ENTIRE SOLUTIONS FOR A TWO-COMPONENT COMPETITION SYSTEM IN A LATTICE

JONG-SHENQ GUO AND CHANG-HONG WU

Abstract. We study entire solutions of a two-component competition system with Lotka-Volterra type nonlinearity in a lattice. It is known that this system has traveling wave front solutions and enjoys comparison principle. Based on these solutions, we construct some new entire solutions which behave as two traveling wave fronts moving towards each other from both sides of x-axis.

1. Introduction

We are concerned with the following Lotka-Volterra competition system in a one-dimensional lattice:

$$\begin{cases} \dfrac{du_j}{dt} = d_1(u_{j+1} + u_{j-1} - 2u_j) + r_1 u_j(1 - b_1 u_j - a_2 v_j), \\[2ex] \dfrac{dv_j}{dt} = d_2(v_{j+1} + v_{j-1} - 2v_j) + r_2 v_j(1 - b_2 v_j - a_1 u_j), \end{cases} \tag{1.1}$$

where $u_j = u_j(t)$, $v_j = v_j(t)$, $t \in \mathbb{R}$, $j \in \mathbb{Z}$, the parameters a_i, b_i, d_i and r_i are all positive numbers for $i = 1, 2$. This model is often used to describe the competing interaction of two species living in a discrete habitat. Here $u_j(t)$ and $v_j(t)$ stand for the populations of two species at time t and niches j, respectively. Thus we only consider that both $u_j(t)$ and $v_j(t)$ are nonnegative. The parameter a_i is the competition coefficient, $1/b_i$ is the carrying capacity, d_i is the diffusion coefficient and r_i is the birth rate of species i, $i = 1, 2$.

By a suitable rescaling,

$$d_1 t \to t, \quad b_1 u_j \to u_j, \quad b_2 v_j \to v_j,$$

and by letting

$$a = r_1/d_1, \quad b = r_2/d_1, \quad d = d_2/d_1, \quad k = a_2/b_2, \; h = a_1/b_1,$$

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the system (1.1) becomes the following system:

$$
\begin{cases} \dfrac{du_j}{dt} = (u_{j+1} + u_{j-1} - 2u_j) + au_j(1 - u_j - kv_j), \\ \\ \dfrac{dv_j}{dt} = d(v_{j+1} + v_{j-1} - 2v_j) + bv_j(1 - v_j - hu_j), \end{cases} \tag{1.2}
$$

where $t \in \mathbb{R}$, $j \in \mathbb{Z}$ and $a, b, d, h, k > 0$. Henceforth we shall consider the system (1.2) throughout this paper.

For solutions $(u_j, v_j) \equiv (u, v)$ for all $j \in \mathbb{Z}$ of (1.2), the system (1.2) is reduced to

$$
\frac{du}{dt} = au(1 - u - kv), \quad \frac{dv}{dt} = bv(1 - v - hu).
$$

Then by a phase plane analysis we have the following asymptotic behaviors as $t \to +\infty$:

(i) If $0 < k < 1 < h$, then $\lim_{t\to+\infty}(u, v)(t) = (1, 0)$ (the species u wins).
(ii) If $0 < h < 1 < k$, then $\lim_{t\to+\infty}(u, v)(t) = (0, 1)$ (the species v wins).
(iii) If $h, k > 1$, then $\lim_{t\to+\infty}(u, v)(t) = (0, 1)$ or $(1, 0)$ (depending on the initial data).
(iv) If $0 < h, k < 1$, then $\lim_{t\to+\infty}(u, v)(t) = ((1 - k)/(1 - hk), (1 - h)/(1 - hk))$ (two species coexist).

Note that the case (ii) can be reduced to the case (i) by exchanging the roles of u and v.

When $0 < k < 1 < h$, the species u is stronger than v, hence the species u invades v and eventually v will be extinct. It is interesting to know how the stronger species invades the weaker one. To understand the invading phenomenon between two species, the study of entire solutions is an important issue. Here an entire solution of (1.2) means a classical solution defined for all $(j, t) \in \mathbb{Z} \times \mathbb{R}$.

A solution $\{(u_j, v_j)\}$ of (1.2) is called a *traveling wave* (*front*) *solution* of (1.2) connecting $(0, 1)$ and $(1, 0)$ with speed c, if

$$
(u_j(t), v_j(t)) = (U(\xi), V(\xi)), \quad \xi := j + ct
$$

for some function (U, V) satisfying

$$
\begin{cases} cU'(\xi) = D_2[U(\xi)] + aU(\xi)[1 - U(\xi) - kV(\xi)], \ \xi \in \mathbb{R}, \\ \\ cV'(\xi) = dD_2[V(\xi)] + bV(\xi)[1 - V(\xi) - hU(\xi)], \ \xi \in \mathbb{R}, \\ \\ (U, V)(-\infty) = (0, 1), \ (U, V)(+\infty) = (1, 0), \\ \\ 0 \le U, V \le 1 \text{ on } \mathbb{R}, \end{cases} \tag{1.3}
$$

where $D_2[w(\xi)] := w(\xi + 1) + w(\xi - 1) - 2w(\xi)$ for $w = U, V$. The existence and uniqueness of traveling wave solution of (1.2) has been established in [6] for the case (i). Note that traveling wave solutions connecting $(0, 1)$ and $(1, 0)$ are entire solutions which provide the invading phenomenon. The purpose of this article is to establish the existence of *two-front* *entire solutions* of (1.2) which behave as two traveling fronts moving towards each other from both sides of space axis. This provides another invasion way of the stronger species to the weaker one.

In fact, the study of two-front entire solutions of reaction-diffusion equations can be traced back to the works of Hamel-Nadirashvili [7] and Yagisita [18] (also see [3], [5], [1], [14]). Among other things, these works established the existence of entire solutions with some combinations of two traveling wave solutions. Here, again, an entire solution means a classical solution defined for all $(x,t) \in \mathbb{R}^2$. Recently, Morita-Tachibana [15] extend the results of scalar equations to a competition system. More precisely, under some conditions, they prove that there are two-front entire solutions which behave as a traveling waves solution $(\phi(x+c_1t), \psi(x+c_1t))$ in the right x-axis and $(\phi(-x+c_2t), \psi(-x+c_2t))$ in the left x-axis when $t \to -\infty$ for the following competition system:

$$
\begin{cases} u_t = u_{xx} + u(1-u-kv),\ (x,t) \in \mathbb{R}^2, \\ v_t = dv_{xx} + bv(1-v-hu),\ (x,t) \in \mathbb{R}^2 \end{cases} \tag{1.4}
$$

for the cases (i) through (iii). For the study of traveling wave solutions to (1.4), we refer the reader to, e.g., [17, 4, 2, 16, 8, 9, 10, 11, 12, 13].

Motivated by the work of [15], it is very natural to expect that (1.2) also has two-front entire solutions based on the existence of traveling wave solutions. Therefore, we are looking for a solution $\{(u_j(t), v_j(t))\}$ which is defined for all $j \in \mathbb{Z}$ and $t \in \mathbb{R}$ and is a combination of two traveling wave front solutions of (1.2). For this, we embed the system (1.2) into a larger one:

$$
\begin{cases} u_t(x,t) = D_2[u(x,t)] + au(x,t)[1-u(x,t)-kv(x,t)], \\ v_t(x,t) = dD_2[v(x,t)] + bv(x,t)[1-v(x,t)-hu(x,t)], \end{cases} \tag{1.5}
$$

where $(x,t) \in \mathbb{R}^2$ and $D_2[w(x,t)] := w(x+1,t) + w(x-1,t) - 2w(x,t)$ for $w = u, v$. Note that the traveling wave front solution of (1.2) and (1.5) are identical. In this paper, we shall only focus on the case (i) and make the following assumption

(**A1**) $0 < k < 1 < h$, $a > 0$, $b > 0$ and $d > 0$.

In [15], the following assumption is crucial in constructing two-front entire solutions, namely, there is a positive number η_0 such that

$$
\frac{U(\xi)}{1-V(\xi)} \geq \eta_0 \quad \text{for all} \quad \xi \leq 0. \tag{1.6}
$$

Also, they provide some conditions via the eigenvalues of the linearized system around equilibria $(0,1)$ and $(1,0)$ to assure (1.6) holds. Fortunately, the condition (1.6) also holds for our lattice dynamical system (1.2). Indeed, it is proved in [6] that the limit

$$l := \lim_{\xi \to -\infty} U(\xi)/[1 - V(\xi)]$$

exists and is equal to either 0 or a positive number. Moreover, under the extra condition $0 < d \leq 1$, we can be sure that $l > 0$ (see [6, Remark 3.1]). From now on, we shall only consider the traveling wave solutions satisfying (1.6).

Since the comparison principle also holds for our competition system, we can apply the same argument as in [15] to construct two-front entire solutions by the help of a pair of super- and subsolution. We establish the following result.

Theorem 1. *Assume* (**A1**). *Let* (c_i, U_i, V_i) *be a solution of* (1.3) *satisfying* (1.6) *and let* θ_i *be a given constant,* $i = 1, 2$. *Then there exists an entire solution* $(u(x,t), v(x,t)) \in (0,1) \times (0,1)$ *of* (1.5) *such that*

$$\lim_{t \to -\infty} \sup_{x \geq (c_2 - c_1)t/2} \{|u(x,t) - U_1(x + c_1 t + \theta_1)| + |v(x,t) - V_1(x + c_1 t + \theta_1)|\} = 0, \tag{1.7}$$

$$\lim_{t \to -\infty} \sup_{x \leq (c_2 - c_1)t/2} \{|u(x,t) - U_2(-x + c_2 t + \theta_2)| + |v(x,t) - V_2(-x + c_2 t + \theta_2)|\} = 0, \tag{1.8}$$

$$\lim_{t \to +\infty} \sup_{x \in \mathbb{R}} \{|1 - u(x,t)| + |v(x,t)|\} = 0. \tag{1.9}$$

We organize this article as follows. In the next section, we recall some results from [6] on the asymptotic behaviors of traveling waves. With these asymptotic behaviors, the main theorem will be proven in Section 3.

2. Preliminaries

For convenience, we set $w(x,t) := 1 - v(x,t)$. Thus (1.5) becomes the following **(P)**:

$$u_t = D_2[u] + au[1 - u - k(1 - w)],$$
$$w_t = dD_2[w] + b(1 - w)(hu - w).$$

Then, by setting $W := 1 - V$, (1.3) is equivalent to

$$\begin{cases} cU' = D_2[U] + aU[1 - U - k(1 - W)], \\ cW' = dD_2[W] + b(1 - W)(hU - W), \\ (U, W)(-\infty) = (0,0), \quad (U, W)(+\infty) = (1,1), \\ 0 \leq U, W \leq 1. \end{cases} \tag{2.1}$$

In [6], we proved that there is a minimal speed $c_{\min} > 0$ such that (2.1) admits a solution (U, W) if and only if $c \geq c_{\min}$. Thus, both c_1 and c_2 in Theorem 1 are positive. Moreover, any wave profile is strictly monotone. That is, $U_i' > 0$ and $W_i' > 0$ on $\mathbb{R}$ for $i = 1, 2$. On the other hand, we also derive the asymptotic behavior of the traveling wave front of (2.1).

Define

$$\Phi_1(c, \lambda) := c\lambda - [(e^{\lambda} + e^{-\lambda} - 2) + a(1 - k)]. \tag{2.2}$$

It is easy to see that for each

$$c > c_* := \min_{\lambda > 0} \left\{ \frac{e^{\lambda} + e^{-\lambda} - 2 + a(1-k)}{\lambda} \right\} > 0,$$

the equation $\Phi_1(c, \lambda) = 0$ has exactly two real roots $\lambda_i(c)$, $i = 1, 2$, with $0 < \lambda_1(c) < \lambda_2(c)$; for $c = c_*$, $\Phi_1(c, \lambda) = 0$ has a unique real root $\lambda_* > 0$. Next, we also define

$$\Psi_1(c, \lambda) := c\lambda - d(e^{\lambda} + e^{-\lambda} - 2) - b(1 - h),$$
$$\Psi_2(c, \lambda) := c\lambda - (e^{\lambda} + e^{-\lambda} - 2) + a.$$

For any $c > 0$, $\Psi_1(c, \lambda) = 0$ has only one negative root, denoted by $\nu_1(c)$; $\Psi_2(c, \lambda) = 0$ also has only one negative root for any $c > 0$, denoted by $\nu_2(c)$.

Lemma 2.1. *Assume* **(A1)** *and let* (c, U, W) *be a solution of* (2.1) *satisfying* (1.6). *Then*

$$\lim_{\xi \to -\infty} \frac{W'(\xi)}{W(\xi)} = \Lambda(c) = \lim_{\xi \to -\infty} \frac{U'(\xi)}{U(\xi)}, \tag{2.3}$$

$$\lim_{\xi \to +\infty} \frac{W'(\xi)}{1 - W(\xi)} = -\nu_1(c), \tag{2.4}$$

$$\lim_{\xi \to +\infty} \frac{U'(\xi)}{1 - U(\xi)} = -\nu_0(c), \tag{2.5}$$

$$\frac{1 - W(\xi)}{1 - U(\xi)} \le K \text{ for all } \xi \in \mathbb{R} \text{ for some } K > 0, \tag{2.6}$$

where $\Lambda(c) \in \{\lambda_1(c), \lambda_2(c)\}$ *and* $\nu_0(c) \in \{\nu_1(c), \nu_2(c)\}$.

The proof of Lemma 2.1 can be found in [6, Lemmas 3.2 and 3.4 through 3.7]. As a consequence, we get the following estimates which we need in the proof of Theorem 1.

Lemma 2.2. *Assume* (**A1**) *and let* (c_i, U_i, W_i) *be a solution of* (2.1) *satisfying* (1.6), $i = 1, 2$. *Then there exist positive numbers* μ_1, m *and* M *such that*

$$0 < U_i(\xi) \le M e^{\mu_1 \xi}, \qquad \textit{for all } \xi \le 1, \tag{2.7}$$

$$0 < W_i(\xi) \le M e^{\mu_1 \xi}, \qquad \textit{for all } \xi \le 1, \tag{2.8}$$

$$m \le \frac{U_i'(\xi)}{U_i(\xi)} \le M, \qquad \textit{for all } \xi \le 1, \tag{2.9}$$

$$m \le \frac{W_i'(\xi)}{W_i(\xi)} \le M, \qquad \textit{for all } \xi \le 1, \tag{2.10}$$

$$m \le \frac{U_i'(\xi)}{1 - U_i(\xi)} \le M, \qquad \textit{for all } \xi \ge -1, \tag{2.11}$$

$$m \le \frac{W_i'(\xi)}{1 - W_i(\xi)} \le M, \qquad \textit{for all } \xi \ge -1. \tag{2.12}$$

Proof. (2.7) through (2.10) follow from (2.3). By (2.4) and (2.5), we obtain (2.11) and (2.12). □

Note that (2.11) and $(U_i, U_i')(-\infty) = (0, 0)$ imply that there exists $\eta > 0$ such that

$$\frac{1 - U_i(\xi + s)}{1 - U_i(\xi)} \le \eta, \ i = 1, 2 \tag{2.13}$$

for all $\xi \in \mathbb{R}$ and $s \in [-1, 1]$, since we have

$$\frac{1 - U(\xi + s)}{1 - U(\xi)} = \exp\left\{ - \int_{\xi}^{\xi + s} \frac{U'(x)}{1 - U(x)} dx \right\}, \ i = 1, 2.$$

Similarly, there exists $\gamma > 0$ such that

$$\frac{U_i(\xi + s)}{U_i(\xi)} \le \gamma, \ i = 1, 2 \tag{2.14}$$

for all $\xi \in \mathbb{R}$ and $s \in [-1, 1]$. Note that, in (2.13) and (2.14), U_i can be replaced by W_i, $i = 1, 2$.

3. Proof of Theorem 1

The following three lemmas are key steps in the proof of Theorem 1.

Lemma 3.1. *Let* (c_i, U_i, W_i) *be a solution of* (2.1) *satisfying* (1.6), $i = 1, 2$, *and define*

$$A(y, p) := U_1(y + p) W_2(-y + p)[1 - U_2(-y + p)][1 - W_1(y + p)],$$
$$B(y, p) := U_2(-y + p) W_1(-y + p)[1 - U_1(y + p)][1 - W_2(-y + p)],$$
$$C(y, p) := U_1'(y + p)[1 - U_2(-y + p)] + U_2'(-y + p)[1 - U_1(y + p)].$$

Then there exists $N > 0$ such that, for any given $p < 0$,

$$A(y,p) \leq Ne^{\mu_1 p} C(y,p) \quad \text{for every} \quad y \in \mathbb{R}, \tag{3.1}$$

$$B(y,p) \leq Ne^{\mu_1 p} C(y,p) \quad \text{for every} \quad y \in \mathbb{R}. \tag{3.2}$$

Proof. Since the proofs of (3.1) and (3.2) are similar, we only show (3.1). We divide $\mathbb{R}$ into four intervals, $(-\infty, p]$, $[p, 0]$, $[0, -p]$ and $[-p, +\infty)$. For $y \in (-\infty, p]$, by using (2.6), (2.11) and (2.7), we obtain

$$\begin{aligned}
\frac{A(y,p)}{C(y,p)} &\leq \frac{A(y,p)}{U_2'(-y+p)[1-U_1(y+p)]} \\
&\leq \frac{1-W_1(y+p)}{1-U_1(y+p)} \frac{1-U_2(-y+p)}{U_2'(-y+p)} W_2(-y+p) U_1(y+p) \\
&\leq N_1 U_1(y+p) \leq N_2 e^{\mu_1 p}
\end{aligned}$$

for some N_1, $N_2 > 0$.

For $y \in [p, 0]$, by (2.6), (1.6), (2.9) and (2.7), we have

$$\begin{aligned}
\frac{A(y,p)}{C(y,p)} &\leq \frac{A(y,p)}{U_2'(-y+p)[1-U_1(y+p)]} \\
&= \frac{1-W_1(y+p)}{1-U_1(y+p)} \frac{W_2(-y+p)}{U_2(-y+p)} \frac{U_2(-y+p)}{U_2'(-y+p)} [1-U_2(-y+p)] U_1(y+p) \\
&\leq N_3 U_1(y+p) \leq N_4 e^{\mu_1 p}
\end{aligned}$$

for some N_3, $N_4 > 0$.

For $y \in [0, -p]$, using (2.9) and (2.8), we obtain

$$\frac{A(y,p)}{C(y,p)} \leq \frac{A(y,p)}{U_1'(y+p)[1-U_2(-y+p)]} \leq N_5 W_2(-y+p) \leq N_6 e^{\mu_1 p}$$

for some N_5, $N_6 > 0$.

For $y \in [-p, +\infty)$, by (2.6), (2.11) and (2.8), we have

$$\begin{aligned}
\frac{A(y,p)}{C(y,p)} &\leq \frac{A(y,p)}{U_1'(y+p)[1-U_2(-y+p)]} \\
&= \frac{1-W_1(y+p)}{1-U_1(y+p)} \frac{1-U_1(y+p)}{U_1'(y+p)} U_1(y+p) W_2(-y+p) \\
&\leq N_7 W_2(-y+p) \leq N_8 e^{\mu_1 p}
\end{aligned}$$

for some N_7, $N_8 > 0$. Then (3.1) follows by taking $N = \max\{N_2, N_4, N_6, N_8\}$ and the lemma follows. □

Lemma 3.2. *Let (c_i, U_i, W_i) be a solution of* (2.1) *satisfying* (1.6), $i = 1, 2$. *We define*

$$\begin{aligned}
D(y,p) &:= [U_1(y+1+p) - U_1(y+p)][U_2(-y+p) - U_2(-y-1+p)], \\
E(y,p) &:= [U_1(y+p) - U_1(y-1+p)][U_2(-y+1+p) - U_2(-y+p)]
\end{aligned}$$

while we define $C(y,p)$ as in Lemma 3.1. Then there exists $N_0>0$ such that, for any given $p<0$, we have

$$D(y,p)\le N_0e^{\mu_1 p}C(y,p) \ \text{ for every } \ y\in\mathbb{R}, \tag{3.3}$$

$$E(y,p)\le N_0e^{\mu_1 p}C(y,p) \ \text{ for every } \ y\in\mathbb{R}. \tag{3.4}$$

Proof. Since the proofs (3.3) and (3.4) are similar, we only prove (3.3). For $y\ge -p$, there are $\eta_1(y),\eta_2(y)\in(0,1)$ and $L_1>0$ such that

$$\begin{aligned}
&\frac{D(y,p)}{C(y,p)}=\frac{U_1'(y+\eta_1+p)U_2'(-y-\eta_2+p)}{C(y,p)}\le\frac{U_1'(y+\eta_1+p)}{U_1'(y+p)}\frac{U_2'(-y-\eta_2+p)}{1-U_2(-y+p)}\\
\le\ &\left\{\frac{U_1'(y+\eta_1+p)}{1-U_1(y+\eta_1+p)}\frac{1-U_1(y+\eta_1+p)}{1-U_1(y+p)}\frac{1-U_1(y+p)}{U_1'(y+p)}\right\}\frac{U_2'(-y-\eta_2+p)}{1-U_2(0)}\\
\le\ &L_1U_2'(-y-\eta_2+p),
\end{aligned}$$

where the last inequality follows from (2.11) and (2.13). It then follows from (2.7) and (2.9) that

$$\frac{D(y,p)}{C(y,p)}\le L_1U_2'(-y-\eta_2+p)\le L_1L_2Me^{\mu_1 p}.$$

For $y\in[0,-p]$, there exists $L_3>0$ such that

$$\begin{aligned}
&\frac{D(y,p)}{C(y,p)}=\frac{U_1'(y+\eta_1+p)U_2'(-y-\eta_2+p)}{C(y,p)}\\
\le\ &\left\{\frac{U_1'(y+\eta_1+p)}{U_1(y+\eta_1+p)}\frac{U_1(y+\eta_1+p)}{U_1(y+p)}\frac{U_1(y+p)}{U_1'(y+p)}\right\}\frac{U_2'(-y-\eta_2+p)}{1-U_2(0)}\\
\le\ &L_3U_2'(-y-\eta_2+p),
\end{aligned}$$

where the last inequality follows from (2.9) and (2.14). Again, by (2.7) and (2.9), we obtain $D(y,p)/C(y,p)\le L_4Me^{\mu p}$ for some $L_4>0$. Thus, (3.3) holds for all $y\ge 0$.

For $y\le 0$, we divide it to the cases $y\in[p,0]$ and $y\in(-\infty,p]$. By using the same argument, (3.3) also holds for all $y\le 0$. Thus, we complete the proof of this lemma. □

Lemma 3.3. *Let (c_i,U_i,W_i) be a solution of (2.1) satisfying (1.6), $i=1,2$. Then there exists $N_1>0$ such that, for any given $p<0$, we have*

$$F(y,p),\ G(y,p),\ H(y,p)\le N_1e^{\mu_1 p}I(y,p) \ \text{ for every } \ y\in\mathbb{R},$$

where

$$\begin{aligned}
F(y,p)&:=[W_1(y+1+p)-W_1(y+p)][W_2(-y+p)-W_2(-y-1+p)],\\
G(y,p)&:=[W_1(y+p)-W_1(y-1+p)][W_2(-y+1+p)-W_2(-y+p)],\\
H(y,p)&:=W_1(y+p)W_2(-y+p)[1-W_1(y+p)][1-W_2(-y+p)],\\
I(y,p)&:=W_1'(y+p)[1-W_2(-y+p)]+W_2'(-y+p)[1-W_1(y+p)].
\end{aligned}$$

Proof. This lemma is proved by using Lemma 2.2. Since the proof is similar to those of Lemmas 3.1 and 3.2, we omit the details. $\square$

By the transformation $y = x + (c_1 - c_2)t/2$, we define $(u(y,t), w(y,t)) := (u(x,t), w(x,t))$. Then **(P)** becomes **(Q)**:

$$u_t + \left(\frac{c_1 - c_2}{2}\right) u_y = D_2[u] + f(u,w),\ (y,t) \in \mathbb{R}^2,$$
$$w_t + \left(\frac{c_1 - c_2}{2}\right) w_y = dD_2[w] + g(u,w),\ (y,t) \in \mathbb{R}^2,$$

where $f(u,w) := au[1 - u - k(1-w)]$ and $g(u,w) := b(1-w)(hu - w)$.

We call (u^-, w^-) a *subsolution* of **(Q)** for $(y,t) \in \mathbb{R} \times [T_1, T_2]$ if $\mathcal{F}_1(u^-, w^-) \le 0$ and $\mathcal{F}_2(u^-, w^-) \le 0$ for all $(y,t) \in \mathbb{R} \times [T_1, T_2]$, where

$$\mathcal{F}_1(u,w) := u_t + \left(\frac{c_1 - c_2}{2}\right) u_y - D_2[u] - f(u,w),$$
$$\mathcal{F}_2(u,w) := w_t + \left(\frac{c_1 - c_2}{2}\right) w_y - dD_2[w] - g(u,w).$$

Similarly, a *supersolution* (u^+, w^+) is defined by reversing the above inequalities.

Next, we introduce the following initial value problem:

$$p'(t) = \left(\frac{c_1 + c_2}{2}\right) + Le^{\mu_1 p(t)},\ t \le 0,$$
$$p(0) = p_0 < 0,$$

where $\mu_1 > 0$ is defined in Lemma 2.2 and $L > 0$ is to be determined. Then the solution can be easily obtained as

$$p(t) = p_0 + \left(\frac{c_1 + c_2}{2}\right) t - \frac{1}{\mu_1} \ln \left\{ 1 + \frac{2L}{(c_1 + c_2)} e^{\mu_1 p_0} (1 - e^{(c_1+c_2)\mu_1 t/2}) \right\} < 0,\ t \le 0.$$

Note that

$$\lim_{t \to -\infty} \left\{ p(t) - \left(\frac{c_1 + c_2}{2}\right) t \right\} = -\frac{1}{\mu_1} \ln \left\{ e^{-\mu_1 p_0} + \frac{2L}{c_1 + c_2} \right\} < 0.$$

The following equalities are useful in the subsequent estimates:

$$(3.5)\begin{cases} f(u_1 + u_2 - u_1 u_2, w_1 + w_2 - w_1 w_2) - (1-u_2) f(u_1, w_1) - (1-u_1) f(u_2, w_2) \\ = a(u_1 + u_2 - u_1 u_2)[(1-u_1)(1-u_2) - k(1-w_1)(1-w_2)] \\ \quad -au_1(1-u_2)[1 - u_1 - k(1-w_1)] - au_2(1-u_1)[1 - u_2 - k(1-w_2)] \\ = a\{-u_1 u_2 (1-u_2)(1-u_2) - k(u_1 + u_2 - u_1 u_2)(1-w_1)(1-w_2) \\ \quad +ku_1(1-u_2)(1-w_1) + ku_2(1-u_1)(1-w_2)\}, \end{cases}$$

$$
\begin{aligned}
(3.6)\qquad & g(u_1+u_2-u_1u_2, w_1+w_2-w_1w_2)-(1-w_2)g(u_1,w_1)-(1-w_1)g(u_2,w_2)\\
&= bw_1w_2(1-w_1)(1-w_2)-bhu_1u_2(1-w_1)(1-w_2).
\end{aligned}
$$

Proof of Theorem 1. Without loss of generality, it suffices to consider the case when $\theta_1=\theta_2=\delta$, where

$$
\delta := -\frac{1}{\mu_1}\ln\left\{e^{-\mu_1 p_0}+\frac{2L}{c_1+c_2}\right\}<0, \tag{3.7}
$$

$\mu_1>0$ is defined in Lemma 2.2 and $L>0$ is to be determined. Indeed, the case for general θ_1 and θ_2 can be reduced by a suitable space and time shift to the case $\theta_1=\theta_2=\delta$. The detail can be seen in [5].

We now claim that (u^+,w^+) defined by

$$
\begin{aligned}
u^+(y,t) &:= U_1(y+p(t))+U_2(-y+p(t))-U_1(y+p(t))U_2(-y+p(t)),\\
w^+(y,t) &:= W_1(y+p(t))+W_2(-y+p(t))-W_1(y+p(t))W_2(-y+p(t))
\end{aligned}
$$

is a supersolution of **(Q)** for $y\in\mathbb{R}$ and $t\le 0$, where (U_i,W_i) solves (2.1), $i=1,2$.

Note that

$$
\begin{aligned}
&\mathcal{F}_1(u^+,w^+)\\
=\;& p'(t)[(1-U_2)U_1'+(1-U_1)U_2']+\left(\frac{c_1-c_2}{2}\right)[(1-U_2)U_1'-(1-U_1)U_2']\\
&-D_2[U_1]-D_2[U_2]+D_2[U_1U_2]-f(U_1+U_2-U_1U_2,W_1+W_2-W_1W_2)\\
=\;& \left\{p'(t)-\left(\frac{c_1+c_2}{2}\right)\right\}[(1-U_2)U_1'+(1-U_1)U_2']\\
&+[U_1(y+1+p(t))-U_1(y+p(t))][U_2(-y-1+p(t))-U_2(-y+p(t))]\\
&+[U_1(y+p(t))-U_1(y-1+p(t))][U_2(-y+p(t))-U_2(-y+1+p(t))]\\
&-f(U_1+U_2-U_1U_2,W_1+W_2-W_1W_2)\\
&+(1-U_2)f(U_1,W_1)+(1-U_1)f(U_2,W_2).
\end{aligned}
$$

We now estimate the last three terms of the above equality. From (3.5),

$$
\begin{aligned}
&f(U_1+U_2-U_1U_2,W_1+W_2-W_1W_2)-(1-U_2)f(U_1,W_1)-(1-U_1)f(U_2,W_2)\\
\le\;& ak[U_1(1-U_2)(1-W_1)+U_2(1-U_1)(1-W_2)-(U_1+U_2-U_1U_2)(1-W_1)(1-W_2)]\\
=\;& ak[U_1W_2(1-W_1)(1-U_2)+U_2W_1(1-U_1)(1-W_2)-U_1U_2(1-W_1)(1-W_2)]\\
\le\;& ak[U_1W_2(1-W_1)(1-U_2)+U_2W_1(1-U_1)(1-W_2)].
\end{aligned}
$$

It follows that

$$
\begin{aligned}
& \mathcal{F}_1(u^+, w^+) \\
\geq\ & Le^{\mu_1 p(t)}[(1-U_2)U_1' + (1-U_1)U_2'] - D(y,p(t)) - E(y,p(t)) \\
& -ak[U_1W_2(1-W_1)(1-U_2) + U_2W_1(1-U_1)(1-W_2)] \\
=\ & C(y,p(t))\left\{Le^{\mu_1 p(t)} - \frac{D(y,p(t))+E(y,p(t))}{C(y,p(t))} - ak\frac{A(y,p(t))+B(y,p(t))}{C(y,p(t))}\right\},
\end{aligned}
$$

where A, B, C, D and E are defined as in Lemmas 3.1 and 3.2. Therefore, by Lemmas 3.1 and 3.2, there exist $N>0$ such that

$$\mathcal{F}_1(u^+, w^+) \geq C(y,p(t))\{Le^{\mu_1 p(t)} - 2Ne^{\mu_1 p(t)} - 2akNe^{\mu_1 p(t)}\}$$

for all $y \in \mathbb{R}$ and $t \leq 0$. Therefore, by choosing $L \geq 2N + 2akN$, we obtain $\mathcal{F}_1(u^+, w^+) \geq 0$ for all $y \in \mathbb{R}$ and $t \leq 0$. Next, by Lemma 3.3, (3.6) and by choosing $L \gg 1$, we can derive that $\mathcal{F}_2(u^+, w^+) \geq 0$ for $y \in \mathbb{R}$ and $t \leq 0$ by the same argument as above. Hence (u^+, w^+) is a supersolution of **(Q)** for a fixed large $L > 0$.

Similarly, the pair (u^-, w^-) defined by

$$
\begin{aligned}
u^-(y,t) &:= \max\{U_1(y + \frac{c_1+c_2}{2}t + \delta), U_2(-y + \frac{c_1+c_2}{2}t + \delta)\}, \\
w^-(y,t) &:= \max\{W_1(y + \frac{c_1+c_2}{2}t + \delta), W_2(-y + \frac{c_1+c_2}{2}t + \delta)\}
\end{aligned}
$$

is a subsolution of **(Q)**, where δ is defined in (3.7) and L is fixed as in the supersolution. Note that $u^-(y,t) \leq u^+(y,t)$ and $w^-(y,t) \leq w^+(y,t)$ for all $y \in \mathbb{R}$ and $t \leq 0$. Moreover, we have

$$\lim_{t\to-\infty}\sup_{y\in\mathbb{R}}[u^+(y,t) - u^-(y,t)] = 0 = \lim_{t\to-\infty}\sup_{y\in\mathbb{R}}[w^+(y,t) - w^-(y,t)].$$

Since our system enjoys the comparison principle, we can apply the method in [15] to find a solution $(u(y,t), w(y,t))$ such that $u^- \leq u \leq u^+$ and $w^- \leq w \leq w^+$ for all $y \in \mathbb{R}$ and $t \leq 0$. Then the asymptotic behaviors (1.7) and (1.8) hold, since $(u(y,t), w(y,t))$ is still a solution after time shift. Finally, note that the subsolution (u^-, w^-) is defined for all $t \in \mathbb{R}$ and

$$\lim_{t\to+\infty}\sup_{y\in\mathbb{R}}[1 - u^-(y,t)] = 0 = \lim_{t\to+\infty}\sup_{y\in\mathbb{R}}[1 - w^-(y,t)].$$

Therefore, since (u, w) can be extended to all $t > 0$, we can derive (1.9) and the proof of Theorem 1 is completed. □

References

[1] X. Chen and J.-S. Guo, *Existence and uniqueness of entire solutions for a reaction-diffusion equation*, J. Differential Equations **212** (2005), 62–84.

[2] C. Conley and R. Gardner, *An application of the generalized Morse index to traveling wave solutions of a competitive reaction diffusion model*, Indiana Univ. Math. J. **33** (1984) 319–343.

[3] Y. Fukao, Y. Morita and H. Ninomiya, *Some entire solutions of the Allen-Cahn equation*, Taiwanese J. Math. **8** (2004), 15–32.
[4] R. A. Gardner, *Existence and stability of traveling wave solutions of competition models: a degree theoretic*, J. Differential Equations **44** (1982), 343–364.
[5] J.-S. Guo and Y. Morita, *Entire solutions of reaction-diffusion equations and an application to discrete diffusive equations*, Discrete Contin. Dyn. Syst. **12** (2005), 193–212.
[6] J.-S. Guo and C.-H. Wu, *Traveling wave front for a two-component lattice dynamical system arising in competition models*, preprint.
[7] F. Hamel and N. Nadirashvili, *Entire solutions of the KPP equation*, Comm. Pure Appl. Math. **52** (1999), 1255–1276.
[8] Y. Kan-on, *Parameter dependence of propagation speed of travelling waves for competition-diffusion equations*, SIAM J. Appl. Math. **26** (1995), 340–363.
[9] Y. Kan-on, *Existence of standing waves for competition-diffusion equations*, Japan J. Indust. Appl. Math. **13** (1996), 117–133.
[10] Y. Kan-on, *Fisher wave fronts for the Lotka-Volterra competition model with diffusion*, Nonlinear Anal. **28** (1997), 145–164.
[11] Y. Kan-on, *Instability of stationary solutions for a Lotka-Volterra competition model with diffusion*, J. Math. Anal. Appl. **208** (1997), 158–170.
[12] Y. Kan-on and Q. Fang, *Stability of monotone travelling waves for competition-diffusion equations*, J. Indust. Appl. Math. **13** (1996), 343–349.
[13] Y. Kan-on and E. Yanagida, *Existence of nonconstant stable equilibria in competition-diffusion equations*, Hiroshima Math. J. **23** (1993), 193–221.
[14] Y. Morita and H. Ninomiya, *Entire solutions with merging fronts to reaction-diffusion equations*, J. Dynam. Differential Equations **18** (2006), 841–861.
[15] Y. Morita and K. Tachibana, *An entire solution to the Lotka-Volterra competition-diffusion equations*, SIAM J. Math. Anal, **40** (2009), 2217–2240.
[16] A. Okubo, P. K. Maini, M. H. Williamson and J. D. Murray, *On the spatial spread of the grey squirrel in Britain*, Proc. R. Soc. Lond. B **238** (1989), 113–125.
[17] M. M. Tang and P. C. Fife, *Propagating fronts for competing species equations with diffusion*, Arch. Rational Mech. Anal. **73** (1980) 69–77.
[18] H. Yagisita, *Backward global solutions characterizing annihilation dynamics of travelling fronts*, Publ. Res. Inst. Math. Sci. **39** (2003), 117–164.

Department of Mathematics, National Taiwan Normal University, 88, S-4, Ting Chou Road, Taipei 11677, Taiwan; and Taida Institute of Mathematical Sciences, National Taiwan University, 1, S-4, Roosevelt Road, Taipei 10617 Taiwan
E-mail address: `jsguo@ntnu.edu.tw`

Department of Mathematics, National Taiwan Normal University, 88, S-4, Ting Chou Road, Taipei 11677, Taiwan
E-mail address: `896400012@ntnu.edu.tw`