one-dimensional and is divided into niches or regions. Here $u_j(t)$ and $v_j(t)$ stand for the populations at time t and niches j of two species u, v, respectively. With a certain normalization, we assume that the birth rates of species u, v are given by $1, r$, the carrying capacities are equal to 1, and the diffusion coefficients of species u, v are given by $1, d$. Here all constants are positive. The constants h, k are inter-specific competition coefficients.

In general, there are three distribution patterns of species in ecology: random, uniform and aggregated dispersion. For the aggregated dispersion, LDS model is more suitable than continuous PDE model to describe the phenomenon of two competition species. On the other hand, if we consider spatial scaling by setting $u_j(t) := u(j\Delta x, t)$ and $v_j(t) := v(j\Delta x, t)$, where

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Δx is the spatial mesh size, then by taking $\Delta x \to 0$ we obtain the continuous model. The PDE model is realized under an assumption that census tracts can be viewed as infinitesimal.

Lattice dynamical systems can be found in many applications, such as material science, image processing, pattern recognition, chemical reaction, biological system and so on. This can be seen from the survey papers by Chow [4] and Mallet-Paret [21] or the books of Fife [6] and of Shorrocks and Swingland [23]. On the other hand, lattice dynamical systems can also be considered as a discrete version of PDEs. For example, the problem (1.1) can be thought as a spatial discretization of the following diffusing Lotka-Volterra competition model:

$$\begin{cases} u_t = u_{xx} + u(1-u-kv), \\ v_t = dv_{xx} + rv(1-v-hu), \end{cases} \tag{1.2}$$

where $x \in \mathbb{R}$, $t \in \mathbb{R}$, $u = u(x,t)$ and $v = v(x,t)$. The variables $u = u(x,t)$ and $v = v(x,t)$ stand for the population densities of two species, so we only consider $u \geq 0$ and $v \geq 0$.

In the case of diffusion free, (1.2) becomes an ODE system which has at least three equilibrium solutions $(u,v) = (0,0),(0,1)$ and $(1,0)$. Moreover, when $0 < h,k < 1$ or $h,k > 1$, there exists the fourth equilibrium solution

$$e_4 := \left(\frac{1-k}{1-hk}, \frac{1-h}{1-hk}\right).$$

In fact, for any given initial data, we can classify the asymptotic behavior of the solutions into four cases as follows:

(A) If $0 < k < 1 < h$, then $\lim_{t\to+\infty}(u,v)(t) = (1,0)$ (the species u wins).
(B) If $0 < h < 1 < k$, then $\lim_{t\to+\infty}(u,v)(t) = (0,1)$ (the species v wins).
(C) If $h,k > 1$, then $\lim_{t\to+\infty}(u,v)(t) = (0,1)$ or $(1,0)$ (depending on the initial data).
(D) If $0 < h,k < 1$, then $\lim_{t\to+\infty}(u,v)(t) = e_4$ (two species coexist).

Not that the case (B) can be reduced to the case (A) by exchanging the roles of u and v.

From the biological point of view, it is interesting to see whether one species is stronger than the other. The superior species shall invade the inferior one so that the inferior species will be eventually extinct. To describe such an invading phenomenon, the traveling front plays an important role. Here traveling fronts are C^2 bounded functions with the special form

$$u(x,t) = u(\xi), \quad v(x,t) = v(\xi), \quad \xi = x + ct,$$

which connect two different equilibria from $\{(0,1),(1,0),(0,0),e_4\}$, where $c \in \mathbb{R}$ is called the wave speed. Then (1.2) becomes the following system of ODEs

$$\begin{cases} cu_\xi = u_{\xi\xi} + u(1-u-kv), \ \xi \in \mathbb{R} \\ cv_\xi = dv_{\xi\xi} + rv(1-v-hu), \ \xi \in \mathbb{R}. \end{cases}$$

The study of traveling front for Lotka-Volterra competition model with diffusion has attracted a lot of attention for past years. There are many interesting studies on the existence of positive traveling front solutions of (1.2) which connect two different equilibria. We list some known results as follows.

For case (A), Okubo, Maini, Williamson and Murray [22] showed that a positive wave connecting $(0, 1)$ and $(1, 0)$ exists if and only if its wave speed is larger than or equal to $2\sqrt{1-k}$, when $r = d = 1$ and $h + k = 2$. Hosono [11] showed that the existence of positive waves of (1.2) for small $d > 0$ by using the singular perturbation method. Kan-on [15] proved that a monotone wave connecting $(0, 1)$ and $(1, 0)$ exists if and only if its wave speed is larger than or equal to a constant (depending on r, d, h, k) which is the so called the *minimal wave speed*. Moreover, the minimal wave speed of (1.2) is always larger than or equal to $2\sqrt{1-k}$.

For case (C), there exists a unique wave speed such that a traveling front connecting $(0, 1)$ and $(1, 0)$ exists and is unique up to translations. Gardner [7] and Conley and Gardner [5] determined the wave speed implicitly by a topological method. This case is also called the case of strong competition. Both equilibria $(0, 1)$ and $(1, 0)$ are stable and so we have the bistable nonlinearity.

On the other hand, it is also very interesting to determine whether two species can live together. This is case (D). This case is the so-called co-existence case with weak competition. For case (D), Tang and Fife [24] proved that there exists a positive constant c_0 such that a positive wave front connecting $(0, 0)$ and e_4 exists if and only if the wave speed is larger than or equal to c_0. For more works about the study of traveling wave solutions of (1.2), we refer to [13, 14, 15, 16, 17, 18] and the references cited therein.

The purpose of this paper is to study the cases (A) and (B) (monostable case) for lattice dynamical system (1.1). Since the case (B) can be reduced to the case (A) by exchanging the roles of u_j and v_j, therefore we shall only consider the case (A). We are interested in traveling front solutions of (1.1) in the special form $u_j(t) = U(\xi)$ and $v_j(t) = V(\xi)$, $\xi = j + ct$, where $c \in \mathbb{R}$ is called wave speed, U, V are called wave profiles. Since we are looking for fronts connecting $(0, 1)$ and $(1, 0)$, therefore, our problem is to find $(c, U, V) \in \mathbb{R} \times C^1(\mathbb{R}) \times C^1(\mathbb{R})$ such that

$$(1.3) \quad \begin{cases} cU' = D_2[U] + U(1 - U - kV), \; cV' = dD_2[V] + rV(1 - V - hU) \text{ on } \mathbb{R}, \\ (U, V)(-\infty) = (0, 1), \quad (U, V)(+\infty) = (1, 0), \quad 0 \leq U, V \leq 1 \text{ on } \mathbb{R}, \end{cases}$$

where $D_2[\phi](\xi) := \phi(\xi + 1) + \phi(\xi - 1) - 2\phi(\xi)$, $0 < k < 1 < h$ and $r, d > 0$.

In the sequel, we shall always assume the following assumption:

$$(\mathbf{A1}) \quad 0 < k < 1 < h, \; d > 0 \text{ and } r > 0.$$

First, we prove the following theorem on the existence of traveling fronts.

Theorem 1. *Assume* (**A1**). *Then there exists a positive constant c_{min} such that the problem* (1.3) *admits a solution* (c, U, V) *satisfying* $U'(\cdot) > 0$ *and* $V'(\cdot) < 0$ *on* $\mathbb{R}$ *if and only if* $c \geq c_{min}$.

Due to this theorem, we call the positive constant c_{min} as the minimal wave speed. Next, we prove the monotonicity and uniqueness of wave profiles as follows.

Theorem 2. *Assume* (**A1**). *Then all wave profiles are strictly monotone.*

Theorem 3. *Assume* (**A1**). *If* $d \leq 1$, *then the wave profile is unique up to translations for a given wave speed* $c \geq c_{min}$.

The proofs of these two theorems rely on the analysis of asymptotic behaviors of wave tails. We put this complicated analysis in §3.

Finally, we give the characterization of the minimal wave speed by those parameters r, d, h, k. A plausible lower bound of the minimal wave speed is given by

$$c_* = c_*(k) := \min_{\lambda > 0} \left\{ \frac{(e^{\lambda} + e^{-\lambda} - 2) + (1 - k)}{\lambda} \right\} > 0.$$

Indeed, by linearizing the U-equation in (1.3) around the unstable equilibrium $(0, 1)$ gives the following characteristic equation

$$\Phi(c, \lambda) := c\lambda - [(e^{\lambda} + e^{-\lambda} - 2) + (1 - k)] = 0. \tag{1.4}$$

It is easily to see that (1.4) has a real root if and only if $c \geq c_*$. In fact, we have the following characterization of minimal wave speed.

Theorem 4. *Assume* (**A1**). *Then* $c_{min} \geq c_*$. *In particular,* $c_{min} = c_*$ *when* $d \leq 1$ *and*

$$(h, k, r) \in \{hk \leq 1, r > 0\} \cup \left\{ hk > 1, 0 < r \leq \frac{1 - k}{hk - 1} \right\}. \tag{1.5}$$

From Theorem 4 we know that under condition (1.5), the minimal wave speed of (1.3) can be characterized exactly, i.e., $c_{min} = c_*$. Condition (1.5) is similar to a condition which appears in studying the spreading speed of the PDE model (1.2) in [19]. Indeed, Lewis, Li and Weinberger [19] proved that the spreading speed of (1.2) is equal to $2\sqrt{1 - k}$ when

$$(h, k, r) \in \{hk \leq 1, r > 0\} \cup \left\{ hk > 1, 0 < r \leq (2 - d)\frac{1 - k}{hk - 1} \right\} \tag{1.6}$$

for $d \in (0, 2]$. They called the spreading speed of the PDE model (1.2) is *linearly determined*. See also [25, 19, 20]. It is known that, for certain homogeneous PDE models, the spreading speed is the same as the minimal wave speed. In a forthcoming paper [9], we shall genearlize Theorem 4 to a wider range of parameters than (1.6) so that the minimal wave speed for (1.2) can be characterized to be equal to $2\sqrt{1 - k}$ from a different view point than that of

[20, 19]. Note that, by a numerical simulation, in [12] Hosono conjectured that $2\sqrt{1-k}$ is the minimal wave speed of (1.2) under the condition

$$(h,k,r) \in \{hk \le 1, r > 0\} \cup \{hk > 1, 0 < r \le r_*\}$$

for some $r_* = r_*(h,k,d) > 0$ for certain $d > 0$.

On the other hand, the minimal wave speed can be thought as the invasion speed. Under condition (1.5), Theorem 4 shows that the species u will accelerate their invasion speed when k decreases, since $c_*(k)$ is increasing as k decreases. Moreover, when k is quite close to 0 (i.e., v almost cannot threaten u), we have

$$c_* = c_*(k) \approx \min_{\lambda>0} \left\{ \frac{(e^{\lambda} + e^{-\lambda} - 2) + 1}{\lambda} \right\} := \hat{c}.$$

Note that $\hat{c}$ is the minimal wave speed of the following one component lattice dynamical system with KPP nonlinearity

$$u_j' = (u_{j+1} + u_{j-1} - 2u_j) + u_j(1 - u_j),$$

see, e.g., [2]. Also, $c_*(k) \le \hat{c}$ for all $k \in (0,1)$. This tells us that the competition indeed makes the invasion speed slower and we may almost neglect the influence from species v to u when $k \ll 1$.

We now briefly describe the organization and main ideas of this paper as follows. In the next section, we give a proof of Theorem 1 which is motivated by the work [2] in which a traveling front solution can be constructed by using a sequence of truncated problems with the help of a super-solution. For extending this method from a single equation to a system, the key point here is to choose suitable translations of truncated solutions for both components so that the limit functions are not trivial (i.e., not identically equal to 0 or 1). It turns out that, due to the nonlinearity of our system, we only need to work on the V-component so that $V_i(0)$ takes a fixed value in $(0,1)$ along a suitable approximated sequence $\{V_i\}$. For more details, see §2.

In preparation of proving monotonicity and uniqueness of wave profiles, we study the asymptotic behavior of wave tails in §3. Based on a fundamental theory (see Proposition 2 in §3) developed in [2] (see also [3]), the limit of U'/U as $\xi \to -\infty$ can be easily computed. This also gives an upper bound estimate of the minimal wave speed. However, it is not trivial to compute the limit of $V'/(1-V)$ as $\xi \to -\infty$. The main difficulty here is the lack of exact information about the limit of $U/(1-V)$ as $\xi \to -\infty$, which is needed in applying Proposition 2. Hence a new idea is developed here to overcome this difficulty. Similarly, we can compute the limits of V'/V and $U'/(1-U)$ as $\xi \to +\infty$. This is the first part of §3.

Although the above asymptotic limits are sufficient for the proof of monotonicity theorem, we need more precise information about the wave tails for the uniqueness of wave profiles.

In order to derive more precise asymptotically exponential tails of wave profiles, we use the bilateral Laplace transform for both components U and $1-V$. A modified version of Ikehara's Theorem is applied (cf., e.g., [1] and [10]). This is the second part of §3.

Based on these asymptotic behaviors, we then show the monotonicity of wave profiles and the uniqueness of wave profiles for a given wave speed in §4. Finally, in §5, under the assumption (1.5), a super-solution can be constructed so that Theorem 4 can be proved. Some discussions on the minimal wave speed shall also be given at the end of this paper.

2. Existence of traveling front

To study the existence of traveling front, it is more convenient to work on (U,W), where $W:=1-V$. Then (1.3) is equivalent to

$$\begin{cases} cU' = D_2[U] + U[1-U-k(1-W)], \\ cW' = dD_2[W] + r(1-W)(hU-W), \\ (U,W)(-\infty) = (0,0), \quad (U,W)(+\infty) = (1,1), \\ 0 \le U, W \le 1, \end{cases} \tag{2.1}$$

where $0<k<1<h$ and $r,d>0$ are always assumed.

We first give some properties of solutions of (2.1).

Lemma 2.1. *If (c,U,W) is a solution of* (2.1)*, then $0<U(\cdot),W(\cdot)<1$ in $\mathbb{R}$ and $c>0$.*

Proof. For a contradiction, we assume that there exists $\xi_0\in\mathbb{R}$ such that $U(\xi_0)=0$. Without loss of generality, we may assume ξ_0 is the right-most point such that $U(\xi_0)=0$. Such ξ_0 exists, since $U(+\infty)=1$. Since $0\le U,W\le 1$ and $U'(\xi_0)=0$, from the first equation of (2.1) it follows that $U(\xi_0+1)=U(\xi_0-1)=0$, a contradiction with the definition of ξ_0. Thus $U(\cdot)>0$ in $\mathbb{R}$. Similarly, we have $W(\cdot)<1$ in $\mathbb{R}$. Then it is easy to derive $U(\cdot)<1$ and $W(\cdot)>0$ in $\mathbb{R}$ by using $U(\cdot)>0$ and $W(\cdot)<1$ in $\mathbb{R}$.

Next, since $(U,W)(-\infty)=(0,0)$, there exists $N\gg 1$ such that

$$1-U(\cdot)-k(1-W(\cdot)) > \frac{1}{2}(1-k) > 0$$

on $(-\infty,-N)$. Integrating the first equation of (2.1) over $(-\infty,\xi)$ and using the boundary conditions, we obtain

$$cU(\xi) = \int_{\xi}^{\xi+1} U(s)ds - \int_{\xi-1}^{\xi} U(s)ds + \int_{-\infty}^{\xi} U[1-U-k(1-W)](s)ds. \tag{2.2}$$

Hence for $\xi<-N$ we have

$$|c|+1 \ge cU(\xi) - \int_{\xi}^{\xi+1} U(s)ds + \int_{\xi-1}^{\xi} U(s)ds \ge \frac{1}{2}(1-k)\int_{-\infty}^{\xi} U(s)ds.$$

This implies that $R(\xi) := \int_{-\infty}^{\xi} U(s)ds$ is well-defined for all $\xi < +\infty$. For $x < -N$, since R is increasing, by integrating over $(-\infty, x)$ we deduce from (2.2) that

$$cR(x) = \int_{x}^{x+1} R(\xi)d\xi - \int_{x-1}^{x} R(\xi)d\xi + \int_{-\infty}^{x}\int_{-\infty}^{\xi} U[1-U-k(1-W)](s)dsd\xi > 0.$$

Thus $c > 0$ and the lemma follows. $\square$

Now, let c be a fixed (arbitrary) positive constant. For a positive constant μ (to be specified later), we define

$$H_1(U,W)(\xi) = \mu U(\xi) + \frac{1}{c}D_2[U](\xi) + \frac{1}{c}U(\xi)[1-U(\xi)-k(1-W(\xi))],$$
$$H_2(U,W)(\xi) = \mu W(\xi) + \frac{d}{c}D_2[W](\xi) + \frac{r}{c}(1-W(\xi))(hU(\xi)-W(\xi)).$$

It is easy to see that if (U,W) is a solution of (2.1), then

$$U(\xi) = T_1(U,W)(\xi) := e^{-\mu\xi}\int_{-\infty}^{\xi} e^{\mu s}H_1(U,W)(s)ds,$$
$$W(\xi) = T_2(U,W)(\xi) := e^{-\mu\xi}\int_{-\infty}^{\xi} e^{\mu s}H_2(U,W)(s)ds.$$

Conversely, if (U,W) satisfies the above integral equations, then it satisfies the differential equations of (2.1). By choosing $\mu > 0$ sufficiently large, we see that the integrals are well-defined in $\mathbb{R}$ and have the monotonic property, i.e.,

$$\begin{aligned}
&0 \le U_1(\cdot) \le U_2(\cdot) \le 1,\ 0 \le W_1(\cdot) \le W_2(\cdot) \le 1 \text{ in } \mathbb{R}\\
\Rightarrow\ &H_1(U_1,W_1)(\cdot) \le H_1(U_2,W_2)(\cdot),\ H_2(U_1,W_1)(\cdot) \le H_2(U_2,W_2)(\cdot) \text{ in } \mathbb{R}\\
\Rightarrow\ &T_1(U_1,W_1)(\cdot) \le T_1(U_2,W_2)(\cdot),\ T_2(U_1,W_1)(\cdot) \le T_2(U_2,W_2)(\cdot) \text{ in } \mathbb{R}.
\end{aligned}$$

To see this, we may write

$$\begin{aligned}
(2.3)\quad c[H_1(U_1,W_1)(s) - H_1(U_1,W_1)(s)] &= \{c\mu - 2 + (1-k) - (U_1+U_2)(s)\}(U_1-U_2)(s)\\
&\quad + (U_1-U_2)(s+1) + (U_1-U_2)(s-1) + k(U_1W_1 - U_2W_2)(s),
\end{aligned}$$

for example, then it is easy to derive the above monotonic property.

Following [2], for each $n \in \mathbb{N}$, we consider the following truncated problem:

$$cU' = D_2[U] + U[1-U-k(1-W)],\ \forall\ \xi \in [-n,0], \tag{2.4}$$
$$cW' = dD_2[W] + r(1-W)(hU-W),\ \forall\ \xi \in [-n,0] \tag{2.5}$$

with the boundary conditions:

$$U(\xi) = W(\xi) = 1,\ \forall\ \xi \in (0,+\infty), \tag{2.6}$$
$$U(\xi) = W(\xi) = \varepsilon,\ \forall\ \xi \in (-\infty,-n], \tag{2.7}$$

where $\varepsilon \in [0,1)$. Via the integrating factor $e^{\mu\xi}$, (2.4) and (2.5) can be reduced to the integral equations

$$U(\xi) = T_1^n(U,W)(\xi),\ W(\xi) = T_2^n(U,W)(\xi),\ \forall\, \xi \in [-n,0], \tag{2.8}$$

where

$$T_1^n(U,W)(\xi) := e^{-\mu\xi}\int_{-\infty}^{-n} \varepsilon\mu e^{\mu s} ds + e^{-\mu\xi}\int_{-n}^{\xi} e^{\mu s} H_1(U,W)(s) ds,$$
$$T_2^n(U,W)(\xi) := e^{-\mu\xi}\int_{-\infty}^{-n} \varepsilon\mu e^{\mu s} ds + e^{-\mu\xi}\int_{-n}^{\xi} e^{\mu s} H_2(U,W)(s) ds$$

for all $\xi \in [-n,0]$.

Due to $\mu > 0$ large enough, T_1^n and T_2^n also have the monotonic property, i.e.,

$$0 \le U_1(\cdot) \le U_2(\cdot) \le 1 \quad \text{and} \quad 0 \le W_1(\cdot) \le W_2(\cdot) \le 1 \quad \text{in } \mathbb{R}$$

imply that

$$T_1^n(U_1,W_1)(\cdot) \le T_1^n(U_2,W_2)(\cdot), \quad T_2^n(U_1,W_1)(\cdot) \le T_2^n(U_2,W_2)(\cdot) \quad \text{on } [-n,0].$$

From this, we can prove the following lemma.

Lemma 2.2. *For each $n \in \mathbb{N}$ and $\varepsilon \in [0,1)$, there exists a unique function $(U^{n,\varepsilon}, W^{n,\varepsilon})$ from $\mathbb{R}$ to $[\varepsilon,1] \times [\varepsilon,1]$ that satisfies* (2.6)-(2.8) *and has the following properties:*

(1) *$U^{n,\varepsilon}(\cdot)$, $W^{n,\varepsilon}(\cdot) \in C^1((-n,0)) \cap C((-\infty,0])$.*
(2) *$(U^{n,\varepsilon})'(\cdot) > 0$ and $(W^{n,\varepsilon})'(\cdot) > 0$ on $(-n,0)$ for any $\varepsilon \in [0,1)$.*
(3) *$\dfrac{d}{d\varepsilon}U^{n,\varepsilon}(\xi) \ge e^{-\mu(\xi+n)}$ and $\dfrac{d}{d\varepsilon}W^{n,\varepsilon}(\xi) \ge e^{-\mu(\xi+n)}$ for $\xi \in [-n,0]$.*

Proof. Given $n \in \mathbb{N}$ and $\varepsilon \in [0,1)$. First, if $\varepsilon \le U(\cdot) \le 1$ and $\varepsilon \le W(\cdot) \le 1$, then

$$\mu\varepsilon + \frac{\varepsilon}{c}(1-k)(1-\varepsilon) = H_1(\varepsilon,\varepsilon)(\cdot) \le H_1(U,W)(\cdot) \le H_1(1,1)(\cdot) = \mu,$$
$$\mu\varepsilon + \frac{r\varepsilon}{c}(h-1)(1-\varepsilon) = H_2(\varepsilon,\varepsilon)(\cdot) \le H_2(U,W)(\cdot) \le H_2(1,1)(\cdot) = \mu$$

on $\mathbb{R}$. Thus, we obtain

$$\varepsilon \le \varepsilon + \frac{\varepsilon}{c\mu}(1-k)(1-\varepsilon)(1-e^{-\mu(\xi+n)}) \le T_1^n(U,W)(\xi) \le 1-(1-\varepsilon)e^{-\mu(\xi+n)} \le 1,$$
$$\varepsilon \le \varepsilon + \frac{r\varepsilon}{c\mu}(h-1)(1-\varepsilon)(1-e^{-\mu(\xi+n)}) \le T_2^n(U,W)(\xi) \le 1-(1-\varepsilon)e^{-\mu(\xi+n)} \le 1$$

for all $\xi \in [-n,0]$.

Now, we define inductively

$$\begin{cases} (U_0^{n,\varepsilon}(\xi), W_0^{n,\varepsilon}(\xi)) := (1,1),\ \xi \in (-n,\infty), \\ (U_0^{n,\varepsilon}(\xi), W_0^{n,\varepsilon}(\xi)) := (\varepsilon,\varepsilon),\ \xi \in (-\infty,-n]. \end{cases}$$

Also, for all $j \in \mathbb{N}$, we define

$$(U_j^{n,\varepsilon}(\cdot), W_j^{n,\varepsilon}(\cdot)) = (\ T_1^n(U_{j-1}^{n,\varepsilon}, W_{j-1}^{n,\varepsilon})(\cdot)\ ,\ T_2^n(U_{j-1}^{n,\varepsilon}, W_{j-1}^{n,\varepsilon})(\cdot)\ ) \text{ on } [-n,0]$$
$$(U_j^{n,\varepsilon}(\cdot), W_j^{n,\varepsilon}(\cdot)) = (1,1) \ \text{ in } (0,\infty),\ (U_j^{n,\varepsilon}(\cdot), W_j^{n,\varepsilon}(\cdot)) = (\varepsilon,\varepsilon) \ \text{ in } (-\infty,-n].$$

Since

$$U_1^{n,\varepsilon}(\cdot) = T_1^n(1,1)(\cdot) \le 1 = U_0^{n,\varepsilon}(\cdot) \text{ on } [-n,0],$$
$$W_1^{n,\varepsilon}(\cdot) = T_2^n(1,1)(\cdot) \le 1 = W_0^{n,\varepsilon}(\cdot) \text{ on } [-n,0],$$

it follows from the monotone property of T_1^n and T_2^n that

$$\varepsilon \le T_1^n(U_j^{n,\varepsilon}, W_j^{n,\varepsilon})(\cdot) \le T_1^n(U_{j-1}^{n,\varepsilon}, U_{j-1}^{n,\varepsilon})(\cdot) \le 1,$$
$$\varepsilon \le T_2^n(U_j^{n,\varepsilon}, W_j^{n,\varepsilon})(\cdot) \le T_2^n(U_{j-1}^{n,\varepsilon}, W_{j-1}^{n,\varepsilon})(\cdot) \le 1$$

on $[-n,0]$ for all $j \in \mathbb{N}$. From these iterations, for any given $\xi \in [-n,0]$, we obtain $\{U_j^{n,\varepsilon}(\xi)\}$ and $\{W_j^{n,\varepsilon}(\xi)\}$ are non-increasing in j. Therefore, the limit

$$(U_*(\xi), W_*(\xi)) := (\lim_{j\to+\infty} U_j^{n,\varepsilon}(\xi), \lim_{j\to+\infty} W_j^{n,\varepsilon}(\xi))$$

exists. By applying Lebesgue's Dominated Convergence Theorem, we have

$$U_*(\xi) = T_1^n(U_*, W_*)(\xi),\ W_*(\xi) = T_2^n(U_*, W_*)(\xi) \ \ \forall\ \xi \in [-n,0].$$

We now define $U_*(\xi) = W_*(\xi) = 1$ for all $\xi > 0$ and $U_*(\xi) = W_*(\xi) = \varepsilon$ for all $\xi < -n$. Then it is not hard to see $U_*(\cdot)$ and $W_*(\cdot) \in C^1((-n,0)) \cap C((-\infty,0])$.

Next, we prove the uniqueness. For this, we let $(U^*, W^*) : \mathbb{R} \mapsto [\varepsilon,1]\times[\varepsilon,1]$ be another solution of (2.6)-(2.8). Thus $U^*(\cdot), W^*(\cdot) \in C^1((-n,0))$. Claim that $U^* \equiv U_*$ and $W^* \equiv W_*$ on $[-n,0]$. Since $U^* \le U_0$, $W^* \le W_0$ and by the monotone property of T_1^n and T_2^n, we have

$$U^*(\cdot) = T_1^n(U^*, W^*)(\cdot) \le T_1^n(U_0, W_0) = U_1(\cdot) \text{ on } [-n,0],$$
$$W^*(\cdot) = T_2^n(U^*, W^*)(\cdot) \le T_2^n(U_0, W_0) = W_1(\cdot) \text{ on } [-n,0].$$

By induction, we obtain that $U^* \le U_*$ and $W^* \le W_*$. This also implies that $U^*(\cdot), W^*(\cdot) \in C((-\infty,0])$.

To derive the reverse inequalities, we define

$$\eta^* := \inf\{\eta > 0 | U^*(\xi) \ge U_*(\xi - y),\ W^*(\xi) \ge W_*(\xi - y)\ \forall\ \xi \in [-n+y, 0],\ y \ge \eta\}.$$

Note that η^* is well-defined and $0 \le \eta^* \le n$, since $U^*(0) \ge \varepsilon = U_*(-n)$ and $W^*(0) \ge \varepsilon = W_*(-n)$. By continuity, $U^*(\cdot) \ge U_*(\cdot - \eta^*)$ and $W^*(\cdot) \ge W_*(\cdot - \eta^*)$ on $[-n+\eta^*, 0]$. Moreover, due to the boundary conditions we have $U^*(\cdot) \ge U_*(\cdot - \eta^*)$ and $W^*(\cdot) \ge W_*(\cdot - \eta^*)$ in $\mathbb{R}$. Hence $H_i(U^*, W^*)(\cdot) \ge H_i(U_*, W_*)(\cdot - \eta^*)$ in $\mathbb{R}$ for $i = 1, 2$.

For any $\xi \in [-n+\eta^*, 0]$, we have

$$\begin{aligned}
& U^*(\xi) - U_*(\xi - \eta^*) \\
= \; & T_1^n(U^*, W^*)(\xi) - T_1^n(U_*, W_*)(\xi - \eta^*) \\
\geq \; & \int_{-n-\xi}^{0} e^{\mu s} H_1(U^*, W^*)(s+\xi) ds - \int_{-n-\xi+\eta^*}^{0} e^{\mu s} H_1(U^*, W^*)(s+\xi) ds \\
& + \int_{-\infty}^{-n-\xi} \mu e^{\mu s} \varepsilon ds - \int_{-\infty}^{-n-\xi+\eta^*} \mu e^{\mu s} \varepsilon ds \\
\geq \; & \int_{-n-\xi}^{-n-\xi+\eta^*} e^{\mu s} \{H_1(U^*, W^*)(s+\xi) - \mu\varepsilon\} ds \\
\geq \; & \int_{-n-\xi}^{-n-\xi+\eta^*} e^{\mu s} \{H_1(\varepsilon, \varepsilon)(s+\xi) - \mu\varepsilon\} ds \\
\geq \; & \frac{\varepsilon}{c}(1-k)(1-\varepsilon) \int_{-n-\xi}^{-n-\xi+\eta^*} e^{\mu s} ds.
\end{aligned}$$

Similarly, we can calculate that

$$W^*(\xi) - W_*(\xi - \eta^*) \geq \frac{r\varepsilon}{c}(h-1)(1-\varepsilon) \int_{-n-\xi}^{-n-\xi+\eta^*} e^{\mu s} ds$$

for any $\xi \in [-n+\eta^*, 0]$. Using these two estimates, we are ready to show that $\eta^* = 0$.

Indeed, if $\varepsilon \in (0,1)$, then $\eta^* > 0$ implies $U^*(\xi) - U_*(\xi - \eta^*) > 0$ and $W^*(\xi) - W_*(\xi - \eta^*) > 0$ for any $\xi \in [-n+\eta^*, 0]$. Then, by continuity, we may find $0 < \delta \ll 1$ such that

$$U^*(\xi) - U_*(\xi - (\eta^* - \delta)) > 0, \quad W^*(\xi) - W_*(\xi - (\eta^* - \delta)) > 0$$

for all $\xi \in [-n+\eta^*, 0]$. This contradicts the definition of η^* and so $\eta^* = 0$.

If $\varepsilon = 0$, then, for any $\xi \in [-n+\eta^*, 0]$, $\eta^* > 0$ leads to

$$\begin{aligned}
U^*(\xi) - U_*(\xi - \eta^*) \; & \geq \; \int_{-n-\xi}^{-n-\xi+\eta^*} e^{\mu s} H_1(U^*, W^*)(s+\xi) ds \\
& > \; \int_{-n-\xi}^{-n-\xi+\eta^*} e^{\mu s} H_1(0,0)(s+\xi) ds = 0,
\end{aligned}$$

by using the fact that $0 < U^*(\cdot), W^*(\cdot) < 1$ on $(-n, 0)$. Also, we have $W^*(\xi) - W_*(\xi - \eta^*) > 0$ for any $\xi \in [-n+\eta^*, 0]$. This contradicts the definition of η^* again. Thus when $\varepsilon = 0$ we also have $\eta^* = 0$. This completes the proof of the uniqueness.

Now, we prove (2). Due to the uniqueness and $\eta^* = 0$, we have $U_*(\xi) \geq U_*(\xi - s)$ and $W_*(\xi) \geq W_*(\xi - s)$ for all $s \geq 0$ and $\xi \in \mathbb{R}$. Thus $U_*'(\xi) \geq 0$ and $W_*'(\xi) \geq 0$ for $\xi \in (-n, 0)$. It follows from the monotonic property of H_i that

$$H_i(U_*, W_*)(s) \leq H_i(U_*, W_*)(\xi) \; \forall \; s \leq \xi$$

for $i = 1, 2$. Moreover, by differentiating $U_* = T_1^n(U_*, W_*)$ and $W_* = T_2^n(U_*, W_*)$, it is easy to see that $(U_*)'(\xi) > 0$ and $(W_*)'(\xi) > 0$ for all $\xi \in (-n, 0)$ and $\varepsilon \in [0, 1)$.

Finally, we prove (3). For given $0 \le \varepsilon_1 < \varepsilon_2 < 1$, by the construction of U^{n,ε_1} and U^{n,ε_2}, it is easy to know $U^{n,\varepsilon_2}(\xi) \ge U^{n,\varepsilon_1}(\xi)$ for all $\xi \in [-n, 0]$. Moreover,

$$\begin{aligned}
& U^{n,\varepsilon_2}(\xi) - U^{n,\varepsilon_1}(\xi) \\
= \ & e^{-\mu\xi} \int_{-n}^{\xi} \mu e^{\mu s} \{H_1(U^{n,\varepsilon_2}(\xi), W^{n,\varepsilon_2})(s) - H_2(U^{n,\varepsilon_1}(\xi), W^{n,\varepsilon_1})(s)\} ds \\
& + (\varepsilon_2 - \varepsilon_1) e^{-\mu\xi} \int_{-\infty}^{-n} \mu e^{\mu s} ds \\
\ge \ & (\varepsilon_2 - \varepsilon_1) e^{-\mu(\xi+n)} \ \forall\ \xi \in [-n, 0].
\end{aligned}$$

This implies

$$\frac{d}{d\varepsilon} U^{n,\varepsilon}(\cdot) \ge e^{-\mu(\xi+n)} \text{ for all } \xi \in [-n, 0].$$

Similarly, we also have

$$\frac{d}{d\varepsilon} W^{n,\varepsilon}(\cdot) \ge e^{-\mu(\xi+n)} \text{ for all } \xi \in [-n, 0].$$

Therefore, the proof of this lemma is completed. □

In order to derive the existence of solutions of (2.1), we first recall the following Helly's Lemma.

Proposition 1 (Helly's Lemma). *Let $\{U_n(\cdot)\}_{n\in\mathbb{N}}$ be a sequence of uniformly bounded and non-decreasing functions in $\mathbb{R}$. Then there exist a subsequence $\{U_{n_i}(\cdot)\}$ of $\{U_n(\cdot)\}$ and a non-decreasing function U such that $U_{n_i}(\cdot) \to U(\cdot)$ as $i \to +\infty$ pointwise in $\mathbb{R}$.*

Now, we define the notion of super-solutions. Given a constant $c > 0$. A continuous function (U^+, W^+) from $\mathbb{R}$ to $(0, 1]$ is called a *super-solution* of (2.1), if $W^+(\cdot)$ is a non-constant function, $U^+(+\infty) = W^+(+\infty) = 1$ and both U^+ and W^+ are differentiable a.e. in $\mathbb{R}$ such that

$$\begin{cases} c(U^+)' \ge D_2[U^+] + U^+[1 - U^+ - k(1 - W^+)] \ \text{ a.e. in } \mathbb{R}, \\ c(W^+)' \ge dD_2[W^+] + r(1 - W^+)(hU^+ - W^+) \ \text{ a.e. in } \mathbb{R}. \end{cases} \tag{2.9}$$

Hereafter we say that a vector-valued function (U, W) is non-decreasing in $\mathbb{R}$ if both U and W are non-decreasing in $\mathbb{R}$.

Then we have the following lemma.

Lemma 2.3. *If there exists a super-solution (U^+, W^+) satisfying $U^+(\cdot) = W^+(\cdot) = 1$ on $[0, +\infty)$ for a given $c > 0$, then (2.1) admits a solution (c, U, W) with $U'(\cdot) > 0$ and $W'(\cdot) > 0$ in $\mathbb{R}$.*

Proof. First, we choose $n_0 > 0$ such that $W^+(-n_0) = \varepsilon_0$ for some $\varepsilon_0 \in (0,1)$. This ε_0 exists, since W^+ is a non-constant function. Then for each $n > 2n_0$ we shall claim that there exists a unique $\varepsilon = \varepsilon(n) \in (0,1)$ such that $W^{n,\varepsilon(n)}(-n/2) = \varepsilon_0$.

To see this, we first prove $W^{n,0}(-n/2) < \varepsilon_0$ for any $n > 2n_0$. For this, we define

$$\eta^* := \inf\{\eta > 0 |\ U^+(\xi) \geq U^{n,0}(\xi - \eta),\ W^+(\xi) \geq W^{n,0}(\xi - \eta),\ \forall\, \xi \in (-\infty, 0]\}.$$

Note that η^* is well-defined and $\eta^* \in [0, n]$, since $U^+(\cdot) = W^+(\cdot) = 1$ on $[0,+\infty)$ and $W^{n,0}(\cdot) = U^{n,0}(\cdot) = 0$ on $(-\infty, -n]$. By continuity, $U^+(\xi) \geq U^{n,0}(\xi - \eta^*)$ and $W^+(\xi) \geq W^{n,0}(\xi - \eta^*)$ for all $\xi \in (-\infty, 0]$. This implies that $H_i(U^+, W^+)(\xi) \geq H_i(U^{n,0}, W^{n,0})(\xi - \eta^*)$ for $\xi \in (-\infty, 0]$ and $i = 1, 2$. We claim that $\eta^* = 0$. Indeed, we have

$$\begin{aligned}
& W^+(\xi) - W^{n,0}(\xi - \eta^*) \\
\geq\ & T_2(U^+, W^+)(\xi) - T_2^n(U^{n,0}, W^{n,0})(\xi - \eta^*) \\
=\ & \int_{-\infty}^{0} e^{\mu s} H_2(U^+, W^+)(s + \xi) ds - \int_{-n-\xi+\eta^*}^{0} e^{\mu s} H_2(U^{n,0}, W^{n,0})(s + \xi - \eta^*) ds \\
\geq\ & \int_{-\infty}^{-n-\xi+\eta^*} e^{\mu s} H_2(U^{n,0}, W^{n,0})(s + \xi) ds > 0
\end{aligned}$$

for all $\xi \in [-n + \eta^*, 0]$. Similarly, $U^+(\xi) - U^{n,0}(\xi - \eta^*) > 0$ for all $\xi \in [-n + \eta^*, 0]$. A similar argument as in the proof of Lemma 2.2 leads that $\eta^* = 0$. Hence $W^+(\cdot) \geq W^{n,0}(\cdot)$ on $(-\infty, 0]$ and so we have

$$W^{n,0}(-\frac{n}{2}) < W^{n,0}(-n_0) \leq W^+(-n_0) = \varepsilon_0$$

for any $n > 2n_0$. By using Lemma 2.2 and noting that $W^{\varepsilon,n}$ is continuous in ε, we conclude that there exists a unique $\varepsilon = \varepsilon(n) \in (0, \varepsilon_0] \subset (0,1)$ such that $W^{n,\varepsilon(n)}(-n/2) = \varepsilon_0$.

We now consider the sequence of functions $\{U^{n,\varepsilon(n)}(-n/2 + \cdot), W^{n,\varepsilon(n)}(-n/2 + \cdot)\}_{n > 2n_0}$ in $\mathbb{R}$. By Helly's Lemma, there exists a sequence $\{U^{n_i,\varepsilon(n_i)}(-n_i/2 + \cdot), W^{n_i,\varepsilon(n_i)}(-n_i/2 + \cdot)\}$ and a non-decreasing function (U, W) from $\mathbb{R}$ to $[0,1] \times [0,1]$ such that $n_i \to +\infty$ and

$$\left(\ U^{n_i,\varepsilon(n_i)}(-\frac{n_i}{2} + \cdot), W^{n_i,\varepsilon(n_i)}(-\frac{n_i}{2} + \cdot)\ \right) \to (U(\cdot), W(\cdot)) \text{ as } \ i \to +\infty.$$

By Lebesgue's Dominated Convergence Theorem, we obtain

$$U(\xi) = T_1(U, W)(\xi), \quad W(\xi) = T_2(U, W)(\xi) \text{ for all } \xi \in \mathbb{R}.$$

Moreover, $0 \leq U,\ W \leq 1$ in $\mathbb{R}$ and $U,\ W \in C^1(\mathbb{R})$.

Next, it remains to prove that (U, W) satisfies the boundary conditions. Since both U and W are non-decreasing in $\mathbb{R}$ and $0 \leq U,\ W \leq 1$ in $\mathbb{R}$, both $U(\pm\infty)$ and $W(\pm\infty)$ exist. By using $U = T_1(U,W)$, $W = T_2(U,W)$ and L'Hospital's rule, we have

$$\begin{aligned}
\lim_{\xi\to\pm\infty} U(\xi) &= \lim_{\xi\to\pm\infty} T_1(U,W)(\xi) \\
&= \lim_{\xi\to\pm\infty} \left\{ U(\xi) + \frac{1}{c\mu}\Big[D_2[U](\xi) + U(\xi)(1-U(\xi)-k(1-W(\xi))\Big]\right\}, \\
\lim_{\xi\to\pm\infty} W(\xi) &= \lim_{\xi\to\pm\infty} T_2(U,W)(\xi) \\
&= \lim_{\xi\to\pm\infty} \left\{ W(\xi) + \frac{1}{c\mu}\Big[dD_2[U](\xi) + r(1-W(\xi))(hU(\xi)-W(\xi))\Big]\right\}.
\end{aligned}$$

This implies that

$$\begin{cases} U(\pm\infty)(1-U(\pm\infty)-k(1-W(\pm\infty))) = 0, \\ (1-W(\pm\infty))(hU(\pm\infty)-W(\pm\infty)) = 0. \end{cases}$$

Hence $U(\pm\infty), W(\pm\infty) \in \{0,1\}$.

Note that $W(0) = \varepsilon_0 \in (0,1)$, since $W^{n_i,\varepsilon(n_i)}(-n_i/2) = \varepsilon_0 \in (0,1)$ for all i. Also, since W is non-decreasing in $\mathbb{R}$, we have $W(-\infty) = 0$ and $W(+\infty) = 1$. Note that $W(-\infty) = 0$ implies $U(-\infty) = 0$. On the other hand, $(U(-\infty), U(+\infty)) = (0,0)$ implies that $U \equiv 0$. By integrating the second equation of (2.1) over $(-\infty,+\infty)$, and noting that

$$\int_{-\infty}^{+\infty} D_2[W](s)ds = 0,$$

we have

$$0 < c = -r\int_{-\infty}^{+\infty} W(s)(1-W(s))ds < 0,$$

a contradiction. Thus $(U(-\infty), U(+\infty)) = (0,1)$.

Finally, by differentiating $U = T_1(U,W)$ and $W = T_2(U,W)$, using $U', W' \geq 0$ it follows that $U'(\xi) > 0$ and $W'(\xi) > 0$ for all $\xi \in \mathbb{R}$. This proves the lemma. □

In the following lemma, we add the monotonicity condition and remove the condition $U^+(\cdot) = W^+(\cdot) = 1$ on $[0,+\infty)$ posed in Lemma 2.3.

Lemma 2.4. *If there exists a super-solution (U^+, W^+) of (2.1) with $(U^+)', (W^+)' > 0$ for a given $c > 0$, then* (2.1) *admits a solution (c, U, W) with $U'(\cdot) > 0$ and $W'(\cdot) > 0$ in $\mathbb{R}$.*

Proof. Suppose (U^+, W^+) is a super-solution of (2.1) with $(U^+)', (W^+)' > 0$ for given $c > 0$.

To find a solution of (2.1), we shall apply Lemma 2.3. For any $0 < \delta \ll 1$, we define

$$(U_\delta^+(\xi), W_\delta^+(\xi)) := (\min\{1, (1+\delta)U^+(\xi)\}, \min\{1, (1+\delta)W^+(\xi)\}),$$

for all $\xi \in \mathbb{R}$. Then it is easy to see that $U_\delta^+(\cdot) \equiv 1$ on $[M_1, +\infty)$, $U_\delta^+ < 1$ on $(-\infty, M_1)$, $W_\delta^+(\cdot) \equiv 1$ on $[M_2, +\infty)$ and $W_\delta^+ < 1$ on $(-\infty, M_2)$ for some $M_i = M_i(\delta) \gg 1$, $i = 1, 2$. We claim that $(U_\delta^+(\xi), W_\delta^+(\xi))$ is a super-solution of the problem (P_δ):

$$\begin{cases} cU' = D_2[U] + f_\delta(U,W), \\ cW' = dD_2[W] + g_\delta(U,W), \\ (U,W)(-\infty) = (0,0), (U,W)(+\infty) = (1,1), \\ 0 \le U, W \le 1, \end{cases}$$

where

$$f_\delta(U,W) := \min\left\{U\Big(1 - \frac{U}{1+\delta} - \frac{k(1+\delta-W)}{1+\delta}\Big), U(1-U-k(1-W))\right\},$$

$$g_\delta(U,W) := \min\left\{r(hU-W)\Big(1-\frac{W}{1+\delta}\Big), r(hU-W)(1-W)\right\}.$$

To see this, without loss of generality we may assume that $M_1 \le M_2$. The case when $M_1 = M_2$ is trivial. So we assume that $M_1 < M_2$. Clearly, the condition (2.9) holds when $\xi < M_1$ and $\xi > M_2$. Suppose that $\xi \in (M_1, M_2)$. Then

$$U_\delta^+(\xi) = 1, (U_\delta^+)'(\xi) = 0, W_\delta^+(\xi) = (1+\delta)W^+(\xi), (W_\delta^+)'(\xi) = (1+\delta)(W^+)'(\xi).$$

Moreover, we have

$$\begin{aligned} D_2[U_\delta^+](\xi) &= U_\delta^+(\xi+1) + U_\delta^+(\xi-1) - 2U_\delta^+(\xi) \\ &= 1 + U_\delta^+(\xi-1) - 2 \le 0, \\ f_\delta(U_\delta^+, W_\delta^+)(\xi) &\le U_\delta^+(\xi)\{1 - U_\delta^+(\xi) - k[1 - W_\delta^+(\xi)]\} \\ &= -k[1 - W_\delta^+(\xi)] \le 0, \\ (W_\delta^+)'(\xi) &= (1+\delta)(W^+)'(\xi) \\ &\ge (1+\delta)\{dD_2[W^+](\xi) + r[hU^+(\xi) - W^+(\xi)][1 - W^+(\xi)]\} \\ &\ge dD_2[W_\delta^+](\xi) + r[hU_\delta^+(\xi) - W_\delta^+(\xi)][1 - W_\delta^+(\xi)], \end{aligned}$$

where the facts $W_\delta^+(\xi+1) \le (1+\delta)W^+(\xi+1)$ and $U_\delta^+(\xi) \le (1+\delta)U^+(\xi)$ are used. Hence (2.9) holds for $\xi \in (M_1, M_2)$. We conclude that $(U_\delta^+(\xi), W_\delta^+(\xi))$ is a super-solution of problem (P_δ).

Next, setting

$$\hat{U}_\delta^+(\xi) = U_\delta^+(\xi + M_2), \quad \hat{W}_\delta^+(\xi) = W_\delta^+(\xi + M_2),$$

Then $\hat{U}_\delta^+(\cdot) = \hat{W}_\delta^+(\cdot) = 1$ on $[0, +\infty)$ and $(\hat{U}_\delta^+, \hat{W}_\delta^+)$ is a super-solution of problem (P_δ). Thus we can apply Lemma 2.3 to obtain a solution (U_δ, W_δ) of (P_δ) with $U_\delta' > 0$ and $W_\delta' > 0$ in $\mathbb{R}$.

Now, let $\{c, U_{\delta_i}, W_{\delta_i}\}$ be a sequence of monotone increasing solutions of (P_{δ_i}) such that $W_{\delta_i}(0) = 1/2$ for all i and $\delta_i \downarrow 0$ as $i \to \infty$. By Helly's Lemma, there exists a subsequence $\{c, U_{\delta_{i_j}}, W_{\delta_{i_j}}\}$ and a monotone non-decreasing function (U_0, W_0) such that $(c, U_{\delta_{i_j}}, W_{\delta_{i_j}}) \to (c, U_0, W_0)$ as $j \to \infty$ pointwise in $\mathbb{R}$. Note that $0 \le U_0, W_0 \le 1$ in $\mathbb{R}$ and $W_0(0) = 1/2$. By the same argument as in Lemma 2.3, we can derive that (c, U_0, W_0) satisfies (2.1) such that $U_0' > 0$, $W_0' > 0$ in $\mathbb{R}$. Thus the lemma follows. $\square$

Next, we shall find a super-solution of (2.1) for $c \gg 1$.

Lemma 2.5. *For $c > 0$ large enough, (U^+, W^+) is a super-solution of* (2.1)*, where*

$$U^+(\xi) = W^+(\xi) = \min\{1, e^{\xi}\}.$$

Proof. By choosing

$$c \geq c_1 := \max\left\{(e + e^{-1} - 2) + (1 - k),\ d(e + e^{-1} - 2) + r(h - 1)\right\},$$

it is easy to check that (U^+, W^+) is a super-solution of (2.1). $\square$

Now, we are ready to give a proof of Theorem 1.

Proof of Theorem 1. First, by Lemmas 2.3 and 2.5, (2.1) admits a solution (c, U, W) with $U' > 0$ and $W' > 0$ in $\mathbb{R}$ for all $c \geq c_1$. It follows that the constant

$$c_{min} := \inf\{c > 0 \mid \text{ (2.1) has a solution } (c, U, W) \text{ with } U' > 0 \text{ and } W' > 0 \text{ in } \mathbb{R}\}$$

is well-defined. Since a monotone front with speed c_0 gives a super-solution of (2.1) for any $c > c_0$, Lemma 2.4 implies that (2.1) has a solution (c, U, W) with $U' > 0$ and $W' > 0$ in $\mathbb{R}$ for any $c > c_{min}$.

We now claim that, for $c = c_{min}$, (2.1) has a solution (c, U, W) with $U' > 0$ and $W' > 0$ in $\mathbb{R}$. For this, we let $\{c_i, U_i, W_i\}$ be a sequence of solutions of (2.1) for $c = c_i$ such that $W_i(0) = 1/2$, $U_i', W_i' > 0$ in $\mathbb{R}$ for all $i \in \mathbb{N}$ and $c_i \downarrow c_{min}$ as $i \to \infty$. By the same argument as in Lemma 2.4, (2.1) has a solution (c, U_*, W_*) with $U_*' > 0$ and $W_*' > 0$ in $\mathbb{R}$ when $c = c_{min}$.

Finally, the constant c_{min} is positive, by Lemma 2.1. This proves the theorem. $\square$

3. Asymptotic behavior of wave profile

In this section, we shall study the asymptotic behavior of wave profile as $\xi \to \pm\infty$. The following fundamental theory (cf. [2, 3]) plays an important role in this section.

Proposition 2. *Let $c > 0$ be a constant and $B(\cdot)$ be a continuous function having finite $B(\pm\infty) := \lim_{x\to\pm\infty} B(x)$. Let $z(\cdot)$ be a measurable function satisfying*

$$cz(x) = e^{\int_x^{x+1} z(s)ds} + e^{\int_x^{x-1} z(s)ds} + B(x),\ \forall x \in \mathbb{R}.$$

Then z is uniformly continuous and bounded. In addition, $\omega^{\pm} = \lim_{x\to\pm\infty} z(x)$ exist and are real roots of the characteristic equation

$$c\omega = e^{\omega} + e^{-\omega} + B(\pm\infty).$$

We shall apply this proposition to $z = U'/U$ or W'/W. First, we give some basic properties of solutions of (1.3).

Lemma 3.1. *Let* (c, U, V) *be a solution of* (1.3). *Then*
(i) U'/U *is uniformly bounded in* $\mathbb{R}$.
(ii) $U(\xi+s)/U(\xi)$ *is uniformly bounded in* $\xi \in \mathbb{R}$ *for* $s \in [-1, 1]$.

Proof. Although this lemma follows from Proposition 2 directly by setting $z := U'/U$, we also give a proof here because this technique will be used later. Choose $\mu > 4/c$, then $U'(\cdot) + \mu U(\cdot) > 0$ in $\mathbb{R}$. By an integration over $[\xi - s, \xi]$ with $s > 0$, we have

$$U(\xi - s) \le U(\xi)e^{\mu s}, \ \forall\, \xi \in \mathbb{R} \text{ and } s > 0. \tag{3.1}$$

From this inequality, we have

$$U(\xi + \frac{1}{2}) = U(\eta + 1 + \xi - \frac{1}{2} - \eta) \le U(\eta + 1)e^{\mu/2}, \ \forall \eta \in [\xi - \frac{1}{2}, \xi]. \tag{3.2}$$

Now due to $(U, W) \to (0, 0)$ as $\xi \to -\infty$, there exists $N \gg 1$ such that

$$U(1 - U - k(1 - W)) \ge 0 \text{ on } (-\infty, -\mathrm{N}]. \tag{3.3}$$

Next, by integrating the first equation of (2.1) over $(-\infty, \xi)$, $\xi \le -N$ and using (3.1)-(3.3), we have

$$\begin{aligned} cU(\xi) &\ge \int_{-\infty}^{\xi} D_2[U](s)ds = \int_{\xi-1}^{\xi} U(s+1)ds - \int_{\xi-1}^{\xi} U(s)ds \\ &\ge \int_{\xi-\frac{1}{2}}^{\xi} U(s+1)ds - e^{\mu}U(\xi) \\ &\ge \frac{1}{2}e^{-\mu/2}U(\xi + \frac{1}{2}) - e^{\mu}U(\xi). \end{aligned}$$

Hence we obtain

$$\frac{U(\xi + 1/2)}{U(\xi)} \le 2e^{\mu/2}(c + e^{\mu}), \ \forall\, \xi \in (-\infty, -N].$$

This implies that

$$\frac{U(\xi + 1)}{U(\xi)} \le 4e^{\mu}(c + e^{\mu})^2, \ \forall\, \xi \in (-\infty, -N].$$

Combining with the fact $\lim_{\xi\to\infty} U(\xi+1)/U(\xi) = 1$, we obtain that $U(\xi+1)/U(\xi)$ is bounded in $\mathbb{R}$. Also, (3.1) implies that $U(\xi - 1)/U(\xi)$ is bounded in $\mathbb{R}$. Hence by the first equation of (2.1) we conclude that U'/U is bounded in $\mathbb{R}$.

From (3.1) we have

$$\frac{U(\xi + s)}{U(\xi)} \le \frac{U(\xi + 1)}{U(\xi)} e^{(1-s)\mu},$$

for all $s < 1$. It follows that $U(\xi + s)/U(\xi)$ is uniformly bounded for $\xi \in \mathbb{R}$ and $s \in [-1, 1]$. The lemma follows. $\square$

Let the assumption (**A1**) be reinforced and $c>0$. Then the equation

$$c\lambda=(e^{\lambda}+e^{-\lambda}-2)+(1-k) \tag{3.4}$$

has a real root if and only if $c\geq c_*$. Moreover, it has exactly two real positive roots, say, $0<\lambda_1(c)\leq\lambda_2(c)$ for $c\geq c_*$.

Lemma 3.2. *Let* (c,U,V) *be a solution of* (1.3)*. Then* $c\geq c_*$ *and*

$$\lim_{\xi\to-\infty}\frac{U'(\xi)}{U(\xi)}=\Lambda(c)\in\{\lambda_1(c),\lambda_2(c)\}.$$

Proof. Set $z(\xi):=U'(\xi)/U(\xi)$. By the first equation of (2.1), ρ satisfies

$$cz(\xi)-[e^{\int_\xi^{\xi+1}z(s)ds}+e^{\int_\xi^{\xi-1}z(s)ds}-2]+[1-U(\xi)-k(1-W(\xi))]=0,\ \forall\,\xi\in\mathbb{R}.$$

Hence the lemma follows from Proposition 2. □

Next, to study the asymptotic behavior of $V'(\xi)/(1-V(\xi))$ as $\xi\to-\infty$ by applying Proposition 2, it is required to determine the limit of $U/(1-V)$ in advance. In the sequel, we say that a function $U(\cdot)$ is *eventually monotone* for $\xi<0$ ($\xi>0$) if U has no extreme points on $(-\infty,-n]$ (or $[n,+\infty)$) for some $n\gg1$.

Lemma 3.3. *Let* (c,U,V) *be a solution of* (1.3)*. Then* $U(\cdot)/W(\cdot)$ *is bounded in* $\mathbb{R}$*.*

Proof. Assume that U/W is unbounded. Since $U(\cdot)/W(\cdot)\geq0$ in $\mathbb{R}$ and

$$\lim_{\xi\to\infty}U(\xi)/W(\xi)=1,$$

there are only two possibilities as follows.

Case 1. U/W is eventually monotone for $\xi<0$ and $\lim_{\xi\to-\infty}U(\xi)/W(\xi)=+\infty$.

Case 2. There exists a sequence $\{\xi_n\}$ of extreme points of U/W such that

$$\xi_n\to-\infty \text{ and } \frac{U(\xi_n)}{W(\xi_n)}\nearrow+\infty \text{ as } n\to+\infty.$$

Note that by the second equation of (2.1) and the proof of Lemma 3.1(i), we can derive that $W(\xi\pm1)/W(\xi)$ is bounded for all $\xi\in\mathbb{R}$.

For **Case 1**, there exists $\xi_0\gg1$ such that

$$\frac{U(\cdot)}{W(\cdot)}>\frac{1}{h}\ \text{ on } (-\infty,-\xi_0].$$

On the other hand, from

$$c\frac{W'}{W}=d\frac{D_2[W]}{W}+r(1-W)(\frac{hU}{W}-1), \tag{3.5}$$

and taking $\xi\to-\infty$, we have $\lim_{\xi\to-\infty}W'(\xi)/W(\xi)=+\infty$. Then

$$\frac{W(\xi+1)}{W(\xi)}=\exp\left\{\int_\xi^{\xi+1}\frac{W'(s)}{W(s)}ds\right\}\to+\infty$$

as $\xi \to -\infty$. This contradicts that $W(\xi+1)/W(\xi)$ is bounded in $\mathbb{R}$.

Suppose that **Case 2** holds. Then there exists a sequence $\{\xi_n\}$ such that

$$\xi_n \to -\infty,\ \frac{U(\xi_n)}{W(\xi_n)} \nearrow +\infty \text{ as } n \to +\infty,\ \text{and } (U/W)'(\xi_n) = 0 \text{ for all } n.$$

From

$$\left(\frac{U}{W}\right)' = \left(\frac{U'}{U} - \frac{W'}{W}\right)\frac{U}{W} \tag{3.6}$$

and recalling from Lemma 3.2 that $\lim_{\xi\to-\infty} U'(\xi)/U(\xi) = \Lambda$, we obtain $W'(\xi_n)/W(\xi_n) \to \Lambda$ as $n \to \infty$. But, by (3.5) again, we have $D_2[W](\xi_n)/W(\xi_n) \to -\infty$ as $n \to \infty$, a contradiction. Hence U/W is bounded in $\mathbb{R}$ and the lemma is proved. □

Note that, for any $c > 0$, the equation

$$c\lambda = d(e^\lambda + e^{-\lambda} - 2) - r$$

has exactly two real roots, one is positive and the other is negative, say $\zeta(c) < 0 < \nu(c)$.

The proof of the following crucial lemma for the asymptotic behavior of wave profiles is highly nontrivial. Some new ideas are introduced.

Lemma 3.4. *Let (c, U, V) be a solution of (1.3).*
(i) *If $\liminf_{\xi\to-\infty} U(\xi)/W(\xi) = 0$, then $\lim_{\xi\to-\infty} U(\xi)/W(\xi) = 0$ and*

$$\lim_{\xi\to-\infty} \frac{W'(\xi)}{W(\xi)} = \nu(c) \le \Lambda(c) = \lim_{\xi\to-\infty} \frac{U'(\xi)}{U(\xi)}, \tag{3.7}$$

where $\nu(c)$ is the unique positive root of $c\lambda = d(e^\lambda + e^{-\lambda} - 2) - r$.
(ii) *If $\liminf_{\xi\to-\infty} U(\xi)/W(\xi) > 0$, then*

$$\lim_{\xi\to-\infty} \frac{U(\xi)}{W(\xi)} = \frac{1}{rh}\left\{(1-d)(e^{\Lambda(c)} + e^{-\Lambda(c)} - 2) + (1-k)\right\} + \frac{1}{h} > 0,$$

$$\lim_{\xi\to-\infty} \frac{W'(\xi)}{W(\xi)} = \Lambda(c) = \lim_{\xi\to-\infty} \frac{U'(\xi)}{U(\xi)},$$

where $\Lambda(c)$ is a positive root of $c\lambda = (e^\lambda + e^{-\lambda} - 2) + (1-k)$.

Proof. We first prove part (i). We shall divide our discussions into two cases.

Case 1. U/W has infinitely many local minimal points $\{z_n\}$ in $(-\infty, 0)$ such that $z_n \downarrow -\infty$ as $n \to \infty$.

Case 2. U/W is eventually monotone for $\xi < 0$.

For **Case 1**, we define $\{\xi_n\} \subset \{z_n\}$ to be the sequence of local minimal points of $g := U/W$ in $(-\infty, 0)$ such that $\xi_n < \xi_{n-1}$, $g(\xi_n) < g(\xi_{n-1})$ for all $n \in \mathbb{N}$, and $g(\xi) \ge g(\xi_{n-1})$ for any minimal point ξ of g in (ξ_n, ξ_{n-1}) (if it exists). Clearly, $\lim_{n\to+\infty} \xi_n = -\infty$. It follows from $\liminf_{\xi\to-\infty} U(\xi)/W(\xi) = 0$ that

$$\lim_{n\to+\infty} \frac{U(\xi_n)}{W(\xi_n)} = 0 \tag{3.8}$$

and without loss of generality (by dropping some finite number of ξ_n) we may assume that

$$\frac{U(\xi_n)}{W(\xi_n)} \leq \frac{U(\xi_n+1)}{W(\xi_n+1)} \tag{3.9}$$

holds for all $n \geq 1$. Moreover, due to $(U/W)'(\xi_n) = 0$ for all $n \in \mathbb{N}$, (3.6) and Lemma 3.2, it follows that

$$\lim_{n\to+\infty} \frac{W'(\xi_n)}{W(\xi_n)} = \lim_{n\to+\infty} \frac{U'(\xi_n)}{U(\xi_n)} = \Lambda \tag{3.10}$$

Next, we divide this case into two subcases.

Subcase 1-1. Suppose that

$$\frac{U(\xi_n)}{W(\xi_n)} > \frac{U(\xi_n-1)}{W(\xi_n-1)},\ \forall\ n \gg 1. \tag{3.11}$$

We shall first claim that W'/W is bounded in $\mathbb{R}$ under the condition (3.11).

For this, we first set $A_n := [\ \xi_n - 1, \xi_n + 1\ ]$. If W'/W is unbounded, then there exists a sequence $\{x_n\}$ such that $\lim_{n\to+\infty} x_n = -\infty$ and $\lim_{n\to+\infty} W'(x_n)/W(x_n) = +\infty$. Note that there exists $\mu > 0$ such that

$$W'(\xi) + \mu W(\xi) > 0,\ \forall\ \xi \in \mathbb{R}. \tag{3.12}$$

This implies that

$$\frac{W(\xi-1)}{W(\xi)} \leq e^{\mu},\ \forall\ \xi \in \mathbb{R}. \tag{3.13}$$

By the second equation of (2.1), (3.13) and the boundedness of U/W, we have

$$\lim_{n\to+\infty} \frac{W(x_n+1)}{W(x_n)} = +\infty.$$

Next, we claim that for all $n \gg 1$, $x_n \in A_m$ for some $m = m(n) \in \mathbb{N}$. Indeed, if not, then we can choose $i \gg 1$ such that $x_i \in (\xi_j + 1, \xi_{j-1} - 1)$ for some large j and

$$\frac{W'(x_i)}{W(x_i)} > \kappa := \sup_{\xi\in\mathbb{R}} \frac{U'(\xi)}{U(\xi)}.$$

By the choice of ξ_j and (3.11), we can easily see that $(U/W)'(\xi) \geq 0$ for any $\xi \in (\xi_j+1, \xi_{j-1}-1)$, if $(\xi_j+1, \xi_{j-1}-1) \neq \emptyset$. Otherwise, there exists a minimal point $\xi \in (\xi_j+1, \xi_{j-1}-1)$ such that $(U/W)(\xi) \leq (U/W)(\xi_{j-1}-1) < (U/W)(\xi_j)$, a contradiction. Due to (3.6), we obtain

$$\frac{U'(x_i)}{U(x_i)} \geq \frac{W'(x_i)}{W(x_i)} > \kappa,$$

a contradiction. Hence $x_n \in A_m$ for some $m = m(n) \in \mathbb{N}$.

Finally, we choose n large enough such that

$$\ln \frac{W(x_n+1)}{W(x_n)} > 4\kappa + 3\mu \tag{3.14}$$

and $x_n \in A_m$ for some large $m \in \mathbb{N}$. By (3.11) and the definition of ξ_m, we have

$$\frac{U(\xi_m - 1)}{W(\xi_m - 1)} \le \frac{U(\xi_m)}{W(\xi_m)} \le \frac{U(\xi_m + 2)}{W(\xi_m + 2)}.$$

This implies

$$\exp\left\{\int_{\xi_m - 1}^{\xi_m + 2}\left[\frac{U'(s)}{U(s)} - \frac{W'(s)}{W(s)}\right] ds\right\} \ge 1.$$

Set $E := (\xi_m - 1, \xi_m + 2)\backslash(x_n, x_n + 1)$. Then by (3.12) and (3.14)

$$\begin{aligned} 3\kappa &\ge \int_{\xi_m - 1}^{\xi_m + 2}\frac{U'(s)}{U(s)}ds \ge \int_{\xi_m - 1}^{\xi_m + 2}\frac{W'(s)}{W(s)}ds \\ &\ge \int_{x_n}^{x_n + 1}\frac{W'(s)}{W(s)}ds + \int_E \frac{W'(s)}{W(s)}ds \\ &\ge \ln\frac{W(x_n + 1)}{W(x_n)} - 3\mu > 4\kappa, \end{aligned}$$

a contradiction. Hence W'/W is bounded in $\mathbb{R}$ and so $W(\xi + s)/W(\xi)$ is uniformly bounded in $\xi \in \mathbb{R}$ and $s \in [-1, 1]$ under the condition (3.11).

Now, we are ready to show that $\lim_{\xi \to -\infty} U(\xi)/W(\xi) = 0$. Otherwise, recalling that $(U/W)'(\xi) \ge 0$ for any $\xi \in (-\infty, \xi_1] \setminus \bigcup_{m=1}^{\infty} A_m$ (by the definition of ξ_j and (3.11)), we can choose a sequence $\{y_n\}$ such that $y_n \in A_n$ for all n, $\lim_{n \to +\infty} U(y_n)/W(y_n) = M$ for some $M > 0$ and $\lim_{n \to +\infty} y_n = -\infty$. But, this implies that

$$\frac{W(\xi_n)}{W(y_n)} = \frac{W(\xi_n)}{U(\xi_n)}\frac{U(\xi_n)}{U(y_n)}\frac{U(y_n)}{W(y_n)} \to +\infty \text{ as } n \to +\infty,$$

by (3.8) and $U(\xi_n)/U(y_n) \ge \beta > 0$ for some constant $\beta > 0$ and for all n. Note that the latter lower bound estimate follows from Lemma 3.1(ii). This contradicts that $W(\xi + s)/W(\xi)$ is uniformly bounded in $\xi \in \mathbb{R}$ and $s \in [-1, 1]$. Thus we conclude that $\lim_{\xi \to -\infty} U(\xi)/W(\xi) = 0$ and so (3.7) follows from Proposition 2 again. Note that, by (3.10), we also have $\nu(c) = \Lambda(c)$.

Subcase 1-2. Suppose that (3.11) dose not hold. Then we can choose a subsequence $\{\xi_{n_j}\}$ of $\{\xi_n\}$ such that

$$\frac{U(\xi_{n_j})}{W(\xi_{n_j})} \le \frac{U(\xi_{n_j} - 1)}{W(\xi_{n_j} - 1)}, \ \forall\ j. \tag{3.15}$$

By (3.9), (3.15) and the second equation of (2.1), we know

$$\begin{aligned} c\frac{W'(\xi_{n_j})}{W(\xi_{n_j})} &= d\frac{W(\xi_{n_j} + 1)}{U(\xi_{n_j} + 1)}\frac{U(\xi_{n_j} + 1)}{U(\xi_{n_j})}\frac{U(\xi_{n_j})}{W(\xi_{n_j})} + d\frac{W(\xi_{n_j} - 1)}{U(\xi_{n_j} - 1)}\frac{U(\xi_{n_j} - 1)}{U(\xi_{n_j})}\frac{U(\xi_{n_j})}{W(\xi_{n_j})} \\ &\quad -2d + r[1 - W(\xi_{n_j})]\left(\frac{hU(\xi_{n_j})}{W(\xi_{n_j})} - 1\right) \\ &\le d\frac{U(\xi_{n_j} + 1)}{U(\xi_{n_j})} + d\frac{U(\xi_{n_j} - 1)}{U(\xi_{n_j})} - 2d + r[1 - W(\xi_{n_j})]\left(\frac{hU(\xi_{n_j})}{W(\xi_{n_j})} - 1\right). \end{aligned}$$

Letting $j \to +\infty$, we obtain

$$c\Lambda \le d(e^{\Lambda} + e^{-\Lambda} - 2) - r. \tag{3.16}$$

Now, set $M := \limsup_{\xi\to-\infty} U(\xi)/W(\xi)$. Then $0 \le M < +\infty$, since U/W is bounded in $\mathbb{R}$. We claim that $M = 0$. For a contradiction, we assume that $M > 0$. Then we can choose a sequence $\{x_n\}$ of local maximal points of U/W such that $x_n \to -\infty$ and $U(x_n)/W(x_n) \to M$ as $n \to +\infty$. For any $\varepsilon > 0$, we have

$$\frac{W(x_n \pm 1)}{W(x_n)} = \frac{W(x_n \pm 1)}{U(x_n \pm 1)} \frac{U(x_n \pm 1)}{U(x_n)} \frac{U(x_n)}{W(x_n)} \ge \frac{1}{M+\varepsilon} \frac{U(x_n \pm 1)}{U(x_n)} \frac{U(x_n)}{W(x_n)}$$

for all large enough n. Recall from (3.6) that

$$\frac{U'(x_n)}{U(x_n)} = \frac{W'(x_n)}{W(x_n)},\ \forall\, n \in \mathbb{N}.$$

Dividing the second equation of (2.1) by W and letting $n \to +\infty$, we obtain

$$\begin{aligned} c\Lambda &= \lim_{n\to\infty} \left\{ d\frac{D_2[W](x_n)}{W(x_n)} \right\} + r(hM - 1) \\ &\ge d\Big[\frac{M}{M+\varepsilon}(e^{\Lambda} + e^{-\Lambda}) - 2\Big] + r(hM - 1). \end{aligned}$$

Letting $\varepsilon \to 0$, we deduce that

$$c\Lambda \ge d[(e^{\Lambda} + e^{-\Lambda}) - 2] + r(hM - 1), \tag{3.17}$$

using the fact $M > 0$. It follows from (3.16) and (3.17) that $M = 0$, a contradiction. Thus we obtain that

$$\lim_{\xi\to-\infty} U(\xi)/W(\xi) = 0.$$

Then (3.7) follows from Proposition 2. Also, we have $\nu = \Lambda$ by (3.10).

For **Case 2**, we have $\lim_{\xi\to-\infty} U(\xi)/W(\xi)$ exists and is equal to 0, since by assumption $\liminf_{\xi\to-\infty} U(\xi)/W(\xi) = 0$. Note that $(U/W)'(\xi) \ge 0$ for all $-\xi \gg 1$ and so $U'/U \ge W'/W$ for all $-\xi \gg 1$. Thus (3.7) follows from Proposition 2 and so part (i) is proved.

Now we prove part (ii). We also divide it into two cases as part (i).

Case 1. U/W has infinitely many extreme points for $\xi < 0$. Set

$$M := \limsup_{\xi\to-\infty} \frac{U(\xi)}{W(\xi)}, \quad m := \liminf_{\xi\to-\infty} \frac{U(\xi)}{W(\xi)}.$$

Then $0 < m \le M < +\infty$ because of Lemma 3.3. Choose a sequence $\{x_n\}$ ($\{y_n\}$) of local maximal (minimal, respectively) points of U/W such that $x_n \to -\infty$ ($y_n \to -\infty$, resp.) and $U(x_n)/W(x_n) \to M$ as $n \to +\infty$ ($U(y_n)/W(y_n) \to m$ as $n \to +\infty$, resp.). For any $\varepsilon > 0$, we have

$$\frac{W(x_n \pm 1)}{W(x_n)} = \frac{W(x_n \pm 1)}{U(x_n \pm 1)} \frac{U(x_n \pm 1)}{U(x_n)} \frac{U(x_n)}{W(x_n)} \ge \frac{1}{M+\varepsilon} \frac{U(x_n \pm 1)}{U(x_n)} \frac{U(x_n)}{W(x_n)}$$

for all large enough n. Note that

$$\frac{U'(x_n)}{U(x_n)} = \frac{W'(x_n)}{W(x_n)}, \ \forall\, n \in \mathbb{N}.$$

By the second equation of (2.1), we have

$$\begin{aligned} c\Lambda &= \lim_{n\to\infty}\left\{ d\frac{D_2[W](x_n)}{W(x_n)}\right\} + r(hM-1) \\ &\geq d\Big[\frac{M}{M+\varepsilon}(e^{\Lambda}+e^{-\Lambda})-2\Big] + r(hM-1), \end{aligned}$$

Hence

$$c\Lambda \geq d(e^{\Lambda}+e^{-\Lambda}-2) + r(hM-1), \tag{3.18}$$

since $\varepsilon > 0$ is arbitrarily. Similarly, we also have

$$c\Lambda \leq d(e^{\Lambda}+e^{-\Lambda}-2) + r(hm-1). \tag{3.19}$$

By (3.18), (3.19) and noting that $M \geq m$, we obtain $M = m$ and so

$$\lim_{\xi\to-\infty}\frac{U(\xi)}{W(\xi)} = \frac{1}{rh}\left\{(1-d)(e^{\Lambda(c)}+e^{-\Lambda(c)}-2)+(1-k)\right\} + \frac{1}{h} > 0.$$

Also, note that

$$\frac{W(\xi\pm1)}{W(\xi)} = \frac{W(\xi\pm1)}{U(\xi\pm1)}\frac{U(\xi\pm1)}{U(\xi)}\frac{U(\xi)}{W(\xi)} \to e^{\pm\Lambda} \ \text{ as } \xi\to-\infty,$$

which implies that $\lim_{\xi\to-\infty} W'(\xi)/W(\xi) = \Lambda$.

Case 2. U/W is eventually monotone for $\xi < 0$. Then the limit $l := \lim_{\xi\to-\infty} U(\xi)/W(\xi)$ exists and $l > 0$. Note that

$$\frac{W(\xi\pm1)}{W(\xi)} = \frac{W(\xi\pm1)}{U(\xi\pm1)}\frac{U(\xi\pm1)}{U(\xi)}\frac{U(\xi)}{W(\xi)} \to \frac{1}{l}\cdot e^{\pm\Lambda}\cdot l = e^{\pm\Lambda}$$

as $\xi \to -\infty$. Hence by the second equation of (2.1) we know $\lim_{\xi\to-\infty} W'(\xi)/W(\xi)$ exists. Now, integrating (3.6) over $[\xi, \xi+1]$ gives

$$\frac{U(\xi+1)}{W(\xi+1)} = \frac{U(\xi)}{W(\xi)}\exp\left\{\int_{\xi}^{\xi+1}\left[\frac{U'(s)}{U(s)} - \frac{W'(s)}{W(s)}\right]ds\right\}.$$

Letting $\xi \to -\infty$, we deduce that $\lim_{\xi\to-\infty} W'(\xi)/W(\xi) = \Lambda$. Finally, by taking $\xi \to -\infty$ in the second equation of (2.1), we obtain

$$l = \frac{1}{rh}\left\{(1-d)(e^{\Lambda(c)}+e^{-\Lambda(c)}-2)+(1-k)\right\} + \frac{1}{h}.$$

Hence the lemma follows. □

Remark 3.1. *When $0 < d \leq 1$ and let (c, U, V) be a solution of (1.3). Since*

$$(e^{\lambda}+e^{-\lambda}-2) + (1-k) > d(e^{\lambda}+e^{-\lambda}-2) - r, \ \forall\, \lambda > 0,$$

we must have $\nu(c) > \Lambda(c)$. Thus, by Lemma 3.4, we can conclude that

$$\lim_{\xi\to-\infty} \frac{U(\xi)}{1-V(\xi)} = \frac{1}{rh}\left\{(1-d)(e^{\Lambda(c)}+e^{-\Lambda(c)}-2)+(1-k)\right\}+\frac{1}{h} > 0,$$

for any $0 < d \le 1$.

To study the asymptotic behavior of U and V as $\xi \to +\infty$, we set $Z = 1-U$. Then (1.3) becomes

$$\begin{cases} cZ' = D_2[Z] + (1-Z)(kV-Z), \\ cV' = dD_2[V] + rV(1-V-h(1-Z)), \\ (Z,V)(-\infty) = (1,1), (Z,V)(+\infty) = (0,0), \\ 0 \le Z, V \le 1. \end{cases} \tag{3.20}$$

Given any $c > 0$, the equation

$$c\lambda = d(e^{\lambda}+e^{-\lambda}-2)+r(1-h) \tag{3.21}$$

has exactly two real roots, one is positive and the other is negative. We denote the negative root by $\nu_1 = \nu_1(c)$. Also, the equation

$$c\lambda = (e^{\lambda}+e^{-\lambda}-2)-1 \tag{3.22}$$

has exactly two real roots, one is positive and the other is negative. We denote the negative root by $\nu_2 = \nu_2(c)$.

Lemma 3.5. *Let (c, Z, V) be a solution of* (3.20). *Then*

$$\lim_{\xi\to\infty} \frac{V'(\xi)}{V(\xi)} = \nu_1(c) < 0.$$

Proof. Let $z(\xi) = V'(\xi)/V(\xi)$. By using Proposition 2, $\lim_{\xi\to\infty}\{V'(\xi)/V(\xi)\}$ exists and the limit is a real root of (3.21). From $V(+\infty) = 0$, it follows that the limit is non-positive and so it is $\nu_1 < 0$. Hence the lemma follows. $\square$

Lemma 3.6. *Let (c, Z, V) be a solution of* (3.20).
(i) *If $\liminf_{\xi\to+\infty} V(\xi)/Z(\xi) > 0$, then $\lim_{\xi\to+\infty} Z'(\xi)/Z(\xi) = \nu_1(c)$ and*

$$\lim_{\xi\to+\infty} \frac{V(\xi)}{Z(\xi)} = \frac{1}{k}\{(d-1)(e^{\nu_1(c)}+e^{-\nu_1(c)}-2)+r(1-h)\}+\frac{1}{k} > 0.$$

(ii) *If $\liminf_{\xi\to+\infty} V(\xi)/Z(\xi) = 0$, then*

$$\lim_{\xi\to+\infty} V(\xi)/Z(\xi) = 0, \quad \lim_{\xi\to+\infty} Z'(\xi)/Z(\xi) = \nu_2(c).$$

Proof. First, by using a similar argument as in the proof of Lemma 3.3, it is not hard to see that V/Z is bounded in $\mathbb{R}$. Using the same argument as in the proof Lemma 3.4(ii), conclusion (i) can be easily proved. We shall not repeat it here.

When $\liminf_{\xi\to+\infty} V(\xi)/Z(\xi) = 0$, we divide our discussions into two cases.
Case 1. V/Z is eventually monotone for $\xi > 0$.
Case 2. V/Z has infinitely many extreme points for $\xi > 0$.

For **Case 1**, we have $\lim_{\xi\to+\infty}\{V(\xi)/Z(\xi)\} = 0$. By applying Proposition 2, we have $\lim_{\xi\to+\infty}\{Z'(\xi)/Z(\xi)\}$ exists and is equal to $\nu_2(c) < 0$, since $Z(+\infty) = 0$.

For **Case 2**, we first set

$$\limsup_{\xi\to+\infty} \frac{V(\xi)}{Z(\xi)} = M \geq 0.$$

We now prove that $M = 0$. If $M > 0$, similar to (3.18), we have the inequality

$$c\nu_1 \geq (e^{\nu_1} + e^{-\nu_1} - 2) + (kM - 1),$$

where $\nu_1 < 0$. It follows that $kM - 1 < 0$. Hence

$$(1 - Z(\xi))(kV(\xi) - Z(\xi)) < 0, \ \forall\, \xi \gg 1. \tag{3.23}$$

We now prove Z'/Z is bounded in $\mathbb{R}$. Since there exists $\mu > 0$ large enough such that $Z' + \mu Z \geq 0$ in $\mathbb{R}$, we have $Z(\xi - s)/Z(\xi) \leq e^{\mu s}$ for all $\xi \in \mathbb{R}$, $s > 0$. Since V/Z is bounded in $\mathbb{R}$, from the first equation of (3.20) we can see that Z'/Z is bounded in $\mathbb{R}$ if and only if $Z(\xi + 1)/Z(\xi)$ is bounded in $\mathbb{R}$. Assume for contradiction that Z'/Z is unbounded in $\mathbb{R}$. By (3.23), we can choose $N \gg 1$ such that

$$(1 - Z(\xi))(kV(\xi) - Z(\xi)) < 0, \ \forall\, \xi \geq N.$$

Next, we choose $\xi_0 > N$ such that $Z(\xi_0 + 1)/Z(\xi_0) > e^{\mu}$. Since $Z(+\infty) = 0$, we may find $x_0 \geq \xi_0$ such that

$$Z(x_0) = \max\{Z(\xi)|\ \xi \in [\xi_0, \infty)\}.$$

Since $Z(\xi) \leq Z(\xi_0)e^{\mu} \leq Z(\xi_0 + 1) \leq Z(x_0)$ for all $\xi \in [\xi_0 - 1, \xi_0]$, we have $Z(x_0) \geq Z(x_0 - 1)$. Also, noting that $x_0 > N$, we obtain

$$\begin{aligned} 0 = cZ'(x_0) \ &= \ (Z(x_0 + 1) - Z(x_0)) + (Z(x_0 - 1) - Z(x_0)) \\ &\quad + (1 - Z(\xi))(kV(\xi) - Z(\xi)) < 0, \end{aligned}$$

a contradiction. Hence Z'/Z is bounded in $\mathbb{R}$.

Finally, by using the same argument as in the proof of Lemma 3.4(i), it follows that $\lim_{\xi\to+\infty} V(\xi)/Z(\xi) = 0$, which gives a contradiction with $M > 0$. Thus, we conclude that $M = 0$ and then $\lim_{\xi\to+\infty} V(\xi)/Z(\xi) = 0$. By using Proposition 2 the lemma follows. ☐

Although Lemmas 3.2, 3.4, 3.5 and 3.6 are sufficient for the proof of the monotonicity of wave profile, in order to study the uniqueness, we shall need more precise information on the wave tails. Namely, we need to show that U and $1 - V$ have exponential tails as $\xi \to -\infty$ which are stronger than the existence of limits of U'/U and $V'/(1 - V)$ as $\xi \to -\infty$. Due to the above lemmas, the bilateral Laplace transform for U and $1-V$ are well-defined in some strip. Then a modified version of Ikehara's Theorem (cf. [1]) can be applied to study the tail behaviors of wave profiles (see also [10]).

Proposition 3 (**Ikehara's Theorem**). *Let U be a positive non-decreasing function in $\mathbb{R}$, and define*

$$F(\lambda) := \int_{-\infty}^{0} e^{-\lambda\xi} U(\xi) d\xi.$$

If F can be written as $F(\lambda) = H(\lambda)/(\alpha-\lambda)^{q+1}$ for some constants $q > -1, \alpha > 0$, and H analytic in the strip $0 < \mathrm{Re}\lambda \leq \alpha$, then

$$\lim_{\xi\to-\infty} \frac{U(\xi)}{|\xi|^q e^{\alpha\xi}} = \frac{H(\alpha)}{\Gamma(\alpha+1)}.$$

In the sequel, we let

$$\begin{aligned} I(\lambda) &:= \int_{-\infty}^{+\infty} e^{-\lambda\xi}[kU(\xi)W(\xi) - U^2(\xi)]d\xi, \\ J(\lambda) &:= \int_{-\infty}^{+\infty} e^{-\lambda\xi}[W^2(\xi) - hU(\xi)W(\xi)]d\xi. \end{aligned}$$

Then we have the following lemma on the asymptotically exponential tails.

Lemma 3.7. *Assume* (**A1**) *and let (c, U, V) be a solution of* (1.3)*. Let $\Lambda(c)$ be the constant defined in Lemma 3.2 and $\nu(c) > 0$ be the unique positive root of $c\lambda = d(e^{\lambda} + e^{-\lambda} - 2) - r$. Then there exists a constant $p \in \{0, 1\}$ with $p = 1$, if $I(\Lambda) \neq 0$, and $p = 0$, if $I(\Lambda) = 0$, such that the following statements hold.*

(1) *There exist η_0, $\eta_1 \in \mathbb{R}$ depending on U and V such that*

$$\lim_{\xi\to-\infty} \frac{U(\xi+\eta_0)}{e^{\Lambda(c)\xi}} = 1 \quad \text{if} \quad c > c_*, \ \lim_{\xi\to-\infty} \frac{U(\xi+\eta_1)}{|\xi|^p e^{\Lambda(c)\xi}} = 1 \quad \text{if} \quad c = c_*.$$

(2) *For $c > c_*$, there exist η_2, η_3, $\eta_4 \in \mathbb{R}$ depending on U and V such that*

$$\begin{aligned} &\lim_{\xi\to-\infty} \frac{1 - V(\xi+\eta_2)}{e^{\Lambda(c)\xi}} = 1 \quad \text{if} \quad \nu(c) > \Lambda(c), \\ &\lim_{\xi\to-\infty} \frac{1 - V(\xi+\eta_3)}{|\xi| e^{\Lambda(c)\xi}} = 1 \quad \text{if} \quad \nu(c) = \Lambda(c), \\ &\lim_{\xi\to-\infty} \frac{1 - V(\xi+\eta_4)}{e^{\nu(c)\xi}} = 1 \quad \text{if} \quad \nu(c) < \Lambda(c). \end{aligned}$$

(3) *For $c = c_*$, there exist η_5, η_6, $\eta_7 \in \mathbb{R}$ depending on U and V such that*

$$\lim_{\xi\to-\infty} \frac{1 - V(\xi + \eta_5)}{|\xi|^p e^{\Lambda(c)\xi}} = 1 \quad \text{if} \quad \nu(c) > \Lambda(c),$$
$$\lim_{\xi\to-\infty} \frac{1 - V(\xi + \eta_6)}{|\xi|^{p+1} e^{\Lambda(c)\xi}} = 1 \quad \text{if} \quad \nu(c) = \Lambda(c),$$
$$\lim_{\xi\to-\infty} \frac{1 - V(\xi + \eta_7)}{e^{\nu(c)\xi}} = 1 \quad \text{if} \quad \nu(c) < \Lambda(c).$$

Proof. Recall $W := 1 - V$. By Lemmas 3.2 and 3.4, we can define

$$\mathcal{L}(\lambda, U) := \int_{-\infty}^{+\infty} e^{-\lambda\xi} U(\xi) d\xi \ \text{ for } \lambda \in \mathbb{C} \text{ with } 0 < Re\lambda < \Lambda,$$
$$\mathcal{L}(\lambda, W) := \int_{-\infty}^{+\infty} e^{-\lambda\xi} W(\xi) d\xi \ \text{ for } \lambda \in \mathbb{C} \text{ with } 0 < Re\lambda < \sigma := \min\{\Lambda, \nu\}.$$

It follows from (2.1) that

$$\Phi(c, \lambda)\mathcal{L}(\lambda, U) = I(\lambda) \tag{3.24}$$

for $\lambda \in \mathbb{C}$ with $0 < Re\lambda < \Lambda$ and

$$\Psi(c, \lambda)\mathcal{L}(\lambda, W) = \frac{rhI(\lambda)}{\Phi(c, \lambda)} + rJ(\lambda) \tag{3.25}$$

for $\lambda \in \mathbb{C}$ with $0 < Re\lambda < \sigma$.

To prove this lemma, we first give some facts as follows.

(a) The only root of $\Phi(c, \lambda)$ on $\{Re\lambda = \Lambda\}$ is $\lambda = \Lambda$. Indeed, let $\Lambda + i\beta$ be any root of $\Phi(c, \lambda)$. Then we have

$$c\beta - e^{\Lambda} \sin\beta + e^{-\Lambda} \sin\beta = 0, \quad \cos\beta = 1.$$

Thus it follows that $\beta = 0$. Similarly, the only root of $\Psi(c, \lambda)$ on $\{Re\lambda = \nu\}$ is $\lambda = \nu$.

(b) The functions $I(\lambda), J(\lambda)$ are analytic in the strip $0 < Re\lambda < \Lambda + \sigma$, $0 < Re\lambda < 2\sigma$, respectively. Indeed, this follows from Lemmas 3.2 and 3.4 which imply that

$$kU(\xi)W(\xi) - U^2(\xi) = O(e^{\beta\xi}) \text{ as } \xi \to -\infty, \ \forall\ \beta \in (0, \Lambda + \sigma),$$
$$W^2(\xi) - hU(\xi)W(\xi) = O(e^{\beta\xi}) \text{ as } \xi \to -\infty, \ \forall\ \beta \in (0, 2\sigma).$$

To prove (1), we rewrite (3.24) as

$$F(\lambda) := \int_{-\infty}^{0} e^{-\lambda\xi} U(\xi) d\xi = \frac{I(\lambda)}{\Phi(c, \lambda)} - \int_0^{+\infty} e^{-\lambda\xi} U(\xi) d\xi.$$

We shall prove that $H(\lambda)$ is analytic in the strip $0 < \mathrm{Re}\lambda \le \Lambda$, where

$$H(\lambda) := \frac{I(\lambda)}{\Phi(c, \lambda)/(\Lambda - \lambda)^{q+1}} - (\Lambda - \lambda)^{q+1} \int_0^{+\infty} e^{-\lambda\xi} U(\xi) d\xi, \tag{3.26}$$

where $q = 0$ for $c > c_*$ and $q = p$ for $c = c_*$. Note that when $0 < \text{Re}\lambda < \Lambda$, H can be written as $F(\lambda)/(\Lambda - \lambda)^{q+1}$ which implies H is analytic on $0 < \text{Re}\lambda < \Lambda$. The analyticity of H on $\{\text{Re}\lambda = \Lambda\}$ follows from (a), (b) and (3.26). Hence H is analytic on the strip $0 < \text{Re}\lambda \leq \Lambda$.

To proceed further, we suppose first that U is non-decreasing. Then, from the above discussions, we can apply Ikehara's Theorem to obtain

$$\lim_{\xi\to-\infty} U(\xi)/|\,\xi|^q e^{\Lambda(c)\xi} = H(\Lambda)/\Gamma(\Lambda+1),$$

where $q = 0$ for $c > c_*$ and $q = p$ for $c = c_*$. We next claim that $H(\Lambda) \neq 0$.

For the case $c > c_*$, note that $H(\Lambda) = I(\Lambda)/g(c,\Lambda)$, where g satisfies $g(c,\Lambda) \neq 0$ and $g(c,\lambda)(\Lambda - \lambda) = \Phi(c,\lambda)$ on $0 < \text{Re}\lambda < \Lambda + \sigma$. If $H(\Lambda) = 0$, then $I(\Lambda) = 0$ and so the singularity $\lambda = \Lambda$ of $\mathcal{R}(c,\lambda) := I(\lambda)/\Phi(c,\lambda)$ is removable. Thus $\mathcal{R}(c,\lambda)$ is analytic on $0 < \text{Re}\lambda < \Lambda + \varepsilon$ for some sufficiently small $\varepsilon > 0$. From (3.24) we conclude that $\mathcal{L}(\lambda, U)$ is well-defined on $0 < \text{Re}\lambda < \Lambda + \varepsilon$. On the other hand, $\lim_{\xi\to-\infty} U'(\xi)/U(\xi) = \Lambda$ implies $\mathcal{L}(\lambda, U)$ must diverge for $\text{Re}\lambda > \Lambda$ which leads to a contradiction. Therefore, $H(\Lambda) \neq 0$ and so (1) holds for $c > c_*$.

For the case $c = c_*$, since $\Lambda = \Lambda(c_*)$ is a double root of $\Phi(c_*, \lambda) = 0$, we need to distinguish two cases. If $I(\Lambda) \neq 0$, then $H(\Lambda) \neq 0$ by choosing $p = 1$. On the other hand, if $I(\Lambda) = 0$, then $\lambda = \Lambda$ must be a simple root of $I(\lambda) = 0$. Otherwise, $\mathcal{L}(\lambda, U)$ will be well-defined on $\text{Re}\lambda = \Lambda(c)$ which leads a contradiction as above. Then $H(\Lambda) \neq 0$ by choosing $p = 0$. Thus (1) holds for $c = c_*$.

For general U, we replace $U(\xi)$ by $\widetilde{U}(\xi) := e^{\mu\xi}U(\xi)$ for some $\mu > 4/c$. Then $\widetilde{U}'(\xi) > 0$ in $\mathbb{R}$ and so Ikehara's Theorem can be applied. Thus we can derive that

$$\lim_{\xi\to-\infty} \widetilde{U}(\xi)/|\,\xi|^q e^{(\Lambda(c)+\mu)\xi} > 0,$$

where $q = 0$ for $c > c_*$ and $q = p$ for $c = c_*$. Hence (1) has been proved.

Next, we shall prove (2) and (3). The argument is similar as (1) and we need to use (a) and (b). To prove (2), we rewrite (3.25) as

$$F(\lambda) := \int_{-\infty}^{0} e^{-\lambda\xi} W(\xi)d\xi = \frac{rhI(\lambda)}{\Phi(c,\lambda)\Psi(c,\lambda)} + \frac{rJ(\lambda)}{\Psi(c,\lambda)} - \int_{0}^{+\infty} e^{-\lambda\xi} W(\xi)d\xi. \tag{3.27}$$

Note that

$$H(\lambda) = (\sigma - \lambda)^{q+1} F(\lambda), \tag{3.28}$$

on $0 < \text{Re}\lambda < \sigma$ where $q = 0$ for $\nu \neq \Lambda$ and $q = 1$ for $\nu = \Lambda$. Hence H is analytic in the strip $0 < \text{Re}\lambda < \sigma$. From (a) and (b), we conclude that H is also analytic on $\{\text{Re}\lambda = \sigma\}$. Then by Ikehara's Theorem (if necessary we replace $W(\xi)$ by $e^{\mu\xi}W(\xi)$ for some $\mu \gg 1$), we derive $\lim_{\xi\to-\infty} W(\xi)/|\,\xi|^q e^{\sigma\xi} = H(\sigma)/\Gamma(\sigma+1)$ where $q = 0$ for $\nu \neq \Lambda$ and $q = 1$ for $\nu = \Lambda$. Note that $\sigma = \nu$ if $\nu < \Lambda$; $\sigma = \Lambda$ if $\nu \geq \Lambda$

Next, we shall prove $H(\sigma) \neq 0$ by a contradiction argument. For $\nu \geq \Lambda$ (i.e., $\sigma = \Lambda$), by (3.28), $H(\Lambda) = 0$ implies that $I(\Lambda) = 0$, since the second term and the third term of right-hand side of (3.28) become zero when $\lambda = \Lambda$. But, this contradicts the fact $I(\Lambda) \neq 0$ and so $H(\sigma) \neq 0$ for $\nu \geq \Lambda$.

For $\nu < \Lambda$ (i.e., $\sigma = \nu$), note that

$$H(\lambda) = \frac{r\int_{-\infty}^{+\infty} e^{-\lambda\xi}[hU(\xi)(1-W(\xi)) + W^2(\xi)]d\xi}{\Psi(c,\lambda)/(\nu-\lambda)} - (\nu-\lambda)\int_0^{+\infty} e^{-\lambda\xi}W(\xi)d\xi.$$

If $H(\nu) = 0$, then $\int_{-\infty}^{+\infty} e^{-\lambda\xi}[hU(\xi)(1-W(\xi)) + W^2(\xi)]d\xi = 0$ which implies

$$hU(\xi)[1-W(\xi)] + W^2(\xi) = 0, \ \forall\, \xi \in \mathbb{R}.$$

This leads $U(\cdot) \equiv W(\cdot) \equiv 0$ in $\mathbb{R}$, and so we reach a contradiction. Thus, $H(\nu) \neq 0$ so that (2) holds. The same argument can be used to show (3), we omit the detail here. $\square$

4. Monotonicity and uniqueness of wave profiles

Due to Lemmas 3.2, 3.4, 3.5 and 3.6, we know U and W are increasing in $\mathbb{R}\setminus[-N,N]$ for some $N \gg 1$. Before showing Theorem 2, we give *the strong comparison principle* as follows.

Lemma 4.1. *Let* (c, U_1, W_1) *and* (c, U_2, W_2) *be two solutions of* (2.1) *satisfying* $U_1 \leq U_2$ *and* $W_1 \leq W_2$ *in* $\mathbb{R}$. *Then we have* (i) *either* $U_1(\cdot) < U_2(\cdot)$ *in* $\mathbb{R}$ *or* $U_1 \equiv U_2$*; and* (ii) *either* $W_1(\cdot) < W_2(\cdot)$ *in* $\mathbb{R}$ *or* $W_1 \equiv W_2$.

Proof. The argument of (i) and (ii) are similar, so we only prove (i). If there exists $\xi_0 \in \mathbb{R}$ such that $U_1(\xi_0) = U_2(\xi_0)$, then

$$0 = U_1(\xi_0) - U_2(\xi_0) = e^{-\mu\xi_0}\int_{-\infty}^{\xi_0} e^{\mu s}\left\{H_1(U_1,W_1)(s) - H_1(U_2,W_2)(s)\right\}ds.$$

Due to $U_1 \leq U_2$ and $W_1 \leq W_2$ in $\mathbb{R}$, we have $H_1(U_1,W_1)(s) \leq H_1(U_2,W_2)(s)$ for all s. Hence we have $H_1(U_1,W_1)(s) = H_1(U_2,W_2)(s)$ for all $s \leq \xi_0$ Therefore, it follows from (2.3) that $U_1(s+1) = U_2(s+1)$ for all $s \leq \xi_0$. Replacing ξ_0 by ξ_0+1 and repeating the above procedure we can derive $U_1(s) = U_2(s)$ for all $s \leq \xi_0 + 2$. Hence $U_1 \equiv U_2$ in $\mathbb{R}$ by repeating the above argument infinitely many times. The lemma follows. $\square$

Now, we shall use the sliding method to prove the theorem on monotonicity.

Proof of Theorem 2. By Lemmas 3.2, 3.4, 3.5 and 3.6, we may take $N \gg 1$ such that $U' > 0$ and $W' > 0$ in $\mathbb{R}\setminus[-N,N]$. Since $U(+\infty) = W(+\infty) = 1$ and $U(-\infty) = W(-\infty) = 0$, the set

$$A := \{\eta > 0 |\ U(\xi+s) \geq U(\xi),\ W(\xi+s) \geq W(\xi),\ \forall\ s \geq \eta,\ \xi \in \mathbb{R}\}$$

is not empty. Hence $\eta^* := \inf A$ is well-defined. By continuity, we have

$$U(\xi+\eta^*) \geq U(\xi),\ W(\xi+\eta^*) \geq W(\xi),\ \forall\ \xi \in \mathbb{R}.$$

We now prove that $\eta^* = 0$. For a contradiction, we suppose that $\eta^* > 0$. By Lemma 4.1, we have

$$U(\xi+\eta^*) > U(\xi),\ W(\xi+\eta^*) > W(\xi),\ \forall\ \xi \in \mathbb{R}.$$

Due to the continuity of U and W, there exists $\eta_0 \in (0, \eta^*)$ such that

$$U(\xi+\eta) > U(\xi),\ W(\xi+\eta) > W(\xi),\ \forall\ \xi \in [-N-\eta^*, N],\ \eta \in [\eta_0, \eta^*].$$

Also, it follows from $U' > 0$ and $W' > 0$ in $\mathbb{R} \setminus [-N, N]$ that

$$U(\xi+\eta) \geq U(\xi),\ W(\xi+\eta) \geq W(\xi),\ \forall\ \xi \in \mathbb{R} \setminus [-N-\eta^*, N],\ \eta \in [\eta_0, \eta^*].$$

Thus, $U(\xi+\eta) \geq U(\xi)$ and $W(\xi+\eta) \geq W(\xi)$, $\forall\ \xi \in \mathbb{R}$ and $\eta > \eta_0$. This contradicts the definition of η^*. Hence $\eta^* = 0$ and it follows that $U' \geq 0$ and $W' \geq 0$ in $\mathbb{R}$. By differentiating $U = T_1(U, W)$ and $W = T_2(U, W)$, it is not hard to obtain that $U' > 0$ and $W' > 0$ in $\mathbb{R}$. $\square$

Now we prove that waves profiles of (1.3) of a given wave speed are unique up to translations. That is, for a given $c > 0$ and any two solutions (c, U_1, W_1) and (c, U_2, W_2) of (2.1), there exists an $\eta \in \mathbb{R}$ such that $U_1(\cdot) = U_2(\cdot + \eta)$ and $W_1(\cdot) = W_2(\cdot + \eta)$. Our strategy is to apply the sliding method (cf. [2]). Due to the exponential tail behaviors, the left-hand tails of wave profiles can be controlled. To control the right-hand tail behaviors of wave profiles, the following key lemma shall be used. Its proof also relies on the use of the strong comparison principle (Lemma 4.1).

Lemma 4.2. *Let (c, U_1, W_1) and (c, U_2, W_2) be two solutions of (2.1). If there exists $q > 0$ such that $(1+q)U_1(\cdot - \kappa q) \geq U_2(\cdot)$ and $(1+q)W_1(\cdot - \kappa q) \geq W_2(\cdot)$ in $\mathbb{R}$, then $U_1(\cdot) \geq U_2(\cdot)$ and $W_1(\cdot) \geq W_2(\cdot)$ in $\mathbb{R}$, where $\kappa = \kappa(U_1, W_1)$ is defined by*

$$\kappa := \max_{\xi \in (-\infty, N_0]} \left\{ \frac{U_1(\xi)}{U_1'(\xi)}, \frac{W_1(\xi)}{W_1'(\xi)} \right\}$$

with $N_0 \gg 1$ such that $U_1(\xi)/W_1(\xi) > \max\{k, 1/h\}$ for all $\xi > N_0$.

Proof. Define

$$\mathcal{F}_U(q, \xi) := (1+q)U_1(\xi - \kappa q) - U_2(\xi),$$
$$\mathcal{F}_W(q, \xi) := (1+q)W_1(\xi - \kappa q) - W_2(\xi).$$

By assumption, the following quantity

$$q^* := \inf\{q > 0 |\, \mathcal{F}_U(q, \xi) \geq 0 \text{ and } \mathcal{F}_W(q, \xi) \geq 0\ \forall\ \xi \in \mathbb{R}\}$$

is well-defined. We claim that $q^* = 0$. If not, then $q^* > 0$. By continuity, we have $\mathcal{F}_U(q^*,\xi) \geq 0$ and $\mathcal{F}_W(q^*,\xi) \geq 0$ for all $\xi \in \mathbb{R}$. Note that

$$\frac{d}{dq}\mathcal{F}_U(q,\xi) = U_1(\xi-\kappa q)\left\{1-(1+q)\kappa\frac{U_1'(\xi-\kappa q)}{U_1(\xi-\kappa q)}\right\} < 0,$$

for all $\xi \leq N_0 + \kappa q$. Similarly, $\dfrac{d}{dq}\mathcal{F}_W(q,\xi) < 0$ for all $\xi \leq N_0 + \kappa q$. Note that

$$\mathcal{F}_U(q^*,+\infty) = \mathcal{F}_W(q^*,+\infty) = q^* > 0.$$

Thus there is $\xi_0 > N_0 + \kappa q^*$ such that one of the followings:

$$\frac{d}{d\xi}\mathcal{F}_U(q^*,\xi_0) = 0 = \mathcal{F}_U(q^*,\xi_0),\ \mathcal{F}_U(q^*,\xi) \geq 0 \text{ and } \mathcal{F}_W(q^*,\xi) \geq 0\ \forall\ \xi \in \mathbb{R}, \tag{4.1}$$

$$\frac{d}{d\xi}\mathcal{F}_W(q^*,\xi_0) = 0 = \mathcal{F}_W(q^*,\xi_0),\ \mathcal{F}_U(q^*,\xi) \geq 0 \text{ and } \mathcal{F}_W(q^*,\xi) \geq 0\ \forall\ \xi \in \mathbb{R} \tag{4.2}$$

must happen. If (4.1) occurs, i.e.,

$$\begin{aligned}&(1+q^*)U_1(\bar{\xi}_0) = U_2(\xi_0),\ (1+q^*)U_1'(\bar{\xi}_0) = U_2'(\xi_0),\\ &(1+q^*)U_1(\bar{\xi}) \geq U_2(\xi),\ (1+q^*)W_1(\bar{\xi}) \geq W_2(\xi),\ \forall\ \xi \in \mathbb{R},\end{aligned}$$

where $\bar{\xi} := \xi - \kappa q^*$, then by the first equation of (2.1) we obtain

$$\begin{aligned}&(1+q^*)D_2[U_1](\bar{\xi}_0) + (1+q^*)U_1(\bar{\xi}_0)[1-U_1(\bar{\xi}_0)-k(1-W_1(\bar{\xi}_0))\,]\\ &= D_2[U_2](\xi_0) + U_2(\xi_0)[1-U_2(\xi_0)-k(1-W_2(\xi_0))\,].\end{aligned}$$

This implies

$$U_1(\bar{\xi}_0) + kW_2(\xi_0) \geq U_2(\xi_0) + kW_1(\bar{\xi}_0).$$

Therefore, we obtain that

$$U_1(\bar{\xi}_0)/W_1(\bar{\xi}_0) \leq k.$$

This contradicts the fact that $U_1(\xi)/W_1(\xi) > k$ for $\xi > N_0$. This tells us that (4.2) must occur, i.e.,

$$\begin{aligned}&(1+q^*)W_1(\bar{\xi}_0) = W_2(\xi_0),\ (1+q^*)W_1'(\bar{\xi}_0) = W_2'(\xi_0),\\ &(1+q^*)W_1(\bar{\xi}) \geq W_2(\xi),\ (1+q^*)U_1(\bar{\xi}) \geq U_2(\xi),\ \forall\ \xi \in \mathbb{R}.\end{aligned}$$

But, by the second equation of (2.1), we can conclude from (4.2) that

$$\begin{aligned}&d(1+q^*)D_2[W_1](\bar{\xi}_0) + r(1+q^*)(1-W_1(\bar{\xi}_0))(hU_1(\bar{\xi}_0)-W_1(\bar{\xi}_0))\\ &= dD_2[W_2](\xi_0) + r(1-W_2(\xi_0))(hU_2(\xi_0)-W_2(\xi_0)),\end{aligned}$$

which implies

$$\begin{aligned}W_2(\xi_0) - W_1(\bar{\xi}_0) - hU_2(\xi_0) &\geq \frac{h}{W_2(\xi_0)}\{(1+q^*)U_1(\bar{\xi}_0) - U_2(\xi_0)\} - hU_1(\bar{\xi}_0)\\ &\geq h\{(1+q^*)U_1(\bar{\xi}_0) - U_2(\xi_0) - U_1(\bar{\xi}_0)\}.\end{aligned}$$

Then it follows that

$$U_1(\bar{\xi}_0)/W_1(\bar{\xi}_0) \le 1/h,$$

a contradiction again, since $U_1(\xi)/W_1(\xi) > 1/h$ for $\xi > N_0$. Therefore, we have $q^* = 0$ which implies $U_1(\cdot) \ge U_2(\cdot)$ and $W_1(\cdot) \ge W_2(\cdot)$ in $\mathbb{R}$. Then the lemma follows. $\square$

We are ready to prove Theorem 3.

Proof of Theorem 3. Assume (**A1**) and $d \in (0,1]$. For given two solutions (c, U_1, W_1) and (c, U_2, W_2), we may assume that $U_1(0) = U_2(0) = 1/2$ by a suitable translation. Moreover, by Lemma 3.7 and exchanging U_1 and U_2 (if it is necessary), we may assume

$$\lim_{\xi \to -\infty} U_1(\xi)/U_2(\xi) \ge 1. \tag{4.3}$$

Note that $d \in (0,1]$ implies $\nu > \Lambda$. By Remark 3.1 and Lemma 3.7 again, it is not hard to see that

$$\lim_{\xi \to -\infty} W_1(\xi)/W_2(\xi) \ge 1. \tag{4.4}$$

Thus, for any $n_0 > 0$ we have $U_1(\cdot) > U_2(\cdot - n_0)$ and $W_1(\cdot) > W_2(\cdot - n_0)$ on $(-\infty, -\xi_0]$ for some $\xi_0 \gg 1$. Also, since $W_i(+\infty) = U_i(+\infty) = 1$, $i = 1, 2$, there exists $x_0 \gg 1$ such that $2U_1(\cdot - \kappa) \ge U_2(\cdot - x_0)$ and $2W_1(\cdot - \kappa) \ge W_2(\cdot - x_0)$ in $\mathbb{R}$. It follows from Lemma 4.2 that $U_1(\cdot) \ge U_2(\cdot - x_0)$ and $W_1(\cdot) \ge W_2(\cdot - x_0)$ in $\mathbb{R}$. Thus, we can define

$$\eta^* := \inf\{\eta > 0 |\, U_1(\xi) \ge U_2(\xi - \eta) \text{ and } W_1(\xi) \ge W_2(\xi - \eta)\ \forall\, \xi \in \mathbb{R}\}.$$

We now claim $\eta^* = 0$. If $\eta^* > 0$, by using Lemma 3.7, (4.3) and (4.4), there exists $\xi_1 > 0$ such that

$$U_1(\cdot - \eta^*/2) > U_2(\cdot - \eta^*) \text{ and } W_1(\cdot - \eta^*/2) > W_2(\cdot - \eta^*) \text{ on } (-\infty, -\xi_1]. \tag{4.5}$$

Note that $U_1(+\infty) = W_1(+\infty) = 1$ and $U_1'(+\infty) = W_1'(+\infty) = 0$, there is $\xi_2 \gg 1$ such that

$$\frac{d}{dq}(1+q)U_1(\xi - 2\kappa q) = U_1(\xi - 2\kappa q) - 2\kappa(1+q)U_1'(\xi - 2\kappa q) > 0,$$
$$\frac{d}{dq}(1+q)W_1(\xi - 2\kappa q) = W_1(\xi - 2\kappa q) - 2\kappa(1+q)W_1'(\xi - 2\kappa q) > 0,$$

for all $\xi \ge \xi_2$ and $q \in [0,1]$. Thus, we have

$$\begin{cases} (1+q)U_1(\xi - 2\kappa q) \ge U_1(\xi) \ge U_2(\xi - \eta^*), \\ (1+q)W_1(\xi - 2\kappa q) \ge W_1(\xi) \ge W_2(\xi - \eta^*), \end{cases} \tag{4.6}$$

for all $\xi \ge \xi_2$ and for all $q \in [0,1]$.

Finally, we treat the interval $[-\xi_1, \xi_2]$. Note that $U_1(\cdot) \geq U_2(\cdot - \eta^*)$ and $W_1(\cdot) \geq W_2(\cdot - \eta^*)$ in $\mathbb{R}$. Then Lemma 4.1 implies that $U_1(\cdot) > U_2(\cdot - \eta^*)$ and $W_1(\cdot) > W_2(\cdot - \eta^*)$ in $\mathbb{R}$. By continuity, there exists $\varepsilon \in (0, \min\{1, \eta^*/4\kappa\})$ such that

$$U_1(\cdot - 2\kappa\varepsilon) > U_2(\cdot - \eta^*) \text{ and } W_1(\cdot - 2\kappa\varepsilon) > W_2(\cdot - \eta^*) \text{ on } [-\xi_1, \xi_2]. \tag{4.7}$$

From (4.5), (4.6) and (4.7), we conclude that

$$(1+\varepsilon)U_1(\cdot - 2\kappa\varepsilon) \geq U_2(\cdot - \eta^*) \text{ on } \mathbb{R},$$
$$(1+\varepsilon)W_1(\cdot - 2\kappa\varepsilon) \geq W_2(\cdot - \eta^*) \text{ on } \mathbb{R}$$

It follows from Lemma 4.2 that $U_1(\cdot - \kappa\varepsilon) \geq U_2(\cdot - \eta^*)$ and $W_1(\cdot - \kappa\varepsilon) \geq W_2(\cdot - \eta^*)$ in $\mathbb{R}$. This contradicts the definition of η^*. Hence $\eta^* = 0$ and we derive that $U_1(\cdot) \geq U_2(\cdot)$ and $W_1(\cdot) \geq W_2(\cdot)$ in $\mathbb{R}$. From Lemma 4.1 and $U_1(0) = U_2(0) = 1/2$, it follows that $U_1(\cdot) \equiv U_2(\cdot)$ and $W_1(\cdot) \equiv W_2(\cdot)$ in $\mathbb{R}$. Then the theorem is proved. □

Remark 4.1. *The restriction $d \in (0,1]$ is to make sure that $\lim_{\xi\to-\infty} W_1(\xi)/W_2(\xi) \geq 1$ when $\lim_{\xi\to-\infty} U_1(\xi)/U_2(\xi) \geq 1$ and $U_1(x_0) = U_2(x_0)$ for some x_0. Otherwise, (4.5) may not hold.*

5. Characterization of the minimal wave speed

In this section, we first give a proof of Theorem 4. Then we shall discuss some implications of Theorem 4 to the derivation of the minimal wave speed of PDE model (1.2).

Proof of Theorem 4. By Lemma 3.2, we have $c_{min} \geq c_*$. We now show that $c_{min} = c_*$ when conditions (**A1**), $d \leq 1$ and (1.5) hold.

For each $c \geq c_*$, we define

$$(U^+(\xi), W^+(\xi)) = (\ \min\{1, e^{\lambda_1(c)\xi}\},\ \min\{1, e^{\lambda_1(c)\xi}/k\}\ )$$

where $\lambda_1(c)$ is the smaller root of (3.4). We claim that $(U^+(\xi), W^+(\xi))$ is a super-solution of (2.1) for the given c.

For $\xi > 0$, since $U^+(\xi) = W^+(\xi) = 1$, it is easy to see that (2.9) holds.

For $(\ln k)/\lambda_1 < \xi < 0$, we have $(U^+(\xi), W^+(\xi)) = (e^{\lambda_1\xi}, 1)$ and so

$$\left\{c(U^+)' - D_2[U^+] - U^+(1 - U^+ - k(1 - W^+))\right\}(\xi) \geq ke^{\lambda_1\xi}(\frac{1}{k}e^{\lambda_1\xi} - 1) \geq 0,$$
$$\left\{c(W^+)' - dD_2[W^+] - r(1 - W^+)(hU^+ - W^+)\right\}(\xi) = d[1 - W^+(\xi - 1)] \geq 0.$$

For $\xi < (\ln k)/\lambda_1$, since $(U^+(\xi), W^+(\xi)) = (e^{\lambda_1\xi}, e^{\lambda_1\xi}/k)$, we have

$$\begin{aligned} & \left\{c(U^+)' - D_2[U^+] - U^+(1 - U^+ - k(1 - W^+))\right\}(\xi) \\ \geq\ & e^{\lambda_1\xi}\left\{c\lambda_1 - (e^{\lambda_1} + e^{-\lambda_1} - 2) - (1 - k)\right\} = 0. \end{aligned}$$

Also, when $0 < d \le 1$, we have

$$\begin{aligned} & \left\{c(W^+)' - dD_2[W^+] - r(1-W^+)(hU^+ - W^+)\right\}(\xi) \\ \ge\ & \frac{1}{k}e^{\lambda_1 \xi}\left\{c\lambda_1 - d(e^{\lambda_1} + e^{-\lambda_1} - 2) + r(1 - \frac{e^{\lambda_1 \xi}}{k})(1-hk)\right\} \\ \ge\ & \frac{1}{k}e^{\lambda_1 \xi}\left\{(1-k) + r(1 - \frac{e^{\lambda_1 \xi}}{k})(1-hk)\right\} \ge 0. \end{aligned}$$

The last inequality holds for any $h > 1$, $0 < k < 1$, $r > 0$, and

$$(h, k, r) \in \{hk \le 1, r > 0\} \cup \left\{hk > 1, 0 < r \le \frac{1-k}{hk-1}\right\}.$$

Thus (U^+, W^+) is a super-solution of (2.1) with $U^+(\cdot) = W^+(\cdot) = 1$ on $[0, +\infty)$. By Lemma 2.3, (2.1) admits a solution (U, W) with $U' > 0$ and $W' > 0$ in $\mathbb{R}$. Thus we have derived that $c_{min} \ge c_*$ when conditions (**A1**), $d \le 1$ and (1.5) hold. Hence $c_{min} = c_*$ and this theorem follows. $\square$

We now give some implications of Theorem 4 at the end of this paper. In the numerical computation, the solution of a partial differential equation can be approximated by a finite difference scheme. In particular, the diffusing Lotka-Volterra competition model (1.2) can be approximated by the following spatial discretized system:

$$\left\{ \begin{aligned} \hat{u}_j'(t) &= \frac{\hat{u}_{j+1}(t) + \hat{u}_{j-1}(t) - 2\hat{u}_j(t)}{\tau^2} + \hat{u}_j(t)(1 - \hat{u}_j(t) - k\hat{v}_j(t)), \\ \hat{v}_j'(t) &= \frac{d(\hat{v}_{j+1}(t) + \hat{v}_{j-1}(t) - 2\hat{v}_j(t))}{\tau^2} + r\hat{v}_j(t)(1 - \hat{v}_j(t) - h\hat{u}_j(t)), \end{aligned} \right. \tag{5.1}$$

where $j \in \mathbb{Z}$, $t \in \mathbb{R}$, $\hat{u}_j(t) := u(j\tau, t)$, $\hat{v}_j(t) := v(j\tau, t)$ and τ is the spatial mesh size. Replacing (3.4) by

$$c\lambda = (e^\lambda + e^{-\lambda} - 2)\tau^{-2} + (1-k)$$

and by checking carefully the proof of Theorem 2, we can see that the minimal wave speed of (5.1) is given by

$$\begin{aligned} c_*(k;\tau) &= \min_{\lambda > 0}\left\{\frac{(e^\lambda + e^{-\lambda} - 2)\tau^{-2} + (1-k)}{\lambda}\right\} \\ &= \min_{\lambda > 0}\left\{\frac{(e^{\lambda\tau} + e^{-\lambda\tau} - 2)\tau^{-2} + (1-k)}{\lambda\tau}\right\} \end{aligned} \tag{5.2}$$

under the assumptions (**A1**), $d \le 1$ and (1.5).

In fact, we can show that

$$\tau c_*(k;\tau) \to 2\sqrt{1-k} \quad \text{as } \tau \to 0, \tag{5.3}$$

where $c_*(k;\tau)$ is given by (5.2), without the assumptions (**A1**), $d \le 1$ and (1.5).

To show (5.3), we show that

$$2\sqrt{1-k} \le \liminf_{\tau\to 0}[\tau c_*(k;\tau)] \le \limsup_{\tau\to 0}[\tau c_*(k;\tau)] \le 2\sqrt{1-k}.$$

We first prove that $\limsup_{\tau\to 0}[\tau c_*(k;\tau)] \le 2\sqrt{1-k}$. Note that

$$\begin{aligned}\tau c_*(k,\tau) &= \min_{\lambda>0}\left\{\frac{(e^{\lambda\tau}+e^{-\lambda\tau}-2)\tau^{-2}+(1-k)}{\lambda}\right\}\\ &\le \frac{(e^{\lambda\tau}+e^{-\lambda\tau}-2)\tau^{-2}+(1-k)}{\lambda}\end{aligned}$$

for any $\lambda > 0$. By using l'Hospital's rule,

$$\lim_{\tau\to 0}\left\{\frac{e^{\lambda\tau}+e^{-\lambda\tau}-2}{\lambda\tau^2}\right\} = \lim_{\tau\to 0}\left\{\frac{\lambda^2 e^{\lambda\tau}+\lambda^2 e^{-\lambda\tau}}{2\lambda}\right\} = \lambda.$$

It follows that

$$\limsup_{\tau\to 0}[\tau c_*(k;\tau)] \le \lambda + \frac{1-k}{\lambda},\ \forall\ \lambda > 0.$$

Thus we obtain that

$$\limsup_{\tau\to 0}[\tau c_*(k;\tau)] \le \min_{\lambda>0}\{\lambda + \frac{1-k}{\lambda}\} = 2\sqrt{1-k}.$$

It remains to show that $\liminf_{\tau\to 0}[\tau c_*(k;\tau)] \ge 2\sqrt{1-k}$. We now set

$$\liminf_{\tau\to 0}\tau c_*(\tau,a,k) = l.$$

Then $l \in [0, 2\sqrt{1-k}\ ]$. Choose a sequence $\{\tau_n\}$ such that $\tau_n \downarrow 0$ and $\tau_n c_*(\tau_n,a,k) \to l$ as $n \to +\infty$. For each n, we can find a unique $\lambda_n > 0$ such that

$$\begin{aligned}\tau_n c_*(k;\tau_n) &= \min_{\lambda>0}\left\{\frac{\tau_n^{-2}(e^{\lambda\tau_n}+e^{-\lambda\tau_n}-2)+(1-k)}{\lambda}\right\}\\ &= \frac{\tau_n^{-2}(e^{\lambda_n\tau_n}+e^{-\lambda_n\tau_n}-2)+(1-k)}{\lambda_n}.\end{aligned}\tag{5.4}$$

We shall prove that there exist $M > m > 0$ such that $m < \lambda_n < M$ for all n. By (5.4),

$$\tau_n c_*(k;\tau_n) \ge \frac{1-k}{\lambda_n} > 0,\ \forall\ n \in \mathbb{N}.$$

Since $\tau_n c_*(k;\tau_n) \to l$ as $n \to +\infty$, there exists a positive constant m such that $\lambda_n > m$. On the other hand, by the definition of λ_n,

$$\frac{d}{d\lambda}\left\{\frac{\tau_n^{-2}(e^{\lambda\tau_n}+e^{-\lambda\tau_n}-2)+(1-k)}{\lambda}\right\}\Big|_{\lambda=\lambda_n} = 0,\ \forall\ n \in \mathbb{N}.$$

Thus we obtain

$$\tau_n^{-2}(e^{\lambda_n\tau_n}+e^{-\lambda_n\tau_n}-2)+(1-k) = (\tau_n^{-1}e^{\lambda_n\tau_n}-\tau_n^{-1}e^{-\lambda_n\tau_n})\lambda_n$$

which leads (5.4) to

$$\tau_n c_*(k;\tau_n) = \frac{e^{\lambda_n\tau_n}-e^{-\lambda_n\tau_n}}{\tau_n},\ \forall\ n \in \mathbb{N}.\tag{5.5}$$

From (5.5), we can conclude that $\lambda_n\tau_n \to 0$ as $n \to +\infty$. Otherwise, there exists a subsequence $\{n_j\}$ of $\{n\}$ such that $\lambda_{n_j}\tau_{n_j} \to \eta$ as $j \to +\infty$ for some $\eta > 0$. Then by taking $j \to +\infty$ in (5.5) we have $l = +\infty$ which contradicts $l \in [0, 2\sqrt{1-k}\ ]$. Thus, $\lambda_n\tau_n \to 0$ as $n \to +\infty$. By the fact

$$\lim_{x\to 0}\frac{e^x - 1 - x}{x^2} = \frac{1}{2},$$

we have

$$e^{\pm\lambda_n\tau_n} - 1 > \pm\lambda_n\tau_n + \frac{1}{4}(\lambda_n\tau_n)^2 \tag{5.6}$$

for all sufficiently large n. By (5.4), (5.6) and $\tau_n c_*(k;\tau_n) \to l$ as $n \to +\infty$, there exists sufficiently large $N > 0$ such that

$$\begin{aligned} l + 1 &> \frac{1}{\lambda_n\tau_n^2}\{e^{\lambda_n\tau_n} + e^{-\lambda_n\tau_n} - 2\} \quad (\text{since } 1-k>0) \\ &\geq \frac{1}{\lambda_n\tau_n^2}\{-\lambda_n\tau_n + \frac{1}{4}(-\lambda_n\tau_n)^2 + \lambda_n\tau_n + \frac{1}{4}(\lambda_n\tau_n)^2\} \\ &\geq \frac{1}{2}\lambda_n,\ \forall\, n \geq N. \end{aligned}$$

Therefore, we can find $M > 0$ such that $\lambda_n \leq M < +\infty$ for all n.

From the above discussion, we have proved that there exist $M > m > 0$ such that $m < \lambda_n < M$ for all n. So there is a subsequence $\{\lambda_{n_i}\}$ of $\{\lambda_n\}$ and $\beta > 0$ such that $\lambda_{n_i} \to \beta$ as $i \to +\infty$. Replacing λ_n by λ_{n_i} in (5.4) and letting $i \to +\infty$, we obtain

$$l = \beta + \frac{(1-k)}{\beta} \geq 2\sqrt{1-k}.$$

Hence (5.3) is derived.

From (5.3) and the conjecture of Hosono [11], we expect that the minimal wave speed for (1.2) can be characterized for a wider range of parameters than (1.5) (cf. [9]). We also refer to the work [8] in which the authors treat the discrete version of a reaction diffusion equation with KPP nonlinearity in the periodic media and showed that the discretized minimal wave speed converges to the continuous minimal wave speed as the mesh size tends to zero.

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