FRONT PROPAGATION FOR A TWO-DIMENSIONAL PERIODIC MONOSTABLE LATTICE DYNAMICAL SYSTEM

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ABSTRACT. We study the traveling wave front solutions for a two-dimensional periodic lattice dynamical system with monostable nonlinearity. We first show that there is a minimal speed such that a traveling wave solution exists if and only if its speed is above this minimal speed. Then we prove that any wave profile is strictly monotone. Finally, we derive the convergence of discretized minimal speed to the continuous minimal speed.

1. INTRODUCTION

Many mathematical models, such as chemical kinetic and biological invasions, are often described by reaction-diffusion equations (see, e.g., [12]). A typical example is

$$u_t = \nabla \cdot (A(x)\nabla u) + f(x,u),\ x \in \mathbb{R}^n, t > 0. \tag{1.1}$$

In this paper, we are mainly concerned with the wave propagation in periodic media, i.e., the case when the diffusion matrix A and the reaction term f are periodic in x. The study of wave propagation in reaction-diffusion equations in periodic media can be traced back to the work of Gärtner and Freidlin [18] in 1979. See also the papers by Freidlin [14], Shigesada, Kawasaki and Teramoto [24], Hudson and Zinner [21], Berestycki, Hamel and Roques [5, 6] and the references cited therein. For reaction-diffusion-convection equations in quite general domains with KPP type nonlinearity ([22]), we refer the reader to, e.g., [3, 4].

Recently, in [19], the authors study the traveling waves for one dimensional spatial discrete version of (1.1) in periodic media. Among other things, they proved that a traveling front solution exists if and only if the wave speed is above a positive minimal speed. In this paper, we shall extend the work [19] in one dimensional case to the two dimensional spatial discrete version of (1.1) in periodic media in which the diffusion matrix is assumed to be

$$A(x) = \begin{bmatrix} p(x) & 0 \\ 0 & q(x) \end{bmatrix},\ x \in \mathbb{R}^2.$$

Date: June 29, 2009. Corresponding Author: J.-S. Guo.

This work was partially supported by the National Science Council of the Republic of China under the grant NSC 96-2119-M-003-001. The authors would like to thank the referees for many helpful comments.

2000 Mathematics Subject Classification. Primary: 34K05, 34A34; Secondary: 34K60, 34E05.

Key words and phrases: Lattice dynamical system, monostable, traveling wave, wave speed, wave profile.

More precisely, we shall study the following problem (P) for a two-dimensional lattice dynamical system:

$$u'_{i,j}(t) = p_{i+1,j}u_{i+1,j}(t) + p_{i,j}u_{i-1,j}(t) + q_{i,j+1}u_{i,j+1}(t) + q_{i,j}u_{i,j-1}(t) \tag{1.2}$$
$$-D_{i,j}u_{i,j}(t) + f(i,j,u_{i,j}(t)), \quad t \in \mathbb{R}, \ (i,j) \in \mathbb{Z}^2,$$

$$u_{i+N,j}(t + \frac{Nr}{c}) = u_{i,j}(t) = u_{i,j+N}(t + \frac{Ns}{c}), \quad t \in \mathbb{R}, (i,j) \in \mathbb{Z}^2, \ c \neq 0, \tag{1.3}$$

$$\lim_{ri+sj \to -\infty} u_{i,j}(t) = 1, \quad \lim_{ri+sj \to +\infty} u_{i,j}(t) = 0, \quad t \in \mathbb{R}, \tag{1.4}$$

$$0 \leq u_{i,j}(t) \leq 1, \quad t \in \mathbb{R}, (i,j) \in \mathbb{Z}^2, \tag{1.5}$$

where

$$D_{i,j} := (p_{i+1,j} + p_{i,j} + q_{i,j+1} + q_{i,j}), \quad (i,j) \in \mathbb{Z}^2,$$
$$p_{i+N,j} = p_{i,j} = p_{i,j+N}, \quad q_{i+N,j} = q_{i,j} = q_{i,j+N}, \quad (i,j) \in \mathbb{Z}^2,$$
$$f(i+N,j,s) = f(i,j,s) = f(i,j+N,s), \quad (i,j) \in \mathbb{Z}^2, s \in [0,1]$$

for some positive integer N. Here c is the unknown wave speed and $(r,s) := (\cos\theta, \sin\theta)$ with $\theta \in [0, 2\pi)$ represents the direction of movement of wave. A solution of (P) is called a traveling wave in the direction θ and $u(\cdot) = \{u_{i,j}(\cdot)\}$ is called the wave profile.

We shall make the following further assumptions.

- (**A1**) The coefficients $p_{i,j}$ and $q_{i,j}$ are bounded from above and below by two positive constants for all $(i,j) \in \mathbb{Z}^2$.
- (**A2**) $f(i,j,0) = f(i,j,1) = 0 < f(i,j,s)$ for all $(i,j,s) \in \mathbb{Z}^2 \times (0,1)$.
- (**A3**) $f(i,j,s) \leq f'_s(i,j,0)s$ for all $(i,j,s) \in \mathbb{Z}^2 \times [0,1]$.
- (**A4**) There exists $\alpha > 0$ and $\beta \geq 0$ such that $f(i,j,s) \geq f'_s(i,j,0)s - \beta s^{1+\alpha}$ for all $(i,j,s) \in \mathbb{Z}^2 \times [0,1]$.
- **(A5)** There exists $\rho \in (0,1)$ such that $f(i,j,s_2) \leq f(i,j,s_1)$, if $\rho < s_1 < s_2 < 1$, $\forall (i,j)$.

Hereafter $f'_s(i,j,s) := (\partial f/\partial s)(i,j,s)$. Note that, by (**A3**), $f'_s(i,j,0) > 0$ for all $(i,j) \in \mathbb{Z}^2$. Also, the assumption **(A5)** is valid if we have $f'_s(i,j,1) < 0$ for all $(i,j) \in \mathbb{Z}^2$.

Although the equation (1.2) is a spatial discrete version of (1.1) in two space dimension, it can also arise directly in many biological models (cf., e.g., [25]). For related works to (1.2) in homogeneous media with monostable or bistable nonlinearity for one dimensional lattice dynamical system, we refer the reader to ([8],[9],[10],[11],[15],[16],[20],[28],[29]) and the references cited therein. The two dimensional lattice dynamical system was treated in [17] for the homogeneous media. In this paper, we extend the work [17] to the periodic media.

To state our main results, we first introduce the linear operator $\mathcal{L}_\lambda : \mathbb{R}^{N^2} \to \mathbb{R}^{N^2}$ by

$$(1.6)\qquad \begin{aligned}(\mathcal{L}_\lambda v)_{i,j} \;:=\; & p_{i+1,j}e^{-r\lambda}v_{i+1,j} + p_{i,j}e^{r\lambda}v_{i-1,j} + q_{i,j+1}e^{-s\lambda}v_{i,j+1} + q_{i,j}e^{s\lambda}v_{i,j-1} \\ & -D_{i,j}v_{i,j} + f_s'(i,j,0)v_{i,j}, \quad i,j = 1,2,\cdots,N,\end{aligned}$$

where $v := (v_{1,1}, v_{1,2}, \cdots, v_{N,N}) \in \mathbb{R}^{N^2}$ with $v_{0,j} := v_{N,j}$, $v_{i,0} := v_{i,N}$, $v_{N+1,j} := v_{1,j}$ and $v_{i,N+1} := v_{i,1}$ for $i,j = 1,\cdots,N$. We shall show that the largest real eigenvalue of the operator $\mathcal{L}_\lambda$ exists, which we denote it by $M(\lambda)$. Moreover, the constant

$$c_* := \min_{\lambda>0} \frac{M(\lambda)}{\lambda}$$

exists and is positive. Indeed, this constant c_* is the minimum speed as shown in the following theorems.

We now state our main results as follows.

Theorem 1. *Assume* **(A1)-(A4)**. *For each $c \geq c_*$, the problem (P) admits a solution.*

Theorem 2. *Assume* **(A1)-(A4)**. *If (P) has a solution with $c \neq 0$, then $c \geq c_*$.*

Theorem 3. *Assume* **(A1)-(A5)**. *Let $u := \{u_{i,j}\}$ be a solution of (P) with $c \neq 0$. Then the wave profile $u(\cdot)$ is strictly increasing in t.*

Although some of the proofs of Theorems 1-3 are similar to the work [19] for the one dimensional lattice dynamical system, there are certain different ideas in this paper from those in [19]. For example, for the existence of traveling wave solutions, we use a different approach from the one used in [19]. For a solution (c,u) of (P) with $c > 0$, we introduce the following transformation

$$(1.7)\qquad w_{i,j}(\xi) := u_{i,j}(t), \quad \xi := ct - ri - sj.$$

Then we apply the monotone iteration method (cf. [1, 27, 9]) to the new system of equations satisfied by $w_{i,j}$ to derive the existence of traveling waves. The super-sub-solutions constructed in [19] are useful in applying this method. It turns out that this approach is much simpler than the method used in [19]. Indeed, the transformation (1.7) is reminiscent of the so-called moving coordinates (cf. [13]). For the proof of the monotonicity of wave profile, the transformation (1.7) is also proved to be very useful. By using w variable, the proof of monotonicity becomes more transparent.

It is also interesting to see the dependence of the direction θ for the minimum speed. For the continuous version, the authors in [3] announced that the minimum speed depends on θ for reaction-diffusion-advection equation in the periodic framework. But, for the homogeneous case (with KPP assumption) the minimum speed of planar waves for reaction-diffusion equation is independent of θ. In the discrete version we found that, even in the homogeneous case, the minimum speed depends on θ. To see this, we recall the minimum speed for the homogeneous case with $(r,s)=(\cos\theta,\sin\theta)$ (cf. [17]):

$$c_*(\theta)=\min_{\lambda>0}\{\frac{e^{-\lambda r}+e^{\lambda r}+e^{-\lambda s}+e^{-\lambda s}-4+f'(0)}{\lambda}\}.$$

Take, for example, $\theta=0,\pi/4$. Then it is easy to check that $c_*(0)>c_*(\pi/4)$. Therefore, the minimum speed depends on the direction θ. Indeed, this phenomenon was also observed before in [7] for the discrete bistable case.

Finally, from the numerical point of view, it is very important to see whether the discretized minimum speeds converge to the continuous minimum speed as the mesh size tends to zero. The answer to this question for 1D periodic case is positive (cf. [19]). Here we shall extend this result to the 2D case.

For this, we assume the following.

(1) p,q and f are periodic with period $L>0$, i.e.,

$$\begin{aligned}
p(x_1+L,x_2)&=p(x_1,x_2)=p(x_1,x_2+L),\\
q(x_1+L,x_2)&=q(x_1,x_2)=q(x_1,x_2+L),\\
f(x_1+L,x_2,s)&=f(x_1,x_2,s)=f(x_1,x_2+L,s).
\end{aligned}$$

(2) $p,q\in C^{1,\delta}(\mathbb{R}^2)$ for some $\delta>0$ and

$$0<\inf_{\mathbb{R}^2}p\le\sup_{\mathbb{R}^2}p<+\infty,\ 0<\inf_{\mathbb{R}^2}q\le\sup_{\mathbb{R}^2}q<+\infty.$$

(3) the nonlinearity $f:\mathbb{R}^2\times[0,1]$ is monostable with KPP assumption (i.e., f satisfies **(A2),(A3),(A5)** with (i,j) replacing by $x\in\mathbb{R}^2$) and there exists $\alpha>0$ and $\beta\ge 0$ such that $f(x_1,x_2,s)\ge f_s'(x_1,x_2,0)s-\beta s^{1+\alpha}$ for all $(x_1,x_2,s)\in\mathbb{R}^2\times[0,1]$.

Then it is known from [2] that (1.1) has a pulsating traveling wave solution if and only if

$$\gamma\ge\gamma_*:=\min_{\lambda>0}\frac{k(\lambda)}{\lambda}>0,$$

where $k(\lambda)$ is the principal eigenvalue of the operator $\mathcal{P}_\lambda$, where

$$\mathcal{P}_\lambda\phi:=\nabla\cdot(A\nabla\phi)-2\lambda e^TA\nabla\phi+[-\lambda\nabla\cdot(Ae)+\lambda^2e^TAe+f_s'(x_1,x_2,0)]\phi,\ e:=(r,s)^T,$$

acting on the set

$$E:=\{\phi\in C^2(\mathbb{R}^2)\mid\phi(x_1+L,x_2)=\phi(x_1,x_2)=\phi(x_1,x_2+L)\}.$$

We use the following discretized problem to approximate (1.1):

$$\begin{aligned}
u_{i,j}'(t)\ =\ &\frac{1}{h^2}\{p((i+\frac{1}{2})h,jh)[u_{i+1,j}(t)-u_{i,j}(t)]-p((i-\frac{1}{2})h,jh)[u_{i,j}(t)-u_{i-1,j}(t)]\\
&+q(ih,(j+\frac{1}{2})h)[u_{i,j+1}(t)-u_{i,j}(t)]-q(ih,(j-\frac{1}{2})h)[u_{i,j}(t)-u_{i,j-1}(t)]\}\\
&+f(ih,jh,u_{i,j}(t)),\quad t\in\mathbb{R},(i,j)\in\mathbb{Z}^2,
\end{aligned}\tag{1.8}$$

where $u_{i,j}(t) := u(ih, jh, t)$ with $h := L/N$ the mesh size for $N \in \mathbb{N}$. If we define

$$\begin{aligned} p^h_{i,j} &:= \frac{1}{h^2} p((i-\frac{1}{2})h, jh) = \frac{N^2}{L^2} p((i-\frac{1}{2})h, jh), \\ q^h_{i,j} &:= \frac{1}{h^2} q(ih, (j-\frac{1}{2})h) = \frac{N^2}{L^2} q(ih, (j-\frac{1}{2})h), \\ f^h(i,j,s) &:= f(ih, jh, s), \end{aligned}$$

then it is easy to check $p^h_{i+N,j} = p^h_{i,j} = p^h_{i,j+N}$, $q^h_{i+N,j} = q^h_{i,j} = q^h_{i,j+N}$ and $f^h(i+N,j,s) = f^h(i,j,s) = f^h(i,j+N,s)$. For each $N \in \mathbb{N}$, by Theorems 1 and 2, we know that (1.8) has a traveling wave solution if and only if $c \geq c_*(h)$.

Theorem 4. *Under the above notation, we have*

$$hc_*(h) \to \gamma_* \text{ as } N \to +\infty, \text{ where } h = \frac{L}{N}.$$

This paper is organized as follows. In §2, we first give some basic properties of solutions of (P) and study the eigenvalue problem for the operator $\mathcal{L}_\lambda$ to characterize the minimum speed c_*. Then we use the monotone iteration method with the help of a pair of super-sub-solutions to prove **Theorem 1**. In §3, we first give a comparison principle and then give a proof of **Theorem 2**. Next, we prove **Theorem 3** by a sliding method in §4. Finally, we follow a method of [19] to drive **Theorem 4** in §5.

Although, in this paper, we treat only the case with monostable nonlinearity in a two-dimensional lattice, our methods can be easily generalized to some other cases. For example, the existence and monotonicity of traveling wave in the case of monostable nonlinearity can be generalized to general N-dimensional lattice by taking the following transformation with moving coordinates:

$$w_{i_1,..,i_N}(\xi) := u_{i_1,...,i_N}(t), \quad \xi := ct - \sum_{k=1}^{N} e_k \cdot i_k,$$

for a given direction of movement of wave $e := (e_1, ..., e_N)$.

The uniqueness of traveling wave in the periodic monostable case is still an open problem, due to lack of the information on asymptotic behaviors of wave profiles at tails. For other nonlinearities, such as the bistable case, we refer the reader to the works [10] and [26]. It is interesting to see whether the method of [10] can be generalized to the ignition type nonlinearity. We leave it as an open problem.

2. Existence

In this section, we shall prove **Theorem 1**. First, we have some basic properties as follows.

Lemma 2.1. *Let $u = \{u_{i,j}\}$ be a solution of (P) with $c \neq 0$. Then*
(i) $0 < u_{i,j}(t) < 1$ *for all* $(i,j,t) \in \mathbb{Z}^2 \times \mathbb{R}$.
(ii) $c > 0$.
(iii) $\lim_{t\to\infty} u_{i,j}(t) = 1$, $\lim_{t\to-\infty} u_{i,j}(t) = 0$, *and* $\lim_{t\to\pm\infty} u'_{i,j}(t) = 0$ *for all* $(i,j) \in \mathbb{Z}^2$.

Proof. First, we show $u_{i,j}(t) > 0$ for all $(i,j,t) \in \mathbb{Z}^2 \times \mathbb{R}$. For a contradiction, we suppose that there exists $(I,J,T) \in \mathbb{Z}^2 \times \mathbb{R}$ such that $u_{I,J}(T) = 0$. Then $u'_{I,J}(T) = 0 = f(I,J,u_{I,J}(T))$. By (1.2), we obtain

$$0 = p_{I+1,J}u_{I+1,J}(T) + p_{I,J}u_{I-1,J}(T) + q_{I,J+1}u_{I,J+1}(T) + q_{I,J}u_{I,J-1}(T).$$

Then $u_{I\pm1,J}(T) = u_{I,J\pm1}(T) = 0$ due to $p_{i,j}, q_{i,j}$ and $u_{i,j}(T)$ are nonnegative for all i,j. Also, by induction, we have $u_{i,j}(T) = 0$ for all $(i,j) \in \mathbb{Z}^2$. This contradicts $\lim_{ri+sj\to-\infty} u_{i,j}(t) = 1$ and so $u_{i,j}(t) > 0$. Similarly, using the same argument, we obtain $u_{i,j}(t) < 1$.

Next, we claim that $c > 0$. Integrating (1.2) over $[a,b]$ with $-\infty < a < b < \infty$, we obtain

$$\begin{aligned} u_{i,j}(b) - u_{i,j}(a) &= \int_a^b \{p_{i+1,j}u_{i+1,j}(t) + p_{i,j}u_{i-1,j}(t) + q_{i,j+1}u_{i,j+1}(t) + q_{i,j}u_{i,j-1}(t) \\ &\quad -D_{i,j}u_{i,j}(t) + f(i,j,u_{i,j}(t))\}dt. \end{aligned}$$

Sum over $i,j = 1$ to N, (1.3) and by the periodicity of $p_{i,j}$ and $q_{i,j}$, we have

$$\begin{aligned} \sum_{i,j=1}^{N} [u_{i,j}(b) - u_{i,j}(a)] &= \sum_{j=1}^{N} p_{1,j}\Big\{ \int_b^{b+\frac{Nr}{c}} u_{N,j}(t)dt - \int_a^{a+\frac{Nr}{c}} u_{N,j}(t)dt \\ &\quad + \int_a^{a+\frac{Nr}{c}} u_{N+1,j}(t)dt - \int_b^{b+\frac{Nr}{c}} u_{N+1,j}(t)dt \Big\} \\ &\quad + \sum_{i=1}^{N} q_{i,1}\Big\{ \int_b^{b+\frac{Ns}{c}} u_{i,N}(t)dt - \int_a^{a+\frac{Ns}{c}} u_{i,N}(t)dt \\ &\quad + \int_a^{a+\frac{Ns}{c}} u_{i,N+1}(t)dt - \int_b^{b+\frac{Ns}{c}} u_{i,N+1}(t)dt \Big\} \\ &\quad + \sum_{i,j=1}^{N} \int_a^b f(i,j,u_{i,j}(t))dt. \end{aligned}$$

From (1.3) and (1.4), we have the following:

$$\text{If } c > 0, \text{ then } \lim_{t\to\infty} u_{i,j}(t) = 1, \text{ and } \lim_{t\to-\infty} u_{i,j}(t) = 0 \text{ for all } (i,j,t) \in \mathbb{Z}^2 \times \mathbb{R};$$
$$\text{if } c < 0, \text{ then } \lim_{t\to\infty} u_{i,j}(t) = 0, \text{ and } \lim_{t\to-\infty} u_{i,j}(t) = 1 \text{ for all } (i,j,t) \in \mathbb{Z}^2 \times \mathbb{R}.$$

Letting $b \to +\infty$ and $a \to -\infty$, we obtain

$$N^2 \operatorname{sgn}(c) = \sum_{i,j=1}^{N} [u_{i,j}(+\infty) - u_{i,j}(-\infty)] = \sum_{i,j=1}^{N} \int_{-\infty}^{+\infty} f(i,j,u_{i,j}(t))dt > 0.$$

Hence $c > 0$, $u_{i,j}(+\infty) = 1$ and $u_{i,j}(-\infty) = 0$ for all $(i,j) \in \mathbb{Z}^2$. Moreover, by (1.2), we have $u'_{i,j}(\pm\infty) = 0$ for all $(i,j) \in \mathbb{Z}^2$. This completes the proof. $\square$

In order to characterize the minimum speed c_*, we recall from (1.6) the linear operator $\mathcal{L}_\lambda : \mathbb{R}^{N^2} \to \mathbb{R}^{N^2}$ defined by

$$\begin{aligned}(\mathcal{L}_\lambda v)_{i,j} \;:=\; & p_{i+1,j}e^{-r\lambda}v_{i+1,j} + p_{i,j}e^{r\lambda}v_{i-1,j} + q_{i,j+1}e^{-s\lambda}v_{i,j+1} + q_{i,j}e^{s\lambda}v_{i,j-1} \\ & -D_{i,j}v_{i,j} + f'_s(i,j,0)v_{i,j},\end{aligned}$$

where $v := (v_{1,1}, v_{1,2}, \cdots, v_{N,N}) \in \mathbb{R}^{N^2}$. It is always understood that $v_{0,j} := v_{N,j}$, $v_{i,0} := v_{i,N}$, $v_{N+1,j} := v_{1,j}$ and $v_{i,N+1} := v_{i,1}$ for $i,j = 1, \cdots, N$. We also recall the following two results in Krein-Rutman Theorem from [23].

(i) If a linear compact operator A, leaving invariant a cone K, has a nonzero eigenvalue, then it has a positive eigenvalue ρ, not less in modulus than every other eigenvalue, and to this number ρ it corresponds at least one eigenvector $v \in K$ of the operator A.

(ii) Suppose that K is a cone with interior and that A is a compact linear operator which is strongly positive with respect to K. Then A has one and only one unit eigenvector v interior to K such that $Av = \rho v$.

With Krein-Rutman Theorem, we have the following lemma for the spectrum of $\mathcal{L}_\lambda$.

Lemma 2.2. *Let the linear operator $\mathcal{L}_\lambda$ be defined above. Then*
(i) *The operator $\mathcal{L}_\lambda$ has a largest real eigenvalue $M(\lambda)$ for all $\lambda \in \mathbb{R}$.*
(ii) *$M(\cdot)$ is convex in $\mathbb{R}$.*
(iii) *$c_* := \min_{\lambda>0} \dfrac{M(\lambda)}{\lambda}$ exists and is positive.*

Proof. Let

$$K := \{\phi = (\phi_{1,1}, \phi_{1,2}, \cdots, \phi_{N,N}) \in \mathbb{R}^{N^2} \mid \phi_{i,j} > 0, i,j = 1,..,N\}.$$

Note that $\overline{K}$ is a cone. By Krein-Rutman Theorem, for each $\lambda \in \mathbb{R}$, when $\alpha > 0$ large enough, $\mathcal{L}_\lambda + \alpha I$ has a largest positive and simple eigenvalue. Hence $\mathcal{L}_\lambda$ also has a largest real simple eigenvalue, say $M(\lambda)$.

Let $\omega = \omega(\lambda) := (\omega_{1,1}, \omega_{1,2}, \cdots, \omega_{N,N}) \in K$ be an eigenvector of $\mathcal{L}_\lambda$ corresponding to $M(\lambda)$, i.e.,

$$\begin{aligned}M(\lambda)\omega_{i,j} \;=\; & p_{i+1,j}e^{-r\lambda}\omega_{i+1,j} + p_{i,j}e^{r\lambda}\omega_{i-1,j} + q_{i,j+1}e^{-s\lambda}\omega_{i,j+1} + q_{i,j}e^{s\lambda}\omega_{i,j-1} \\ & -D_{i,j}\omega_{i,j} + f'_s(i,j,0)\omega_{i,j},\end{aligned} \tag{2.1}$$

for $i,j=1,\cdots,N$. Set $\omega_{I,J}:=\min\{\omega_{1,1},\omega_{1,2},\cdots,\omega_{N,N}\}$. Then we have

$$
\begin{aligned}
M(\lambda) &= p_{I+1,J}e^{-r\lambda}(\frac{\omega_{I+1,J}}{\omega_{I,J}})+p_{I,J}e^{r\lambda}(\frac{\omega_{I-1,J}}{\omega_{I,J}})+q_{I,J+1}e^{-s\lambda}(\frac{\omega_{I,J+1}}{\omega_{I,J}})\\
&\quad +q_{I,J}e^{s\lambda}(\frac{\omega_{I,J-1}}{\omega_{I,J}})-D_{I,J}+f_s'(I,J,0)\\
&\geq p_{I+1,J}e^{-r\lambda}+p_{I,J}e^{r\lambda}+q_{I,J+1}e^{-s\lambda}+q_{I,J}e^{s\lambda}-D_{I,J}+f_s'(I,J,0).
\end{aligned}
\tag{2.2}
$$

This implies $M(0)\geq f_s'(I,J,0)>0$.

Before we prove that M is convex in $\mathbb{R}$, we first recall

$$
M(\lambda)=\inf_{\phi\in K}\max_{i,j\in\{1,\cdots,N\}}\frac{(\mathcal{L}_\lambda\phi)_{i,j}}{\phi_{i,j}}. \tag{2.3}
$$

Set

$$
\begin{aligned}
&K_{per}:=\{u=\{u_{i,j}\}\,|\,u_{i,j}>0,u_{i+N,j}=u_{i,j}=u_{i,j+N}\text{ for all }(i,j)\in\mathbb{Z}^2\},\\
&g(\lambda,u,i,j):=\frac{(\mathcal{L}_\lambda u)_{i,j}}{u_{i,j}}.
\end{aligned}
$$

Then (2.3) can also be written as

$$
M(\lambda)=\inf_{u\in K_{per}}\max_{(i,j)\in\mathbb{Z}^2}g(\lambda,u,i,j).
$$

Now, we claim that $M(\lambda)$ is convex in $\lambda\in\mathbb{R}$. For any λ_1, $\lambda_2\in\mathbb{R}$, $(u,v)\in K_{per}\times K_{per}$ and $t\in[0,1]$, we set $\lambda:=t\lambda_1+(1-t)\lambda_2$ and $U=\{U_{i,j}\}:=\{u_{i,j}^t v_{i,j}^{1-t}\}$. Since $U\in K_{per}$, we have

$$
M(\lambda)\leq\max_{(i,j)\in\mathbb{Z}^2}g(\lambda,U,i,j).
$$

Since the function e^x is convex in $\mathbb{R}$ and $p_{i,j}$, $q_{i,j}>0$, we can easily show that

$$
g(\lambda,U,i,j)\leq tg(\lambda_1,u,i,j)+(1-t)g(\lambda_2,v,i,j).
$$

Hence we obtain that

$$
\begin{aligned}
M(\lambda) &\leq \max_{(i,j)\in\mathbb{Z}^2}\{tg(\lambda_1,u,i,j)+(1-t)g(\lambda_2,v,i,j)\}\\
&\leq t\max_{(i,j)\in\mathbb{Z}^2}g(\lambda_1,u,i,j)+(1-t)\max_{(i,j)\in\mathbb{Z}^2}g(\lambda_2,v,i,j).
\end{aligned}
$$

Taking the infimum over u,$v\in K_{per}$, it follows that $M(\lambda)\leq tM(\lambda_1)+(1-t)M(\lambda_2)$ for all $t\in[0,1]$. Hence $M(\lambda)$ is convex in $\mathbb{R}$ and then $M(\lambda)$ is continuous in $\mathbb{R}$. Also, by $M(0)>0$ and (2.2), we obtain $\lim_{\lambda\to0^+}\frac{M(\lambda)}{\lambda}=+\infty$ and $\liminf_{\lambda\to+\infty}\frac{M(\lambda)}{\lambda}=+\infty$, so $\min_{\lambda>0}\frac{M(\lambda)}{\lambda}$ exists.

Finally, we prove that $\min_{\lambda>0}\frac{M(\lambda)}{\lambda}$ is positive. If $M'(0)\geq0$, then it follows from the convexity of M that $M(\lambda)>0$ for all $\lambda>0$ and so $\min_{\lambda>0}\frac{M(\lambda)}{\lambda}$ is positive. Indeed we shall prove that $M'(0)=0$.

By Krein-Rutman Theorem, there exists a unique $u_\lambda \in K_{per}$ such that

$$\|u_\lambda\| := \max_{i,j=1,\cdots,N} (u_\lambda)_{i,j} = 1$$

and

$$\begin{aligned} M(\lambda)(u_\lambda)_{i,j} &= p_{i+1,j}e^{-r\lambda}(u_\lambda)_{i+1,j} + p_{i,j}e^{r\lambda}(u_\lambda)_{i-1,j} + q_{i,j+1}e^{-s\lambda}(u_\lambda)_{i,j+1} \\ &\quad + q_{i,j}e^{s\lambda}(u_\lambda)_{i,j-1} - D_{i,j}(u_\lambda)_{i,j} + f'_s(i,j,0)(u_\lambda)_{i,j}, \end{aligned} \tag{2.4}$$

for $i, j = 1, 2, \cdots, N$. Choose $\{\lambda_n\}$ such $\lambda_n \searrow 0$ as $n \to \infty$. Since $\{u_{\lambda_n}\}$ is a bounded sequence in $\mathbb{R}^{N^2}$, there exists a subsequence $\{u_{\lambda_{n_k}}\}$ of $\{u_{\lambda_n}\}$ such that $u_{\lambda_{n_k}} \to z$ as $k \to \infty$ for some $z \in K_{per}$ with $\|z\| = 1$. Now, we replace λ by λ_{n_k} in (2.4) and take $k \to \infty$, then we obtain

$$\begin{aligned} M(0)z_{i,j} &= p_{i+1,j}z_{i+1,j} + p_{i,j}z_{i-1,j} + q_{i,j+1}z_{i,j+1} + q_{i,j}z_{i,j-1} \\ &\quad - D_{i,j}z_{i,j} + f'_s(i,j,0)z_{i,j}, \end{aligned}$$

for $i, j = 1, 2, \cdots, N$. This implies that z is the eigenvector of $\mathcal{L}_0$ corresponding to the eigenvalue $M(0)$ such that $\|z\| = 1$. We then conclude that $u_{\lambda_n} \to z$ as $n \to \infty$ for any sequence $\{\lambda_n\}$ which converges to 0 as $n \to \infty$. Hence $u_\lambda \to z$ as $\lambda \to 0$.

Note that

$$(\mathcal{L}_\lambda u_\lambda)_{i,j} z_{i,j} - (\mathcal{L}_0 z)_{i,j}(u_\lambda)_{i,j} = [M(\lambda) - M(0)](u_\lambda)_{i,j} z_{i,j} \quad \forall (i,j).$$

Summing over $i, j = 1, \cdots, N$, we obtain

$$\begin{aligned} &[M(\lambda) - M(0)] \sum_{i,j=1}^{N} (u_\lambda)_{i,j} z_{i,j} \\ = \; &(e^{-r\lambda} - 1) \sum_{i,j=1}^{N} p_{i,j}(u_\lambda)_{i,j} z_{i-1,j} + (e^{r\lambda} - 1) \sum_{i,j=1}^{N} p_{i+1,j}(u_\lambda)_{i,j} z_{i+1,j} \\ &+ (e^{-s\lambda} - 1) \sum_{i,j=1}^{N} q_{i,j}(u_\lambda)_{i,j} z_{i,j-1} + (e^{s\lambda} - 1) \sum_{i,j=1}^{N} q_{i+1,j}(u_\lambda)_{i,j} z_{i+1,j}. \end{aligned}$$

Dividing it by λ and taking $\lambda \to 0$, also due to periodicity of $z_{i,j}$, we have

$$\begin{aligned} &\left\{ \lim_{\lambda \to 0} \frac{M(\lambda) - M(0)}{\lambda} \right\} \sum_{i,j=1}^{N} (z_{i,j})^2 \\ = \; &-r\Big(\sum_{i,j=1}^{N} p_{i,j} z_{i-1,j} z_{i,j}\Big) + r\Big(\sum_{i,j=1}^{N} p_{i+1,j} z_{i,j} z_{i+1,j}\Big) - s\Big(\sum_{i,j=1}^{N} q_{i,j} z_{i,j-1} z_{i,j}\Big) \\ &+ s\Big(\sum_{i,j=1}^{N} p_{i,j+1} z_{i,j} z_{i,j+1}\Big) = 0. \end{aligned}$$

It follows that $M'(0) = 0$ and the lemma is proved. $\square$

For a given $c>0$, let (c,u) be a solution of (P). We set $\xi := ct - ri - sj$ and introduce

$$w_{i,j}(\xi) := u_{i,j}(t)\mid_{\xi=ct-ri-sj} = u_{i,j}(\frac{\xi+ri+sj}{c}).$$

Then, by (1.3), we have

$$w_{i+N,j}(\xi) = u_{i+N,j}(\frac{\xi+ri+sj+Nr}{c}) = u_{i,j}(\frac{\xi+ri+sj}{c}) = w_{i,j}(\xi),$$
$$w_{i,j+N}(\xi) = u_{i,j+N}(\frac{\xi+ri+sj+Ns}{c}) = u_{i,j}(\frac{\xi+ri+sj}{c}) = w_{i,j}(\xi).$$

It follows that

$$w_{i+N,j}(\xi) = w_{i,j}(\xi) = w_{i,j+N}(\xi) \ \ \forall\, \xi\in\mathbb{R},\ (i,j)\in\mathbb{Z}^2. \tag{2.5}$$

Next, (1.2) becomes

$$\begin{aligned} cw'_{i,j}(\xi) &= p_{i+1,j}w_{i+1,j}(\xi-r) + p_{i,j}w_{i-1,j}(\xi+r) + q_{i,j+1}w_{i,j+1}(\xi-s) \\ &\quad + q_{i,j}w_{i,j-1}(\xi+s) - D_{i,j}w_{i,j}(\xi) + f(i,j,w_{i,j}(\xi)),\ \xi\in\mathbb{R}, (i,j)\in\mathbb{Z}^2. \end{aligned} \tag{2.6}$$

For each $(i,j)\in\mathbb{Z}^2$, by Lemma 2.1(iii),

$$w_{i,j}(-\infty) = 0, \quad w_{i,j}(+\infty) = 1. \tag{2.7}$$

Also, note that

$$0 \le w_{i,j}(\xi) \le 1 \ \ \forall\ (i,j,\xi)\in\mathbb{Z}^2\times\mathbb{R}. \tag{2.8}$$

We shall denote the problem (P') by the problem (2.5)-(2.8). From the above discussion, we see that (c,w) is a solution of (P'), if (c,u) is a solution of (P). Conversely, if (c,w) solves (P'), by defining $u_{i,j}(t) := w_{i,j}(ct-ri-sj)$ for all $(i,j,t)\in\mathbb{Z}^2\times\mathbb{R}$, then (c,u) solves (P). Therefore, we have established the following proposition.

Proposition 2.3. *The problem (P) admits a solution (c,u) if and only if the problem (P') admits a solution (c,w).*

Now, we define the operator H and the set Γ by

$$\begin{aligned} H(w_{i,j})(\xi) &:= \nu w_{i,j}(\xi) + \frac{1}{c}[\ p_{i+1,j}w_{i+1,j}(\xi-r) + p_{i,j}w_{i-1,j}(\xi+r) + q_{i,j+1}w_{i,j+1}(\xi-s) \\ &\quad + q_{i,j}w_{i,j-1}(\xi+s) - D_{i,j}w_{i,j}(\xi) + f(i,j,w_{i,j}(\xi))\ ], \\ \Gamma &:= \{\{w_{i,j}\}\mid w_{i,j}(-\infty)=0\le w_{i,j}(\xi)\le 1\,, w_{i+N,j}(\xi)=w_{i,j}(\xi)=w_{i,j+N}(\xi) \\ &\quad \forall\,(i,j,\xi)\in\mathbb{Z}^2\times\mathbb{R}\}, \end{aligned}$$

where the constant $\nu > \{\max_{i,j}|D_{i,j}| + \max_{i,j}\max_{s\in[0,1]}|f'_s(i,j,s)|\}/c$.

Due to the choice of ν, the following proposition can be easily derived.

Proposition 2.4. *Let H be defined as above. Then we have*
(i) *If $w_{i,j}(\xi) \geq v_{i,j}(\xi)$ for all $(i,j,\xi) \in \mathbb{Z}^2 \times \mathbb{R}$, then $H(w_{i,j})(\xi) \geq H(v_{i,j})(\xi)$ for all $(i,j,\xi) \in \mathbb{Z}^2 \times \mathbb{R}$.*
(ii) *If $w_{i,j}(\cdot)$ is non-decreasing in $\mathbb{R}$ for all $(i,j) \in \mathbb{Z}^2$, then $H(w_{i,j})(\cdot)$ is also non-decreasing in $\mathbb{R}$ for all $(i,j) \in \mathbb{Z}^2$.*

Next, by the integrating factor $e^{\nu\xi}$, (2.6) becomes

$$w_{i,j}(\xi) = e^{-\nu\xi} \int_{-\infty}^{\xi} e^{\nu x} H(w_{i,j}(x))dx \quad \forall\ (i,j,t) \in \mathbb{Z}^2 \times \mathbb{R}.$$

We define

$$T^c(w_{i,j})(\xi) := e^{-\nu\xi} \int_{-\infty}^{\xi} e^{\nu x} H(w_{i,j}(x))dx.$$

Then, due to Proposition 2.4(i), we have the following important property:

$$T^c(w_{i,j}) \leq T^c(v_{i,j}) \ \text{ if } \ w_{i,j} \leq v_{i,j} \quad \forall (i,j) \in \mathbb{Z}^2. \tag{2.9}$$

Moreover, we have

Lemma 2.5. *A pair $(c,w) \in \mathbb{R}^+ \times \Gamma$ with $w_{i,j}(+\infty) = 1$ satisfies $w_{i,j} = T^c(w_{i,j})$ for all $(i,j) \in \mathbb{Z}^2$ if and only if it solves (P').*

Proof. It follows from some direct calculations. □

We call $\phi^{\pm} = \{\phi^{\pm}_{i,j}\}$ a **super/sub-solution** of (P'), if
(i) $\phi^+_{i,j}$ is non-decreasing and

$$\begin{aligned} c(\phi^+_{i,j})'(\xi) \ &\geq \ p_{i+1,j}\phi^+_{i+1,j}(\xi - r) + p_{i,j}\phi^+_{i-1,j}(\xi + r) + q_{i,j+1}\phi^+_{i,j+1}(\xi - s) \\ &\quad + q_{i,j}\phi^+_{i,j-1}(\xi + s) - D_{i,j}\phi^+_{i,j}(\xi) + f(i,j,\phi^+_{i,j}(\xi)) \end{aligned} \tag{2.10}$$

a.e. in $\mathbb{R}$ for all $(i,j) \in \mathbb{Z}^2$;
(ii) $\phi^-_{i,j}$ is differentiable a.e., $\phi^-_{i,j} \not\equiv 0$ and

$$\begin{aligned} c(\phi^-_{i,j})'(\xi) \ &\leq \ p_{i+1,j}\phi^-_{i+1,j}(\xi - r) + p_{i,j}\phi^-_{i-1,j}(\xi + r) + q_{i,j+1}\phi^-_{i,j+1}(\xi - s) \\ &\quad + q_{i,j}\phi^-_{i,j-1}(\xi + s) - D_{i,j}\phi^-_{i,j}(\xi) + f(i,j,\phi^-_{i,j}(\xi)) \end{aligned}$$

a.e. in $\mathbb{R}$ for all $(i,j) \in \mathbb{Z}^2$.

Lemma 2.6. *Given $c > 0$. Let $w^{\pm} \in \Gamma$ be a super/sub-solution of (P') such that $w^-_{i,j}(\xi) \leq w^+_{i,j}(\xi)$ for all $(i,j,\xi) \in \mathbb{Z}^2 \times \mathbb{R}$. Then there exists $w \in \Gamma$ such that $w_{i,j}(+\infty) = 1$ and $w_{i,j} = T^c(w_{i,j})$ for all $(i,j) \in \mathbb{Z}^2$.*

Proof. Define $w^1_{i,j} := T^c(w^+_{i,j})$ for $(i,j) \in \mathbb{Z}^2$. Then $w^1_{i+N,j}(\xi) = w^1_{i,j}(\xi) = w^1_{i,j+N}(\xi)$ for all $(i,j,\xi) \in \mathbb{Z}^2 \times \mathbb{R}$. By the definition of super-solution, we have $w^1_{i,j}(\cdot) \le w^+_{i,j}(\cdot)$ in $\mathbb{R}$. This implies that $w^1_{i,j}(-\infty) = 0 \le w^1_{i,j}(\xi) \le 1$ for all $(i,j,\xi) \in \mathbb{Z}^2 \times \mathbb{R}$. Hence $\{w^1_{i,j}\} \in \Gamma$. Also, by (2.9) and the definition of sub-solution, we obtain

$$w^-_{i,j} \le T^c(w^-_{i,j}) \le T^c(w^+_{i,j}) = w^1_{i,j}.$$

Moreover,

$$(w^1_{i,j})'(\xi) = e^{-\nu\xi} \int_{-\infty}^{\xi} e^{\nu x} \{H(w_{i,j}(\xi)) - H(w_{i,j}(x))\} dx \ge 0,$$

by Proposition 2.4(ii).

Next, we define $w^{n+1}_{i,j} = T^c(w^n_{i,j})$ for each $n \in \mathbb{N}$. Then, for any $n \in \mathbb{N}$, by a similar argument as above, we have $\{w^n_{i,j}\} \in \Gamma$,

$$0 \le w^-_{i,j} \le w^{n+1}_{i,j} \le w^n_{i,j} \le w^+_{i,j} \le 1 \text{ and } (w^n_{i,j})' \ge 0 \text{ in } \mathbb{R}\ \forall\ (i,j) \in \mathbb{Z}^2.$$

Hence $w_{i,j}(\xi) := \lim_{n\to\infty} w^n_{i,j}(\xi)$ exists and $0 \le w_{i,j}(\cdot) \le 1$ in $\mathbb{R}$. Applying Lebesgue's Dominated Convergence Theorem, we obtain $w_{i,j}(\xi) = T^c(w_{i,j})(\xi)$ in $\mathbb{R}$ for all $(i,j) \in \mathbb{Z}^2$. Moreover, $w'_{i,j} \ge 0$ in $\mathbb{R}$.

Finally, we claim $w_{i,j}(-\infty) = 0$ and $w_{i,j}(+\infty) = 1$. Since $w'_{i,j}(\xi) \ge 0$ and $0 \le w_{i,j}(\xi) \le 1$ for all $(i,j,\xi) \in \mathbb{Z}^2 \times \mathbb{R}$, $w_{i,j}(\pm\infty)$ exists. By $w^+_{i,j}(-\infty) = 0$ and $0 \le w_{i,j} \le w^+_{i,j}$, it follows that $w_{i,j}(-\infty) = 0$. To derive $w_{i,j}(+\infty) = 1$, using l'Hospital's rule, for any $(i,j) \in \mathbb{Z}^2$,

$$\begin{aligned}
\lim_{\xi\to\infty} w_{i,j}(\xi) &= \lim_{\xi\to\infty} T^c(w_{i,j})(\xi) \\
&= \lim_{\xi\to\infty} \{w_{i,j}(\xi) + \frac{1}{c\nu}[p_{i+1,j} w_{i+1,j}(\xi - r) + p_{i,j} w_{i-1,j}(\xi + r) \\
&\quad + q_{i,j+1} w_{i,j+1}(\xi - s) + q_{i,j} w_{i,j-1}(\xi + s) - D_{i,j} w_{i,j}(\xi) + f(i,j,w_{i,j}(\xi))]\}.
\end{aligned}$$

This implies that

$$\begin{aligned}
&p_{i+1,j} w_{i+1,j}(+\infty) + p_{i,j} w_{i-1,j}(+\infty) + q_{i,j+1} w_{i,j+1}(+\infty) \\
&\qquad + q_{i,j} w_{i,j-1}(+\infty) - D_{i,j} w_{i,j}(+\infty) = -f(i,j,w_{i,j}(+\infty)).
\end{aligned}$$

Let

$$\gamma := \min_{(i,j)\in\mathbb{Z}^2} \{w_{i,j}(+\infty)\} = w_{I,J}(+\infty)$$

for some $(I,J) \in \mathbb{Z}^2$. Then

$$\begin{aligned}
0 &\le p_{I+1,J}[w_{I+1,J}(+\infty) - \gamma] + p_{I,J}[w_{I-1,J}(+\infty) - \gamma] \\
&\quad + q_{I,J+1}[w_{I,J+1}(+\infty) - \gamma] + q_{I,J}[w_{I,J-1}(+\infty) - \gamma] \\
&= -f(I,J,\gamma) \le 0.
\end{aligned}$$

It follows that $f(I,J,\gamma)=0$ and so $\gamma=0$ or 1. Also, note that $w_{I,J}(\cdot)$ is non-decreasing and $w^-_{I,J}(\cdot)\not\equiv 0$ in $\mathbb{R}$, so there exists $\xi_0\in\mathbb{R}$ such that

$$\gamma=w_{I,J}(+\infty)\geq w_{I,J}(\xi_0)\geq w^-_{I,J}(\xi_0)>0.$$

Hence $\gamma=1$. This implies that $w_{i,j}(+\infty)=1$ for all $(i,j)\in\mathbb{Z}^2$ and the Lemma follows. $\square$

Let $g(\lambda):=M(\lambda)/\lambda$ for $\lambda>0$. It follows from Lemma 2.2 that there exists a unique $\lambda^*>0$ such that $g(\lambda^*)=c_*$ and $g(\lambda)>c_*$ for all $\lambda\in(0,\lambda^*)$. Moreover, it follows from the convexity of $M(\lambda)$ that g is strictly decreasing in $(0,\lambda^*)$. Therefore, for any $c>c_*$, there exists a unique $\lambda\in(0,\lambda^*)$ such that $g(\lambda)=c$. Also, for any $c>c_*$, we can find $\mu\in(\lambda,\lambda^*)$ such that $\mu<\lambda(1+\alpha)$ and $g(\mu)<c$, where α is the constant defined in **(A4)**.

Now, we fix a $c>c_*$. Let $\{U_{i,j}\}\in K_{per}$ be an eigenvector of $\mathcal{L}$ corresponding to λ and $\{V_{i,j}\}\in K_{per}$ be an eigenvector of $\mathcal{L}$ corresponding to μ. Then we can find a pair of super-sub-solutions as follows.

Lemma 2.7. *Fix a $c>c_*$. Let $w^+=\{w^+_{i,j}\}$ and $w^-=\{w^-_{i,j}\}$ be defined by*

$$\begin{aligned} w^+_{i,j}(\xi) &:= \min\{e^{\lambda\xi}U_{i,j}\ ,\ 1\},\\ w^-_{i,j}(\xi) &:= \max\{e^{\lambda\xi}U_{i,j}-Ae^{\mu\xi}V_{i,j}\ ,\ 0\}, \end{aligned}$$

where $A>0$ is large enough. Then w^+ is a super-solution of (P') and w^- is a sub-solution of (P').

Proof. Since the constant 1 satisfies (2.6), it is enough to show $e^{\lambda\xi}U_{i,j}$ satisfies (2.10) when $e^{\lambda\xi}U_{i,j}<1$. By the assumption

$$0<f(i,j,s)\leq f'_s(i,j,0)s\ \ \forall\ (i,j,s)\in\mathbb{Z}^2\times[0,1],$$

we can conclude that

$$\begin{aligned} & c(w^+_{i,j})'(\xi)-[p_{i+1,j}w^+_{i+1,j}(\xi-r)+p_{i,j}w^+_{i-1,j}(\xi+r)+q_{i,j+1}w^+_{i,j+1}(\xi-s)\\ & +q_{i,j}w^+_{i,j-1}(\xi+s)-D_{i,j}w^+_{i,j}(\xi)+f(i,j,w^+_{i,j}(\xi))]\\ \geq\ & c\lambda e^{\lambda\xi}U_{i,j}-[p_{i+1,j}e^{(\xi-r)\lambda}U_{i+1,j}+p_{i,j}e^{(\xi+r)\lambda}U_{i-1,j}+q_{i,j+1}e^{(\xi-s)\lambda}U_{i,j+1}\\ & +q_{i,j}e^{(\xi+s)\lambda}U_{i,j-1}-D_{i,j}e^{\lambda\xi}U_{i,j}+f'_s(i,j,0)e^{\lambda\xi}U_{i,j}]\\ =\ & e^{\lambda\xi}[c\lambda-M(\lambda)]U_{i,j}=0. \end{aligned}$$

Hence w^+ is a super-solution of (P').

To prove w^- is a sub-solution of (P'), we first choose $A>0$ large enough such that

$$\beta U^{1+\alpha}_{i,j}+A[M(\mu)-c\mu)]V_{i,j}\leq 0$$

for all $(i,j)\in\mathbb{Z}^2$, where α and β are constant defined in the assumption (**A4**), and $w^-_{i,j}(\xi)>0$ implies that $\xi<0$. Then, by (**A4**),

$$\begin{aligned}
& c(w^-_{i,j})'(\xi)-[p_{i+1,j}w^-_{i+1,j}(\xi-r)+p_{i,j}w^-_{i-1,j}(\xi+r)+q_{i,j+1}w^-_{i,j+1}(\xi-s)\\
& +q_{i,j}w^-_{i,j-1}(\xi+s)-D_{i,j}w^-_{i,j}(\xi)+f(i,j,w^-_{i,j}(\xi))]\\
\leq\ & Ae^{\mu\xi}V_{i,j}[M(\mu)-c\mu]+\beta(e^{\lambda\xi}U_{i,j}-Ae^{\mu\xi}V_{i,j})^{1+\alpha}\\
\leq\ & Ae^{\mu\xi}V_{i,j}[M(\mu)-c\mu]+\beta e^{\lambda(1+\alpha)\xi}U_{i,j}^{1+\alpha} \qquad (\text{ since } w^-_{i,j}(\xi)>0)\\
\leq\ & e^{\mu\xi}\{\beta U_{i,j}^{1+\alpha}+A[M(\mu)-c\mu]V_{i,j}\} \qquad (\text{ since } \mu<\lambda(1+\alpha) \text{ and } \xi<0)\\
\leq\ & 0
\end{aligned}$$

for all $(i,j)\in\mathbb{Z}^2$ such that $w^-_{i,j}(\xi)>0$. Hence w^- is a sub-solution of (P') and so the lemma follows. □

Proof of Theorem 1. From Lemma 2.6 and Lemma 2.7, we conclude that (P') has a solution for each $c>c_*$. Also, by Proposition 2.3, it follows that (P) admits a solution for each $c>c_*$.

For $c=c_*$, we first choose a sequence of solution $\{c_k,w^k\}_{k=1}^\infty$ of (P') such that $c_k\downarrow c_*$ and w^k is non-decreasing for all k. Applying Arzela-Ascoli Theorem, there exists a subsequence $\{w^{k_l}\}_{l=1}^\infty$ of $\{w^k\}_{k=1}^\infty$ and $w^*=\{w^*_{i,j}\}$ such that $w^{k_l}(\cdot)\to w^*(\cdot)$ in $\mathbb{R}$ as $l\to\infty$ uniformly on any compact subset of $\mathbb{R}$. Moreover, w^* satisfies $w^*_{i,j}(\xi)=T^{c_*}(w^*_{i,j})(\xi)$ and $w^*_{i+N,j}(\xi)=w^*_{i,j}(\xi)=w^*_{i,j+N}(\xi)$ for all $(i,j,\xi)\in\mathbb{Z}^2\times\mathbb{R}$.

Now, we claim $w^*_{i,j}(+\infty)=1$ and $w^*_{i,j}(-\infty)=0$. Fix $(i,j)\in\mathbb{Z}^2$. By appropriate translation, we may assume $w^{k_l}_{i,j}(0)=1/2$ for any l. Note that $w^*_{i,j}$ is also non-decreasing and $0\leq w^*_{i,j}(\cdot)\leq 1$ in $\mathbb{R}$, then $w^*_{i,j}(\pm\infty)$ exists and is between 0 and 1. Applying Fatou's Lemma, we have

$$\int_{-\infty}^{+\infty} f(i,j,w^*_{i,j}(\xi))d\xi\leq\liminf_{l\to\infty}\int_{-\infty}^{+\infty} f(i,j,w^{k_l}_{i,j}(\xi))d\xi<+\infty.$$

This implies that $f(i,j,w^*(\pm\infty))=0$ and so $w^*_{i,j}(\pm\infty)\in\{0,1\}$ for any $(i,j)\in\mathbb{Z}^2$.

Next, it follows from $w^*_{i,j}(+\infty)=T^{c_*}(w^*_{i,j})(+\infty)$ that

$$\begin{aligned}
& p_{i+1,j}[w_{i+1,j}(+\infty)-w_{i,j}(+\infty)]+p_{i,j}[w_{i-1,j}(+\infty)-w_{i,j}(+\infty)]\\
& +q_{i,j+1}[w_{i,j+1}(+\infty)-w_{i,j}(+\infty)]+q_{i,j}[w_{i,j-1}(+\infty)-w_{i,j}(+\infty)]=0.
\end{aligned}$$

Then $w_{i,j}(+\infty)=w_{i\pm1,j}(+\infty)=w_{i,j\pm1}(+\infty)$ for all i,j due to $p_{i,j},q_{i,j}>0$. Similarly, we also have $w_{i,j}(-\infty)=w_{i\pm1,j}(-\infty)=w_{i,j\pm1}(-\infty)$ for all i,j.

On the other hand, by integrating (2.6) over $(-\infty, +\infty)$ and summing over i, j from 1 to N, we have

$$c_* \sum_{i,j=1}^{N} [w^*_{i,j}(+\infty) - w^*_{i,j}(-\infty)] = \sum_{i,j=1}^{N} \int_{-\infty}^{+\infty} f(i, j, w^*_{i,j}(\xi)) d\xi > 0.$$

The last inequality holds, since $w^*_{i,j}(0) = 1/2$ for some i, j. Then $w^*_{i,j}(+\infty) = 1$ and $w^*_{i,j}(-\infty) = 0$ for any $(i, j) \in \mathbb{Z}^2$, thereby completing the proof of Theorem 1. $\square$

3. Existence of the minimum speed

This section is devoted to the proof of Theorem 2. Throughout this section, the periodicity of f in (i, j) and $p_{i,j}, q_{i,j}$ are in force. First, we define

$$\begin{aligned} \mathcal{F}u(i,j,t) \;:=\; & u'_{i,j}(t) - [\, p_{i+1,j}u_{i+1,j}(t) + p_{i,j}u_{i-1,j}(t) + q_{i,j+1}u_{i,j+1}(t) \\ & + q_{i,j}u_{i,j-1}(t) - D_{i,j}u_{i,j}(t) + f(i,j,u_{i,j}(t))\,]. \end{aligned} \tag{3.1}$$

We have the following comparison principle. The proof is standard so we omit it.

Lemma 3.1. *Let $t_0 \in \mathbb{R}$. Assume that $u(t) = \{u_{i,j}(t)\}$ and $v(t) = \{v_{i,j}(t)\}$ are continuously differentiable on $[t_0, \infty)$ and bounded for $(i, j, t) \in \mathbb{Z}^2 \times [t_0, \infty)$. If*

$$\mathcal{F}u(i,j,t) \geq \mathcal{F}v(i,j,t)\ \forall (i,j,t) \in \mathbb{Z}^2 \times [t_0,\infty), \quad u_{i,j}(t_0) \geq v_{i,j}(t_0)\ \forall (i,j) \in \mathbb{Z}^2,$$

then $u_{i,j}(t) \geq v_{i,j}(t)$ for all $(i, j, t) \in \mathbb{Z}^2 \times [t_0, \infty)$. Moreover, if the condition $u_{i,j}(t_0) \geq v_{i,j}(t_0)$ is replaced by $u_{i,j}(t_0) > v_{i,j}(t_0)$, then $u_{i,j}(t) > v_{i,j}(t)$ for all $(i, j, t) \in \mathbb{Z}^2 \times [t_0, \infty)$.

Remark 3.1. If $\{p_{i,j}\}$ and $\{q_{i,j}\}$ are replaced by $\{p_{i,j}(t)\}$ and $\{q_{i,j}(t)\}$ such that $0 < p_{i,j}(t), q_{i,j}(t) \leq M$ for some $M > 0$ and for all $(i, j, t) \in \mathbb{Z}^2 \times [t_0, \infty)$, then Lemma 3.1 also holds.

Lemma 3.2. *Let $u = \{u_{i,j}\}$ be a solution of (P) with $c \neq 0$. Then for any bounded interval E and any $(m, n) \in \mathbb{Z}^2$ we have*

$$\sup \left\{ \frac{u_{i+m,j+n}(t)}{u_{i,j}(t+\eta)} \,\middle|\, (i,j,t) \in \mathbb{Z}^2 \times \mathbb{R}, \eta \in E \right\} < \infty. \tag{3.2}$$

Moreover, we have

$$\sup \left\{ \frac{|u'_{i,j}(t)|}{u_{i,j}(t)} \,\middle|\, (i,j,t) \in \mathbb{Z}^2 \times \mathbb{R} \right\} < \infty. \tag{3.3}$$

Proof. Recall $(r, s) := (\cos\theta, \sin\theta)$. Without loss of generality, we may assume $r > 0$ and, by (1.3), only consider the case when $E = [0, rN/c]$.

First, we choose any $(i_0, j_0, t_0, \eta_0) \in \mathbb{Z}^2 \times \mathbb{R} \times E$. Let $v = \{v_{i,j}\}$ be the solution of

$$v'_{i,j}(t) = p_{i+1,j}v_{i+1,j}(t) + p_{i,j}v_{i-1,j}(t) + q_{i,j+1}v_{i,j+1}(t) + q_{i,j}v_{i,j-1}(t) - D_{i,j}v_{i,j}(t), \tag{3.4}$$

for $(i,j,t) \in \mathbb{Z}^2 \times [t_0,\infty)$ with the initial condition $v_{i_0+m,j_0+n}(t_0) = u_{i_0+m,j_0+n}(t_0)$ and $v_{i,j}(t_0)=0$ for all $(i,j)\neq(i_0+m,j_0+n)$. Note that this initial value problem is equivalent to the integral equation

$$v_{i,j}(t) \;=\; v_{i,j}(t_0)e^{-\mu(t-t_0)} + \int_{t_0}^{t} e^{-\mu(t-s)}H(v_{i,j})(s)ds,$$

where $\mu > \max_{(i,j)\in\mathbb{Z}^2} D_{i,j}$ and

$$H(v_{i,j}) := (\mu - D_{i,j})v_{i,j} + p_{i+1,j}v_{i+1,j} + p_{i,j}v_{i-1,j} + q_{i,j+1}v_{i,j+1} + q_{i,j}v_{i,j-1}.$$

Furthermore, the existence of v can be derived by using the following Picard's iteration:

$$\begin{aligned} v_{i,j}^{(0)}(t) &:= v_{i,j}(t_0)e^{-\mu(t-t_0)},\ t\geq t_0 \\ v_{i,j}^{(n)}(t) &:= v_{i,j}^{(0)}(t) + \int_{t_0}^{t} e^{-\mu(t-s)}H(v_{i,j}^{(n-1)})(s)ds,\ t\geq t_0,\ n\in\mathbb{N}, \end{aligned}$$

together with the monotonicity of $\{v_{i,j}\}\mapsto\{H(v_{i,j})\}$. Moreover, since $0\leq v_{i,j}^{(0)}(t)\leq 1$ for all $(i,j,t)\in\mathbb{Z}^2\times[t_0,\infty)$ and due to monotonicity of the operator H in v again, we obtain

$$0\leq v_{i,j}^{(n)}(t)\leq 1\ \ \forall\ (i,j,t)\in\mathbb{Z}^2\times[t_0,\infty),\ \forall\ n\in\mathbb{N}.$$

Hence $0\leq v_{i,j}(t)\leq 1$ for all $(i,j,t)\in\mathbb{Z}^2\times[t_0,\infty)$.

Now, since $\mathcal{F}u(i,j,t) = 0 \geq -f(i,j,v_{i,j}) = \mathcal{F}v(i,j,t)$ for any $(i,j,t)\in\mathbb{Z}\times[t_0,\infty)$ and $u_{i,j}(t_0)\geq v_{i,j}(t_0)$, it follows from the comparison principle that

$$u_{i,j}(t)\geq v_{i,j}(t)\ \ \forall\ (i,j,t)\in\mathbb{Z}^2\times[t_0,\infty).$$

In particular,

$$u_{i_0+N,j_0}(t_0+\eta_0+\frac{rN}{c}) \geq v_{i_0+N,j_0}(t_0+\eta_0+\frac{rN}{c}). \tag{3.5}$$

Next, for each $(h,k)\in\mathbb{Z}^2$, let $z(\cdot)=\{z_{i,j}(\cdot\,;h,k)\}$ be the solution of (3.4) for $t\geq 0$ with the initial condition $z_{h+m,k+n}(0;h,k)=1$ and $z_{i,j}(0;h,k)=0$ for $(i,j)\neq(h+m,k+n)$. Note that we also have $0\leq z_{i,j}(t;h,k)\leq 1$ for all $(i,j,t)\in\mathbb{Z}^2\times[0,\infty)$. We claim that

$$z_{h+N,k}(\frac{rN}{c}+\eta_0;h,k)>0\ \ \forall\ (h,k)\in\mathbb{Z}^2. \tag{3.6}$$

For a contradiction, we suppose that there exists $(\bar{h},\bar{k})\in\mathbb{Z}^2$ such that

$$z_{\bar{h}+N,\bar{k}}(\frac{rN}{c}+\eta_0;\bar{h},\bar{k})=0.$$

Then $z'_{\bar{h}+N,\bar{k}}(rN/c+\eta_0;\bar{h},\bar{k})=0$. Therefore, by (3.4), we obtain

$$z_{\bar{h}\pm1+N,\bar{k}}(\frac{rN}{c}+\eta_0;\bar{h},\bar{k})=0=z_{\bar{h}+N,\bar{k}\pm1}(\frac{rN}{c}+\eta_0;\bar{h},\bar{k}).$$

By induction, we can conclude that

$$z_{i,j}(\frac{rN}{c}+\eta_0;\bar{h},\bar{k})=0\ \ \forall\ (i,j)\in\mathbb{Z}^2. \tag{3.7}$$

On the other hand, since z satisfies (3.4), we have

$$z'_{h+m,k+n}(t;h,k) \geq -4Mz_{h+m,k+n}(t;h,k), \quad M := \max_{i,j}\{p_{i,j}, q_{i,j}\}.$$

By integrating over $[0, rN/c + \eta_0]$ and using $z_{h+m,k+n}(0;h,k) = 1$, we obtain

$$z_{h+m,k+n}(\frac{rN}{c} + \eta_0; h, k) \geq \exp\{-4M(\frac{rN}{c} + \eta_0)\} > 0.$$

This contradicts (3.7) and the claim (3.6) follows.

By the periodicity of $p_{i,j}$ and $q_{i,j}$, we have

$$\begin{aligned} z_{(h+N)+N,k}(\frac{rN}{c} + \eta_0; h+N, k) &= z_{h+N,k}(\frac{rN}{c} + \eta_0; h, k) \\ &= z_{h+N,k+N}(\frac{rN}{c} + \eta_0; h, k+N). \end{aligned}$$

Thus the number

$$A := \min\{z_{h+N,k}(\frac{rN}{c} + \eta_0; h, k) \,|\, (h,k) \in \mathbb{Z}^2, \eta_0 \in E\}$$

is well-defined and $A > 0$. Note that the constant A is independent of i_0, j_0, t_0 and η_0.

Finally, since (3.4) is linear and the initial values $v_{i,j}(t_0) = u_{i_0+m,j_0+n}(t_0)z_{i,j}(0; i_0, j_0)$, we have

$$\begin{aligned} v_{i_0+N,j_0}(t_0 + \frac{rN}{c} + \eta_0) &= u_{i_0+m,j_0+n}(t_0)z_{i_0+N,j_0}(\frac{rN}{c} + \eta_0; i_0, j_0) \\ &\geq u_{i_0+m,j_0+n}(t_0)A. \end{aligned}$$

From (3.5) it follows that

$$\frac{u_{i_0+m,j_0+n}(t_0)}{u_{i_0+N,j_0}(t_0 + \frac{rN}{c} + \eta_0)} \leq \frac{u_{i_0+m,j_0+n}(t_0)}{v_{i_0+N,j_0}(t_0 + \frac{rN}{c} + \eta_0)} \leq \frac{1}{A}.$$

Since

$$u_{i_0,j_0}(t_0 + \eta_0) = u_{i_0+N,j_0}(t_0 + \frac{rN}{c} + \eta_0),$$

(3.2) follows. Moreover, (3.3) follows from (1.2) and (3.2). Hence the lemma is proved. $\square$

Proof of Theorem 2. Let (c, u) be a solution of P with $c \neq 0$. By (3.3), the limit

$$\mu_{i,j} := \liminf_{t\to-\infty} \frac{u'_{i,j}(t)}{u_{i,j}(t)}$$

exists and is finite for $(i,j) \in \mathbb{Z}^2$. Also from (1.3) we know $\mu_{i+N} = \mu_{i,j} = \mu_{i,j+N}$. Hence $\mu := \min_{(i,j)\in\mathbb{Z}^2} \mu_{i,j}$ exists and we may assume $\liminf_{t\to-\infty} \frac{u'_{I,J}(t)}{u_{I,J}(t)} = \mu$ for some $I, J \in \{1, \cdots, N\}$.

Given any fixed $(i,j) \in \mathbb{Z}^2$. We consider the sequence of functions $\left\{\frac{u_{i,j}(t+t_n)}{u_{I,J}(t_n)}\right\}$, where $\{t_n\}$ is a sequence such that $\frac{u'_{I,J}(t_n)}{u_{I,J}(t_n)} \to \mu$ and $t_n \to -\infty$ as $n \to \infty$. For each $\kappa \in \mathbb{N}$, by

(3.2), $\left\{\dfrac{u_{i,j}(t+t_n)}{u_{I,J}(t_n)}\right\}$ is uniformly bounded for $t\in[-\kappa,\kappa]$ and $n\in\mathbb{N}$. Next, for $t,\tilde{t}\in[-\kappa,\kappa]$, applying the mean value theorem, there exists some ξ between t and $\tilde{t}$ such that

$$\left|\frac{u_{i,j}(t+t_n)}{u_{I,J}(t_n)}-\frac{u_{i,j}(\tilde{t}+t_n)}{u_{I,J}(t_n)}\right|=\frac{|u'_{i,j}(\xi+t_n)|}{|u_{I,J}(t_n)|}|t-\tilde{t}|.$$

Hence, by Lemma 3.2, we have

$$\left|\frac{u_{i,j}(t+t_n)}{u_{I,J}(t_n)}-\frac{u_{i,j}(\tilde{t}+t_n)}{u_{I,J}(t_n)}\right|=\frac{|u'_{i,j}(\xi+t_n)|}{|u_{i,j}(\xi+t_n)|}\frac{|u_{i,j}(\xi+t_n)|}{|u_{I,J}(t_n)|}|t-\tilde{t}|\le C|t-\tilde{t}|$$

for some positive constant C. It follows that $\left\{\dfrac{u_{i,j}(t+t_n)}{u_{I,J}(t_n)}\right\}$ is equi-continuous on $[-\kappa,\kappa]$. Therefore, by applying Arzela-Ascoli Theorem and using a diagonal process, we conclude that there exists a subsequnece $\left\{\dfrac{u_{i,j}(t+t_{n_k})}{u_{I,J}(t_{n_k})}\right\}$ of $\left\{\dfrac{u_{i,j}(t+t_n)}{u_{I,J}(t_n)}\right\}$ such that $\dfrac{u_{i,j}(t+t_{n_k})}{u_{I,J}(t_{n_k})}\to v_{i,j}(t)$ in $\mathbb{R}$ as $k\to\infty$ uniformly in any compact subset of $\mathbb{R}$. Moreover, the limit $v_{i,j}$ satisfies

$$\begin{aligned}v'_{i,j}(t) &= p_{i+1,j}v_{i+1,j}(t)+p_{i,j}v_{i-1,j}(t)+q_{i,j+1}v_{i,j+1}(t)+q_{i,j}v_{i,j-1}(t)\\ &\quad -D_{i,j}v_{i,j}(t)+f'_s(i,j,0)v_{i,j}(t).\end{aligned}\tag{3.8}$$

Now, we claim $v_{i,j}(t)>0$ for all $(i,j,t)\in\mathbb{Z}^2\times\mathbb{R}$. Note that $v_{i,j}(t)\ge 0$ and $v_{I,J}(0)=1$. If there is $(i_0,j_0)\in\mathbb{Z}^2$ such that $v_{i_0,j_0}(0)=0$, then $v'_{i_0,j_0}(0)=0$. It follows from (3.8) that

$$0=p_{i_0+1,j_0}v_{i_0+1,j_0}(0)+p_{i_0,j_0}v_{i_0-1,j_0}(0)+q_{i_0,j_0+1}v_{i_0,j_0+1}(0)+q_{i_0,j_0}v_{i_0,j_0-1}(0).$$

Hence $v_{i_0\pm1,j_0}(0)=v_{i_0,j_0\pm1}(0)=0$, since $p_{i,j},q_{i,j}>0$. By induction, we obtain that $v_{i,j}(0)=0$ for all $(i,j)\in\mathbb{Z}^2$. This contradicts $v_{I,J}(0)=1$. Therefore, $v_{i,j}(0)>0$ for all $(i,j)\in\mathbb{Z}^2$. Thus the comparison principle implies that $v_{i,j}(t)>0$ for all $(i,j,t)\in\mathbb{Z}^2\times[0,\infty)$. Moreover, since v also satisfies (1.3), $v_{i,j}(t)>0$ for all $(i,j,t)\in\mathbb{Z}^2\times\mathbb{R}$.

Define $z_{i,j}(t):=\dfrac{v'_{i,j}(t)}{v_{i,j}(t)}$, we shall show that $z_{i,j}(t)=\mu$ for all $(i,j,t)\in\mathbb{Z}^2\times\mathbb{R}$, where

$$\mu:=\min_{i,j}\left\{\liminf_{t\to-\infty}\frac{u'_{i,j}(t)}{u_{i,j}(t)}\right\}.$$

Note that $z_{i,j}(t)\ge\mu$ for all $(i,j,t)\in\mathbb{Z}^2\times\mathbb{R}$ by the definition of μ. We write

$$\begin{aligned}z'_{i,j}(t) &= \left[p_{i+1,j}\frac{v_{i+1,j}(t)}{v_{i,j}(t)}\right]z_{i+1,j}(t)+\left[p_{i,j}\frac{v_{i-1,j}(t)}{v_{i,j}(t)}\right]z_{i-1,j}(t)\\ &\quad+\left[q_{i,j+1}\frac{v_{i,j+1}(t)}{v_{i,j}(t)}\right]z_{i,j+1}(t)+\left[q_{i,j}\frac{v_{i,j-1}(t)}{v_{i,j}(t)}\right]z_{i,j-1}(t)\\ &\quad-\left[p_{i+1,j}\frac{v_{i+1,j}(t)}{v_{i,j}(t)}+p_{i,j}\frac{v_{i-1,j}(t)}{v_{i,j}(t)}+q_{i,j+1}\frac{v_{i,j+1}(t)}{v_{i,j}(t)}+q_{i,j}\frac{v_{i,j-1}(t)}{v_{i,j}(t)}\right]z_{i,j}(t).\end{aligned}$$

Let $\hat{z}(\cdot)=\{\hat{z}_{i,j}(\cdot)\}=\{\mu\}$ and note that

$$|z_{i,j}(t)|=\left|\frac{v'_{i,j}(t)}{v_{i,j}(t)}\right|\le\sup_{i,j,t}\left|\frac{v'_{i,j}(t)}{v_{i,j}(t)}\right|<\infty.$$

By the comparison principle (see Remark 3.1) and noting that $z_{i,j}$ satisfies (1.3), we conclude $z_{i,j}(t) = \mu$ for all $(i,j,t) \in \mathbb{Z}^2 \times \mathbb{R}$.

Finally, we prove that there exists $\Lambda > 0$ such that $M(\Lambda) = c\Lambda$. Since $\dfrac{v'_{i,j}(t)}{v_{i,j}(t)} = z_{i,j}(t) = \mu$ for all $(i,j,t) \in \mathbb{Z}^2 \times \mathbb{R}$. we obtain

$$v_{i,j}(t) = v_{i,j}(0)e^{\mu t} = U_{i,j}e^{\mu t-(\mu/c)(ri+sj)},$$

where $U_{i,j} = v_{i,j}(0)e^{(\mu/c)(ri+sj)}$. Then, by using (1.3), it is easy to see that $U_{i,j} \in K_{per}$. By (3.8), $U_{i,j}$ satisfies

$$\begin{aligned}\mu U_{i,j} &= p_{i+1,j}e^{-r\mu/c}U_{i+1,j} + p_{i,j}e^{r\mu/c}U_{i-1,j} + q_{i,j+1}e^{-s\mu/c}U_{i,j+1} \\ &\quad + q_{i,j}e^{s\mu/c}U_{i,j-1} - D_{i,j}U_{i,j} + f'_s(i,j,0)U_{i,j}.\end{aligned}$$

Then $M(\mu/c) \geq \mu$. On the other hand, recalling

$$M(\lambda) = \inf_{\phi \in K_{per}} \max_{(i,j)\in\mathbb{Z}^2} \frac{(\mathcal{L}_\lambda \phi)_{i,j}}{\phi_{i,j}},$$

so we have $\mu \geq M(\mu/c)$. It follows that $M(\Lambda) = c\Lambda$, where $\Lambda := \mu/c$. Therefore, the theorem is proved. $\square$

Therefore, we have proved the sufficient and necessary condition for existence of solution of (P).

4. Monotonicity of wave profile

In this section, we shall prove that any wave profile of (P) is strictly increasing in t under the assumptions **(A1)-(A5)**. Recall

(A5) there exists $\rho \in (0,1)$ such that $f(i,j,s_2) \leq f(i,j,s_1)$, if $\rho < s_1 < s_2 < 1$, $\forall (i,j)$.

First, we have the following lemma.

Lemma 4.1. *If (c,u) is a solution of (P) with $c \neq 0$ and $u'_{i,j}(t) \geq 0$ for all $(i,j,t) \in \mathbb{Z}^2 \times \mathbb{R}$, then $u'_{i,j}(t) > 0$ for all $(i,j,t) \in \mathbb{Z}^2 \times \mathbb{R}$.*

Proof. Differentiating (1.2) with respect to t and using a contradiction argument as in the proof of Lemma 2.1(i), we can easily prove this lemma. The detail is omitted. $\square$

Lemma 4.2. *Let (c,u) be a solution of (P) with $c \neq 0$. Then, given any $\varepsilon \in (0,1)$, there exist constants K_1 and K_2 such that*

(1) *$\varepsilon < u_{i,j}(t) < 1$ for all $(i,j,t) \in \mathbb{Z}^2 \times \mathbb{R}$ with $ct - ri - sj \geq K_1$,*
(2) *$0 < u_{i,j}(t) < \varepsilon$ for all $(i,j,t) \in \mathbb{Z}^2 \times \mathbb{R}$ with $ct - ri - sj \leq K_2$.*

Moreover, there exists a constant K_3 *such that*

$$u'_{i,j}(t) > 0 \text{ for all } (i,j,t) \in \mathbb{Z}^2 \times \mathbb{R} \text{ with } ct - ri - sj \leq K_3.$$

Proof. Given $\varepsilon \in (0,1)$. For each $(i,j) \in \mathbb{Z}^2$, since $u_{i,j}(+\infty) = 1$ and $u_{i,j}(-\infty) = 0$, there exists real numbers $\tau_{i,j}$ and $\kappa_{i,j}$ such that

$$\varepsilon < u_{i,j}(t) < 1 \ \ \forall\, t \geq \tau_{i,j}, \quad 0 < u_{i,j}(t) < \varepsilon \ \ \forall\, t \leq \kappa_{i,j}.$$

Define

$$K_1 := \max_{i,j \in \{1,\cdots,N\}} \{c\tau_{i,j} - ri - sj\}, \quad K_2 := \min_{i,j \in \{1,\cdots,N\}} \{c\kappa_{i,j} - ri - sj\}.$$

Then (1) and (2) follows from (1.3).

Next, recall

$$\mu := \min_{i,j} \left\{ \liminf_{t \to -\infty} \frac{u'_{i,j}(t)}{u_{i,j}(t)} \right\} > 0.$$

For each fixed $i,j \in \{1,..,N\}$, there exists $T_{i,j} < 0$ such that

$$\frac{u'_{i,j}(t)}{u_{i,j}(t)} > \frac{\mu}{2} > 0 \ \ \forall\, t \leq T_{i,j}.$$

Since $u_{i,j}(t) > 0$ for all $(i,j,t) \in \mathbb{Z}^2 \times \mathbb{R}$, we obtain that $u'_{i,j}(t) > 0$ for all $t \leq T_{i,j}$. Now, we define

$$K_3 := \min_{i,j \in \{1,\cdots,N\}} \{cT_{i,j} - ri - sj\}$$

and use (1.3), it follows that $u'_{i,j}(t) > 0$ for all $(i,j,t) \in \mathbb{Z}^2 \times \mathbb{R}$ with $ct - ri - sj \leq K_3$. Hence the lemma is proved. $\square$

As a corollary of Lemma 4.2, by using

$$u_{i,j}(t) = w_{i,j}(ct - ri - sj),$$

we have

Corollary 4.3. *Let* (c, w) *be a solution of* (P') *with* $c \neq 0$. *Then, given any* $\varepsilon \in (0,1)$, *there exist constants* K_1 *and* K_2 *such that* $\varepsilon < w_{i,j}(\xi) < 1$ *for all* $\xi \geq K_1$ *and* $0 < w_{i,j}(\xi) < \varepsilon$ *for all* $\xi \leq K_2$ *for any* $(i,j) \in \mathbb{Z}^2$. *Moreover, there exists a constant* K_3 *such that* $w'_{i,j}(\xi) > 0$ *for all* $\xi \leq K_3$ *for any* $(i,j) \in \mathbb{Z}^2$.

To derive the monotonicity of u in t, we use a sliding method. It is more convenient to consider the function w than u. We set

$$A := \{\tau > 0 \mid w_{i,j}(\xi + T) \geq w_{i,j}(\xi) \ \ \forall\, i,j \in \{1,\cdots,N\},\ \xi \in \mathbb{R},\ T \geq \tau\}.$$

Then we have the following lemma.

Lemma 4.4. $A \neq \varnothing$.

Proof. Let ρ be the constant defined in **(A5)**. Then, by Corollary 4.3, we can choose $D \geq 1$ such that

$$\rho < w_{i,j}(\xi) < 1 \quad \forall\ (i,j) \in \mathbb{Z}^2,\ \xi \geq D, \tag{4.1}$$

$$0 < w_{i,j}(\xi) < \rho, \quad w'_{i,j}(\xi) > 0 \ \ \forall\ (i,j) \in \mathbb{Z}^2,\ \xi \leq -D+1. \tag{4.2}$$

Note that the constant

$$\eta := \min\{w_{i,j}(\xi) \mid \xi \in [-D,D], i,j \in \{1,\cdots,N\}\} \tag{4.3}$$

is positive. Since $w_{i,j}(-\infty) = 0$, we can choose a constant $T_0 \geq 2D$ such that $w_{i,j}(\xi) < \eta$ for all $\xi \leq D - T_0$ for all $i,j \in \{1,\cdots,N\}$. For any $T \geq T_0$ and $\xi \leq -D$, since

$$\begin{aligned}
&\xi + T \geq D, \text{ if } \xi \geq D - T;\\
&\xi + T \in [-D,D], \text{ if } \xi \in [-D-T, D-T];\\
&\xi + T \leq -D, \text{ if } \xi \leq -D - T,
\end{aligned}$$

it follows from (4.1)-(4.3) that

$$\begin{aligned}
&w_{i,j}(\xi+T) > \rho > w_{i,j}(\xi), \quad \text{if } -D \geq \xi \geq D - T;\\
&w_{i,j}(\xi+T) \geq \eta > w_{i,j}(\xi), \quad \text{if } \xi \in [-D-T, D-T];\\
&w_{i,j}(\xi+T) > w_{i,j}(\xi), \quad \text{if } \xi \leq -D - T,
\end{aligned}$$

where the last inequality follows from the fact $w'_{i,j} > 0$ in $(-\infty, -D+1]$. Therefore, we have

$$w_{i,j}(\xi+T) > w_{i,j}(\xi) \quad \forall\ \xi \leq -D, T \geq T_0, i,j \in \{1,\cdots,N\}. \tag{4.4}$$

Now, we set

$$\sigma := \max\{w_{i,j}(\xi) \mid \xi \in [-D, D+1], i,j \in \{1,\cdots,N\}\}.$$

Then $\sigma \in (0,1)$. Since $w_{i,j}(\infty) = 1$, there exists $M \gg 1$ such that $w_{i,j}(\xi) > \sigma$ for all $\xi \geq M$ for all $i,j \in \{1,\cdots,N\}$. Taking a larger T_0 so that $T_0 \geq \max\{2D, D+M\}$, we obtain that

$$w_{i,j}(\xi+T) > \sigma \geq w_{i,j}(\xi) \quad \forall\ \xi \in [-D, D+1], T \geq T_0, i,j \in \{1,\cdots,N\}. \tag{4.5}$$

Finally, for the same T_0 as above, we claim that

$$w_{i,j}(\xi+T) \geq w_{i,j}(\xi) \quad \forall\ \xi \geq D,\ T \geq T_0,\ i,j \in \{1,\cdots,N\}. \tag{4.6}$$

To prove (4.6), we consider the function

$$W_{i,j}(\xi) = W_{i,j}(\xi;\delta) := w_{i,j}(\xi+T) - w_{i,j}(\xi) + \delta, \quad \xi \in \mathbb{R},$$

where the constants δ, T are given so that $\delta \in (0,2)$ and $T \geq T_0$. Since $w_{i,j} < 1$, we have $W_{i,j}(\xi;\delta) > 0$ for all ξ, if $\delta > 1$. Moreover, for any $\delta > 0$, $W_{i,j}(\xi;\delta) > 0$, if $\xi \gg 1$, since $w_{i,j}(\infty) = 1$. Recall that $W_{i,j}(\xi;\delta) > 0$ for any $\delta > 0$, $\xi \leq D+1$, $T \geq T_0$, $i,j \in \{1,\cdots,N\}$.

We claim that $W_{i,j}(\xi;\delta) > 0$ for all $\xi \geq D$, $i,j \in \{1,\cdots,N\}$, for any $\delta > 0$. For contradiction, suppose that there exist $\delta_0 \in (0,1]$, $y \geq D+1$ (by (4.5)), $I,J \in \{1,\cdots,N\}$ such that $W_{I,J}(y;\delta_0) = 0$ and $W_{i,j}(\xi;\delta) > 0$ for any $\xi \geq D$, $\delta \in (\delta_0, 2)$, $i,j \in \{1,\cdots,N\}$. Then $W_{i,j}(\xi;\delta_0) \geq 0$ for all $\xi \geq D$, $i,j \in \{1,\cdots,N\}$. Since $W'_{I,J}(y;\delta_0) = 0$, we have

$$\begin{aligned} 0 &= p_{I+1,J}W_{I+1,J}(y-r) + p_{I,J}W_{I-1,J}(y+r) + q_{I,J+1}W_{I,J+1}(y-s) \\ &\quad + q_{I,J}W_{I,J-1}(y+s) - D_{I,J}W_{I,J}(y) + f(I,J,w_{I,J}(y+T)) - f(I,J,w_{I,J}(y)), \end{aligned}$$

by using (2.6). Hereafter we suppress the dependence of δ_0. Since $W_{I,J}(y) = 0$, we have $w_{I,J}(y+T) = w_{I,J}(y) - \delta_0 < w_{I,J}(y)$. Also, $y+T > y \geq D$, it follows from (4.1) that $w_{I,J}(y) > w_{I,J}(y+T) > \rho$. Then the assumption **(A5)** implies that

$$\begin{aligned} 0 &\geq p_{I+1,J}W_{I+1,J}(y-r) + p_{I,J}W_{I-1,J}(y+r) \\ &\quad + q_{I,J+1}W_{I,J+1}(y-s) + q_{I,J}W_{I,J-1}(y+s). \end{aligned}$$

Note that $r,s \in [-1,1]$. Also, recall that $y \geq D+1$. Hence

$$W_{I+1,J}(y-r) = W_{I-1,J}(y+r) = W_{I,J+1}(y-s) = W_{I,J-1}(y+s) = 0.$$

Without loss of generality, we may assume that $r > 0$. Starting with $W_{I-1,J}(y+r) = 0$, by induction, we can show that $W_{I-K,J}(y+Kr) = 0$ for any $K \in \mathbb{N}$. In particular, for $K = kN$, we have $W_{I,J}(y+kNr) = W_{I-kN,J}(y+kNr) = 0$. But, this is a contradiction to $w_{i,j}(\infty) = 1$, if we let $k \to \infty$. We thus have proved that $W_{i,j}(\xi;\delta) > 0$ for all $\xi \geq D$, $T \geq T_0$, $i,j \in \{1,\cdots,N\}$, $\delta > 0$. Taking $\delta \downarrow 0$, (4.6) follows.

Combining (4.4)-(4.6), we see that $T_0 \in A$. This completes the proof of the lemma. $\square$

Proof of Theorem 3. It follows from Lemma 4.4 that the number $T^* := \inf A$ is well-defined and $T^* \geq 0$. If $T^* = 0$, then we have $w_{i,j}(\xi+T) \geq w_{i,j}(\xi)$ for all $\xi \in \mathbb{R}$, $T > 0$, and $i,j \in \mathbb{Z}$. Hence $w'_{i,j}(\xi) \geq 0$ for all $\xi \in \mathbb{R}$ and $i,j \in \mathbb{Z}$. Since $u'_{i,j}(t) = cw'_{i,j}(ct - ri - sj)$ and $c > 0$, we have $u'_{i,j}(t) \geq 0$ for all $t \in \mathbb{R}$ and $i,j \in \mathbb{Z}$. It then follows from Lemma 4.1 that $u'_{i,j}(t) > 0$ for all $t \in \mathbb{R}$ and $i,j \in \mathbb{Z}$. Hence the theorem is proved. Therefore, it suffices to prove that $T^* = 0$.

To prove $T^* = 0$, we use a contradiction argument. Suppose that $T^* > 0$. Then

$$w_{i,j}(\xi+T^*) \geq w_{i,j}(\xi) \quad \forall\, \xi \in \mathbb{R},\ i,j \in \mathbb{Z}.$$

We shall follow a similar argument as in Lemma 4.4. Set

$$W_{i,j}(\xi;T) := w_{i,j}(\xi+T) - w_{i,j}(\xi), \quad \xi \in \mathbb{R},\ i,j \in \mathbb{Z},\ T > 0.$$

Note that $W_{i,j}(\cdot\,;T^*) \geq 0$. We claim that $W_{i,j}(\xi;T^*) > 0$ for all $\xi \in \mathbb{R}$, $i,j \in \mathbb{Z}$. Otherwise, there exists $(I,J,y) \in \mathbb{Z}^2 \times \mathbb{R}$ such that $W_{I,J}(y;T^*) = 0$. Then $W'_{I,J}(y;T^*) = 0$ and it follows from (2.6) that

$$p_{I+1,J}W_{I+1,J}(y-r;T^*)+p_{I,J}W_{I-1,J}(y+r;T^*)$$
$$+q_{I,J+1}W_{I,J+1}(y-s;T^*)+q_{I,J}W_{I,J-1}(y+s;T^*)=0.$$

Hence

$$W_{I+1,J}(y-r;T^*)=W_{I-1,J}(y+r;T^*)=W_{I,J+1}(y-s;T^*)=W_{I,J-1}(y+s;T^*)=0.$$

This leads to a contradiction by the same argument as in the proof of Lemma 4.4. Hence we obtain that $W_{i,j}(\xi;T^*)>0$ for all $\xi\in\mathbb{R}$, $i,j\in\mathbb{Z}$.

Now, for the constant D defined in (4.1)-(4.2), we set

$$\kappa:=\min\{W_{i,j}(\xi;T^*)\mid \xi\in[-D-T^*,D+1]\}.$$

Then κ is well-defined and $\kappa>0$. Also, by the continuity of $w_{i,j}$, there exists a constant $\tau\in(0,T^*)$ such that $W_{i,j}(\xi;T)>\kappa/2$ for all $\xi\in[-D-T^*,D+1]$, $i,j\in\mathbb{Z}$, $T\in[\tau,T^*]$. Hence for $T\in[\tau,T^*]$ we have

$$w_{i,j}(\xi+T)>w_{i,j}(\xi)\quad\forall\,\xi\in[-D-T^*,D+1],\ i,j\in\mathbb{Z}. \tag{4.7}$$

For $\xi\le -D-T^*$, since $\xi<\xi+T\le -D<-D+1$ for $T\in[\tau,T^*]$, it follows from (4.2) that

$$w_{i,j}(\xi+T)>w_{i,j}(\xi)\quad\forall\,\xi\le -D-T^*,\ i,j\in\mathbb{Z},\ T\in[\tau,T^*]. \tag{4.8}$$

Finally, as in the proof of Lemma 4.4, we can also show that

$$w_{i,j}(\xi+T)\ge w_{i,j}(\xi)\quad\forall\,\xi\ge D,\ i,j\in\mathbb{Z},\ T\in[\tau,T^*]. \tag{4.9}$$

Combining (4.7)-(4.9), we conclude that $\tau\in A$, a contradiction to the definition of T^*. Hence we must have $T^*=0$ and the theorem is proved. □

5. Convergence of the discretized minimal speed

In this section, we shall follow the idea of [19] to prove **Theorem 4**. Since the proof is quite similar to the one given in [19], we shall omit some details.

First note that

$$hc_*(h)=\min_{\lambda>0}\frac{LM^h(\lambda)}{\lambda},\quad (h=\frac{L}{N})$$

where $M^h(\lambda)$ is the largest real number such that there exists $\phi\in K_{per}$ satisfying

$$\begin{aligned} M^h(\lambda)\phi_{i,j} &= p^h_{i+1,j}e^{-r\lambda/N}\phi_{i+1,j}+p^h_{i,j}e^{r\lambda/N}\phi_{i-1,j}+q^h_{i,j+1}e^{-s\lambda/N}\phi_{i,j+1}\\ &\quad +q^h_{i,j}e^{s\lambda/N}\phi_{i,j-1}-D^h_{i,j}\phi_{i,j}+(f^h)'_s(i,j,0)\phi_{i,j}, \end{aligned} \tag{5.1}$$

for all $i,j\in\mathbb{Z}$ with $D^h_{i,j}:=p^h_{i+1,j}+p^h_{i,j}+q^h_{i,j+1}+q^h_{i,j}$.

The proof of the following lemma is similar to that of Lemma 4.1 in [19].

Lemma 5.1. $\limsup_{N\to+\infty}[hc_*(h)] < +\infty$, *where* $h = L/N$.

Next, we set $\gamma := \liminf_{N\to+\infty}[hc_*(h)]$. By Lemma 5.1, $\gamma \in [0,+\infty)$. Let $\{h_k\} = \{L/N_k\}$ be a sequence such that $N_k \to +\infty$ and $h_k c_*(h_k) \to \gamma$ as $k \to +\infty$. For each k, we define $\lambda_k > 0$ such that

$$\frac{LM^{h_k}(\lambda_k)}{\lambda_k} = \min_{\lambda>0} \frac{LM^{h_k}(\lambda)}{\lambda} = h_k c_*(h_k).$$

Lemma 5.2. *There are two positive numbers* A *and* B *such that*

$$0 < A \le \lambda_k \le B < +\infty.$$

Proof. By (2.2), $M^{h_k}(\lambda_k) \ge \min_{i,j}(f^{h_k})'_s(i,j,0) \ge \min_{\mathbb{R}^2} f'_s(x,y,0) > 0$. If there exists $\{\lambda_{k_j}\}$ such that $\lambda_{k_j} \to 0$ as $j \to +\infty$, then $\gamma = +\infty$, a contradiction. This proves a uniformly positive lower bound for $\{\lambda_k\}$.

To find an upper bound B, we set

$$\limsup_{k\to+\infty} \frac{\lambda_k}{N_k} := \kappa \in [0,+\infty].$$

Then by the same argument as the proof of Lemma 4.2 in [19] we can conclude that $\kappa = 0$ and so

$$\lim_{k\to+\infty} \frac{\lambda_k}{N_k} = 0.$$

By using the fact $\lim_{x\to 0}(e^x - 1 - x)/x^2 = 1/2$, we have

$$e^{\pm r\lambda_k/N_k} - 1 > \pm\frac{r\lambda_k}{N_k} + \frac{1}{4}(\pm\frac{r\lambda_k}{N_k})^2 \text{ and } e^{\pm s\lambda_k/N_k} - 1 > \pm\frac{s\lambda_k}{N_k} + \frac{1}{4}(\pm\frac{s\lambda_k}{N_k})^2$$

for all sufficiently large k. Also, as in (2.2), we obtain that

$$\begin{aligned}\frac{LM^{h_k}(\lambda_k)}{\lambda_k} \quad > \quad & \frac{L}{\lambda_k}\{p^{h_k}_{I_k+1,J_k}(-\frac{r\lambda_k}{N_k} + \frac{1}{4}(\frac{r\lambda_k}{N_k})^2) + p^{h_k}_{I_k,J_k}(\frac{r\lambda_k}{N_k} + \frac{1}{4}(\frac{r\lambda_k}{N_k})^2) \\ & + q^{h_k}_{I_k,J_k+1}(-\frac{s\lambda_k}{N_k} + \frac{1}{4}(\frac{s\lambda_k}{N_k})^2) + q^{h_k}_{I_k,J_k}(\frac{s\lambda_k}{N_k} + \frac{1}{4}(\frac{s\lambda_k}{N_k})^2)\}\end{aligned}$$

for all k large enough. Hence we can find two positive constants C_1 and C_2 such that

$$\lambda_k \le C_1 \frac{M^{h_k}(\lambda_k)}{\lambda_k} + C_2$$

for all sufficiently large k. Since

$$\frac{LM^{h_k}(\lambda_k)}{\lambda_k} \to \gamma \text{ as } k \to +\infty,$$

we obtain an upper bound estimate for λ_k. Hence the lemma follows. $\square$

Recall the operator $\mathcal{P}$ and the set E defined in §1:

$$\begin{aligned}
\mathcal{P}_\lambda \phi &:= \nabla\cdot(A\nabla\phi) - 2\lambda e^T A\nabla\phi + [-\lambda\nabla\cdot(Ae) + \lambda^2 e^T A e + f_s'(x_1,x_2,0)]\phi,\\
E &:= \{\phi\in C^2(\mathbb{R}^2) \mid \phi(x_1+L,x_2) = \phi(x_1,x_2) = \phi(x_1,x_2+L)\}.
\end{aligned}$$

By Lemma 5.2, there is a number $\Lambda\in(0,+\infty)$ such that $\lambda_k\to\Lambda$ as $k\to+\infty$ (up to some subsequence of $\{\lambda_k\}$). Thus, we have

$$M^{h_k}(\lambda_k)\to\gamma\Lambda/L \text{ as } k\to+\infty. \tag{5.2}$$

Now, we define a function space

$$H^1_{per} := \{\psi\in H^1_{loc}(\mathbb{R}^2) \mid \psi(x_1+L,x_2) = \psi(x_1,x_2) = \psi(x_1,x_2+L)\}$$

with the H^1 norm in $(0,L)\times(0,L)$.

Lemma 5.3. *There exists $\phi\in E$ such that $\phi>0$ and $\mathcal{P}_\mu\phi = \mu\gamma\phi$, where $\mu := \Lambda/L > 0$.*

Proof. For each k, there exists $u^k\in K_{per}$ such that

$$\begin{aligned}
M^{h_k}(\lambda_k)u^k_{i,j} &= p^{h_k}_{i+1,j}e^{-r\lambda_k/N_k}u^k_{i+1,j} + p^{h_k}_{i,j}e^{r\lambda_k/N_k}u^k_{i-1,j} + q^{h_k}_{i,j+1}e^{-s\lambda_k/N_k}u^k_{i,j+1}\\
&\quad + q^{h_k}_{i,j}e^{s\lambda_k/N_k}u^k_{i,j-1} - D^{h_k}_{i,j}u^k_{i,j} + (f^{h_k})_s'(i,j,0)u^k_{i,j},\ \forall i,j.
\end{aligned} \tag{5.3}$$

With this u^k, we define $\phi_k:\mathbb{R}^2\to\mathbb{R}$ by

$$\begin{aligned}
\phi_k(x_1,x_2) &= u^k_{i,j} \text{ if } (x_1,x_2) = (ih_k, jh_k) \text{ for some } i,j\in\mathbb{Z};\\
\phi_k(x_1,x_2) &= \Big(\frac{u^k_{i+1,j}-u^k_{i,j}}{h_k}\Big)x_1 + \Big(\frac{u^k_{i,j+1}-u^k_{i,j}}{h_k}\Big)x_2\\
&\quad + u^k_{i,j} - i(u^k_{i+1,j}-u^k_{i,j}) - j(u^k_{i,j+1}-u^k_{i,j}),\\
&\quad \text{if } x_1\ge ih_k,\ x_2\ge jh_k \text{ and } x_1+x_2\le(i+j+1)h_k \text{ for some } i,j\in\mathbb{Z};\\
\phi_k(x_1,x_2) &= \Big(\frac{u^k_{i+1,j+1}-u^k_{i,j+1}}{h_k}\Big)x_1 + \Big(\frac{u^k_{i+1,j+1}-u^k_{i+1,j}}{h_k}\Big)x_2\\
&\quad + u^k_{i+1,j+1} - (i+1)(u^k_{i+1,j+1}-u^k_{i,j+1}) - (j+1)(u^k_{i+1,j+1}-u^k_{i+1,j}),\\
&\quad \text{if } x_1\le(i+1)h_k,\ x_2\le(j+1)h_k \text{ and } x_1+x_2\ge(i+j+1)h_k\\
&\quad \text{for some } i,j\in\mathbb{Z}.
\end{aligned}$$

Set $D := (0,L)\times(0,L)$. Since (5.3) is linear, without loss of generality, we may assume that $||\phi_k||^2_{L^2(D)} = 1$ for all k. Note that $\phi_k\in H^1_{per}$ for all k.

We claim that $\{\phi_k\}$ is bounded in H^1_{per}. Multiplying (5.3) by $u^k_{i,j}h_k^2$ and summing over $i,j=1,...,N_k$, due to the periodicity of $d^{h_k}_{i,j}$ and $u^k_{i,j}$, we have

$$
\begin{aligned}
& h_k^2\sum_{i,j=1}^{N_k} M^{h_k}(\lambda_k)(u^k_{i,j})^2 \\
= \ & h_k^2\Big\{\sum_{i,j=1}^{N_k} p^{h_k}_{i,j}u^k_{i-1,j}u^k_{i,j}(e^{-r\lambda_k/N_k}+e^{r\lambda_k/N_k}) \\
& +\sum_{i,j=1}^{N_k} q^{h_k}_{i,j}u^k_{i,j-1}u^k_{i,j}(e^{-s\lambda_k/N_k}+e^{s\lambda_k/N_k}) \\
& -\sum_{i,j=1}^{N_k} p^{h_k}_{i,j}[(u^k_{i-1,j})^2+(u^k_{i,j})^2]-\sum_{i,j=1}^{N_k} q^{h_k}_{i,j}[(u^k_{i,j-1})^2+(u^k_{i,j})^2] \\
& +\sum_{i,j=1}^{N_k}(f^{h_k})'_s(ih_k,jh_k,0)(u^k_{i,j})^2\Big\}.
\end{aligned}
\tag{5.4}
$$

It follows from the definition of ϕ_k and (5.4) that

$$
\begin{aligned}
& \int_D [(\phi_k)_{x_1}]^2+[(\phi_k)_{x_2}]^2dx_1dx_2 \\
\leq \ & C_1(h_k)^2\Big\{\sum_{i,j=1}^{N_k} p^{h_k}_{i,j}u^k_{i-1,j}u^k_{i,j}[e^{-r\lambda_k/N_k}+e^{r\lambda_k/N_k}-2] \\
& +\sum_{i,j=1}^{N_k} q^{h_k}_{i,j}u^k_{i,j-1}u^k_{i,j}[e^{-s\lambda_k/N_k}+e^{s\lambda_k/N_k}-2] \\
& +\sum_{i,j=1}^{N_k}(f^{h_k})'_s(ih_k,jh_k,0)(u^k_{i,j})^2-\sum_{i,j=1}^{N_k} M^{h_k}(\lambda_k)(u^k_{i,j})^2\Big\} \\
\leq \ & C_2(h_k)^2\sum_{i,j=1}^{N_k}(u^k_{i,j})^2,
\end{aligned}
\tag{5.5}
$$

for some positive constants C_1 and C_2. On the other hand, by the definition of ϕ_k we can easily calculate that

$$1=||\phi_k||^2_{L^2(D)}\geq C_3(h_k)^2\sum_{i,j=1}^{N_k}(u^k_{i,j})^2 \quad \text{for all } k$$

for some positive constant C_3. It follows from (5.5) that $\{\phi_k\}$ is uniformly bounded in H^1_{per}. Up to some subsequence, there exists $\phi\in H^1_{per}$ such that $\phi_k\rightharpoonup\phi$ in H^1_{per} weakly and $\phi_k\to\phi$ in L^2. Note that we have $\phi\not\equiv 0$, since $||\phi||^2_{L^2(D)}=1$. Also, without loss of generality, we may assume $\phi_k\to\phi$ a.e. for the same sequence.

Finally, we claim that

$$\begin{aligned}(5.6)\ \mu\gamma\int_D \phi\psi &= -\int_D p\phi_{x_1}\psi_{x_1} - \int_D q\phi_{x_2}\psi_{x_2} - \mu\int_D (rp\phi_{x_1} + sq\phi_{x_2})\psi + \mu\int_D rp\phi\psi_{x_1} \\ &\quad +\mu\int_D sq\phi\psi_{x_2} + \mu^2\int_D (r^2p + s^2q)\phi\psi + \int_D f'_s(x_1, x_2, 0)\phi\psi\end{aligned}$$

for any smooth test function ψ. Multiplying (5.3) by $h_k^2\psi(ih_k, jh_k)$, we write

$$\sum_{i,j=1}^{N_k} h_k^2 M^{h_k}(\lambda_k)\phi_k(ih_k, jh_k)\psi(ih_k, jh_k) = A_k + B_k + C_k + D_k,$$

where

$$\begin{aligned}A_k &:= \sum_{i,j=1}^{N_k} h_k^2 (f^{h_k})'_s(ih_k, jh_k, 0)\phi_k(ih_k, jh_k)\psi(ih_k, jh_k),\\ B_k &:= -\sum_{i,j=1}^{N_k} h_k^2 p_{i,j}^{h_k}(\phi_k(ih_k, jh_k) - \phi_k((i-1)h_k, jh_k))(\psi(ih_k, jh_k) - \psi((i-1)h_k, jh_k))\\ &\quad -\sum_{i,j=1}^{N_k} h_k^2 q_{i,j}^{h_k}(\phi_k(ih_k, jh_k) - \phi_k(ih_k, (j-1)h_k))(\psi(ih_k, jh_k) - \psi(ih_k, (j-1)h_k)),\\ C_k &:= \sum_{i,j=1}^{N_k} h_k^2 p_{i,j}^{h_k}[(e^{-r\lambda_k/N_k} - 1)\phi_k(ih_k, jh_k)\psi((i-1)h_k, jh_k)\\ &\quad +(e^{r\lambda_k/N_k} - 1)\phi_k((i-1)h_k, jh_k)\psi(ih_k, jh_k)],\\ D_k &:= \sum_{i,j=1}^{N_k} h_k^2 q_{i,j}^{h_k}[(e^{-s\lambda_k/N_k} - 1)\phi_k(ih_k, jh_k)\psi(ih_k, (j-1)h_k)\\ &\quad +(e^{s\lambda_k/N_k} - 1)\phi_k(ih_k, (j-1)h_k)\psi(ih_k, jh_k)].\end{aligned}$$

Passing to the limit, we can derive (cf. [19]) that

$$\begin{aligned}&\sum_{i,j=1}^{N_k} h_k^2 M^{h_k}(\lambda_k)\phi_k(ih_k, jh_k)\psi(ih_k, jh_k) \to \mu\gamma\int_D \phi\psi dx_1dx_2 \text{ as } k\to+\infty,\\ &A_k \to \int_D f'_s(x,0)\phi\psi dx_1dx_2 \text{ as } k\to+\infty,\\ &B_k \to -\int_D p\phi_{x_1}\psi_{x_1}dx_1dx_2 - \int_D q\phi_{x_2}\psi_{x_2}dx_1dx_2 \text{ as } k\to+\infty,\\ &C_k \to \mu\int_D pr(-\phi_{x_1}\psi + \phi\psi_{x_1})dx_1dx_2 + \mu^2r^2\int_D p\phi\psi dx_1dx_2 \quad \text{as } k\to+\infty,\\ &D_k \to \mu\int_D qs(-\phi_{x_2}\psi + \phi\psi_{x_2})dx_1dx_2 + \mu^2s^2\int_D q\phi\psi dx_1dx_2 \text{ as } k\to+\infty.\end{aligned}$$

Therefore, (5.6) is proved. Moreover, by p, $q \in C^{1,\delta}(\mathbb{R}^2)$ and using the elliptic regularity theory, $\phi \in C^{2,\delta}(\mathbb{R}^2)$ such that $\mathcal{P}_\mu \phi = \mu\gamma\phi$. Finally, by the strong maximum principle, we have $\phi > 0$. Hence the lemma follows. □

Proof of Theorem 4. By Lemma 5.3, we conclude that $k(\mu) = \gamma\mu$ and so

$$\liminf_{N\to+\infty}[hc_*(h)] = \gamma \geq \min_{\lambda>0} k(\lambda)/\lambda := \gamma_*.$$

On the other hand, we define

$$\kappa := \limsup_{N\to+\infty}[hc_*(h)].$$

Note that $0 \leq \kappa < +\infty$, by Lemma 5.1. Then there exists a sequence $\{h_k\} = \{L/N_k\}$ such that $N_k \to +\infty$ and $h_k c_*(h_k) \to \kappa$ as $k \to +\infty$.

Now, we choose any positive real number ν. Then we know

$$h_k c_*(h_k) := \min_{\lambda>0} \frac{LM^{h_k}(\lambda)}{\lambda} \leq \frac{LM^{h_k}(\nu)}{\nu}. \tag{5.7}$$

As in the proof of Lemma 5.1, we know $\left\{\dfrac{LM^{h_k}(\nu)}{\nu}\right\}$ is uniformly bounded in k. Since

$$M^{h_k}(\nu) \geq \min_{\mathbb{R}^2} f_s'(x, y, 0) > 0 \ \text{ for all } \ k,$$

there is a positive real number ρ such that (up to some subsequence)

$$\frac{LM^{h_k}(\nu)}{\nu} \to \rho \ \text{ as } \ k \to +\infty.$$

Next, following the same argument as in the proof of Lemma 5.3, we can derive that

$$k(\nu/L) = \rho\nu/L.$$

Then, by taking $k \to +\infty$ in (5.7), we obtain

$$\kappa \leq \rho = \frac{k(\nu/L)}{\nu/L}.$$

Since $\nu > 0$ is arbitrary, we have

$$\limsup_{N\to+\infty}[hc_*(h)] := \kappa \leq \gamma_*.$$

This completes the proof of Theorem 4. □

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