UNIQUENESS AND STABILITY OF TRAVELING WAVES FOR PERIODIC MONOSTABLE LATTICE DYNAMICAL SYSTEM

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Abstract. We study the traveling waves for a lattice dynamical system with monostable nonlinearity in periodic media. It is well-known that there exists a minimal wave speed such that a traveling wave exists if and only if the wave speed is above this minimal wave speed. In this paper, we first derive a stability theorem for certain waves of non-minimal speed. Moreover, we show that wave profiles of a given speed are unique up to translations.

1. Introduction

We consider the following lattice dynamical system for unknown $u = \{u_j\}_{j\in\mathbb{Z}}$:

$$u_j' = d_{j+1}u_{j+1} + d_j u_{j-1} - (d_{j+1} + d_j)u_j + f_j(u_j),\ j \in \mathbb{Z}, \tag{1.1}$$

where $f_j \in C^{1+\alpha}[0,1]$ for some $\alpha \in (0,1)$ for $j \in \mathbb{Z}$, $f_{j+N} = f_j$ and $d_{j+N} = d_j > 0$ for all $j \in \mathbb{Z}$ for some positive integer N. The equation (1.1) can be regarded as a spatial discrete version of the following reaction-diffusion equation

$$u_t = (d(x)u_x)_x + f(x,u),$$

where $d(x)$ and $f(x,u)$ are periodic in x. In biology, let u_j denote the density of a certain species in a periodic patchy environment. Assuming the species at site j can only interact with those at the nearby sites, then the equation (1.1) describes the rate of change of density of this species at each site j. It is equal to the sum of the source $f_j(u_j)$ at site j and the fluxes $q_{j\pm1}$ from sites $j \pm 1$ to site j:

$$q_{j+1} := d_{j+1}[u_{j+1} - u_j],\ q_{j-1} := d_j[u_{j-1} - u_j],$$

where d_j, d_{j+1} are the diffusion constants. See [8, 15, 16] for more references and details.

It is trivial that for a given initial data $\{u_j(0)\} \in [0,1]$ there exists a unique solution u to (1.1) for $t \ge 0$ such that $0 \le u_j(t) \le 1$ for all $t \ge 0$ and $j \in \mathbb{Z}$. We are interested in the wave propagation phenomenon. In particular, we are interested in special solutions U of (1.1) for $t \in \mathbb{R}$ satisfying the following conditions:

$$U_j(t + N/c) = U_{j-N}(t),\ t \in \mathbb{R}, j \in \mathbb{Z}, \tag{1.2}$$

$$U_j(t) \to 1 \text{ as } j \to -\infty,\ \ U_j(t) \to 0 \text{ as } j \to +\infty, \quad \text{locally in } t \in \mathbb{R}, \tag{1.3}$$

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for some nonzero constant c. We shall call a solution (c, U) of (1.1)-(1.3) as a traveling wave solution. The constant c is the wave speed and U is the profile. In this paper, we shall always assume that

$$f_j(0) = f_j(1) = 0 \quad \forall\, j \in \mathbb{Z}. \tag{1.4}$$

The study of traveling wave for lattice dynamical system has attracted a lot of attention for past years, see, e.g., the works [1, 2, 3, 4, 6, 7, 10, 11, 12, 13, 14, 17, 18, 19, 20]. The main concerns are existence, uniqueness, and stability of traveling waves. Typically, there are two different nonlinearities, namely, monostable and bistable cases. In the monostable case, we have

$$f_j'(1) < 0 < f_j'(0)\ \forall\, j \in \mathbb{Z}, \quad f_j(s) > 0\ \forall\, s \in (0,1), j \in \mathbb{Z}. \tag{1.5}$$

For the bistable case, we have $f_j'(0) < 0$ and $f_j'(1) < 0$ for all $j \in \mathbb{Z}$. If $N = 1$, then $f_{j+1} = f_j$ and $d_{j+1} = d_j$ for all j. This is the so-called homogeneous media case. In general, if $N > 1$, then it is called the periodic case.

In this paper, we shall focus on the periodic monostable case. We refer the reader to the work [5] and the references cited therein for the periodic bistable case. In [5], the existence, uniqueness and stability of traveling waves for periodic bistable case are studied in detail.

The existence of traveling waves for monostable case in periodic media was first obtained by Hudson and Zinner [11, 12] under the extra assumption

$$f_j'(0)s - Ms^{1+\alpha} \le f_j(s) \le f_j'(0)s,\ \forall\, s \in [0,1], j \in \mathbb{Z}, \tag{1.6}$$

for some constants $M > 0$ and $\alpha \in (0,1)$. Recently, one of the authors and Hamel [10] gave a different approach to prove the existence of traveling waves for all speeds $c \ge c^*$ for some positive minimal speed c^*. Moreover, it is also shown in [10] that the condition $c \ge c^*$ is not only a sufficient condition but also a necessary condition for the existence of traveling waves.

For reader's convenience, we recall some properties of traveling wave from [10]. Let (c, U) be a traveling wave solution of (1.1)-(1.3) with $c \ne 0$. Then we have $0 < U_j(t) < 1$ for all $(j,t) \in \mathbb{Z} \times \mathbb{R}$; $U_j(t) \to 0$ as $t \to -\infty$; $U_j(t) \to 1$ as $t \to \infty$; $U_j'(t) > 0$ for all $t \in \mathbb{R}$ and $U_j'(t) \to 0$ as $t \to \pm\infty$.

The aim of this paper is to study the uniqueness and stability of traveling waves in the periodic monostable case. Hence we shall always assume that (1.4), (1.5) and (1.6) hold.

Recall from [10] that for each $\lambda \in \mathbb{R}$ there exists a unique $v = \{v_j\}$ with $\max_{j\in\mathbb{Z}} v_j = 1$ and $v_{j+N} = v_j > 0$ for all $j \in \mathbb{Z}$ such that

$$M(\lambda)v_j = d_{j+1}e^{-\lambda}v_{j+1} + d_j e^{\lambda} v_{j-1} - (d_{j+1} + d_j)v_j + f_j'(0)v_j \tag{1.7}$$

for all $j \in \mathbb{Z}$, where $M(\lambda)$ is the largest eigenvalue of (1.7). Moreover, there exists $\lambda^* > 0$ such that $c^* = M(\lambda^*)/\lambda^*$ and the mapping $c = M(\lambda)/\lambda : (0, \lambda^*) \mapsto c \in (c^*, \infty)$ is strictly decreasing.

We shall focus our attention on those traveling waves (c, U), $c > c^*$, of (1.1)-(1.3) satisfying

$$\lim_{j-ct\to\infty} \frac{U_j(t)}{e^{-\lambda(j-ct)}v_j} = 1, \tag{1.8}$$

for some $\lambda > 0$ such that $M(\lambda) = c\lambda$ and $\{v_j\}$ is the unique eigenvector of (1.7) corresponding to λ such that $\max_{j\in\mathbb{Z}} v_j = 1$ and $v_{j+N} = v_j > 0$ for all $j \in \mathbb{Z}$.

We now state our stability theorem as follows.

Theorem 1.1. *Suppose that there exists a traveling wave (c, U) with $c > c^*$ such that (1.8) holds for some $\lambda \in (0, \lambda^*)$. Let u be the solution of (1.1) for $t \geq 0$ with the initial value $\{u_j(0)\}$ satisfying*

$$0 \leq u_j(0) \leq 1, \quad u_j(0) \leq e^{-\lambda \cdot j} v_j \quad \forall j \in \mathbb{Z}, \tag{1.9}$$

$$\liminf_{j\to-\infty} u_j(0) > 0, \quad \lim_{j\to\infty} \frac{u_j(0)}{e^{-\lambda \cdot j} v_j} = 1. \tag{1.10}$$

Then

$$\lim_{t\to\infty} \sup_j \{|[u_j(t)/U_j(t)] - 1|\} = 0.$$

The proof of Theorem 1.1 is based on a method in [3] with some nontrivial modifications. In [3], a lattice dynamical system in homogeneous media is studied. There the proof of stability theorem is through a related continuum equation by extending the spatial variable from $j \in \mathbb{Z}$ to $x \in \mathbb{R}$. But, here we shall only use the original equation (1.1) to prove the stability theorem. Moreover, there is only one wave profile for the homogeneous case in [3]. In our periodic lattice dynamical system, there are N wave profiles. This makes the stability analysis more complicated. To overcome this difficulty, we introduce the following transformation

$$W_j(x) := U_j([j - x]/c), \quad \text{equivalently} \quad U_j(t) = W_j(j - ct), \tag{1.11}$$

which is very useful in the periodic framework. Indeed, this transformation is reminiscent of a similar transformation in the case of partial differential equation (cf. [9]).

By adapting a method used in [4], we have the following uniqueness theorem.

Theorem 1.2. *Suppose that (c, U) and $(c, \overline{U})$ are two traveling wave solutions of (1.1)-(1.3) such that*

$$\lim_{j-ct\to\infty} \frac{U_j(t)}{e^{-\lambda(j-ct)} v_j} = h, \quad \lim_{j-ct\to\infty} \frac{\overline{U}_j(t)}{e^{-\lambda(j-ct)} v_j} = \bar{h} \tag{1.12}$$

for some positive constants λ, h and $\bar{h}$ such that $M(\lambda) = c\lambda$, where $\{v_j\}$ is the eigenvector of (1.7) corresponding to λ such that $v_j = v_{j+N} > 0$ for all j and $\max\{v_j\} = 1$. Then there exists $\xi \in \mathbb{R}$ such that $U_j(t) = \overline{U}_j(t + \xi)$ for all $j \in \mathbb{Z}$, $t \in \mathbb{R}$.

This paper is organized as follows. We shall give the proof of Theorem 1.1 in Section 2. The proof of Theorem 1.2 is given in Section 3. In this paper, we shall use both functions U_j and W_j defined in (1.11) alternatively from time to time.

2. Stability of traveling wave

This section is devoted to the proof of Theorem 1.1. First, we call a continuous function w a super-solution of (1.1) in an interval I, if w is differentiable a.e. such that

$$w_j'(t) \geq \mathcal{A}[w_j](t) + f_j(w_j(t)) \quad \text{a.e. for } t \in I, \forall\, j \in \mathbb{Z}, \tag{2.1}$$

where

$$\mathcal{A}[w_j](t) := d_{j+1} w_{j+1}(t) + d_j w_{j-1}(t) - (d_j + d_{j+1}) w_j(t).$$

The notion of sub-solution is defined similarly by reversing the inequality in (2.1).

Based on a traveling wave (c, U), we can construct the following super/sub-solution.

Lemma 2.1. *For each $\delta \in (0,1)$ and $\eta \in (0, \inf_{s\in(-\delta,\delta)}[f(1-s)/s])$, there exists $l = l(\delta,\eta) > 0$ such that for any $\epsilon \in [0,\delta]$ the function $w^{\pm} := \{w_j^{\pm}\}$ defined by*

$$w_j^{\pm}(t) := (1 \pm \epsilon e^{-\eta t}) U_j(t \mp l\epsilon e^{-\eta t}), \quad (j,t) \in \mathbb{Z} \times [0,\infty)$$

is a super/sub-solution of (1.1).

Proof. We consider only the case of super-solution. The case of sub-solution is similar.

Set $w_j(t) := (1+q)U_j(s)$, $s := t - l\epsilon e^{-\eta t}$ and $q := \epsilon e^{-\eta t}$. Then we compute

$$\begin{aligned} & w_j'(t) - \mathcal{A}[w_j](t) - f_j(w_j(t)) \\ = \ & -\eta q U_j(s) + (1+q)(1+l\eta q)U_j'(s) - (1+q)[U_j'(s) - f_j(U_j(s))] - f_j((1+q)U_j(s)) \\ = \ & -\eta q U_j(s) + l\eta q(1+q)U_j'(s) + (1+q)f_j(U_j(s)) - f_j((1+q)U_j(s)). \end{aligned}$$

Notice that

$$\begin{aligned} (1+q)f_j(U_j) - f_j((1+q)U_j) &= \int_0^q [f_j(U_j) - U_j f_j'((1+p)U_j)]dp \\ &= q f_j(U_j) - U_j f_j(1+q) - U_j \int_0^q [f_j'((1+p)U_j) - f_j'(1+p)]dp \\ &\geq -U_j f_j(1+q) - U_j \int_0^q [f_j'((1+p)U_j) - f_j'(1+p)]dp. \end{aligned}$$

Since $f_j \in C^{1+\alpha}([0,1])$, f_j can be suitably extended so that $f_j \in C^{1+\alpha}([-1,2])$. Then we have

$$\left| \int_0^q [f_j'((1+p)U_j) - f_j'(1+p)]dp \right| \leq 2qK(1-U_j)^{\alpha},$$

where

$$K := \max_{j\in\mathbb{Z}} \max_{-1\leq s<t\leq 2} \frac{|f_j'(t) - f_j'(s)|}{|t-s|^{\alpha}}.$$

It follows that

$$\begin{aligned} & \frac{1}{q}\{w_j'(t) - \mathcal{A}[w_j](t) - f_j(w_j(t))\} \\ \geq \ & l\eta(1+q)U_j'(s) - \eta U_j(s) - [f_j(1+q)/q]U_j(s) - 2K(1-U_j(s))^{\alpha} U_j(s). \end{aligned}$$

Now, for a given $\delta > 0$, we set

$$\eta_\delta := \inf_{j\in\mathbb{Z}, -\delta<s<\delta} [f_j(1-s)/s].$$

Note that $\eta_\delta > 0$, since $f_j'(1) < 0$. Choose $\eta \in (0, \eta_\delta)$. Since $\lim_{j-ct\to-\infty} U_j(t) = 1$, there exists ξ_0 such that $2K(1-U_j(s))^{\alpha} \leq \eta_\delta - \eta$ for all $j - cs \leq \xi_0$. Recall from Lemma 2.5 of [10] that $U_j' > 0$ in $\mathbb{R}$ for all $j \in \mathbb{Z}$. Hence

$$w_j'(t) - \mathcal{A}[w_j](t) - f_j(w_j(t)) \geq 0 \quad \forall j - cs \leq \xi_0. \tag{2.2}$$

On the other hand, since $U_j(t) \to 0$ as $j - ct \to \infty$, it follows from Lemma 2.4 of [10] that

$$\liminf_{j-ct\to\infty} \frac{U_j'(t)}{U_j(t)} = \liminf_{(j,t)\in\mathbb{Z}\times\mathbb{R}, U_j(t)\to 0} \frac{U_j'(t)}{U_j(t)} > 0.$$

Hence, if we choose

$$l := \frac{2K}{\eta(1-\delta)} \sup_{j-cs\geq\xi_0} \frac{U_j(s)}{U_j'(s)},$$

then $l > 0$ and we obtain

$$w_j'(t) - \mathcal{A}[w_j](t) - f(w_j(t)) \geq 0 \quad \forall j - cs \geq \xi_0. \tag{2.3}$$

Combining (2.2) and (2.3), we obtain that $w := \{w_j\}$ is a super-solution of (1.1). □

Recall the following standard comparison principle. Since the proof is standard, we omit it here (see also [3]).

Proposition 2.2. *Given two bounded continuous functions u, v on $[t_0, \infty)$ for some $t_0 \geq 0$ such that u, v are differentiable a.e. in $[t_0, \infty)$. Suppose that*

$$u_j'(t) - \mathcal{A}[u_j](t) - f(u_j(t)) \geq v_j'(t) - \mathcal{A}[v_j](t) - f(v_j(t)) \quad \forall\ t \geq t_0,\ j \in \mathbb{Z},$$

and $u_j(t_0) \geq v_j(t_0)$ for all $j \in \mathbb{Z}$. Then $u_j(t) \geq v_j(t)$ for all $t \geq t_0$, $j \in \mathbb{Z}$. Moreover, if, besides the above assumptions, $u_k(t_0) > v_k(t_0)$ for some $k \in \mathbb{Z}$, then $u_j(t) > v_j(t)$ for all $t > t_0$, $j \in \mathbb{Z}$.

Given any $c > c^*$. Let $\lambda \in (0, \lambda^*)$ be such that $M(\lambda) = c\lambda$ and let $\{v_j\}$ be the eigenvector of (1.7) corresponding to λ such that $\max_{j\in\mathbb{Z}} v_j = 1$ and $v_{j+N} = v_j > 0$ for all $j \in \mathbb{Z}$. Then it is easy to check that the function $\bar{u} = (\bar{u}_j)_{j\in\mathbb{Z}}$ defined by

$$\bar{u}_j(t) = \min\{e^{-\lambda(j-ct)}v_j, 1\} \quad \forall\ (j,t) \in \mathbb{Z} \times \mathbb{R} \tag{2.4}$$

is a super-solution of (1.1). Moreover, we can choose $\mu \in (\lambda, \lambda^*)$ such that $\mu < (1+\alpha)\lambda$ and $M(\mu) < c\mu$, where α is the constant defined in (1.6). Let $\{w_j\}$ be the eigenvector of (1.7) corresponding to μ such that $\max_{j\in\mathbb{Z}} w_j = 1$ and $w_{j+N} = w_j > 0$ for all $j \in \mathbb{Z}$. Then the function $\underline{u} = (\underline{u}_j)_{j\in\mathbb{Z}}$ defined by

$$\underline{u}_j(t) = \max\{e^{-\lambda(j-ct)}v_j - Ae^{-\mu(j-ct)}w_j, 0\} \quad \forall\ (j,t) \in \mathbb{Z} \times \mathbb{R} \tag{2.5}$$

is a sub-solution of (1.1), if A is large enough.

Note that the traveling wave solution, denoted by $\{U_j\}$, obtained by an iteration starting from the above super-sub-solutions satisfies (1.8) for some $\lambda \in (0, \lambda^*)$. To see this, we first note from [10] that

$$\underline{u}_j(t) \leq U_j(t) \leq \bar{u}_j(t) \quad \forall\ (j,t) \in \mathbb{Z} \times \mathbb{R}. \tag{2.6}$$

For $j - ct \gg 1$, we have

$$\bar{u}_j(t) = e^{-\lambda(j-ct)}v_j, \quad \underline{u}_j(t) = e^{-\lambda(j-ct)}v_j - Ae^{-\mu(j-ct)}w_j. \tag{2.7}$$

Writing

$$\underline{u}_j(t) = e^{-\lambda(j-ct)}v_j[1 - Ae^{-(\mu-\lambda)(j-ct)}w_j/v_j]$$

and using the fact $\mu \in (\lambda, \lambda^*)$, then (1.8) follows from (2.6) and (2.7).

From now on, we assume that u is the solution of (1.1) for $t \geq 0$ with the initial value $\{u_j(0)\}$ satisfying (1.9) and (1.10) for a traveling wave (c, U) with $c > c^*$ satisfying (1.8) for some $\lambda \in (0, \lambda^*)$. Also, for a a given $c > c^*$, we fix the corresponding $\lambda, \mu, A, v_j, w_j$ defined as above in the following.

Lemma 2.3. *For any $\epsilon > 0$, there exists a constant $\xi_1(\epsilon) > 1$ such that*

$$u_j(t - 2\epsilon) \leq U_j(t) \leq u_j(t + 2\epsilon) \qquad \forall\ j - ct \geq \xi_1(\epsilon),\ t \geq 2\epsilon. \tag{2.8}$$

Proof. Given any $\epsilon > 0$. First, we derive the second inequality in (2.8). By (1.10), there exists j_0 depending on ϵ such that

$$e^{-\lambda(j+c\epsilon)}v_j < u_j(0) < e^{-\lambda(j-c\epsilon)}v_j \quad \forall j \geq j_0. \tag{2.9}$$

Choose $A \geq e^{(\mu-\lambda)(j_0+c\epsilon)}[\max\{v_j\}/\min\{w_j\}]$ large enough so that (2.5) is a sub-solution of (1.1). Then

$$e^{-\lambda(j+c\epsilon)}v_j - Ae^{-\mu(j+c\epsilon)}w_j \leq 0 \quad \forall j \leq j_0. \tag{2.10}$$

Hence, from (2.9) and (2.10),

$$u_j(0) \geq \max\{e^{-\lambda(j+c\epsilon)}v_j - Ae^{-\mu(j+c\epsilon)}w_j, 0\} \quad \forall j \in \mathbb{Z}.$$

By the comparison principle,

$$u_j(t) \geq e^{-\lambda(j-c(t-\epsilon))}v_j - Ae^{-\mu(j-c(t-\epsilon))}w_j \quad \forall j \in \mathbb{Z},\ t \geq 0,$$

i.e.,

$$u_j(t+\epsilon) \geq e^{-\lambda(j-ct)}v_j - Ae^{-\mu(j-ct)}w_j \quad \forall j \in \mathbb{Z},\ t \geq 0. \tag{2.11}$$

Moreover, by (1.8), there exists a constant $x_1(\epsilon) > 1$ such that

$$e^{-\lambda(j-c(t+\epsilon))}v_j - Ae^{-\mu(j-c(t+\epsilon))}w_j \geq U_j(t) \quad \forall j - ct \geq x_1(\epsilon). \tag{2.12}$$

From (2.11) and (2.12) it follows that

$$u_j(t+2\epsilon) \geq U_j(t) \quad \forall j - ct \geq x_1(\epsilon),\ t \geq 0.$$

Next, we derive the first inequality in (2.8). By (1.9), we have

$$u_j(0) \leq \min\{e^{-\lambda \cdot j}v_j, 1\} \quad \forall j \in \mathbb{Z}.$$

By comparison,

$$u_j(t) \leq \min\{e^{-\lambda(j-ct)}v_j, 1\} \quad \forall j \in \mathbb{Z},\ t \geq 0. \tag{2.13}$$

On the other hand, from (1.8), we have

$$\lim_{j-ct\to\infty} \frac{U_j(t)}{e^{-\lambda(j-c(t-2\epsilon))}v_j} = e^{2\lambda c\epsilon} > 1.$$

Hence there exists a constant $x_2(\epsilon) > 1$ such that

$$e^{-\lambda(j-c(t-2\epsilon))}v_j < U_j(t) \quad \forall j - ct \geq x_2(\epsilon).$$

From (2.13) it follows that

$$u_j(t-2\epsilon) \leq \min\{e^{-\lambda(j-c(t-2\epsilon))}v_j, 1\} \leq U_j(t) \quad \forall j - ct \geq x_2(\epsilon),\ t \geq 2\epsilon.$$

Then the lemma follows by taking $\xi_1(\epsilon) = \max\{x_1(\epsilon), x_2(\epsilon)\}$. □

Next, we have the following positivity lemma.

Lemma 2.4. *There exist continuous functions $\{\psi_j\}_{j\in\mathbb{Z}}$ from $(0,1] \times (0,\infty)$ to $(0,1)$ such that if $u_k(0) > 0$ for some $k \in \mathbb{Z}$ then $u_{k+n}(t) \geq \psi_n(u_k(0),t) > 0$ for all $n \in \mathbb{Z}$, $t > 0$.*

Proof. Note that $0 \le u_j(t) \le 1$ for all $t \ge 0$ for all $j \in \mathbb{Z}$. Choose $\sigma > 0$ so that $\sigma > 2\max\{d_j\}$. From (1.1) it follows that

$$\begin{aligned} u_j(t) &= e^{-\sigma t}u_j(0) + \int_0^t e^{\sigma(s-t)}\{d_{j+1}u_{j+1}(s) + d_j u_{j-1}(s) \\ &\quad +[\sigma - (d_{j+1}+d_j)]u_j(s) + f(u_j(s))\}ds. \end{aligned} \tag{2.14}$$

This gives $u_k(t) \ge e^{-\sigma t}u_k(0) > 0$ for all $t > 0$.

Set $q := \min\{d_j\}$. Then $q > 0$. Moreover, from (2.14) it follows that

$$u_j(t) \ge q\int_0^t e^{\sigma(s-t)}u_{j\pm1}(s)ds.$$

Set $\psi_0(y,t) := ye^{-\sigma t}$ and define recursively

$$\psi_{-n}(y,t) = \psi_n(y,t) := q\int_0^t e^{\sigma(s-t)}\psi_{n-1}(y,s)ds, \quad y \in (0,1],\ t > 0,\ n \in \mathbb{N}.$$

The lemma follows. □

Note that

$$\psi_{\pm n}(y,t) = \frac{yq^n t^n e^{-\sigma t}}{n(n-1)\cdots 1}$$

for all $n \in \mathbb{N}$.

Lemma 2.5. *There exist constants $\delta \in (0,1)$, $\eta > 0$, $l > 0$, $z_0 > 0$ and $t_0 \ge 4$ such that*

$$(1-\delta e^{-\eta t})U_j(t - z_0 + l\delta e^{-\eta t}) \le u_j(t) \le (1+\delta e^{-\eta t})U_j(t + z_0 - l\delta e^{-\eta t})$$

for all $j \in \mathbb{Z}$, $t \ge t_0$.

Proof. We first consider the lower bound of u_j. Fix a $t_0 \ge 4$. From Lemma 2.3 with $\epsilon = 1$, there exists a constant $\xi_1(1)$ such that

$$u_j(t_0) \ge U_j(t_0 - 2) \quad \forall j - ct_0 \ge \xi_1(1).$$

Since $\liminf_{j\to-\infty} u_j(0) > 0$, there exist $j_0 \in \mathbb{Z}$ and $\delta_0 > 0$ such that $u_j(0) > \delta_0$ for all $j \le j_0$. By Lemma 2.4, there exist $\delta \in (0,1)$ and $\eta \in (0,\eta_\delta)$ such that

$$u_j(t_0) \ge 1 - \delta e^{-\eta t_0} \quad \forall j - ct_0 \le \xi_1(1).$$

Thus

$$\begin{aligned} u_j(t_0) &\ge (1-\delta e^{-\eta t_0})U_j(t_0 - 2) \\ &= (1-\delta e^{-\eta t_0})U_j(t_0 - (2 + l\delta e^{-\eta t_0}) + l\delta e^{-\eta t_0}) \quad \forall\, j \in \mathbb{Z}, \end{aligned}$$

where $l = l(\delta,\eta) > 0$ is the constant defined in Lemma 2.1. It follows from the comparison principle that

$$u_j(t) \ge (1-\delta e^{-\eta t})U_j(t - z_* + l\delta e^{-\eta t}) \quad \forall\, t \ge t_0,\ j \in \mathbb{Z}, \tag{2.15}$$

where $z_* = 2 + l\delta e^{-\eta t_0}$.

For the upper bound, again by Lemma 2.3, we have

$$u_j(t_0) \le U_j(t_0 + 2) \quad \forall j - ct_0 \ge \xi_1(1).$$

For $j - ct_0 \le \xi_1(1)$, we consider the function

$$W_j(x) := U_j((j-x)/c), \quad j \in \mathbb{Z},\ x \in \mathbb{R}. \tag{2.16}$$

Then, by (1.2), $W_j = W_{j+N}$ for all $j \in \mathbb{Z}$, $W_j(\infty) = 0$ and $W_j(-\infty) = 1$. Therefore, we can choose $\hat{x} \gg 1$ such that $W_j(x) \geq 1/(1 + \delta e^{-\eta t_0})$ for all $j \in \mathbb{Z}$ for all $x \leq -\hat{x}$. Choose a large enough $\hat{t}$ so that $j - c(t_0 + 2 + \hat{t}) \leq -\hat{x}$ for all j with $j - ct_0 \leq \xi_1(1)$. Then

$$U_j(t_0 + 2 + \hat{t}) = W_j(j - c[t_0 + 2 + \hat{t}])$$

and so

$$u_j(t_0) \leq 1 \leq (1 + \delta e^{-\eta t_0})U_j(t_0 + 2 + \hat{t}) \quad \forall j - ct_0 \leq \xi_1(1).$$

Hence, using $U_j' > 0$, we obtain that

$$u_j(t_0) \leq (1 + \delta e^{-\eta t_0})U_j(t_0 + 2 + \hat{t}) \quad \forall j \in \mathbb{Z}.$$

By the comparison principle, we deduce that

$$u_j(t) \leq (1 + \delta e^{-\eta t})U_j(t + z^* - l\delta e^{-\eta t}) \quad \forall\, t \geq t_0,\ j \in \mathbb{Z}, \tag{2.17}$$

where $z^* = 2 + \hat{t} + l\delta e^{-\eta t_0}$. The lemma follows by combining (2.15) and (2.17). □

Lemma 2.6. *Let δ, l be two positive constants. Then there exists a positive constant M_0 depending on δ and l such that for all $\epsilon \in (0, \delta]$*

$$(1 - \epsilon)U_j(t + 3l\epsilon) \leq U_j(t) \leq (1 + \epsilon)U_j(t - 3l\epsilon) \quad \forall j - ct \leq -M_0.$$

Proof. Recall the definition of W in (2.16). Note that $W_j'(\pm\infty) = 0$ and $W_j(-\infty) = 1$ for all $j \in \mathbb{Z}$. We compute that

$$\frac{d}{ds}\{(1 + s)W_j(x + 3cls)\} = W_j(x + 3cls) + 3cl(1 + s)W_j'(x + 3cls).$$

Hence, noting that $W_j = W_{j+N}$ for all j, there exists $M_0 > 0$ such that

$$\frac{d}{ds}\{(1 + s)W_j(x + 3cls)\} > 0 \quad \forall\, x \leq -M_0,\ j \in \mathbb{Z},\ s \in [-\delta, \delta].$$

This implies that

$$\frac{d}{ds}\{(1 + s)U_j(t - 3ls)\} > 0 \quad \forall\, s \in [-\delta, \delta],\ j - ct \leq -M_0.$$

Hence the lemma is proved. □

In the sequel, the constants δ, l, η, M_0 are fixed as in Lemmas 2.5 and 2.6.

Lemma 2.7. *Let $z > 0$, $t_1 \geq 0$ and $M \in \mathbb{R}$. Suppose that $w_j^{\pm}(\cdot; t_1)$ is the solution of (1.1) for $t \geq 0$ with initial value:*

$$w_j^{\pm}(0; t_1) = U_j(t_1 \pm z)\phi(j - ct_1 - M) + U_j(t_1 \pm 2z)[1 - \phi(j - ct_1 - M)] \quad \forall j \in \mathbb{Z}, \tag{2.18}$$

where $\phi(s) = 0$ for $s \leq 0$ and $\phi(s) = 1$ for $s > 0$. Then there exists $\epsilon \in (0, \min\{\delta, z/(3l)\})$, depending only on M and z (independent of t_1), such that

$$\begin{aligned} w_j^{+}(1; t_1) &\leq (1 + \epsilon)U_j(t_1 + 1 + 2z - 3l\epsilon), \\ w_j^{-}(1; t_1) &\geq (1 - \epsilon)U_j(t_1 + 1 - 2z + 3l\epsilon) \end{aligned}$$

for all $j \in \mathbb{Z}$ with $j - ct_1 \leq M + c(1 + 2z)$.

Proof. First, we consider w_j^+. Note that $w_j^+(0;t_1) = U_j(t_1+2z)$ for all $j - ct_1 \le M$ and $w_j^+(0;t_1) = U_j(t_1+z) < U_j(t_1+2z)$ for all $j - ct_1 > M$. By the strong comparison principle,

$$w_j^+(1;t_1) < U_j(t_1+1+2z) \quad \forall\, j \in \mathbb{Z}. \tag{2.19}$$

Consider first when $t_1 \in [0,T)$, where $T := N/c$. Then by the equi-continuity of $\{w_j^+(\cdot\,;t_1)\}$ in $[0,\infty)$ and $\{U_j\}$ in $\mathbb{R}$, there exists $\epsilon \in (0, \min\{\delta, z/(3l)\})$ such that for any initial time $t_1 \in [0,T)$

$$w_j^+(1;t_1) < U_j(t_1+1+2z-3l\epsilon) \quad \text{ if } \; j - c(t_1+1+2z) \in [-M_0, M]. \tag{2.20}$$

For $t_1 \ge T$, we can rewrite $t_1 = t_0 + kT$ for some $k \in \mathbb{N}$ and $t_0 \in [0,T)$. From (2.18) we have

$$\begin{aligned} w_j^+(0;t_1) &= U_j(t_0+kT+z)\phi(j - c(t_0+kT) - M) \\ &\quad + U_j(t_0+kT+2z)[1 - \phi(j - c(t_0+kT) - M)] \\ &= U_{j-kN}(t_0+z)\phi(j - kN - ct_0 - M) \\ &\quad + U_{j-kN}(t_0+2z)[1 - \phi(j - kN - ct_0 - M) \\ &= w_{j-kN}^+(0;t_0). \end{aligned}$$

Hence $w_{j+kN}^+(t;t_1) = w_j^+(t;t_0)$ for all $t \ge 0$. In particular,

$$w_{j+kN}^+(1;t_1) = w_j^+(1;t_0). \tag{2.21}$$

For any integer j_1 with $j_1 - c(t_1+1+2z) \in [-M_0, M]$, i.e.,

$$j_1 \in [-M_0 + c(t_0+1+2z) + kN, M + c(t_0+1+2z) + kN],$$

we can write $j_1 = j_0 + kN$ for a unique integer j_0 such that

$$j_0 - c(t_0+1+2z) \in [-M_0, M].$$

Hence, by (2.21) and (2.20) with t_1 replaced by t_0 and $j = j_0$, we have

$$w_{j_1}^+(1;t_1) = w_{j_0}^+(1;t_0) < U_{j_0}(t_0+1+2z-3l\epsilon) = U_{j_1}(t_1+1+2z-3l\epsilon)$$

for any integer j_1 with $j_1 - c(t_1+1+2z) \in [-M_0, M]$. Here the periodicity of U was used.

Moreover, it follows from Lemma 2.6 that

$$U_j(t_1+1+2z) \le (1+\epsilon)U_j(t_1+1+2z-3l\epsilon) \quad \forall j - c(t_1+1+2z) \le -M_0. \tag{2.22}$$

This proves the inequality for $w_j^+(\cdot\,;t_1)$ for all $t_1 \ge 0$.

The case for w_j^- is similar. Hence the lemma follows. □

Proof of Theorem 1.1. Define $z^\pm := \inf A^\pm$, where

$$A^+ := \{z \ge 0 \mid \limsup_{t\to\infty} \sup_j [u_j(t)/U_j(t+2z)] \le 1\},$$
$$A^- := \{z \ge 0 \mid \liminf_{t\to\infty} \inf_j [u_j(t)/U_j(t-2z)] \ge 1\}.$$

From Lemma 2.5, $z_0/2 \in A^\pm$. Hence $z^\pm$ are well defined and $z^\pm \in [0, z_0/2]$. It suffices to prove that $z^+ = z^- = 0$.

For contradiction, we suppose that $z^+ > 0$. Recall the constant $\xi_1(z^+/2)$ defined in Lemma 2.3. Let $\epsilon \in (0, \min\{\delta, z^+/(3l)\})$ be the constant obtained in Lemma 2.7 with $z = z^+$ and $M := \xi_1(z^+/2) + cz^+$. Since $z^+ \in A^+$, we have

$$\limsup_{t\to\infty} \sup_j \frac{u_j(t)}{U_j(t+2z^+)} \le 1.$$

Hence there exists $t_0 \ge 4$ such that

$$\sup_j \frac{u_j(t_0)}{U_j(t_0+2z^+)} \le 1 + \hat{\epsilon},$$

where

$$\hat{\epsilon} := \epsilon e^{-K} \min_{j\in\{1,2,\dots,N\}} W_j(M + 3cl\epsilon), \quad U_j(t) = W_j(j - ct),$$

and $K := \max\{\|f_j'\|_{L^\infty}\}$. Then

$$u_j(t_0) \le U_j(t_0 + 2z^+) + \hat{\epsilon} \quad \forall\, j \in \mathbb{Z}.$$

Now, let $w_j^\pm(\cdot\,; t_0)$ be the solution of (1.1) for $t \ge 0$ with initial value given by

$$w_j^\pm(0; t_0) = U_j(t_0 \pm z)\phi(j - ct_0 - M) + U_j(t_0 \pm 2z)[1 - \phi(j - ct_0 - M)] \quad \forall j \in \mathbb{Z}.$$

Then $w_j^+(0; t_0) = U_j(t_0 + 2z^+)$ for all $j - ct_0 \le M$ and so

$$u_j(t_0) \le w_j^+(0; t_0) + \hat{\epsilon} \quad \forall\, j - ct_0 \le M.$$

Moreover, from Lemma 2.3, $u_j(t_0) \le U_j(t_0 + z^+)$ if $j - c(t_0 + z^+) \ge \xi_1(z^+/2)$. Since $j - c(t_0 + z^+) \ge \xi_1(z^+/2)$ if $j - ct_0 \ge M$, we obtain from (2.18) that

$$u_j(t_0) \le w_j^+(0; t_0) + \hat{\epsilon} \quad \forall\, j - ct_0 \ge M.$$

We conclude that

$$u_j(t_0) \le w_j^+(0; t_0) + \hat{\epsilon} \quad \forall\, j \in \mathbb{Z}.$$

It is easy to check that $\{w_j^+(t; t_0) + \hat{\epsilon}e^{Kt}\}$ is a super-solution of (1.1). By comparison, $u_j(t_0 + 1) \le w_j^+(1; t_0) + \hat{\epsilon}e^K$ for all $j \in \mathbb{Z}$. Then, by Lemma 2.7,

$$u_j(t_0 + 1) \le (1 + \epsilon)U_j(t_0 + 1 + 2z^+ - 3l\epsilon) + \hat{\epsilon}e^K \quad \text{if } \; j - ct_0 \le M + c(1 + 2z^+).$$

It follows from the choice of $\hat{\epsilon}$ and $W_j' < 0$ that

$$u_j(t_0 + 1) \le (1 + 2\epsilon)U_j(t_0 + 1 + 2z^+ - 3l\epsilon) \quad \text{if } \; j - ct_0 \le M + c(1 + 2z^+).$$

On the other hand, from Lemma 2.3,

$$u_j(t_0 + 1) \le U_j(t_0 + 1 + z^+) \quad \text{if } \; j - c(t_0 + 1 + z^+) \ge \xi_1(z^+/2).$$

Since $0 < \epsilon < z^+/(3l)$ and $U_j' > 0$, we obtain that

$$u_j(t_0 + 1) < (1 + 2\epsilon)U_j(t_0 + 1 + 2z^+ - 3l\epsilon) \quad \text{if } \; j - ct_0 \ge M + c.$$

Hence

$$u_j(t_0 + 1) \le (1 + 2\epsilon)U_j(t_0 + 1 + 2z^+ - 3l\epsilon) \quad \forall\, j \in \mathbb{Z}.$$

By comparison,

$$u_j(t + t_0 + 1) \le (1 + 2\epsilon e^{-\eta t})U_j(t + t_0 + 1 + 2z^+ - 2l\epsilon - l\epsilon e^{-\eta t}) \quad \forall\, t \ge 0,\; j \in \mathbb{Z}. \tag{2.23}$$

By taking $t \to \infty$ in (2.23), we obtain that $z^+ - l\epsilon \in A^+$ which contradicts the definition of z^+. Hence we must have $z^+ = 0$.

Similarly, we can also prove that $z^- = 0$. This completes the proof of Theorem 1.1. $\square$

3. Uniqueness of wave profile

In this section, we shall study the uniqueness of wave profiles for a given wave speed and give a proof of Theorem 1.2.

Suppose that (c, U) and $(c, \overline{U})$ are two traveling wave solutions of (1.1)-(1.3) such that (1.12) holds for some positive constants λ, h and $\bar{h}$ such that $M(\lambda) = c\lambda$, where $\{v_j\}$ is the eigenvector of (1.7) corresponding to λ such that $v_j = v_{j+N} > 0$ for all j and $\max\{v_j\} = 1$. By a suitable translation, we may assume that $h = \bar{h} = 1$. Therefore, (1.8) holds for both (c, U) and $(c, \overline{U})$. Then, using (1.1) and (1.7), it is easy to show that

$$\lim_{j-ct\to\infty} \frac{U_j'(t)}{U_j(t)} = \Lambda = \lim_{j-ct\to\infty} \frac{\overline{U}_j'(t)}{\overline{U}_j(t)}, \quad \Lambda := M(\lambda) = c\lambda. \tag{3.1}$$

First, we consider the function

$$g_j(s, u) := f_j([1+s]u) - (1+s)f_j(u), \quad s \ge 0,\ u \in [0,1].$$

Then $dg_j(s,u)/ds = uf_j'([1+s]u) - f_j(u)$. Since $f_j'(1) < 0$ and $f_j(1) = 0$ for all j, by the periodicity of f_j, there exists $\epsilon_0 \in (0,1)$ such that

$$f_j([1+\epsilon]u) < (1+\epsilon)f_j(u) \quad \forall\ u \in (1-\epsilon_0, 1] \tag{3.2}$$

for any $\epsilon \in (0, \epsilon_0]$, where we have extended $f_j(u)$ to be negative for all $u \in (1,2]$.

We next define the number

$$l_0 = l_0(U) := \sup\{W_j(x)/|cW_j'(x)| : W_j(x) \le 1-\epsilon_0,\ j \in \mathbb{Z}\} \tag{3.3}$$

for a wave profile $\{W_j\}$. Note that $l_0 \in (0,\infty)$, since $W_j(x), W_j'(x) \to 0$ as $x \to \infty$,

$$-c \lim_{x\to\infty} \frac{W_j'(x)}{W_j(x)} = \lim_{j-ct\to\infty} \frac{U_j'(t)}{U_j(t)} = \Lambda > 0,$$

and $W_j' < 0$ for all $j \in \mathbb{Z}$.

Lemma 3.1. *Let (c, U) and $(c, \overline{U})$ be two traveling wave solutions of (1.1)-(1.3). Let ϵ_0 and $l_0 = l_0(U)$ be the constants defined in (3.2) and (3.3). If there exists a constant $\epsilon \in (0, \epsilon_0]$ such that $(1+\epsilon)U_j(t - l_0\epsilon) \ge \overline{U}_j(t)$ for all $t \in \mathbb{R}$, $j \in \mathbb{Z}$, then $U_j(t) \ge \overline{U}_j(t)$ for all $t \in \mathbb{R}$, $j \in \mathbb{Z}$.*

Proof. To prove the lemma, it is equivalent to prove that if

$$(1+\epsilon)W_j(x + cl_0\epsilon) \ge \overline{W}_j(x) \quad \forall\ x \in \mathbb{R},\ j \in \mathbb{Z}, \tag{3.4}$$

for some $\epsilon \in (0, \epsilon_0]$, then $W_j(x) \ge \overline{W}_j(x)$ for all $x \in \mathbb{R}$, $j \in \mathbb{Z}$. For this, we define

$$w_j(q, x) := (1+q)W_j(x + cl_0 q) - \overline{W}_j(x), \quad q > 0,\ x \in \mathbb{R},$$
$$q^* := \inf\{q > 0 \mid w_j(q,x) \ge 0\ \forall\ x \in \mathbb{R}, j \in \mathbb{Z}\}.$$

By continuity, $w_j(q^*, x) \ge 0$ for all $x \in \mathbb{R}$, $j \in \mathbb{Z}$.

We claim that $q^* = 0$. For contradiction, we suppose that $q^* \in (0, \epsilon_0]$. Since, by the definition of l_0,

$$\frac{d}{dq} w_j(q^*, x) = W_j(x + cl_0 q^*) + cl_0(1 + q^*)W_j'(x + cl_0 q^*) < 0$$

for all x with $W_j(x + cl_0 q^*) \le 1 - \epsilon_0$ and $j \in \mathbb{Z}$, we can find $x_0 \in \mathbb{R}$ and $k \in \{1, \cdots, N\}$ with $W_k(y_0) > 1 - \epsilon_0$, $y_0 := x_0 + cl_0 q^*$, such that

$$w_k(q^*, x_0) = \frac{dw_k}{dx}(q^*, x_0) = 0,$$

i.e.,

$$(1 + q^*)W_k(y_0) = \overline{W}_k(x_0), \quad (1 + q^*)W_k'(y_0) = \overline{W}_k'(x_0).$$

Then, using (3.2), we have

$$\begin{aligned} 0 &= c\overline{W}_k'(x_0) + d_{k+1}\overline{W}_{k+1}(x_0 + 1) + d_k\overline{W}_{k-1}(x_0 - 1) \\ &\quad -(d_k + d_{k+1})\overline{W}_k(x_0) + f_k(\overline{W}_k(x_0)) \\ &\le (1 + q^*)\{cW_k'(y_0) + d_{k+1}W_{k+1}(y_0 + 1) + d_kW_{k-1}(y_0 - 1) \\ &\quad -(d_k + d_{k+1})W_k(y_0)\} + f_k([1 + q^*]W_k(y_0)) \\ &= -(1 + q^*)f_k(W_k(y_0)) + f_k([1 + q^*]W_k(y_0)) < 0, \end{aligned}$$

a contradiction. Hence $q^* = 0$ and so $W_j(x) \ge \overline{W}_j(x)$ for all $x \in \mathbb{R}$ and $j \in \mathbb{Z}$. □

In the sequel, we fix the constants ϵ_0, l_0 as above. Recall from the proof of Lemma 2.6 that there exists $M_0(\epsilon_0, l_0) > 0$ such that

$$(1 - q)U_j(t + 2l_0 q) \le U_j(t) \le (1 + q)U_j(t - 2l_0 q) \quad \forall\ j - ct \le -M_0, \tag{3.5}$$

for all $q \in (0, \epsilon_0]$.

Proof of Theorem 1.2. By (1.8), we have

$$\lim_{j-ct\to\infty} \frac{U_j(t+1)}{\overline{U}_j(t)} = \lim_{j-ct\to\infty} \left\{ \frac{U_j(t+1)}{e^{-\lambda(j-c(t+1))}v_j} \cdot \frac{e^{-\lambda(j-ct)}v_j}{\overline{U}_j(t)} \cdot e^{\lambda c} \right\} = e^{\lambda c} > 1.$$

Hence there exists x_1 such that $U_j(t+1) > \overline{U}_j(t)$ if $j - ct \ge x_1$. Since $\lim_{j-ct\to-\infty} U_j(t) = 1$, we can find $x_2 \gg 1$ such that

$$U_j(t) \ge 1/(1 + \epsilon_0) \quad \forall\ j - ct \le -x_2.$$

It follows that

$$\overline{U}_j(t) \le 1 \le (1 + \epsilon_0)U_j(t) \quad \forall\ j - ct \le -x_2.$$

Since

$$\eta := \max\{\overline{W}_j(x) \mid x \in [-x_2, x_1],\ j \in \mathbb{Z}\} \in (0, 1)$$

and $W_j(-\infty) = 1$, there exists $x_3 \gg 1$ such that

$$W_j(x) \ge \eta \quad \forall\ x \le -x_3,\ j \in \mathbb{Z}.$$

Set $\hat{t} := (x_1 + x_3)/c$. Then, for $x = j - ct \in [-x_2, x_1]$, we have

$$U_j(t + \hat{t}) = W_j(j - c(t + \hat{t})) = W_j(x - x_1 - x_3) \ge \eta \ge \overline{W}_j(x) = \overline{U}_j(t).$$

Choosing $T := 1 + \hat{t} + l_0\epsilon_0$ and using the monotonicity of wave profile, we conclude that

$$(1+\epsilon_0)U_j(t+T-l_0\epsilon_0) \geq \overline{U}_j(t) \quad \forall\, t \in \mathbb{R},\ j \in \mathbb{Z}.$$

It then follows from Lemma 3.1 that $U_j(t+T) \geq \overline{U}_j(t)$ for all $j \in \mathbb{Z}$ and $t \in \mathbb{R}$.

Now we set

$$\xi^* := \inf\{h > 0 | U_j(t+h) \geq \overline{U}_j(t) \quad \forall\, j \in \mathbb{Z},\ t \in \mathbb{R}\}.$$

Claim that $\xi^* = 0$. If not, then $\xi^* > 0$ and we have $U_j(t+\xi^*) \geq \overline{U}_j(t)$. By (1.8) again, we have

$$\lim_{j-ct\to\infty} \frac{U_j(t+\xi^*/2)}{\overline{U}_j(t)} = \lim_{j-ct\to\infty} \left\{ \frac{U_j(t+\xi^*/2)}{e^{-\lambda(j-c(t+\xi^*/2))}v_j} \cdot \frac{e^{-\lambda(j-ct)}v_j}{\overline{U}_j(t)} \cdot e^{\lambda c\xi^*/2} \right\} = e^{\lambda c \xi^*/2} > 1.$$

Hence there exists x_4 such that

$$U_j(t+\xi^*/2) \geq \overline{U}_j(t) \quad \forall\, j - ct \geq x_4. \tag{3.6}$$

Moreover from (3.5) for any $q \in (0, \epsilon_0]$,

$$(1+q)U_j(t+\xi^*-2l_0q) \geq U_j(t+\xi^*) \geq \overline{U}_j(t) \quad \forall\, j-ct \leq -M := -M_0 + c\xi^* \tag{3.7}$$

Note that $U_j(t+\xi^*) > \overline{U}_j(t)$ for $j - ct \geq x_4$, by (3.6) and the monotonicity of U. It follows from the strong comparison principle that $U_j(t+\xi^*) > \overline{U}_j(t)$ for all $(j,t) \in \mathbb{Z}\times\mathbb{R}$. Hence, by continuity, we can find $\epsilon \in (0, \min\{\epsilon_0, \xi^*/(4l_0)\})$ such that

$$U_j(t+\xi^*-2l_0\epsilon) \geq \overline{U}_j(t) \quad \forall\, j-ct \in [-M, x_4]. \tag{3.8}$$

Combining (3.6), (3.7) and (3.8), we have

$$(1+\epsilon)U_j(t+\xi^*-2l_0\epsilon) \geq \overline{U}_j(t)$$

for all $(j,t) \in \mathbb{Z}\times\mathbb{R}$. Using Lemma 3.1, we obtain that

$$U_j(t+\xi^*-l_0\epsilon) \geq \overline{U}_j(t)$$

for all $(j,t) \in \mathbb{Z}\times\mathbb{R}$. This contradicts the definition of ξ^*. Hence $\xi^* = 0$ and $U_j(t) \geq \overline{U}_j(t)$ for all $(j,t) \in \mathbb{Z}\times\mathbb{R}$.

Interchanging the role of U and $\overline{U}$, we obtain that $U_j(t) \leq \overline{U}_j(t)$ for all $(j,t) \in \mathbb{Z}\times\mathbb{R}$. Hence $U = \overline{U}$. The proof is completed. □

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