ENTIRE SOLUTIONS ORIGINATING FROM TRAVELING FRONTS FOR A TWO-SPECIES COMPETITION-DIFFUSION SYSTEM

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Abstract. This paper deals with entire solutions (classical solutions defined globally in time and space) of a two-species strong competition model. For this system, it is well known that there exist two-front entire solutions which behave as two traveling fronts moving towards each other from both sides of the x-axis. In this paper, in terms of traveling fronts connecting two different constant states from the coexistence state and the two semi-trivial states, we build entire solutions originating from three and four fronts stuck between appropriate super and subsolutions. Moreover, the non-existence of entire solutions originating from more than seven traveling fronts is proved.

1. Introduction

Competition between species is one of the fundamental features in ecology and occurs in virtually every ecosystem in nature. It could often result in the survival of the fittest and may lead to coexistence when species compete for the same resources. A typical competition model that has been studied widely is the classical Lotka-Volterra type diffusion-competition model between two species, which is described as follows:

$$\begin{cases} u_t = d_1 u_{xx} + r_1 u(1 - c_{11}u - c_{12}v), & x \in \mathbb{R},\ t \in \mathbb{R}, \\ v_t = d_2 v_{xx} + r_2 v(1 - c_{22}u - c_{22}v), & x \in \mathbb{R},\ t \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $u(x,t)$ and $v(x,t)$ represent the population density of two competing species at the position x and time t; d_1, d_2 are the diffusion coefficients of the two species; r_1, r_2 are the intrinsic growth rates of the two species; c_{11} and c_{22} stand for self-regulation of each species and c_{12}, c_{21} are the (inter-specific) competition coefficients of species u and v, respectively. All parameters are assumed to be positive.

By a suitable scaling (cf.[29]), system (1.1) can be reduced to the following dimensionless system

$$\begin{cases} u_t = u_{xx} + u(1 - u - kv), & x \in \mathbb{R},\ t \in \mathbb{R}, \\ v_t = Dv_{xx} + rv(1 - hu - v), & x \in \mathbb{R},\ t \in \mathbb{R}. \end{cases} \tag{1.2}$$

The dynamics for the related kinetic system (diffusion free) to (1.2) is well-known. It has at least three non-negative equilibria:

$$(u,v) = \mathbf{E_0} := (0,0), \quad \mathbf{E_u} := (1,0), \quad \mathbf{E_v} := (0,1).$$

Furthermore, if either $h,k > 1$ or $0 < h,k < 1$, there exists a unique positive equilibrium given by

$$\mathbf{E}_* = (u^*, v^*) := \Big(\frac{1-k}{1-hk}, \frac{1-h}{1-hk}\Big). \tag{1.3}$$

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For any given non-negative initial data for the kinetic system (diffusion free) of (1.2), one can classify the asymptotic behavior of solutions into four cases:

(A) If $0 < k < 1 < h$, then $\lim_{t\to+\infty}(u,v)(t) = \mathbf{E_u}$.
(B) If $0 < h < 1 < k$, then $\lim_{t\to+\infty}(u,v)(t) = \mathbf{E_v}$.
(C) If $h,k > 1$, then almost every trajectory tends to either $\mathbf{E_u}$ or $\mathbf{E_v}$ as $t \to +\infty$ depending on the initial data. This is known as the strong competition case.
(D) If $0 < h,k < 1$, then $\lim_{t\to+\infty}(u,v)(t) = \mathbf{E}_*$. This case is called the weak competition case.

Traveling front solutions play a fundamental role in understanding the interaction between species and have been studied intensively over the past four decades. By a traveling front solution, we mean a positive solution of (1.2) in the form $u(t,x) = U(x+ct)$ and $v(t,x) = V(x+ct)$ for some constant c such that the limits $(U,V)(\pm\infty)$ exist and are unequal (in the sense that $U(-\infty) \neq U(+\infty)$ and $V(-\infty) \neq V(+\infty)$), where c is called the wave speed and $(U,V)(\pm\infty) \in \{\mathbf{E_0}, \mathbf{E_u}, \mathbf{E_v}, \mathbf{E}_*\}$. For convenience, we call a traveling front solution an $(\mathbf{E_i}, \mathbf{E_j})$-front if

$$(U,V)(-\infty) = \mathbf{E_i}, \quad (U,V)(+\infty) = \mathbf{E_j}$$

for some $\mathbf{i},\mathbf{j} \in \{\mathbf{0}, \mathbf{u}, \mathbf{v}, *\}$. For the related works regarding system (1.2), we refer to, for example, [6, 9, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 30, 31, 32, 41] and the references cited therein.

Morita and Tachibana [28] established the existence of two-front entire solutions (i.e., classical solutions defined globally in time and space) which behave as two monotone waves approaching from both sides of x-axis. This remarkable result suggests that a simple competition model (1.2) may support rather complicated spatiotemporal patterns that exhibit the invasion process of the superior species. Since there are several different $(\mathbf{E_i}, \mathbf{E_j})$-fronts with $\mathbf{i},\mathbf{j} \in \{\mathbf{0}, \mathbf{u}, \mathbf{v}, *\}$, we may expect that there should exist more complicated mechanisms for the invasion of the superior species.

A possible pattern may be the N ($N \geq 3$) fronts entire solutions which behave as N waves propagating from both sides of x-axis as $t \to -\infty$. In fact, very recently, Chen, Guo, Ninomiya and Yao [5] constructed entire solutions originating from three and four monotone fronts, respectively, for the Allen-Cahn equation. More precisely, they considered

$$u_t = u_{xx} + \tilde{f}(u), \quad x \in \mathbb{R},\ t \in \mathbb{R},$$

where $\tilde{f} \in C^2(\mathbb{R})$ and satisfies some conditions such that its graph is similar to

$$\tilde{f}(u) = u(1-u)(u-a) \quad \text{for some } a \in (0, 1/2).$$

Therein, an entire solution u originates from N traveling fronts $\{(c_j, \phi_j) \,|\, j = 1, \cdots, N\}$ if these wave speeds satisfy $c_1 < c_2 < \cdots < c_N$ and

$$\lim_{t\to-\infty}\left[\sum_{j=1}^{N} \sup_{\omega_{j-1}(t)<x<\omega_j(t)} |u(x,t) - \phi_j(x + c_j t + \theta_j)|\right] = 0$$

for some $\theta_1, \cdots, \theta_N \in \mathbb{R}$, where $\omega_j(t) = -(c_j + c_{j+1})t/2$, $\omega_0(t) := -\infty$ and $\omega_N(t) = \infty$. Among other things, they also showed that entire solutions originating from N fronts for $N \geq 5$ is impossible. Our main goal in this paper is to generalize the work [5] on the Allen-Cahn equation to the two-species strong competition model.

The importance of the study of entire solutions of reaction-diffusion equations is frequently recalled in the literature. Since the pioneering works of Hamel and Nadirashvili [14, 15], there have been tremendous advances in investigating the existence of entire solutions for various models. See, for example, [2, 3, 4, 5, 8, 12, 13, 26, 27, 28, 33, 35, 36, 37, 38, 39, 40, 42] and the references cited therein.

In this paper, we shall focus on the strong competition case and always assume

(H1) $h > 1$ and $k > 1$.

Under assumption **(H1)**, up to symmetry, one can expect that there are $(\mathbf{E_i}, \mathbf{E_j})$-fronts with $\mathbf{i}, \mathbf{j} \in \{\mathbf{u}, \mathbf{v}, *\}$. Waves connecting either $\mathbf{E_u}$ or $\mathbf{E_v}$ to $\mathbf{E_0}$, if they exist, are not of front type. It is possible to have fronts connecting two equilibria $\mathbf{E_0}$ and $\mathbf{E}_*$, since $\mathbf{E}_*$ has a one-dimensional stable manifold.

For the existence of traveling fronts, by the results of Gardner [9], Conley, Gardner [6] and Kan-on [19], we see that an $(\mathbf{E_v}, \mathbf{E_u})$-front of system (1.2) exists. More precisely, there exists a unique (up to translations) bistable traveling front solution $(u, v)(x, t) = (U_1, V_1)(x + c_1 t)$ of (1.2) satisfying

$$(1.4) \qquad \begin{cases} c_1 U_1'(\xi) = U_1''(\xi) + U_1(\xi)(1 - U_1 - kV_1)(\xi), \quad \xi \in \mathbb{R}, \\ c_1 V_1(\xi)' = DV_1''(\xi) + rV_1(\xi)(1 - V_1 - hU_1)(\xi), \quad \xi \in \mathbb{R}, \\ (U_1, V_1)(-\infty) = (0, 1), \quad (U_1, V_1)(+\infty) = (1, 0), \\ U_1' > 0, \quad V_1' < 0 \quad \text{in } \mathbb{R}. \end{cases}$$

Note that the sign of c_1 is not completely understood (depending on the parameters). We refer to [10, 11, 19] for some discussions.

On the other hand, under assumption **(H1)** system (1.2) can support *monostable* traveling front connecting some suitable equilibria. In fact, one can apply the theory of Li, Weinberger and Lewis [25] to show that there exists $c_{2,\max} < 0$ such that a monotone traveling front $(u, v)(x, t) = (U_2, V_2)(x + c_2 t)$ of (1.2) connecting $\mathbf{E_v}$ and $\mathbf{E}_*$ exists if and only if $c_2 \le c_{2,\max}$; similarly, there exists $c_{3,\min} > 0$ such that a monotone traveling front $(u, v)(x, t) = (U_3, V_3)(x + c_3 t)$ of (1.2) connecting $\mathbf{E}_*$ and $\mathbf{E_u}$ exists if and only if $c_3 \ge c_{3,\min}$. More details can be found in section 2.

To construct multiple-front entire solutions for system (1.2), we need the following technical assumption:

(H2) There exists $\ell > 0$ such that

$$\frac{U_i(\xi)}{1 - V_i(\xi)} \ge \ell \quad \text{for all } \xi \le 0 \text{ and } i = 1, 2.$$

This kind of technical assumption was first proposed in [28] and has been often used in the literature in constructing super-sub-solutions for systems. Roughly speaking, **(H2)** is needed when we try to connect (U_1, V_1) and (U_2, V_2) at $\mathbf{E_v}$. On the other hand, since we will also connect (U_3, V_3) with (U_2, V_2) at $\mathbf{E}_*$, a parallel condition to **(H2)** is required. More precisely, we need to find $\nu > 0$ such that

$$\frac{U_3(\xi) - u^*}{v^* - V_3(\xi)} \ge \nu, \quad \frac{u^* - U_2(-\xi)}{V_2(-\xi) - v^*} \ge \nu \quad \text{for all } \xi \le 0.$$

In fact, such a positive constant ν always exists. See Lemma 2.6(6) and (7).

In addition, without loss of generality we may assume that the species u is stronger than the species v so that

(H3) $c_1 > 0$.

Otherwise, we may exchange the roles of u and v. Note that, under **(H3)**, from Lemma 2.3(1) and Lemma 2.4(1) it follows that **(H2)** holds if $\lambda_{ju}^+ < \lambda_{jv}^+$ for $j = 1, 2$.

We now state the main results of this paper as follows.

Theorem 1. *Assume* **(H1)**, **(H2)** *and* **(H3)**. *Let* (c_1, U_1, V_1), (c_2, U_2, V_2) *and* (c_3, U_3, V_3) *be an* $(\mathbf{E_v}, \mathbf{E_u})$*-front,* $(\mathbf{E_v}, \mathbf{E_*})$*-front and* $(\mathbf{E_*}, \mathbf{E_u})$*-front, respectively, such that*

$$-c_1 < c_2 \le c_{2,\max} < 0. \tag{1.5}$$

Then system (1.2) *admits a three-front entire solution* (u, v) *satisfying*

$$\lim_{t\to-\infty}\Big\{\sup_{x<\theta_1(t)}|u(x,t)-U_1(-x+c_1t+\omega)|+\sum_{j=2}^{3}\sup_{\theta_{j-1}(t)<x<\theta_j(t)}|u(x,t)-U_j(x+c_jt+\omega)|\Big\}=0,$$

$$\lim_{t\to-\infty}\Big\{\sup_{x<\theta_1(t)}|v(x,t)-V_1(-x+c_1t+\omega)|+\sum_{j=2}^{3}\sup_{\theta_{j-1}(t)<x<\theta_j(t)}|v(x,t)-V_j(x+c_jt+\omega)|\Big\}=0$$

for some $\omega \in \mathbb{R}$, *where*

$$\theta_1(t) := -\Big(\frac{-c_1+c_2}{2}\Big)t, \quad \theta_2(t) := -\Big(\frac{c_2+c_3}{2}\Big)t, \quad \theta_3(t) \equiv +\infty.$$

Furthermore,

$$\lim_{t\to\infty}\sup_{x\in\mathbb{R}}|u(x,t)-1|+\sup_{x\in\mathbb{R}}|v(x,t)|=0. \tag{1.6}$$

Remark 1.1. *We make some remarks as follows.*

(i) *The proof Theorem 1 is based on the construction of appropriate super and subsolutions. For single equation case* [27, 5], *the authors constructed an auxiliary rational function, say* Q*-function, which can help them further construct a suitable pair of super-sub solutions. We may expect that this idea should be able to work on monotone systems if a vector-valued* $\mathbf{Q}$*-function can be found. Fortunately, we can make it by extending the key function constructed in* [5] *to a vector-valued function. Besides, we need condition* **(H2)** *to establish some crucial estimates which cannot be obtained directly from those in* [5] *since we face a system rather than a single equation.*

(ii) *The crucial estimates rely on the asymptotical behavior of traveling fronts. The asymptotic behavior of* $(\mathbf{E_v}, \mathbf{E_u})$ *fronts has been established (see, e.g.,* [28]*); while the asymptotic behavior of* $(\mathbf{E_v}, \mathbf{E_*})$*-fronts and* $(\mathbf{E_*}, \mathbf{E_u})$*-fronts near* $\mathbf{E_v}$ *and* $\mathbf{E_u}$ *can be done similarly, respectively. However, the asymptotic behavior of fronts near the coexistence state* $\mathbf{E_*}$ *is more complicated than other cases and needs to be investigated carefully. See Lemma 2.4.*

(iii) *We provide a condition such that* **(H1)**, **(H2)**, **(H3)** *and* (1.5) *hold simultaneously using the results of* [18] *and* [31]. *By the main theorem of* [18], *there exists a monotone* $(\mathbf{E_v}, \mathbf{E_*})$*-front if* $c_2 \le -2\sqrt{k-1}$ *under the assumption* $D = 1$. *Next, we shall choose suitable parameters such that an* $(\mathbf{E_v}, \mathbf{E_u})$*-front exists for some* $c_1 > 0$ *in terms of the existence of exact solutions reported in* [31], *where they consider positive exact solutions of the following system:*

$$\begin{cases} c\widehat{U}' = \widehat{U}'' + \widehat{U}(1-\widehat{U}-\hat{k}\widehat{V}), \quad \xi\in\mathbb{R}, \\ c\widehat{V}' = D\widehat{V}'' + \widehat{U}(a-\widehat{V}-\hat{h}\widehat{U}), \quad \xi\in\mathbb{R}, \\ (\widehat{U},\widehat{V})(-\infty) = (0,a), \quad (\widehat{U},\widehat{V})(\infty) = (1,0), \\ \widehat{U}'>0, \quad \widehat{V}'<0 \quad \text{in } \mathbb{R}, \end{cases} \tag{1.7}$$

in which $1/\hat{k} < a < \hat{h}$. *To apply their result, we introduce the change of variables*

$$a = r, \quad \hat{k} = \frac{k}{r}, \quad \hat{h} = rh, \quad \widehat{U} = U_1, \quad \widehat{V} = rV_1.$$

Then system (1.7) is reduced to system (1.4) with $1/k < 1 < h$. From [31, p.261], we see that (1.4) has a solution with $c_1 = (2-k)/\sqrt{2k}$ if the following conditions hold:

$$D = \frac{r}{3k}, \quad rh = 2 + \frac{5r}{3} - k, \quad 6 + 2r - 3k > 0. \tag{1.8}$$

Let us choose $k \approx 1$, $r = 3k$ (such that $D = 1$) and $h = (2+4k)/3k$ such that (1.8) holds. This implies the existence of an $(\mathbf{E_v}, \mathbf{E_u})$-front with speed $c_1 \approx 1/\sqrt{2}$ and so **(H1)** *and* **(H3)** *hold. Also, one can choose $c_2 = -2\sqrt{k-1}$ such that (1.5) hold because $k \approx 1$. Moreover, it is clear that the above choice of D, k, h and r implies that $\lambda_{ju}^+ < \lambda_{jv}^+$ for $j = 1, 2$ (see Lemma 2.3(1) and Lemma 2.4(2)). Hence,* **(H2)** *holds. Consequently, the hypothesis of Theorem 1 is not void.*

Following the approach used in the proof of Theorem 1 and some ideas from [5] with some suitable changes, we can establish another type of three-front and a four-front entire solutions as follows.

Theorem 2. *Assume* **(H1)**, **(H2)** *and* **(H3)**. *Let (c_1, U_1, V_1) be an $(\mathbf{E_v}, \mathbf{E_u})$-front and both (c_2, U_2, V_2) and $(\widehat{c}_2, \widehat{U}_2, \widehat{V}_2)$ be $(\mathbf{E_v}, \mathbf{E_*})$-fronts such that (1.5) holds. Then system (1.2) admits a three-front entire solution (u, v) satisfying*

$$\lim_{t\to-\infty} \Big\{ \sup_{x\le\theta_1(t)} |u(x,t) - U_1(-x + c_1 t + \omega_1)| + \sup_{\theta_1(t)<x<\theta_2(t)} |u(x,t) - U_2(x + c_2 t + \omega_1)| + \sup_{x\ge\theta_2(t)} |u(x,t) - \widehat{U}_2(-x + \widehat{c}_2 t - \omega_2)| \Big\} = 0,$$

$$\lim_{t\to-\infty} \Big\{ \sup_{x\le\theta_1(t)} |v(x,t) - V_1(-x + c_1 t + \omega_1)| + \sup_{\theta_1(t)<x<\theta_2(t)} |v(x,t) - V_2(x + c_2 t + \omega_1)| + \sup_{x\ge\theta_2(t)} |v(x,t) - \widehat{V}_2(-x + \widehat{c}_2 t - \omega_2)| \Big\} = 0$$

for some $\omega_1, \omega_2 \in \mathbb{R}$, where

$$\theta_1(t) := -\Big(\frac{-c_1 + c_2}{2}\Big)t, \quad \theta_2(t) := -\Big(\frac{c_2 - \widehat{c}_2}{2}\Big)t.$$

Furthermore, $(u,v)(\cdot,t) \to (1,0)$ as $t \to \infty$ uniformly in $(-\infty, L)$ for any $L \in \mathbb{R}$.

Theorem 3. *Assume* **(H1)**, **(H2)** *and* **(H3)**. *Let (c_1, U_1, V_1) be an $(\mathbf{E_v}, \mathbf{E_u})$-front and (c_2, U_2, V_2) be an $(\mathbf{E_v}, \mathbf{E_*})$-front such that (1.5) holds. Then system (1.2) admits a four-front entire solution (u, v) satisfying*

$$\lim_{t\to-\infty} \Big\{ \sup_{x\le\theta_1(t)} |u(x,t) - U_1(-x + c_1 t + \omega_1)| + \sup_{\theta_1(t)\le x\le 0} |u(x,t) - U_2(x + c_2 t + \omega_2)| + \sup_{0\le x\le-\theta_1(t)} |u(x,t) - U_2(-x + c_2 t + \omega_2)| + \sup_{x\ge-\theta_1(t)} |u(x,t) - U_1(x + c_1 t + \omega_1)| \Big\} = 0,$$

$$\lim_{t\to-\infty} \Big\{ \sup_{x\le\theta_1(t)} |v(x,t) - V_1(-x + c_1 t + \omega_1)| + \sup_{\theta_1(t)\le x\le 0} |v(x,t) - V_2(x + c_2 t + \omega_2)| + \sup_{0\le x\le-\theta_1(t)} |v(x,t) - V_2(-x + c_2 t + \omega_2)| + \sup_{x\ge-\theta_1(t)} |v(x,t) - V_1(x + c_1 t + \omega_1)| \Big\} = 0$$

for some $\omega_1, \omega_2 \in \mathbb{R}$, where

$$\theta_1(t) := -\Big(\frac{-c_1 + c_2}{2}\Big)t.$$

Furthermore, the long time behavior (1.6) holds true.

Finally, we show that there is no entire solution of (1.2) originating from N fronts if $N \geq 7$.

Theorem 4. *For $N \geq 7$, there does not exist entire solution originating from N fronts $\{(c_j, \overline{U}_j, \overline{V}_j) \mid j = 1, \cdots, N\}$ satisfying*

$$c_1 < c_2 < \cdots < c_N \tag{1.9}$$

such that

$$\lim_{t\to-\infty} \left[\sum_{j=1}^{N} \sup_{\omega_{j-1}(t)<x<\omega_j(t)} \left|u(x,t) - \overline{U}_j(x + c_j t + \theta_j)\right|\right] = 0, \tag{1.10}$$

$$\lim_{t\to-\infty} \left[\sum_{j=1}^{N} \sup_{\omega_{j-1}(t)<x<\omega_j(t)} \left|v(x,t) - \overline{V}_j(x + c_j t + \theta_j)\right|\right] = 0, \tag{1.11}$$

for some $\theta_1, \cdots, \theta_N \in \mathbb{R}$, where $-c_{j+1}t < \omega_j(t) < -c_j t$ for $j = 1, \cdots, N-1$, $\omega_0(t) := -\infty$ and $\omega_N(t) = \infty$.

We organize the rest of this article as follows. Section 2 is divided into four subsections. In §2.1, we show the existence of monostable traveling fronts connecting different equilibria in terms of the theory of Li, Weinberger and Lewis [25]. In §2.2, the asymptotic behavior of traveling fronts are discussed. In §2.3, we construct a suitable $\mathbf{Q}$-function for our system by extending the Q-function constructed in [5] to a vector-valued function and recall some known results. In §2.4, we establish some important estimates. In Section 3, we prove our main results. Finally, Section 4 is the appendix in which a proof of Lemma 2.9 is given and some results from [12] are collected into a lemma for the reader's convenience.

2. Preliminaries

By the standard transformation

$$w(x,t) := 1 - v(x,t),$$

we can transfer (1.2) into the following cooperative system:

$$\begin{cases} u_t = u_{xx} + f(u,w), & x \in \mathbb{R},\ t \in \mathbb{R}, \\ w_t = D w_{xx} + g(u,w), & x \in \mathbb{R},\ t \in \mathbb{R}, \end{cases} \tag{2.1}$$

where

$$f(u,w) = u[1 - u - k(1-w)], \quad g(u,w) := r(1-w)(hu - w). \tag{2.2}$$

Then ODE equilibria of (1.2): $\mathbf{E_v} = (0,1)$, $\mathbf{E_u} = (1,0)$ and $\mathbf{E}_* = (u^*, v^*)$ are transferred into $\mathbf{0} := (0,0)$, $\mathbf{1} := (1,1)$ and the intermediate (coexistence) equilibrium

$$\mathbf{E_I} = (u^*, w^*) := (u^*, 1 - v^*) = \Big(\frac{1-k}{1-hk}, \frac{h(1-k)}{1-hk}\Big), \tag{2.3}$$

respectively. Hereafter, we shall always use $\mathbf{0}$, $\mathbf{1}$ and $\mathbf{E_I}$ to be ODE equilibria of (2.1). Under assumption **(H1)**, $\mathbf{0}$ and $\mathbf{1}$ are stable; while $\mathbf{E_I}$ is unstable in the ODE sense.

2.1. **Monostable fronts.** The existence of monostable traveling fronts for monotone systems has been investigated extensively. Various approaches can be found in the literature. See, for examples, the Leray-Schauder method [34], the continuation method [21], the upper-lower solution method [7], the theory of monotone semiflows [25] and references therein. Here we shall construct monostable fronts connecting different equilibria in terms of the theory of [25].

Let us recall the framework used in [25] as follows. Consider the reaction-diffusion system

$$\begin{cases} \mathbf{u}_t = d\mathbf{u}_{xx} + \mathbf{F}(\mathbf{u}), \\ \mathbf{u}(x,0) = \mathbf{u}_0(x), \quad x \in \mathbb{R}, \end{cases} \tag{2.4}$$

where $\mathbf{u} := (u_1, u_2)$ and $\mathbf{F}$ are 2-vectors; while $d := \text{diagonal}(d_1, d_2)$ is a constant diagonal matrix.

Some notation are listed as follows: $\mathbf{u}(x) \geq \mathbf{v}(x)$ means that $u_i(x) \geq v_i(x)$ for all i and x; $\mathbf{u} \gg \mathbf{v}$ means that $u_i(x) > v_i(x)$ for all i and x. For any given constant 2-vector $\boldsymbol{\beta} \gg \mathbf{0}$, we define the function space

$$\mathcal{C}_{\boldsymbol{\beta}} := \{\mathbf{u}(x) | \, \mathbf{u}(x) \text{ is continuous and } \mathbf{0} \leq \mathbf{u}(x) \leq \boldsymbol{\beta}\}.$$

For system (2.4), one can denote time-t maps Q_t by $Q_t[\mathbf{u}_0](x) := \mathbf{u}(x,t)$, where Q_t takes the initial value of $\mathbf{u}$ to the value of $\mathbf{u}$ at time t. Clearly, the family of Q_t forms a semigroup. Then $\mathbf{u}$ is a traveling front solution $\mathbf{u}(x,t) = \mathbf{U}(x+ct)$ with speed c if and only if $Q_t[\mathbf{U}](x) = \mathbf{U}(x+ct)$ for all $t > 0$ with the limits $\mathbf{U}(\pm\infty)$ exist and are unequal.

The following result can be seen as a special case of [25, Theorem 4.2].

Proposition 1. *Suppose that system* (2.4) *satisfies the following five conditions:*

(i) $\mathbf{F}(\mathbf{0}) = \mathbf{0}$*, and there is a* $\boldsymbol{\beta} \gg \mathbf{0}$ *such that* $\mathbf{F}(\boldsymbol{\beta}) = \mathbf{0}$ *which is minimal in the sense there is no constant* $\mathbf{v}$ *other than* $\mathbf{0}$ *and* $\boldsymbol{\beta}$ *such that* $\mathbf{F}(\mathbf{v}) = \mathbf{0}$ *and* $\mathbf{0} \ll \mathbf{v} \ll \boldsymbol{\beta}$.

(ii) *The system* (2.4) *is cooperative.*

(iii) $\mathbf{F}$ *does not depend explicitly on either* x *or* t*, and the diagonal matrix* d *is constant.*

(iv) $\mathbf{F}(\mathbf{v})$ *is continuous and has uniformly bounded piecewise continuous first partial derivatives for* $\mathbf{0} \ll \mathbf{v} \ll \boldsymbol{\beta}$*, and it is differentiable at* $\mathbf{0}$*. The Jacobian matrix* $\mathbf{F}'(\mathbf{0})$*, whose off-diagonal entries are non-negative, has a positive eigenvalue whose eigenvector has positive components.*

(v) *The mobilities* d_i*, which are the diagonal and only non-zero entries of* d*, are all positive.*

Then there exists $c^* > 0$ *such that for every* $c \geq c^*$*, the system* (2.4) *has a non-decreasing traveling wave solution* $\mathbf{U}(x+ct)$ *of speed* c *with* $\mathbf{U}(+\infty) = \boldsymbol{\beta}$ *and* $\mathbf{U}(-\infty)$ *a zero of* $\mathbf{F}$ *other than* $\boldsymbol{\beta}$*. If there is a traveling wave* $\mathbf{U}(x+ct)$ *with* $\mathbf{U}(+\infty) = \boldsymbol{\beta}$ *such that for at least one component* i $\liminf_{x\to-\infty} U_i(x) = 0$*, then* $c \geq c^*$.

Remark 2.1. *We remark that Proposition 1 is a partial result of Theorem 4.2 of* [25]*. Therein, they considered more general models which can contain advection terms.*

We can apply Proposition 1 to obtain the following result.

Lemma 2.1. *There exists* $c_{3,\min} > 0$ *such that a non-decreasing traveling front* $(u,w)(x,t) = (U_3, W_3)(x + c_3 t)$ *of* (2.1) *connecting* $\mathbf{E_I}$ *and* $\mathbf{1}$ *exists if and only if* $c_3 \geq c_{3,\min}$.

Proof. In order to apply Proposition 1, we define

$$(\hat{u}, \hat{w}) := (u - u^*, w - w^*). \tag{2.5}$$

Putting this form into (2.1), we see that $(\hat{u},\hat{w})$ satisfies

$$\begin{cases} \hat{u}_t = \hat{u}_{xx} + F_1(\hat{u},\hat{w}), & x\in\mathbb{R},\ t\in\mathbb{R}, \\ \hat{w}_t = D\hat{w}_{xx} + F_2(\hat{u},\hat{w}), & x\in\mathbb{R},\ t\in\mathbb{R}, \end{cases} \tag{2.6}$$

where

$$\begin{aligned} F_1(\hat{u},\hat{w}) &:= (\hat{u}+u^*)[1-\hat{u}-u^*-k(1-\hat{w}-w^*)], \\ F_2(\hat{u},\hat{w}) &:= r(1-\hat{w}-w^*)[h(\hat{u}+u^*)-\hat{w}-w^*]. \end{aligned}$$

Define $\mathbf{F} := (F_1,F_2)$ and $\boldsymbol{\beta} := (1-u^*,1-w^*)$. We now check that conditions (i)-(v) of Proposition 1 hold. For (i), clearly, $\mathbf{F}(\mathbf{0})=\mathbf{F}(\boldsymbol{\beta})=\mathbf{0}$. Also, by **(H1)**,

$$\boldsymbol{\beta} = (1-u^*,1-w^*) = \Big(\frac{k(1-h)}{1-hk},\frac{1-h}{1-hk}\Big) \gg 0.$$

Note that (2.1) has only two ODE equilibria $\mathbf{E_I}$ and $\mathbf{1}$ over $\{\mathbf{u}:=(u,w)\,|\,\mathbf{E_I}\le\mathbf{u}\le\mathbf{1}\}$, it follows that there is no constant $\mathbf{v}$ other than $\mathbf{0}$ and $\boldsymbol{\beta}$ such that $\mathbf{F}(\mathbf{v})=0$ and $\mathbf{0}\ll\mathbf{v}\ll\boldsymbol{\beta}$. Hence (i) holds.

Since $\partial F_1/\partial\hat{w}\ge 0$ and $\partial F_2/\partial\hat{u}\ge 0$ in $\{\mathbf{0}\le(\hat{u},\hat{w})\le\boldsymbol{\beta}\}$, we see that (ii) follows. From (2.6), it is easy to see that (iii) and (v) hold. For (iv), one can calculate

$$\mathbf{F}'(\mathbf{0}) = \begin{pmatrix} -u^* & ku^* \\ hr(1-w^*) & -r(1-w^*) \end{pmatrix}.$$

By **(H1)** we have

$$\det\mathbf{F}'(\mathbf{0}) = (1-hk)ru^*(1-w^*) < 0,$$

which means that $\mathbf{F}'(\mathbf{0})$ has a positive eigenvalue. It is easy to check that all conditions in (iv) are satisfied. Hence conditions (i)-(v) hold for system (2.1). Moreover, there is no non-negative ODE equilibrium except $\mathbf{0}$ and $\boldsymbol{\beta}$ in $\mathcal{C}_{\boldsymbol{\beta}}$. By Proposition 1, there exists $c_{3,\min}>0$ such that there is a non-decreasing traveling front $(u,w)(x,t)=(U_3,W_3)(x+c_3t)$ of (2.1) connecting $\mathbf{E_I}$ and $\mathbf{1}$ if and only if $c_3\ge c_{3,\min}$. This completes the proof. □

By Lemma 2.1, for any $c_3\ge c_{3,\min}$, there exists (c_3,U_3,W_3) such that

$$\begin{cases} c_3U_3' = U_3'' + U_3[1-U_3-k(1-W_3)], & \xi\in\mathbb{R}, \\ c_3W_3' = DW_3'' + r(1-W_3)(hU_3-W_3), & \xi\in\mathbb{R}, \\ (U_3,W_3)(-\infty)=\mathbf{E_I}, \quad (U_3,W_3)(+\infty)=\mathbf{1}, \\ U_3'>0, \quad W_3'>0 \quad \text{in } \mathbb{R}. \end{cases} \tag{2.7}$$

Note that the strict monotonicity of U_3 and W_3 follows from the strong maximum principle.

Similarly, one can establish the existence of $(\mathbf{0},\mathbf{E_I})$-fronts to system (2.1). Indeed, instead of using the transformation (2.5), we define

$$(\tilde{u},\tilde{w}) := (-u+u^*,-w+w^*).$$

Again putting this form into (2.1), then $(\tilde{u},\tilde{w})$ satisfies

$$\begin{cases} \tilde{u}_t = \tilde{u}_{xx} + F_1(\tilde{u},\tilde{w}), & x\in\mathbb{R},\ t\in\mathbb{R}, \\ \tilde{w}_t = D\tilde{w}_{xx} + F_2(\tilde{u},\tilde{w}), & x\in\mathbb{R},\ t\in\mathbb{R}, \end{cases}$$

where

$$F_1(\tilde{u},\tilde{w}) := (\tilde{u}-u^*)[1+\tilde{u}-u^*-k(1+\tilde{w}-w^*)],$$
$$F_2(\tilde{u},\tilde{w}) := r(1+\tilde{w}-w^*)[h(\tilde{u}-u^*)-\tilde{w}+w^*].$$

Define $\mathbf{F} := (F_1, F_2)$ and $\boldsymbol{\beta} := (u^*, w^*)$. Using a similar process to that of the proof of Lemma 2.1, we can obtain the following result. The detailed proof is omitted.

Lemma 2.2. *There exists $c_{2,\max} < 0$ such that a non-decreasing traveling front $(u,w)(x,t) = (U_2,W_2)(x+c_2t)$ of (2.1) connecting $\mathbf{0}$ and $\mathbf{E_I}$ exists if and only if $c_2 \le c_{2,\max}$.*

From Lemma 2.2, for any $c_2 \le c_{2,\max}$, there exists (c_2, U_2, W_2) which satisfies

$$\begin{cases} c_2U_2' = U_2'' + U_2[1-U_2-k(1-W_2)], \quad \xi\in\mathbb{R},\\ c_2W_2' = DW_2'' + r(1-W_2)(hU_2-W_2), \quad \xi\in\mathbb{R},\\ (U_2,W_2)(-\infty)=\mathbf{0}, \quad (U,V)(+\infty)=\mathbf{E_I},\\ U_2'>0, \quad W_2'>0 \quad \text{in } \mathbb{R}. \end{cases} \tag{2.8}$$

2.2. **Asymptotic behavior of traveling fronts.** In this subsection, we provide the asymptotic behavior of $(\mathbf{E_v},\mathbf{E_u})$ fronts, $(\mathbf{E_v},\mathbf{E_*})$-fronts and $(\mathbf{E_*},\mathbf{E_u})$-fronts near $\xi=\pm\infty$, respectively. The following result can be found in [28].

Lemma 2.3 ([28] The asymptotic behavior of $(\mathbf{E_v},\mathbf{E_u})$ fronts at $\xi=\pm\infty$)**.** *Let (c_1,U_1,V_1) be a solution of (1.4) with $c_1>0$. Then the following hold:*

(1) *Define λ_{1u}^+ and λ_{1v}^+ as the positive root of*

$$\lambda^2 - c_1\lambda + (1-k) = 0 \text{ and } D\lambda^2 - c_1\lambda - r = 0,$$

respectively. Then

$$\lim_{\xi\to-\infty}\Big(\frac{U_1(\xi)}{e^{\lambda_{1u}^+\xi}}, \frac{1-V_1(\xi)}{|\xi|^{\gamma_1^+}e^{\beta_1^+\xi}}\Big) = (A_{1u}^+, A_{1v}^+)$$

for some positive constants A_{1u}^+ and A_{1v}^+, where

$$\beta_1^+ := \min\{\lambda_{1u}^+,\lambda_{1v}^+\}, \quad \gamma_1^+ = \begin{cases} 0 & \text{if } \lambda_{1u}^+\neq\lambda_{1v}^+,\\ 1 & \text{if } \lambda_{1u}^+=\lambda_{1v}^+. \end{cases}$$

(2) *Define λ_{1u}^- and λ_{1v}^- as the negative root of*

$$\lambda^2 - c_1\lambda - 1 = 0 \text{ and } \quad D\lambda^2 - c_1\lambda + r(1-h) = 0,$$

respectively. Then

$$\lim_{\xi\to+\infty}\Big(\frac{1-U_1(\xi)}{|\xi|^{\gamma_1^-}e^{\beta_1^-\xi}}, \frac{V_1(\xi)}{e^{\lambda_{1v}^-\xi}}\Big) = (A_{1u}^-, A_{1v}^-)$$

for some positive constants A_{1u}^- and A_{1v}^-, where

$$\beta_1^- := \max\{\lambda_{1u}^-,\lambda_{1v}^-\}, \quad \gamma_1^- = \begin{cases} 0 & \text{if } \lambda_{1u}^-\neq\lambda_{1v}^-,\\ 1 & \text{if } \lambda_{1u}^-=\lambda_{1v}^-. \end{cases}$$

Next, we provide the asymptotic behavior of $(\mathbf{E_v},\mathbf{E_*})$-fronts at $\xi=\pm\infty$.

Lemma 2.4 (The Asymptotic behavior of $(\mathbf{E_v}, \mathbf{E}_*)$-fronts at $\xi = \pm\infty$)**.** *Let (c_2, U_2, V_2) be a solution of (2.8) with $c_2 \le c_{2,\max} < 0$. Then the following hold:*

(1) *Define λ_{2u}^+ and λ_{2v}^+ as the positive root of*

$$\lambda^2 - c_2\lambda + (1-k) = 0 \text{ and } D\lambda^2 - c_2\lambda - r = 0,$$

respectively. Then

$$\lim_{\xi\to-\infty}\Big(\frac{U_2(\xi)}{e^{\lambda_{2u}^+\xi}}, \frac{1-V_2(\xi)}{|\xi|^{\gamma_2^+}e^{\beta_2^+\xi}}\Big) = (A_{2u}^+, A_{2v}^+)$$

for some positive constants A_{2u}^+ and A_{2v}^+, where

$$\beta_2^+ := \min\{\lambda_{2u}^+, \lambda_{2v}^+\}, \quad \gamma_2^+ = \begin{cases} 0 & \text{if } \lambda_{2u}^+ \neq \lambda_{2v}^+, \\ 1 & \text{if } \lambda_{2u}^+ = \lambda_{2v}^+. \end{cases}$$

(2) *Define the characteristic equation*

$$P_2(\lambda) := (\lambda^2 - c_2\lambda - u^*)(D\lambda^2 - c_2\lambda - rv^*) - rhku^*v^* = 0$$

Then

$$\lim_{\xi\to+\infty}\Big(\frac{u^* - U_2(\xi)}{|\xi|^{\gamma_2^-}e^{\lambda_2^-\xi}}, \frac{V_2(\xi) - v^*}{|\xi|^{\gamma_2^-}e^{\lambda_2^-\xi}}\Big) = (A_{2u}^-, A_{2v}^-) \tag{2.9}$$

for some positive constants A_{2u}^- and A_{2v}^-, where u^* and v^* are given in (1.3), λ_2^- is some negative zero of P_2, and

$$\gamma_2^- = \begin{cases} 0 & \textit{if } \lambda_2^- \textit{ is a simple zero of } P_2, \\ 0 \textit{ or } 1 & \textit{if } \lambda_2^- \textit{ is a double zero of } P_2. \end{cases}$$

Proof. Since the proof of (1) is similar to that of [28, Lemma 2.3], we omit the proof. We now deal with (2). Let us write

$$P_2(\lambda) = R_1(\lambda)R_2(\lambda) - rhku^*v^*,$$

where

$$R_1(\lambda) := \lambda^2 - c_2\lambda - u^*, \quad R_2(\lambda) := D\lambda^2 - c_2\lambda - rv^*.$$

Note that R_1 (resp., R_2) has one positive zero μ_+ (resp., σ_+) and one negative zero μ_- (resp., σ_-), where

$$\mu_\pm = \frac{c_2 \pm \sqrt{c_2^2 + 4u^*}}{2}, \quad \sigma_\pm = \frac{c_2 \pm \sqrt{c_2^2 + 4Drv^*}}{2D}.$$

To locate the zeros of P_2, we divide $\mathbb{R}$ into six disjoint intervals

$$I_1 := (-\infty, \min\{\mu_-, \sigma_-\}], \quad I_2 : (\min\{\mu_-, \sigma_-\}, \max\{\mu_-, \sigma_-\}],$$
$$I_3 := (\max\{\mu_-, \sigma_-\}, 0), \quad I_4 := [0, \min\{\mu_+, \sigma_+\}),$$
$$I_5 := [\min\{\mu_+, \sigma_+\}, \max\{\mu_+, \sigma_+\}), \quad I_6 := [\max\{\mu_+, \sigma_+\}, \infty).$$

Some simple facts are given as follows:

(i) $P_2(0) = ru^*v^*(1 - hk) < 0$ (since $h, k > 1$) and $P_2(\mu_\pm) = P_2(\sigma_\pm) = -ru^*v^*hk < 0$.
(ii) P_2 is decreasing in I_1 and is increasing in I_6 with $P_2(\pm\infty) = \infty$.
(iii) $P_2(\lambda) < -ru^*v^*hk < 0$ for all $\lambda \in I_2 \cup I_5$.
(iv) P_2 is decreasing in I_4, since $c_2 < 0$ implies $R_j'(\cdot) > 0$ in $[0, \infty)$ for $j = 1, 2$, it follows that $P_2' = R_1'R_2 + R_1R_2' < 0$ in I_4.

By (i) and (ii), we see that P_2 has exactly one negative zero in I_1 (say κ_1) and exactly one positive zero in I_6 (say κ_4). From (i), (iii) and (iv), it follows that P_2 has no zero in $I_2 \cup I_4 \cup I_5$. Also, since P_2 has at most four real zeros, we have the following three possibilities:

(a) P_2 has four simple zeros $\kappa_1 < \kappa_2 < \kappa_3 < 0 < \kappa_4$ with $\kappa_2, \kappa_3 \in I_3$.
(b) P_2 has four zeros $\kappa_1 < \kappa_2 = \kappa_3 < 0 < \kappa_4$ with one double zero $\kappa_2 \in I_3$.
(c) P_2 has only two real zeros, i.e., $\kappa_1 \in I_1$ and $\kappa_4 \in I_6$, and has two conjugate complex zeros.

We now define

$$(X_1, X_2, X_3, X_4)(\xi) = (U_2, U_2', V_2, V_2')(\xi).$$

Then by the first two equations of (2.8), we have

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix}' = \begin{pmatrix} X_2 \\ c_2 X_2 - f(X_1, X_3) \\ X_4 \\ \frac{c_2}{D} X_4 - \frac{1}{D} g(X_1, x_3) \end{pmatrix} \tag{2.10}$$

By considering the linearized system of (2.10) at $(X_1, X_2, X_3, X_4) = (u^*, 0, v^*, 0)$, we have

$$\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ u^* & c_2 & kv^* & 0 \\ 0 & 0 & 0 & 1 \\ \frac{rh}{D} v^* & 0 & \frac{r}{D} v^* & \frac{c_2}{D} \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{pmatrix} := J \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{pmatrix}. \tag{2.11}$$

By cofactor expansion, the characteristic polynomial of J is

$$\det(J - \lambda I) = P_2(\lambda) = R_1(\lambda) R_2(\lambda) - rhku^* v^*.$$

Note that we have located the zeros of P_2 above. See (a)-(c) above.

<u>Case 1</u> : assume that (a) holds. Namely, J has four distinct real eigenvalues $\kappa_1 < \kappa_2 < \kappa_3 < 0 < \kappa_4$. By some simple calculations, we can find an eigenvector

$$\mathbf{v}_i := \Big(1, \kappa_i, \frac{R_1(\kappa_i)}{ku^*}, \kappa_i \frac{R_1(\kappa_i)}{ku^*}\Big)^T \tag{2.12}$$

with respect to eigenvalue κ_i $(i = 1, 2, 3, 4)$. It follows that every solution of (2.11) that approaches to $\mathbf{0}$ as $\xi \to \infty$ can be represented by

$$\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{pmatrix} = \sum_{i=1}^{3} C_i e^{\kappa_i \xi} \mathbf{v}_i$$

for some constants C_i, $i = 1, 2, 3$. By the unstable manifold theorem, as $\xi \to \infty$, we have

$$\begin{pmatrix} U_2(\xi) \\ U_2'(\xi) \\ V_2(\xi) \\ V_2'(\xi) \end{pmatrix} = \begin{pmatrix} X_1(\xi) \\ X_2(\xi) \\ X_3(\xi) \\ X_4(\xi) \end{pmatrix} = \begin{pmatrix} u^* + \sum_{i=1}^3 \widehat{C}_i e^{\kappa_i \xi}. \\ \sum_{i=1}^3 \widehat{C}_i \kappa_i e^{\kappa_i \xi} \\ v^* + \sum_{i=1}^3 \widehat{C}_i \frac{R_1(\kappa_i)}{ku^*} e^{\kappa_i \xi} \\ \sum_{i=1}^3 \widehat{C}_i \kappa_i \frac{R_1(\kappa_i)}{ku^*} e^{\kappa_i \xi}. \end{pmatrix} + h.o.t. \tag{2.13}$$

for some constants $\widehat{C}_i$ $(i = 1, 2, 3)$.

Next, we show that $\widehat{C}_2^2 + \widehat{C}_3^2 \neq 0$. For contradiction we assume that $\widehat{C}_2 = \widehat{C}_3 = 0$. Then from (2.13), we see that

$$\lim_{\xi\to\infty}\frac{U_2'(\xi)}{e^{\kappa_1\xi}}=\kappa_1\widehat{C}_1,\quad \lim_{\xi\to\infty}\frac{V_2'(\xi)}{e^{\kappa_1\xi}}=\frac{R_1(\kappa_1)}{ku^*}\kappa_1\widehat{C}_1. \tag{2.14}$$

Since $R_1(\kappa_1)>0$, we see from (2.14) that U' and V' have the same sign for all large ξ, which contradicts the fact that $U_2'>0$ and $V_2'<0$ in $\mathbb{R}$. Hence, $\widehat{C}_2^2+\widehat{C}_3^2\neq 0$. Then we obtain (2.9) with $\gamma_2^-=0$ and either $\lambda_2^-=\kappa_2$ (if $\widehat{C}_3=0$) or $\lambda_2^-=\kappa_3$ (if $\widehat{C}_3\neq 0$).

Case 2 : assume that (b) holds. Namely, J has real eigenvalues $\kappa_1<\kappa_2=\kappa_3<0<\kappa_4$. Since the dimension of the generalized eigenspace corresponding to eigenvalue $\lambda=\kappa_2$ is two, by some calculations, J has a generalized eigenvector

$$\widehat{\mathbf{v}}:=\Big(1,1+\lambda,\nu,\kappa_2\nu+\frac{R_1(\kappa_2)}{ku^*}\Big)^T$$

for some constant ν. Hence, every solution of (2.11) that approaches to $\mathbf{0}$ as $\xi\to\infty$ can be represented by

$$\begin{pmatrix}Y_1\\Y_2\\Y_3\\Y_4\end{pmatrix}=C_1e^{\kappa_1\xi}\mathbf{v}_1+C_2e^{\kappa_2\xi}\mathbf{v}_2+C_3e^{\kappa_2\xi}(\mathbf{v}_2\xi+\widehat{\mathbf{v}}),$$

where $\mathbf{v}_i$, $i=1,2$, are given in (2.12) and C_i, $i=1,2,3$, are some constants. By the unstable manifold theorem, as $\xi\to\infty$, we have

$$\begin{pmatrix}U_2(\xi)\\U_2'(\xi)\\V_2(\xi)\\V_2'(\xi)\end{pmatrix}=\begin{pmatrix}u^*+\sum_{i=1}^2\widehat{C}_ie^{\kappa_i\xi}+\widehat{C}_3e^{\kappa_2\xi}(\xi+1).\\ \sum_{i=1}^2\widehat{C}_i\kappa_ie^{\kappa_i\xi}+\widehat{C}_3e^{\kappa_2\xi}(\kappa_2\xi+1+\kappa_2)\\ v^*+\sum_{i=1}^2\widehat{C}_i\frac{R_1(\kappa_i)}{ku^*}e^{\kappa_i\xi}+\widehat{C}_3e^{\kappa_2\xi}(\xi\frac{R_1(\kappa_i)}{ku^*}+\kappa_2\nu).\\ \sum_{i=1}^2\widehat{C}_i\kappa_i\frac{R_1(\kappa_i)}{ku^*}e^{\kappa_i\xi}+\widehat{C}_3e^{\kappa_2\xi}(\kappa_2\xi\frac{R_1(\kappa_i)}{ku^*}+\kappa_2\nu+\frac{R_1(\kappa_2)}{ku^*}).\end{pmatrix}+h.o.t.$$

for some constants $\widehat{C}_i$ $(i=1,2,3)$. Similar to the discussion in Case 1, if $\widehat{C}_2^2+\widehat{C}_3^2=0$, U' and V' must have the same sign for all large ξ, which leads to a contradiction since $U_2'>0$ and $V_2'<0$ in $\mathbb{R}$. Hence $\widehat{C}_2^2+\widehat{C}_3^2\neq 0$, which implies (2.9) with $\lambda_2^-=\kappa_2$ and either $\gamma_2^-=0$ (if $\widehat{C}_3=0$) or $\gamma_2^-=1$ (if $\widehat{C}_3\neq 0$).

Case 3 assume that (c) holds. Since J already has two real eigenvalues $\kappa_1<0<\kappa_4$, the complex eigenvalues are conjugate pairs $a\pm bi$ $(b>0)$. By simple calculations, we have $P_2(bi)\neq 0$. It follows that $a\neq 0$. If $a>0$, then J has only one eigenvalue with negative real part. By the unstable manifold theorem, we again have (2.14). Recall that $R_1(\kappa_1)>0$. Again, it contradicts the fact that $U_2'>0$ and $V_2'<0$ in $\mathbb{R}$. Hence, $a>0$ is impossible. If $a<0$, then every solution of (2.11) that approaches to $\mathbf{0}$ as $\xi\to\infty$ is represented by

$$\begin{pmatrix}Y_1\\Y_2\\Y_3\\Y_4\end{pmatrix}=C_1e^{\kappa_1\xi}\mathbf{v}_1+C_2e^{a\xi}\begin{pmatrix}\cos b\xi\\w_2\\w_3\\w_4\end{pmatrix}+C_3e^{a\xi}\begin{pmatrix}\sin b\xi\\z_2\\z_3\\z_4\end{pmatrix},$$

where $\mathbf{v}_1$ is given in (2.12) and $C_i,w_i,z_i\in\mathbb{R}$. Since $a<0$, by the unstable manifold theorem, as $\xi\to\infty$,

$$\begin{cases}U(\xi)=u^*+\widehat{C}_1e^{\kappa_1\xi}+\widehat{C}_2e^{a\xi}\cos b\xi+\widehat{C}_3e^{a\xi}\sin b\xi+h.o.t.,\\ V(\xi)=v^*+\widehat{C}_1\frac{R_1(\kappa_i)}{ku^*}e^{\kappa_1\xi}+\widehat{C}_2e^{a\xi}w_3+\widehat{C}_3e^{a\xi}z_3+h.o.t.,\end{cases} \tag{2.15}$$

for some $\widehat{C}_i \in \mathbb{R}$ for $i = 1, 2, 3$.

We now show that $a = \kappa_1$. For contradiction we assume that $a \neq \kappa_1$. If $a > \kappa_1$, we can obtain $\widehat{C}_2^2 + \widehat{C}_3^2 \neq 0$. Otherwise, from (2.15) we have

$$\lim_{\xi\to\infty} \frac{U(\xi) - u^*}{e^{\kappa_1 \xi}} = \widehat{C}_1, \quad \lim_{\xi\to\infty} \frac{V(\xi) - v^*}{e^{\kappa_1 \xi}} = \widehat{C}_1 \frac{R_1(\kappa_i)}{ku^*}, \tag{2.16}$$

which contradicts the fact that $U < u^*$ and $v^* < V$ in $\mathbb{R}$. Hence we have $\widehat{C}_2^2 + \widehat{C}_3^2 \neq 0$. Then due to the assumption $0 > a > \kappa_1$, $u^* - U(\xi)$ must change sign infinitely many times for all $\xi \gg 1$, which is impossible since $U < u^*$ in $\mathbb{R}$. It follows that $a \leq \kappa_1$. If $a < \kappa_1$, from (2.15) we see that (2.16) holds, which again contradicts the fact that $U < u^*$ and $v^* < V$ in $\mathbb{R}$. Consequently, we must have $a = \kappa_1$. Clearly, (2.15) implies (2.9) with $\lambda_2^- = \kappa_1$ and $\gamma_2^- = 0$. This completes the proof. $\square$

Finally, we provide the asymptotic behavior of $(\mathbf{E}_*, \mathbf{E_u})$-fronts at $\xi = \pm\infty$. Since the proof is similar that of Lemma 2.4, we shall omit the proof here.

Lemma 2.5 (The Asymptotic behavior of $(\mathbf{E}_*, \mathbf{E_u})$-fronts at $\xi = \pm\infty$)**.** *Let (c_3, U_3, V_3) be a solution of (2.8) with $c_3 \geq c_{3,\min} > 0$. Then the following hold:*

(1) *Define the characteristic equation*

$$P_3(\lambda) := (\lambda^2 - c_3\lambda - u^*)(D\lambda^2 - c_3\lambda - rv^*) - rhku^*v^* = 0$$

Then

$$\lim_{\xi\to-\infty} \Big(\frac{U_3(\xi) - u^*}{|\xi|^{\gamma_3^+} e^{\lambda_3^+ \xi}}, \frac{v^* - V_3(\xi)}{|\xi|^{\gamma_3^+} e^{\lambda_3^+ \xi}} \Big) = (A_{3u}^+, A_{3v}^+)$$

for some positive constants A_{32u}^+ and A_{3v}^+, where λ_3^+ is some positive zero of P_3 and

$$\gamma_3^+ = \begin{cases} 0 & \text{if } \lambda_3^+ \text{ is a simple zero of } P_3, \\ 0 \text{ or } 1 & \text{if } \lambda_3^+ \text{ is a double zero of } P_3. \end{cases}$$

(2) *Define λ_{3u}^- and λ_{3v}^- as the negative roots of*

$$\lambda^2 - c_3\lambda - 1 = 0 \text{ and } D\lambda^2 - c_3\lambda + r(1-h) = 0,$$

respectively. Then

$$\lim_{\xi\to+\infty} \Big(\frac{1 - U_3(\xi)}{|\xi|^{\gamma_3^-} e^{\beta_3^- \xi}}, \frac{1 - V_2(\xi)}{|\xi|^{e^{\beta_{3v}^- \xi}}} \Big) = (A_{3u}^-, A_{3v}^-)$$

for some positive constants A_{3u}^- and A_{3v}^-, where

$$\beta_3^- := \max\{\lambda_{3u}^-, \lambda_{3v}^-\}, \quad \gamma_3^- = \begin{cases} 0 & \text{if } \lambda_{3u}^- \neq \lambda_{3v}^-, \\ 1 & \text{if } \lambda_{3u}^- = \lambda_{3v}^-. \end{cases}$$

As a corollary, we have the following estimates on (U_i, W_i), $i = 1, 2, 3$, at $\xi = \pm\infty$.

Lemma 2.6. *There exist $\alpha_i > 0$, $\beta_i > 0$, $i = 1, 2, 3$, and K_j, $j = 1, \cdots, 9$, such that for any given $p \le 0$,*

(1) $|U_i'(x+p)| + |W_i'(x+p)| \le K_1 e^{\alpha_i(x+p)}$ *for $x \le -p$, $i = 1, 2, 3$,*

(2) $|U_i'(x-p)| + |W_i'(x-p)| \le K_2 e^{-\beta_i(x-p)}$ *for $x \ge p$, $i = 1, 2, 3$,*

(3) $\dfrac{|U_1(x-p)|}{|U_1'(x-p)|} + \dfrac{|W_1(x-p)|}{|W_1'(x-p)|} \le K_3$ *for $x \le p$,*

(4) $\dfrac{|1-U_1(x-p)|}{|U_1'(x-p)|} + \dfrac{|1-W_1(x-p)|}{|W_1'(x-p)|} \le K_4$ *for $x \ge p$,*

(5) $\dfrac{|U_2(x+p)|}{|U_2'(x+p)|} + \dfrac{|W_2(x+p)|}{|W_2'(x+p)|} \le K_5$ *for $x \le -p$,*

(6) $\dfrac{|W_2(x+p)-w^*|}{|U_2(x+p)-u^*|} + \dfrac{|U_2(x+p)-u^*|}{|U_2'(x+p)|} + \dfrac{|W_2(x+p)-w^*|}{|W_2'(x+p)|} \le K_6$ *for $x \ge -p$,*

(7) $\dfrac{|W_3(x+p)-w^*|}{|U_3(x+p)-u^*|} + \dfrac{|U_3(x+p)-u^*|}{|U_3'(x+p)|} + \dfrac{|W_3(x+p)-w^*|}{|W_3'(x+p)|} \le K_7$ *for $x \le -p$,*

(8) $\dfrac{|U_3(x+p)-1|}{|U_3'(x+p)|} + \dfrac{|W_3(x+p)-1|}{|W_3'(x+p)|} \le K_8$ *for $x \ge -p$,*

(9) $\dfrac{|1-W_1(x-p)|}{|1-U_1(x-p)|} + \dfrac{|1-W_3(x-p)|}{|1-U_3(x-p)|} \le K_9$ *for $x \in \mathbb{R}$.*

Proof. These results immediately follow from Lemma 2.3, Lemma 2.4 and Lemma 2.5. ☐

2.3. **The construction of Q-function.** In this subsection, we shall introduce the **Q**-function which plays a crucial role in the construction of a pair of super-sub-solution.

The **Q**-function for the two-species system is a vector-valued function with two components, which is defined by $\mathbf{Q}(y, z, \eta) := (Q_1, Q_2)(y, z, \eta)$. For each component, we adopt the form constructed by Chen et al. [5]:

$$Q_i(y, z, \eta) = z + (1-z)\frac{(1-y)z(\eta - a_i) + y(a_i - z)(1-\eta)}{(1-y)z(1-a_i) + (a_i - z)(1-\eta)}, \quad i = 1, 2, \tag{2.17}$$

defined on D_i for $i = 1, 2$, where $a_1 := u^*$, $a_2 := w^*$ and

$$D_i := [0,1] \times [0, a_i] \times [a_i, 1] \setminus (J_i^1 \cup J_i^2 \cup J_i^3), \tag{2.18}$$

$$J_i^1 := \{(y, 0, 1) \,|\, 0 \le y \le 1\},\ J_i^2 := \{(1, z, 1) \,|\, 0 \le z \le a_i\},\ J_i^3 := \{(1, a_i, \eta) \,|\, a_i \le \eta \le 1\}. \tag{2.19}$$

With the same calculations as in [5], the first and second derivatives of Q_i satisfy some properties which are listed in the following two lemmas. See the proof of [5, Lemma 3.1].

Lemma 2.7. *It holds that*

$$Q_{iy} = \frac{a_i(1-z)(a_i - z)(1-\eta)^2}{[(1-y)z(1-a_i) + (a_i - z)(1-\eta)]^2},$$

$$Q_{iz} = \frac{(1-a_i)a_i(1-y)(1-\eta)(\eta - y)}{[(1-y)z(1-a_i) + (a_i - z)(1-\eta)]^2},$$

$$Q_{i\eta} = \frac{a_i(1-a_i)(1-y)^2 z(1-z)}{[(1-y)z(1-a_i) + (a_i - z)(1-\eta)]^2},$$

and

$$\begin{aligned}
Q_{iyy} &= \frac{2(1-a_i)a_i z(1-z)(a_i-z)(1-\eta)^2}{[(1-y)z(1-a_i)+(a_i-z)(1-\eta)]^3},\\
Q_{izz} &= -\frac{2(1-a_i)a_i(1-y)(1-\eta)(\eta-y)[\eta-a_i-y(1-a_i)]}{[(1-y)z(1-a_i)+(a_i-z)(1-\eta)]^3},\\
Q_{i\eta\eta} &= \frac{2(1-a_i)a_i(1-y)^2 z(a_i-z)(1-z)}{[(1-y)z(1-a_i)+(a_i-z)(1-\eta)]^3},\\
Q_{iyz} &= -\frac{(1-a_i)a_i(1-\eta)^2[(y-\eta)z+a_i(1-2y-z+\eta+yz)]}{[(1-y)z(1-a_i)+(a_i-z)(1-\eta)]^3},\\
Q_{iy\eta} &= -\frac{2(1-a_i)a_i(1-y)z(a_i-z)(1-z)(1-\eta)}{[(1-y)z(1-a_i)+(a_i-z)(1-\eta)]^3},\\
Q_{iz\eta} &= -\frac{(1-a_i)a_i(1-y)^2[(\eta-y)z+a_i(-1+\eta+z-2z\eta+yz)]}{[(1-y)z(1-a_i)+(a_i-z)(1-\eta)]^3},
\end{aligned}$$

where $i = 1, 2$.

Lemma 2.8. *Let* $\mathbf{Q} := (Q_1, Q_2)$ *be defined in* (2.17). *Then the following results hold:*

(1) Q_i, $i = 1, 2$, *can be rewritten as*

$$\begin{aligned}
Q_i(y,z,\eta) &= y + (1-y)z\frac{(1-a_i)(\eta-y)}{(1-y)z(1-a_i)+(a_i-z)(1-\eta)},\\
Q_i(y,z,\eta) &= w + (a_i-z)(1-\eta)\frac{y-\eta}{(1-y)z(1-a_i)+(a_i-z)(1-\eta)}.
\end{aligned}$$

(2) *There exist functions* Q_{ij}, $i = 1, 2$, $j = 1, 2, 3$, *such that*

$$\begin{aligned}
Q_{iy}(y,z,\eta) &= (a_i-z)(1-\eta)Q_{i1}(y,z,\eta),\\
Q_{iz}(y,z,\eta) &= (1-y)(1-\eta)Q_{i2}(y,z,\eta),\\
Q_{i\eta}(y,z,\eta) &= (1-y)zQ_{i3}(y,z,\eta).
\end{aligned}$$

(3) *There exist functions* $\widehat{Q}_{ij}(y,z,\eta)$, $i = 1, 2$, $j = 1, \cdots, 16$, *such that*

$$\begin{aligned}
&Q_{iyy}(y,z,\eta) = z\widehat{Q}_{i1} = (a_i-z)\widehat{Q}_{i2} = (1-\eta)\widehat{Q}_{i3},\\
&Q_{izz}(y,z,\eta) = (1-y)\widehat{Q}_{i4} = (1-\eta)\widehat{Q}_{i5} = y\widehat{Q}_{i6} + (\eta-a_i)\widehat{Q}_{i7},\\
&Q_{i\eta\eta}(y,z,\eta) = (1-y)\widehat{Q}_{i8} = z\widehat{Q}_{i9} = (a_i-z)\widehat{Q}_{i10},\\
&Q_{iyz}(y,z,\eta) = (1-\eta)\widehat{Q}_{i11}, \quad Q_{iz\eta}(y,z,\eta) = (1-y)\widehat{Q}_{i12},\\
&Q_{iy\eta}(y,z,\eta) = (1-y)\widehat{Q}_{i13} = z\widehat{Q}_{i14} = (a_i-z)\widehat{Q}_{i15} = (1-\eta)\widehat{Q}_{i16}.
\end{aligned}$$

2.4. **Super-sub-solutions and crucial estimates.** In this section, we extend the approach used in [27, 5] to the current two-species competition model.

For convenience, we introduce notation σ_i and (ϕ_i, ψ_i), $i = 1, 2, 3$, to be wave speeds and wave profiles, respectively, of traveling fronts as follows. Define

(2.20) $$\sigma_1 := -c_1, \quad \sigma_2 := c_2 \le c_{2,\max}, \quad \sigma_3 := c_3 \ge c_{3,\min}$$

and $(\phi_i,\psi_i)(x+\sigma_i t)$ as a traveling fronts of (2.1) satisfying

$$\begin{cases}\sigma_i\phi_i' = \phi_i'' + f(\phi_i,\psi_i), \quad \xi\in\mathbb{R},\\ \sigma_i\psi_i' = D\psi_i'' + g(\phi_i,\psi_i), \quad \xi\in\mathbb{R},\\ (\phi_i,\psi_i)(-\infty) = L_i, \quad (\phi_i,\psi_i)(+\infty) = R_i,\end{cases}$$

for $i=1,2,3$. Here we shall give a detailed argument for the case when

$$(L_1,R_1,L_2,R_2,L_3,R_3) := (\mathbf{1},\mathbf{0},\mathbf{0},\mathbf{E_I},\mathbf{E_I},\mathbf{1}),$$

so that we have

$$(\phi_1,\psi_1)(\xi) = (U_1,W_1)(-\xi), \quad (\phi_i,\psi_i)(\xi) = (U_i,W_i)(\xi) \quad \text{for } i=2,3.$$

Using the boundary conditions of (ϕ_i,ψ_i) at $\pm\infty$ and the strict monotonicity of (ϕ_i,ψ_i), by suitable translations (shifting to the right enough), we may assume that

$$\begin{cases}(\frac{u^*}{2},\frac{w^*}{2}) \le (\phi_1,\psi_1)(0) \le \mathbf{1},\\ \mathbf{0} \le (\phi_2,\psi_2)(0) \le (\frac{u^*}{2},\frac{w^*}{2}),\\ (u^*,w^*) \le (\phi_3,\psi_3)(0) \le (\frac{1+u^*}{2},\frac{1+w^*}{2}).\end{cases} \tag{2.21}$$

Recall that $w(x,t) := 1 - v(x,t)$ and set

$$(u(x,t),w(x,t)) := (U(\xi,t),W(\xi,t)), \quad \xi := x+\bar\sigma t,\ \bar\sigma := \frac{\sigma_1+\sigma_2}{2}.$$

Then we have

$$\begin{cases}U_t = U_{\xi\xi} - \bar\sigma U_\xi + f(U,W), \quad \xi,t\in\mathbb{R},\\ W_t = DW_{\xi\xi} - \bar\sigma W_\xi + g(U,W), \quad \xi,t\in\mathbb{R},\end{cases} \tag{2.22}$$

where f and g are defined by (2.2).

By direct computation, we can easily see that (2.22) has the following traveling front solutions:

$$(U,W)(\xi,t) = (\phi_1,\psi_1)(\xi - s_1 t),\ (\phi_2,\psi_2)(\xi + s_1 t),\ (\phi_3,\psi_3)(\xi + s_2 t),$$

where

$$s_1 := \frac{\sigma_2-\sigma_1}{2} > 0, \quad s_2 := \sigma_3 - \bar\sigma = \frac{2\sigma_3-\sigma_1-\sigma_2}{2} > \frac{-\sigma_1-\sigma_2}{2} > s_1. \tag{2.23}$$

Next, we introduce some auxiliary smooth functions q_i, $i=1,2,3$, defined on $I := (-\infty,0)$ such that

$$q_1(t) \le 0 \le -q_2(t) \le -q_3(t), \quad t\in I.$$

These functions will be given precisely later.

Define

$$\begin{cases}N_1[U,W](\xi,t) := U_t - U_{\xi\xi} + \bar\sigma U_\xi + f(U,W),\\ N_2[U,W](\xi,t) := W_t - DW_{\xi\xi} + \bar\sigma W_\xi + g(U,W).\end{cases}$$

Recall $\mathbf{Q} := (Q_1,Q_2)$ constructed in the previous subsection and put

$$U(\xi,t) = Q_1(\phi_1(\xi - q_1(t)), \phi_2(\xi + q_2(t)), \phi_3(\xi + q_3(t))),$$
$$W(\xi,t) = Q_2(\psi_1(\xi - q_1(t)), \psi_2(\xi + q_2(t)), \psi_3(\xi + q_3(t)))$$

into the operators N_1 and N_2. Then by some calculations we have

$$\begin{aligned} N_1[Q_1(\phi_1,\phi_2,\phi_3),Q_2(\psi_1,\psi_2,\psi_3)] \\ = -Q_{1y}\phi_1'(q_1'-s_1)+Q_{1z}\phi_2'(q_2'-s_1)+Q_{1\eta}\phi_3'(q_3'-s_2) \\ -G_1(\phi_1,\phi_2,\phi_3)-H_1(\Phi_1,\Phi_2,\Phi_3), \end{aligned} \tag{2.24}$$

where

$$\begin{aligned} G_1 &= Q_{1yy}(\phi_1')^2+Q_{1zz}(\phi_2')^2+Q_{1\eta\eta}(\phi_3')^2+2[Q_{1yz}\phi_1'\phi_2'+Q_{1y\eta}\phi_1'\phi_3'+Q_{1z\eta}\phi_2'\phi_3'], \\ H_1 &= f(\mathbf{Q})-Q_{1y}f(\Phi_1)-Q_{1z}f(\Phi_2)-Q_{1\eta}f(\Phi_3), \end{aligned}$$

where, for convenience, we write

$$\Phi_i := (\phi_i,\psi_i), \quad i=1,2,3. \tag{2.25}$$

and $\phi_i=\phi_i(\xi-q_i(t))$, $\psi_i=\psi_i(\xi-q_i(t))$ for $i=1,2,3$. Similarly, we have

$$\begin{aligned} N_2[Q_1(\phi_1,\phi_2,\phi_3),Q_2(\psi_1,\psi_2,\psi_3)] \\ = -Q_{2y}\psi_2'(q_1'-s_1)+Q_{2z}\psi_2'(q_2'-s_1)+Q_{1\eta}\psi_3'(q_3'-s_2) \\ -G_2(\psi_1,\psi_2,\psi_3)-H_2(\Phi_1,\Phi_2,\Phi_3), \end{aligned} \tag{2.26}$$

where

$$\begin{aligned} G_2 &= Q_{2yy}(\psi_1')^2+Q_{2zz}(\psi_2')^2+Q_{2\eta\eta}(\psi_3')^2+2[Q_{2yz}\psi_1'\psi_2'+Q_{2y\eta}\psi_1'\psi_3'+Q_{2z\eta}\psi_2'\psi_3'], \\ H_2 &= g(\mathbf{Q})-Q_{2y}g(\Phi_1)-Q_{2z}g(\Phi_2)-Q_{2\eta}g(\Phi_3). \end{aligned}$$

Set

$$D_H := D_1\times D_2,$$

where D_i is defined in (2.18), $i=1,2$. Then we have

Lemma 2.9. *The following results hold true.*

(i) *It holds that*

$$H_i(\mathbf{1},\mathbf{x}_2,\mathbf{x}_3)=H_i(\mathbf{x}_1,\mathbf{0},\mathbf{x}_3)=H_i(\mathbf{x}_1,\mathbf{E}_I,\mathbf{x}_3)=H_i(\mathbf{x}_1,\mathbf{x}_2,\mathbf{1})=H_i(\mathbf{0},\mathbf{x}_2,\mathbf{E}_I)=0$$

for $i=1,2$.

(ii) *There exist some continuous functions* $\widehat{H}_i(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)$, $i=1,2$, *such that*

$$H_i(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)=\|\mathbf{1}-\mathbf{x}_1\|\times\|\mathbf{x}_2\|\times\|\mathbf{E_I}-\mathbf{x}_2\|\times\|\mathbf{1}-\mathbf{x}_3\|\times\widehat{H}_i(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3),$$

where

$$\widehat{H}_i(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) := \mathbf{x}_1\cdot(\widehat{H}_{i1},\widehat{H}_{i2})+(\mathbf{x}_3-\mathbf{E_I})\cdot(\widehat{H}_{i3},\widehat{H}_{i4}) \tag{2.27}$$

for some continuous functions $\widehat{H}_{ij}$, $i=1,2$, $j=1,2,3,4$, *defined on* D_H, *and* $\|\cdot\|$ *denotes the standard Euclidean norm.*

Lemma 2.9(ii) plays a crucial role in establishing key estimates. The proof is presented in the appendix.

Lemma 2.10. *There exist positive constants M_1 and m_1 such that*

$$
\begin{aligned}
&m_1 \le Q_{1y}(\phi_1(\xi-q_1),\phi_2(\xi+q_2),\phi_3(\xi+q_3)) \le M_1 \quad \textit{for } \xi\in(-\infty,-q_2],\\
&m_1 \le Q_{2y}(\psi_1(\xi-q_1),\psi_2(\xi+q_2),\psi_3(\xi+q_3)) \le M_1 \quad \textit{for } \xi\in(-\infty,-q_2],\\
&m_1 \le Q_{1z}(\phi_1(\xi-q_1),\phi_2(\xi+q_2),\phi_3(\xi+q_3)) \le M_1 \quad \textit{for } \xi\in[q_1,-q_3],\\
&m_1 \le Q_{2z}(\psi_1(\xi-q_1),\psi_2(\xi+q_2),\psi_3(\xi+q_3)) \le M_1 \quad \textit{for } \xi\in[q_1,-q_3],\\
&m_1 \le Q_{1\eta}(\phi_1(\xi-q_1),\phi_2(\xi+q_2),\phi_3(\xi+q_3)) \le M_1 \quad \textit{for } \xi\in[-q_2,\infty),\\
&m_1 \le Q_{2\eta}(\psi_1(\xi-q_1),\psi_2(\xi+q_2),\psi_3(\xi+q_3)) \le M_1 \quad \textit{for } \xi\in[-q_2,\infty),
\end{aligned}
$$

for any given $q_1<0<-q_2\le -q_3$.

Proof. We divide $\mathbb{R}$ into four intervals: $(-\infty,q_1]$, $[q_1,-q_2]$, $[-q_2,-q_3]$ and $[-q_3,\infty)$. By (2.21) and the monotonicity of the wave profile, we see that for all $\xi\in(-\infty,q_1]$,

$$
\begin{cases}
(\frac{u^*}{2},\frac{w^*}{2}) \le (\phi_1,\psi_1)(\xi-q_1) \le \mathbf{1},\\
\mathbf{0} \le (\phi_2,\psi_2)(\xi+q_2) \le (\frac{u^*}{2},\frac{w^*}{2}),\\
(u^*,w^*) \le (\phi_3,\psi_3)(\xi+q_3) \le (\frac{1+u^*}{2},\frac{1+w^*}{2}).
\end{cases}
$$

Similarly, for $\xi\in[q_1,-q_2]$,

$$
\begin{cases}
\mathbf{0} \le (\phi_1,\psi_1)(\xi-q_1) \le (\phi_1,\psi_1)(0),\\
\mathbf{0} \le (\phi_2,\psi_2)(\xi+q_2) \le (\frac{u^*}{2},\frac{w^*}{2}),\\
(u^*,w^*) \le (\phi_3,\psi_3)(\xi+q_3) \le (\frac{1+u^*}{2},\frac{1+w^*}{2}).
\end{cases}
$$

For $\xi\in[-q_2,-q_3]$,

$$
\begin{cases}
\mathbf{0} \le (\phi_1,\psi_1)(\xi-q_1) \le (\phi_1,\psi_1)(0),\\
(\phi_2,\psi_2)(0) \le (\phi_2,\psi_2)(\xi+q_2) \le (u^*,w^*),\\
(u^*,w^*) \le (\phi_3,\psi_3)(\xi+q_3) \le (\frac{1+u^*}{2},\frac{1+w^*}{2}).
\end{cases}
$$

For $\xi\in[-q_3,\infty)$,

$$
\begin{cases}
\mathbf{0} \le (\phi_1,\psi_1)(\xi-q_1) \le (\phi_1,\psi_1)(0),\\
(\phi_2,\psi_2)(0) \le (\phi_2,\psi_2)(\xi+q_2) \le (u^*,w^*),\\
(\phi_3,\psi_3)(0) \le (\phi_3,\psi_3)(\xi+q_3) \le \mathbf{1}.
\end{cases}
$$

From the above inequalities, we can easily check that (ϕ_1,ϕ_2,ϕ_3) (resp., (ψ_1,ψ_2,ψ_3)) is far away from J_1^k (resp., J_2^k) for $k=1,2,3$ (J_i^k is defined by (2.19)). In fact, we see that the denominator and the numerator of Q_{iy}, Q_{iz} and $Q_{i\eta}$ (see Lemma 2.7), $i=1,2$, satisfy

$$
\begin{cases}
\ell_1 \le (1-u^*)(1-\phi_1)\phi_2+(u^*-\phi_2)(1-\phi_3) \le \ell_2, \quad \xi\in\mathbb{R},\\
\ell_3 \le (1-w^*)(1-\psi_1)\psi_2+(w^*-\psi_2)(1-\psi_3) \le \ell_4, \quad \xi\in\mathbb{R},\\
\ell_5 \le (1-\phi_2)(u^*-\phi_2)(1-\phi_3)^2 \le \ell_6, \quad \xi\in(-\infty,-q_2],\\
\ell_7 \le (1-\psi_2)(w^*-\psi_2)(1-\psi_3)^2 \le \ell_8, \quad \xi\in(-\infty,-q_2],\\
\ell_9 \le (1-\omega_1)(1-\omega_3)(\omega_3-\omega_2) \le \ell_{10}, \quad \xi\in[q_1,-q_3], \quad \omega_i=\phi_i,\psi_i,\\
\ell_{11} \le (1-\omega_1)^2\omega_2(1-\omega_2) \le \ell_{12}, \quad \xi\in[-q_2,\infty), \quad \omega_i=\phi_i,\psi_i
\end{cases}
\tag{2.28}
$$

for all q_i satisfying $q_1 < 0 < -q_2 \le -q_3$, where $\ell_1, \cdots, \ell_{12}$ are some positive constants. Hence the proof of Lemma 2.10 is completed. $\square$

By Lemma 2.8, Lemma 2.9 and Lemma 2.10, we obtain the following result.

Lemma 2.11. *Let $\widehat{H}_{ij}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$, $i = 1, 2$, $j = 1, \cdots, 4$, be given in* (2.27) *and Φ_i be defined in* (2.25)*. Then there exists $M_0 > 0$ such that*

$$|\widehat{H}_{ij}(\Phi_1(\xi - q_1), \Phi_2(\xi + q_2), \Phi_3(\xi + q_3))| \le M_0$$

for all $\xi \in \mathbb{R}$ for any given $q_1 \le 0 \le -q_2 \le -q_3$.

Next, we introduce two functions:

$$\begin{aligned}
F_1(\phi_1(\xi - q_1), \phi_2(\xi + q_2), \phi_3(\xi + q_3)) &:= -Q_{1y}\phi_1'(\xi - q_1) + Q_{1z}\phi_2'(\xi + q_2) + Q_{1\eta}\phi_3'(\xi + q_3),\\
F_2(\psi_1(\xi - q_1), \psi_2(\xi + q_2), \psi_3(\xi + q_3)) &:= -Q_{2y}\psi_1'(\xi - q_1) + Q_{2z}\psi_2'(\xi + q_2) + Q_{2\eta}\psi_3'(\xi + q_3),
\end{aligned}$$

where

$$\begin{aligned}
Q_{1\theta} &:= Q_{1\theta}(\phi_1(\xi - q_1), \phi_2(\xi + q_2), \phi_3(\xi + q_3)),\\
Q_{2\theta} &:= Q_{2\theta}(\psi_1(\xi - q_1), \psi_2(\xi + q_2), \psi_3(\xi + q_3)), \quad \theta = y, z, \eta.
\end{aligned}$$

Lemma 2.12. *There exists $\delta \gg 1$ such that*

$$F_1(\phi_1(\xi - q_1), \phi_2(\xi + q_2), \phi_3(\xi + q_3)) > 0$$

for all $\xi \in \mathbb{R}$ for any $q_1 < -\delta < \delta < -q_2 \le -q_3$. Moreover,

$$\begin{aligned}
F_1 &\ge \frac{1}{2} Q_{1y}|\phi_1'(\xi - q_1)|, \quad \xi \in (-\infty, q_1],\\
F_1 &\ge \frac{1}{2}[Q_{1y}|\phi_1'(\xi - q_1)| + Q_{1z}|\phi_2'(\xi + q_2)|], \quad \xi \in [q_1, -q_2],\\
F_1 &\ge \frac{1}{2}[Q_{1z}|\phi_2'(\xi + q_2)| + Q_{1\eta}|\phi_3'(\xi + q_3)|]), \quad \xi \in [-q_2, -q_3],\\
F_1 &\ge \frac{1}{2} Q_{1\eta}|\phi_3'(\xi + q_3)|, \quad \xi \in [-q_3, \infty),
\end{aligned}$$

for any $q_1 < -\delta < \delta < -q_2 \le -q_3$.

Proof. By the form of the second partial derivatives of Q_i (Lemma 2.7) and (2.28), there exists a positive constant C such that for any given $q_1 \le 0 \le -q_2 \le -q_3$,

$$|\widehat{Q}_{1j}(\phi_1(\xi - q_1), \phi_2(\xi + q_2), \phi_3(\xi + q_3))| \le C \quad \text{for all } \xi \in \mathbb{R} \text{ and } i = 1, \cdots 16,$$

where $\widehat{Q}_{1j}$ is defined in Lemma 2.8. It follows that the proof of [5, Lemma 3.3] can be applied directly to show the desired result. We shall not repeat it again. $\square$

Similarly, we have

Lemma 2.13. *There exists $\delta \gg 1$ such that*

$$F_2(\psi_1(\xi - q_1), \psi_2(\xi + q_2), \psi_3(\xi + q_3)) > 0$$

for all $\xi \in \mathbb{R}$ *for any* $q_1 < -\delta < \delta < -q_2 \le -q_3$. *Moreover,*

$$
\begin{aligned}
&F_2 \ge \frac{1}{2} Q_{2y} |\psi_1'(\xi - q_1)|, \quad \xi \in (-\infty, q_1],\\
&F_2 \ge \frac{1}{2} [Q_{2y} |\psi_1'(\xi - q_1)| + Q_{2z} |\psi_2'(\xi + q_2)|], \quad \xi \in [q_1, -q_2],\\
&F_2 \ge \frac{1}{2} [Q_{2z} |\psi_2'(\xi + q_2)| + Q_{2\eta} |\psi_3'(\xi + q_3)|], \quad \xi \in [-q_2, -q_3],\\
&F_2 \ge \frac{1}{2} Q_{2\eta} |\psi_3'(\xi + q_3)|, \quad \xi \in [-q_3, \infty),
\end{aligned}
$$

for any $q_1 < -\delta < \delta < -q_2 \le -q_3$.

Due to the form of $\mathbf{Q}$-function, we have the following key estimates:

Lemma 2.14. *Let* $\delta \gg 1$ *such that Lemma 2.12 holds. Then there exists* $M > 0$ *such that*

$$
\Big| \frac{H_1(\Phi_1, \Phi_2, \Phi_3)}{F_1(\phi_1, \phi_2, \phi_3)} \Big| \le
\begin{cases}
M(|\phi_2'| + |\phi_3'|), & \xi \in (-\infty, 0],\\
M(|\phi_1'| + |\phi_3'|), & \xi \in [0, -\frac{q_2+q_3}{2}],\\
M(|\phi_1'| + |\phi_2'|), & \xi \in [-\frac{q_2+q_3}{2}, \infty),
\end{cases}
$$

for any given $q_1 < -\delta < \delta < -q_2 \le -q_3$.

Proof. Recall from Lemma 2.9 that

$$
\begin{aligned}
&H_i(\Phi_1, \Phi_2, \Phi_3) = \|\mathbf{1} - \Phi_1\| \times \|\Phi_2\| \times \|\mathbf{E_I} - \Phi_2\| \times \|\mathbf{1} - \Phi_3\| \times \widehat{H}_i(\Phi_1, \Phi_2, \Phi_3),\\
&\widehat{H}_i(\Phi_1, \Phi_2, \Phi_3) := \Phi_1 \cdot (\widehat{H}_{i1}, \widehat{H}_{i2}) + (\Phi_3 - \mathbf{E_I}) \cdot (\widehat{H}_{i3}, \widehat{H}_{i4}),
\end{aligned}
$$

where $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \in D_H$.

We first prepare some estimates for later use. From Lemma 2.6(9) we see that for $\xi \in (-\infty, 0]$,

$$
\|\mathbf{1} - \Phi_1\| \le |1 - \phi_1| + |1 - \psi_1| = |1 - \phi_1| \Big(1 + \frac{|1 - \psi_1|}{|1 - \phi_1|}\Big) \le |1 - \phi_1| \Big(1 + K_9\Big). \tag{2.29}
$$

Again, using Lemma 2.6(9) we obtain that for $\xi \in [0, \infty)$,

$$
\|\mathbf{1} - \Phi_3\| \le |1 - \phi_3| + |1 - \psi_3| = |1 - \phi_3| \Big(1 + \frac{|1 - \psi_3|}{|1 - \phi_3|}\Big) \le |1 - \phi_3| \Big(1 + K_9\Big). \tag{2.30}
$$

By **(H2)** and Lemma 2.6(3), for $\xi \in [-q_1, \infty)$,

$$
\|\Phi_1\| \le |\phi_1| + |\psi_1| = |\phi_1| \Big(1 + \frac{|\psi_1|}{|\phi_1|}\Big) \le |\phi_1| \Big(1 + \frac{1}{\ell}\Big) \le K_3 |\phi_1'| \Big(1 + \frac{1}{\ell}\Big). \tag{2.31}
$$

By **(H2)** and Lemma 2.6(5), for $\xi \in (-\infty, -q_2]$,

$$
\|\Phi_2\| \le |\phi_2| + |\psi_2| = |\phi_2| \Big(1 + \frac{|\psi_2|}{|\phi_2|}\Big) \le |\phi_2| \Big(1 + \frac{1}{\ell}\Big) \le K_5 |\phi_2'| \Big(1 + \frac{1}{\ell}\Big). \tag{2.32}
$$

By Lemma 2.6(6), for $\xi \in [-q_2, \infty)$,

$$
\begin{aligned}
\|\Phi_2 - \mathbf{E}_I\| \quad &\le \quad |\phi_2 - u^*| + |\psi_2 - w^*| = \Big(1 + \frac{|\psi_2 - w^*|}{|\phi_2 - u^*|}\Big) \frac{|\phi_2 - u^*|}{|\phi_2'|} |\phi_2'|\\
&\le \quad (1 + K_6) K_6 |\phi_2'|
\end{aligned} \tag{2.33}
$$

By Lemma 2.6(7), for $\xi \in (-\infty, -q_2]$,

$$
\begin{aligned}
\|\Phi_3 - \mathbf{E}_I\| \quad &\le \quad |\phi_3 - u^*| + |\psi_3 - w^*| = \Big(1 + \frac{|\psi_3 - w^*|}{|\phi_3 - u^*|}\Big) \frac{|\phi_3 - u^*|}{|\phi_3'|} |\phi_3'|\\
&\le \quad (1 + K_7) K_7 |\phi_3'|
\end{aligned} \tag{2.34}
$$

By Lemma 2.11, for $\xi \in \mathbb{R}$,

$$|\widehat{H}_i(\Phi_1, \Phi_2, \Phi_3)| \leq \|\Phi_1\|\sqrt{2}M_0 + \|\Phi_3 - \mathbf{E_I}\|\sqrt{2}M_0. \tag{2.35}$$

We now divide our discussion into several cases.

(i) $\xi \in (-\infty, q_1]$. By Lemma 2.10 and Lemma 2.12, we have

$$|F_1| \geq \frac{m_1|\phi_1'|}{2}. \tag{2.36}$$

By (2.34) and (2.35), we have

$$|\widehat{H}_1(\Phi_1, \Phi_2, \Phi_3)| \leq \sqrt{2}\sqrt{2}M_0 + (1+K_7)K_7|\phi_3'|\sqrt{2}M_0. \tag{2.37}$$

Using (2.29), (2.32), (2.36) and (2.37), there exists $C > 0$ such that

$$\Big|\frac{H_1}{F_1}\Big| \leq C\frac{|1-\phi_1|}{|\phi_1'|}|\phi_2'|\Big(1 + |\phi_3'|\Big), \quad \xi \in (-\infty, q_1].$$

By Lemma 2.6(4) and the boundedness of $|\phi_2'|$, there exists $M > 0$ such that

$$\Big|\frac{H_1}{F_1}\Big| \leq M\Big(|\phi_2'| + |\phi_3'|\Big), \quad \xi \in (-\infty, q_1].$$

(ii) $\xi \in [q_1, 0]$. By Lemma 2.10 and Lemma 2.12, we have

$$|F_1| \geq \frac{m_1|\phi_1'|}{2} + \frac{m_1|\phi_2'|}{2}, \quad \xi \in [q_1, -q_2]. \tag{2.38}$$

Then using (2.38) and (2.35), we obtain

$$\begin{aligned}\Big|\frac{H_1}{F_1}\Big| \quad &\leq \quad \frac{\|\mathbf{1}-\Phi_1\|\|\Phi_2\|\|\mathbf{E_I}\|\|\mathbf{1}\|\|\Phi_1\|\sqrt{2M_0^2}}{m_1|\phi_1'|/2} \\ &\quad + \frac{\|\mathbf{1}-\Phi_1\|\|\Phi_2\|\|\mathbf{E_I}\|\|\mathbf{1}\|\|\Phi_3 - \mathbf{E}_I\|\sqrt{2M_0^2}}{m_1|\phi_2'|/2}.\end{aligned}$$

Thanks to (2.29), (2.32) and (2.34), there exists $C > 0$ such that for $\xi \in [q_1, 0]$,

$$\Big|\frac{H_1}{F_1}\Big| \quad \leq \quad C\Big(\frac{|1-\phi_1|}{|\phi_1'|}|\phi_2'| + |\phi_3'|\Big).$$

By Lemma 2.6(3), there exists $M > 0$ such that

$$\Big|\frac{H_1}{F_1}\Big| \leq M\Big(|\phi_2'| + |\phi_3'|\Big), \quad \xi \in [q_1, 0].$$

(iii) $\xi \in [0, -q_2]$. Using (2.38), (2.35) and (2.34),

$$\Big|\frac{H_1}{F_1}\Big| \quad \leq \quad \frac{\|\mathbf{1}-\Phi_1\|\|\Phi_2\|\|\mathbf{E_I}\|\|\mathbf{1}\|\Big(\|\Phi_1\| + (1+K_7)K_7|\phi_3'|\Big)\sqrt{2M_0^2}}{m_1|\phi_2'|/2}$$

Then using (2.31) and (2.32),

$$\Big|\frac{H_1}{F_1}\Big| \quad \leq \quad \frac{\|\mathbf{1}\|\Big[K_5|\phi_2'|(1+\frac{1}{\ell})\Big]\|\mathbf{E_I}\|\|\mathbf{1}\|\Big[K_3|\phi_1'|(1+\frac{1}{\ell}) + (1+K_7)K_7|\phi_3'|\Big]\sqrt{2M_0^2}}{m_1|\phi_2'|/2}$$

Hence there exists $M > 0$ such that

$$\Big|\frac{H_1}{F_1}\Big| \leq M\Big(|\phi_1'| + |\phi_3'|\Big), \quad \xi \in [0, -q_2].$$

(iv) $\xi \in [-q_2, -(\frac{q_2+q_3}{2})]$. By Lemma 2.10 and Lemma 2.12, we have

$$|F_1| \ge \frac{m_1|\phi_2'|}{2} + \frac{m_1|\phi_3'|}{2}, \quad \xi \in [-q_2, -q_3]. \tag{2.39}$$

Then using (2.39), (2.33), (2.30), (2.35) and (2.31),

$$\begin{aligned}
\Big|\frac{H_1}{F_1}\Big| &\le \frac{\|\mathbf{1}\|\|\Phi_2\|(1+K_6)K_6|\phi_2'|(1+K_9)|1-\phi_3|\Big[(1+\frac{1}{\ell})K_3|\phi_1'|\sqrt{2M_0^2} + \|\mathbf{E}_I\|\sqrt{2M_0^2}\Big]}{m_1|\phi_2'|/2} \\
&\le C|1-\phi_3|(|\phi_1'|+1) = C\frac{|1-\phi_3|}{|\phi_3'|}(|\phi_1'|+1)|\phi_3'| \\
&\le CK_8(|\phi_1'||\phi_3'| + |\phi_3'|)
\end{aligned}$$

for some $C > 0$ and we have used Lemma 2.6(8). By the boundedness of ϕ_3', we obtain the existence of M.

(v) $\xi \in [-(\frac{q_2+q_3}{2}), -q_3]$. Using (2.39) and (2.35), we obtain

$$\begin{aligned}
\Big|\frac{H_1}{F_1}\Big| &\le \frac{\|\mathbf{1}\|\|\Phi_2\|\|\mathbf{E}_I-\Phi_2\|\|\mathbf{1}\|\|\Phi_1\|\sqrt{2M_0^2}}{m_1|\phi_2'|/2} \\
&\quad + \frac{\|\mathbf{1}\|\|\Phi_2\|\|\mathbf{E}_I-\Phi_2\|\|\mathbf{1}\|\|\Phi_3-\mathbf{E}_I\|\sqrt{2M_0^2}}{m_1|\phi_3'|/2}.
\end{aligned}$$

Thanks to (2.30), (2.31) and (2.33), there exists $M > 0$ such that

$$\Big|\frac{H_1}{F_1}\Big| \le M\Big(|\phi_2'| + |\phi_3'|\Big), \quad \xi \in [-(\frac{q_2+q_3}{2}), -q_3].$$

(vi) $\xi \in [-q_3, \infty)$. By Lemma 2.10 and Lemma 2.12, we have

$$|F_1| \ge \frac{m_1|\phi_3'|}{2}. \tag{2.40}$$

By (2.31) and (2.35), we have

$$|\widehat{H}_1(\Phi_1,\Phi_2,\Phi_3)| \le K_3|\phi_1'|\Big(1+\frac{1}{\ell}\Big)\sqrt{2}M_0 + \|\mathbf{E}_I\|\sqrt{2}M_0. \tag{2.41}$$

Using (2.40), (2.30), (2.33) and (2.41), there exists $C > 0$ such that

$$\Big|\frac{H_1}{F_1}\Big| \le C\frac{|1-\phi_3|}{|\phi_3'|}|\phi_2'|\Big(|\phi_1'|+1\Big), \quad \xi \in [-q_3, \infty).$$

By the boundedness of $|\phi_2'|$ and Lemma 2.6(8), there exists $M > 0$ such that

$$\Big|\frac{H_1}{F_1}\Big| \le M\Big(|\phi_1'| + |\phi_2'|\Big), \quad \xi \in [-q_3, \infty).$$

Combining **(i)-(vi)**, the proof is completed. □

The following lemma is exactly the same as Lemma 3.4 in [5].

Lemma 2.15. *Let $\delta \gg 1$ such that Lemma 2.12 holds. Then there exists $M > 0$ such that*

$$\Big|\frac{G_1(\phi_1,\phi_2,\phi_3)}{F_1(\phi_1,\phi_2,\phi_3)}\Big| \le \begin{cases} M(|\phi_2'|+|\phi_3'|), & \xi \in (-\infty, 0], \\ M(|\phi_1'|+|\phi_3'|), & \xi \in [0, -\frac{q_2+q_3}{2}], \\ M(|\phi_1'|+|\phi_2'|), & \xi \in [-\frac{q_2+q_3}{2}, \infty), \end{cases}$$

for any given $q_1 < -\delta < \delta < -q_2 \le -q_3$.

Using the arguments in Lemma 2.14 and Lemma 2.15, we can obtain the following result:

Lemma 2.16. *Let $\delta \gg 1$ such that Lemma 2.13 holds. Then there exists $M > 0$ such that*

$$\frac{|H_2(\Phi_1,\Phi_2,\Phi_3)|}{|F_2(\psi_1,\psi_2,\psi_3)|} + \frac{|G_2(\psi_1,\psi_2,\psi_3)|}{|F_2(\psi_1,\psi_2,\psi_3)|} \leq \begin{cases} M(|\psi_2'| + |\psi_3'|), & \xi \in (-\infty, 0], \\ M(|\psi_1'| + |\psi_3'|), & \xi \in [0, -\frac{q_2+q_3}{2}], \\ M(|\psi_1'| + |\psi_2'|), & \xi \in [-\frac{q_2+q_3}{2}, \infty). \end{cases}$$

3. The proof of main results

In this section, we prove the main results by constructing a pair of super-sub-solutions. For this, we consider the following ODEs used in the literature widely (e.g., [12, 27, 5]):

$$\dot{p}_i = s_i + Le^{\kappa p_i},\ t \in (-\infty, 0), \quad p_i(0) = p_0 < 0, \quad i = 1, 2, \tag{3.1}$$

$$\dot{r}_i = s_i - Le^{\kappa r_i},\ t \in (-\infty, 0), \quad r_i(0) = r_0 < 0, \quad i = 1, 2 \tag{3.2}$$

for some large $L > 0$ and small $\kappa > 0$ which are to be determined later, where s_i is defined in (2.23). The fundamental properties of the above ODEs can be found in [12]. For the reader's convenience, we collect some results for later use in the appendix (Lemma A).

Let $\delta > 0$ such that Lemma 2.12 and Lemma 2.13 hold. Also, take

$$r_0 < \min\left\{p_0, \frac{1}{\kappa}\log(\frac{s_1}{L})\right\} < p_0 < -\delta.$$

Then, by Lemma A, we have

$$0 < p_1(t) - r_1(t) = p_2(t) - r_2(t) \leq 2R_0 e^{\kappa s_1 t}, \quad t \leq 0. \tag{3.3}$$

We can construct a pair of super-sub-solutions to system (2.22).

Lemma 3.1. *Define*

$$\begin{aligned} \overline{U}(\xi,t) &= Q_1(\phi_1(\xi - p_1(t)), \phi_2(\xi + p_1(t)), \phi_3(\xi + p_2(t))), \\ \overline{W}(\xi,t) &= Q_2(\psi_1(\xi - p_1(t)), \psi_2(\xi + p_1(t)), \psi_3(\xi + p_2(t))), \\ \underline{U}(\xi,t) &= Q_1(\phi_1(\xi - r_1(t)), \phi_2(\xi + r_1(t)), \phi_3(\xi + r_2(t))), \\ \underline{W}(\xi,t) &= Q_2(\psi_1(\xi - r_1(t)), \psi_2(\xi + r_1(t)), \psi_3(\xi + r_2(t))). \end{aligned}$$

Then there exists $t_0 < 0$ such that $(\overline{U}, \overline{W})$ and $(\underline{U}, \underline{W})$ are a pair of supersolution and subsolution to system (2.22) for $t \leq t_0$, respectively, satisfying

$$(\overline{U}, \overline{W})(\xi,t) \geq (\underline{U}, \underline{W})(\xi,t) \quad \text{for } \xi \in \mathbb{R} \text{ and } t \leq t_0, \tag{3.4}$$

$$\sup_{\xi\in\mathbb{R}} |\overline{U}(\xi,t) - \underline{U}(\xi,t)| + \sup_{\xi\in\mathbb{R}} |\overline{W}(\xi,t) - \underline{W}(\xi,t)| \leq Ce^{\kappa s_1 t} \quad \text{for } t \leq t_0 \tag{3.5}$$

for some positive constants C and κ.

Proof. We first show that there exists $\tau_0 < 0$ such that $(\overline{U}, \overline{W})$ is a pair of supersolution for $t \leq \tau_0$. To do so, we apply Lemma 2.14, Lemma 2.15 and Lemma 2.6(1)(2) with $(q_1, q_2, q_3) = (p_1, p_1, p_2)$ and $K := \max\{K_1, K_2\}$ to conclude that for all $\xi \leq 0$,

$$\begin{aligned} |G_1(\phi_1,\phi_2,\phi_3)| + |H_1(\Phi_1,\Phi_2,\Phi_3)| &\leq F_1(\phi_1,\phi_2,\phi_3)KM\Big(e^{\alpha_2(\xi+p_1)} + e^{\alpha_3(\xi+p_2)}\Big) \\ &\leq F_1(\phi_1,\phi_2,\phi_3)KM\Big(e^{\alpha_2 p_1} + e^{\alpha_3 p_2}\Big). \end{aligned}$$

For $0 \le \xi \le -(p_1+p_2)/2$,

$$\begin{aligned}|G_1(\phi_1,\phi_2,\phi_3)| + |H_1(\Phi_1,\Phi_2,\Phi_3)| &\le F_1(\phi_1,\phi_2,\phi_3)KM\Big(e^{-\beta_1(\xi-p_1)} + e^{\alpha_3(\xi+p_2)}\Big)\\ &\le F_1(\phi_1,\phi_2,\phi_3)KM\Big(e^{\beta_1 p_1} + e^{\alpha_3(p_2-p_1)/2}\Big).\end{aligned}$$

For $\xi \ge -(p_1+p_2)/2$,

$$\begin{aligned}|G_1(\phi_1,\phi_2,\phi_3)| + |H_1(\Phi_1,\Phi_2,\Phi_3)| &\le F_1(\phi_1,\phi_2,\phi_3)KM\Big(e^{-\beta_1(\xi-p_1)} + e^{-\beta_2(\xi+p_1)}\Big)\\ &\le F_1(\phi_1,\phi_2,\phi_3)KM\Big(e^{\beta_1 p_1} + e^{\beta_2(p_2-p_1)/2}\Big).\end{aligned}$$

Combining the above three inequalities with the fact that $p_i(t) = s_i t + o(t)$ as $t \to -\infty$, $i = 1, 2$, and $s_2 > s_1$, one can pick $\kappa = \kappa_1 > 0$ sufficiently small and t_1 with $-t_1 \gg 1$ such that

$$|G_1(\phi_1,\phi_2,\phi_3)| + |H_1(\Phi_1,\Phi_2,\Phi_3)| \le F_1(\phi_1,\phi_2,\phi_3)KMe^{\kappa_1 p_1} \tag{3.6}$$

for all $\xi \in \mathbb{R}$ and $t \le t_1$. Using (2.24), (3.1) and (3.6), one can take $L > KM$ such that

$$\begin{aligned}N_1[\overline{U},\overline{W}] &= -Q_{1y}\phi_1' Le^{\kappa_1 p_1} + Q_{1z}\phi_2' Le^{\kappa_1 p_1} + Q_{1\eta}\phi_3' Le^{\kappa_1 p_1}\\ &\quad -G_1(\phi_1,\phi_2,\phi_3) - H_1(\Phi_1,\Phi_2,\Phi_3)\\ &\ge F_1(\phi_1,\phi_2,\phi_3)(L-KM)e^{\kappa_1 p_1} \ge 0, \quad \xi \in \mathbb{R},\ t \le t_1.\end{aligned} \tag{3.7}$$

Paralleling to the process described above (but replacing Lemma 2.14 and Lemma 2.15 by Lemma 2.16), one can choose $\kappa = \kappa_2 > 0$ sufficiently small and t_2 with $-t_2 \gg 1$ such that

$$|G_2(\psi_1,\psi_2,\psi_3)| + |H_2(\Phi_1,\Phi_2,\Phi_3)| \le F_2(\psi_1,\psi_2,\psi_3)KMe^{\kappa_2 p_1} \tag{3.8}$$

for all $\xi \in \mathbb{R}$ and $t \le t_2$. Hence, by (2.26), (3.1) and (3.8) one can take $L > KM$ such that

$$\begin{aligned}N_2[\overline{U},\overline{W}] &= -Q_{2y}\psi_1' Le^{\kappa_2 p_1} + Q_{2z}\psi_2' Le^{\kappa_2 p_1} + Q_{2\eta}\psi_3' Le^{\kappa_2 p_1}\\ &\quad -G_2(\psi_1,\psi_2,\psi_3) - H_2(\Phi_1,\Phi_2,\Phi_3)\\ &\ge F_2(\psi_1,\psi_2,\psi_3)(L-KM)e^{\kappa_2 p_1} \ge 0, \quad \xi \in \mathbb{R},\ t \le t_2.\end{aligned} \tag{3.9}$$

Combining (3.7) and (3.9) and re-choosing $\kappa := \min\{\kappa_1,\kappa_2\}$, we see that $(\overline{U},\overline{W})$ is a pair of supersolution for $t \le \tau_0 := \min\{t_1,t_2\}$.

Similarly, we can show that there exists $t_3 < 0$ such that $(\underline{U},\underline{W})$ is a subsolution for $t \le t_3$. Indeed, using the above argument, there exists $\kappa_3 > 0$ sufficiently small and t_3 with $-t_3 \gg 1$ such that

$$|G_i(\phi_1,\phi_2,\phi_3)| + |H_i(\Phi_1,\Phi_2,\Phi_3)| \le F_i(\phi_1,\phi_2,\phi_3)KMe^{\kappa_3 p_1} \tag{3.10}$$

for all $\xi \in \mathbb{R}$, $t \le t_3$ and $i = 1, 2$. Using (2.24), (2.26), (3.2) with $\kappa = \kappa_3$ and (3.10), one can take $L > KM$ such that

$$\begin{aligned}N_1[\underline{U},\underline{W}] &= Q_{1y}\phi_1' Le^{\kappa_3 r_1} - Q_{1z}\phi_2' Le^{\kappa_3 r_1} - Q_{1\eta}\phi_3' Le^{\kappa_3 r_1}\\ &\quad -G_1(\phi_1,\phi_2,\phi_3) - H_1(\Phi_1,\Phi_2,\Phi_3)\\ &\le -F_1(\phi_1,\phi_2,\phi_3)(L-KM)e^{\kappa_1 r_1} \le 0, \quad \xi \in \mathbb{R},\ t \le t_3.\end{aligned}$$

and

$$\begin{aligned}N_2[\underline{U},\underline{W}] &= Q_{2y}\psi_1' Le^{\kappa_3 r_1} - Q_{2z}\psi_2' Le^{\kappa_3 r_1} - Q_{1\eta}\psi_3' Le^{\kappa_3 r_1}\\ &\quad -G_2(\psi_1,\psi_2,\psi_3) - H_2(\Phi_1,\Phi_2,\Phi_3)\\ &\le -F_2(\psi_1,\psi_2,\psi_3)(L-KM)e^{\kappa_1 r_1} \le 0, \quad \xi \in \mathbb{R},\ t \le t_3.\end{aligned}$$

It follows that $(\underline{U}, \underline{W})$ is a subsolution for $t \le t_3$.

Taking $t_0 = \min\{\tau_0, t_3\}$ and again re-choosing $\kappa := \min\{\kappa_1, \kappa_2, \kappa_3\}$, then $(\overline{U}, \overline{W})$ and $(\underline{U}, \underline{W})$ are a pair of supersolution and subsolution to system (2.22) for $t \le t_0$, respectively.

For (3.4) and (3.5), we apply the mean value theorem twice to conclude that

$$\overline{U}(\xi, t) - \underline{U}(\xi, t) = C(\xi, t)[p_1(t) - r_1(t)]$$

for some bounded function $C(\xi, t) > 0$ defined for $\xi \in \mathbb{R}$ and $t \le t_0$, where we have used Lemma 2.7, the fact that $p_1 - r_1 = p_2 - r_2$ and $\phi_1' < 0$ and $\phi_i' > 0$, $i = 2, 3$. By (3.3), we have

$$0 < \overline{U}(\xi, t) - \underline{U}(\xi, t) \le 2R_0 \|C\|_{L^\infty} e^{\kappa s_1 t}, \quad t \le t_0. \tag{3.11}$$

Similarly, we can obtain

$$0 < \overline{W}(\xi, t) - \underline{W}(\xi, t) \le C_0 e^{\kappa s_1 t}, \quad t \le t_0. \tag{3.12}$$

for some constant $C_0 > 0$. Therefore, (3.4) and (3.5) follows and then the proof of Lemma 3.1 is completed. □

We are ready to prove Theorem 1.

Proof of Theorem 1. Thanks to Lemma 3.1, we can apply the standard super-sub-solution method to conclude that (2.1) has a unique entire solution $(u(x, t), w(x, t))$ which satisfies

$$(\underline{U}, \underline{W})(x + \bar{\sigma} t, t) \le (u, v)(x, t) \le (\overline{U}, \overline{W})(x + \bar{\sigma} t, t)$$

for $x \in \mathbb{R}$ and $t \le t_0$, where $\underline{U}$, $\underline{W}$, $\overline{U}$ and $\overline{W}$ are defined in Lemma 3.1. We refer to e.g., [28] for the standard process.

Next, we show the behavior of the solution at $t = \pm\infty$. For the asymptotic behavior of the solution as $t \to \infty$, we can apply [1, Theorem 1] to derive (1.6). On the other hand, set

$$\omega := r_0 - \frac{1}{\kappa} \log \Big[1 + \frac{L}{s_1} e^{\kappa r_0}\Big]. \tag{3.13}$$

By Lemma A, we have

$$-R_0 e^{\kappa s_1 t} < r_i(t) - s_i t - \omega \le 0, \quad t \le 0.$$

Hence, together with (3.11) and (3.12), one can use the argument similar to that in [5, Theorem 4.3] to derive the asymptotic behavior of the solution as $t \to -\infty$. Since the process is standard in the literature, we omit the details. This completes the proof. □

Next, we show Theorem 2 as follows.

Proof of Theorem 2. This result can be done by following the proof of Theorem 1 and using some ideas in [5, Theorem 1.2] with some modifications. We only give a sketch of the proof.

Replace (2.20) by

$$\sigma_1 := -c_1, \quad \sigma_2 := c_2 \le c_{2,\max}, \quad \sigma_3 := -\widehat{c}_2 \ge -c_{2,\max} > 0,$$

and let

$$(\phi_1, \psi_1)(\xi) = (U_1, W_1)(-\xi), \quad (\phi_2, \psi_2)(\xi) = (U_2, W_2)(\xi) \quad (\phi_3, \psi_3)(\xi) = (\widehat{U}_2, \widehat{W}_2)(-\xi).$$

Next, inspired by [5], we replace $\mathbf{Q}$-function in (2.17) by

$$Q_i(y, z, \eta) = z + \frac{(1 - y) z (a_i - \eta)(-z) + y (a_i - z) \eta (1 - z)}{(1 - y) z a_i + (a_i - z) \eta}, \quad i = 1, 2, \tag{3.14}$$

defined on D_i for $i = 1, 2$, where $a_1 := u^*$, $a_2 := w^*$ and

$$\begin{aligned}
&D_i := [0,1] \times [0,a_i] \times [a_i,1] \setminus (J_i^1 \cup J_i^2 \cup J_i^3),\\
&J_i^1 := \{(y,0,0) |\, 0 \le y \le 1\},\\
&J_i^2 := \{(1,z,0) |\, 0 \le z \le a\},\\
&J_i^3 := \{(1,a_i,\eta) |\, 0 \le \eta \le a_i\}.
\end{aligned}$$

For super-sub-solutions, we define

$$\begin{aligned}
&\overline{U}(\xi,t) = Q_1(\phi_1(\xi - p_1(t)), \phi_2(\xi + p_1(t)), \phi_3(\xi + p_2(t))),\\
&\overline{W}(\xi,t) = Q_2(\psi_1(\xi - p_1(t)), \psi_2(\xi + p_1(t)), \psi_3(\xi + p_2(t))),\\
&\underline{U}(\xi,t) = Q_1(\phi_1(\xi - r_1(t)), \phi_2(\xi + r_1(t)), \phi_3(\xi + r_2(t))),\\
&\underline{W}(\xi,t) = Q_2(\psi_1(\xi - r_1(t)), \psi_2(\xi + r_1(t)), \psi_3(\xi + r_2(t))),
\end{aligned}$$

where

$$\begin{cases}
\dot{p}_1 = s_1 + Le^{\kappa p_1},\ t \in (-\infty,0), & p_1(0) = p_0 < 0,\\
\dot{r}_1 = s_1 - Le^{\kappa r_1},\ t \in (-\infty,0), & r_1(0) = r_0 < 0,\\
\dot{p}_2 = s_2 - Le^{\kappa p_2},\ t \in (-\infty,0), & p_2(0) = r_0 < 0,\\
\dot{r}_2 = s_2 + Le^{\kappa r_2},\ t \in (-\infty,0), & r_2(0) = p_0 < 0,
\end{cases} \tag{3.15}$$

for some large $L > 0$ and small $\kappa, p_0, r_0 > 0$.

Following the same process as in proving Theorem 1 with some minor modifications, one can show Theorem 2. We leave details to the reader. □

Proof of Theorem 3. The proof can be finished similarly as the proof of Theorem 1 with suitable modifications. So we only give a sketch of the proof and point out the differences as follows.

Recall that

$$\sigma_1 := -c_1, \quad \sigma_2 := c_2 \le c_{2,\max}, \quad \bar{\sigma} := (\sigma_1 + \sigma_2)/2.$$

In order to obtain the asymptotic behavior as $t \approx -\infty$, our supersolution $(\overline{U}, \overline{W})$ has to be even in x for all t with $-t \gg 1$. For this purpose, it is not appropriate to introduce the new variable $\xi := x + \bar{\sigma}t$. We divide our proof into several steps.

Step 1. The construction of a supersolution.

Inspired by [5, Theorem 1.3], we define

$$\begin{aligned}
&U^*(x,t) = Q_1(\phi_1(x + \bar{\sigma}t - p_1(t)), \phi_2(x + \bar{\sigma}t + p_1(t)), u^*),\\
&W^*(x,t) = Q_2(\psi_1(x + \bar{\sigma}t - p_1(t)), \psi_2(x + \bar{\sigma}t + p_1(t)), w^*),
\end{aligned}$$

where Q_i is given in (2.17), $i = 1, 2$, and $p_1(t)$ satisfies (3.1).

In short, we write $\phi_i := \phi_i(x + \bar{\sigma}t - p_1(t))$ for $i = 1, 2$. From (2.24) and (2.26) we see that

$$\begin{aligned}
&N_1[Q_1(\phi_1, \phi_2, u^*), Q_2(\psi_1, \psi_2, w^*)](x,t)\\
= \;&[-Q_{1y}\phi_1' + Q_{1z}\phi_2'](p_1'(t) - s_1) - G_1(\phi_1, \phi_2, u^*) - H_1(\Phi_1, \Phi_2, \mathbf{E_I}),
\end{aligned}$$

Similar to the proof of Lemma 3.1, one can pick $\kappa > 0$ sufficiently small, t_1 with $-t_1 \gg 1$ and a positive constant C_1 such that

$$|G_1(\phi_1, \phi_2, u^*)| + |H_1(\Phi_1, \Phi_2, \mathbf{E_I})| \quad \le \quad F_1(\phi_1, \phi_2, u^*) C_1 e^{\kappa p_1}$$

for all $x \in \mathbb{R}$ and $t \le t_1$. Then one has

$$\begin{aligned} N_1[U^*,W^*](x,t) &= [-Q_{1y}\phi_1' + Q_{1z}\phi_2']Le^{\kappa p_1} - G_1(\phi_1,\phi_2,u^*) - H_1(\Phi_1,\Phi_2,\mathbf{E_I}) \\ &\ge F_1(\phi_1,\phi_2,u^*)(L-C_1)e^{\kappa p_1} \ge 0, \quad \xi\in\mathbb{R},\ t\le t_1, \end{aligned}$$

as long as $L \ge C_1$. Similarly, we have $N_2[U^*,W^*](x,t) \ge 0$ for all $x \in \mathbb{R}$ and t with $-t \gg 1$ as long as L is chosen large enough. Hence, one can choose $L \gg 1$ such that (U^*,W^*) is a supersolution for all t with $-t \gg 1$.

In order to get the asymptotic behavior as $t \to -\infty$, we need to modify the supersolution as an even function in x. For this, we define

$$(\overline{U},\overline{W})(x,t) = \begin{cases} (U^*,W^*)(x,t) & \text{if } x \le 0, \\ (U^*,W^*)(-x,t) & \text{if } x \ge 0. \end{cases}$$

In order to show $(\overline{U},\overline{W})(x,t)$ is a (weak) supersolution, it suffices to show that

$$\overline{U}_x(0^-,t) \le 0, \quad \overline{U}_x(0^+,t) \ge 0, \tag{3.16}$$

$$\overline{W}_x(0^-,t) \le 0, \quad \overline{U}_x(0^+,t) \ge 0 \tag{3.17}$$

for all $-t \gg 1$.

For (3.16), we consider $x \le 0$ and the left derivative $\overline{U}_x(x,t)$. By straightforward calculation,

$$\begin{aligned} \overline{U}_x(x,t) &= Q_{1y}(\phi_1,\phi_2,u^*)\phi_1' + Q_{1z}(\phi_1,\phi_2,u^*)\phi_2' \\ &= \phi_2'\left[\frac{Q_{1y}(\phi_1,\phi_2,u^*)}{\phi_2'}\phi_1' + Q_{1z}(\phi_1,\phi_2,u^*)\right]. \end{aligned} \tag{3.18}$$

Recall from (2.7) that

$$Q_{1y}(\phi_1,\phi_2,u^*) = \frac{u^*(1-\phi_2)(u^*-\phi_2)}{(1-\phi_1)^2(u^*)^2 + (u^*-\phi_2)^2}, \quad Q_{1z}(\phi_1,\phi_2,u^*) = \frac{u^*(1-\phi_1)(u^*-\phi_1)}{(1-\phi_1)^2(u^*)^2 + (u^*-\phi_2)^2}.$$

By Lemma 2.6 and the fact that

$$\phi_1(\bar\sigma t - p_1(t)) \to 0, \quad \phi_2(\bar\sigma t - p_1(t)) \to u^* \quad \text{as } t \to -\infty,$$

we have for $x = 0$,

$$\frac{Q_{1y}(\phi_1,\phi_2,u^*)}{\phi_2'}\phi_1' = \frac{u^*(1-\phi_2)}{(1-\phi_1)^2(u^*)^2 + (u^*-\phi_2)^2}\,\frac{(u^*-\phi_2)}{\phi_2'}\phi_1' \to 0 \quad \text{as } t \to -\infty$$

and $Q_{1z}(\phi_1,\phi_2,u^*) \to 1$ as $t \to -\infty$. Also, using $\phi_2' > 0$, from (3.18) we see that $\overline{U}_x(0^-,t) \le 0$ for all $-t \gg 1$. Since $\overline{U}$ is even in x, $\overline{U}_x(0^+,t) \ge 0$ for all $-t \gg 1$. Hence (3.16) holds. The above process can be applied to show (3.17) and so we omit the details. Therefore, $(\overline{U},\overline{W})(x,t)$ is a (weak) supersolution for all $-t \gg 1$ and then **Step 1** is completed.

Step 2. The construction of a subsolution.

Define

$$\begin{aligned} &\underline{U}(x,t) = Q_1(\phi_1(x+\bar\sigma t - r_1(t)), \phi_2(x+\bar\sigma t + r_1(t)), \phi_2(-x-\bar\sigma t - r_3(t))), \\ &\underline{W}(x,t) = Q_2(\psi_1(x+\bar\sigma t - r_1(t)), \psi_2(x+\bar\sigma t + r_1(t)), \psi_2(-x-\bar\sigma t - r_3(t))), \end{aligned}$$

where Q_i is defined in (3.14), $r_3(t) := r_2(t) - \ell_0$ for some $\ell_0 \gg 1$ and $r_i(t)$ is defined in (3.15) for $i = 1,2$. Similar to the proof of Lemma 3.1 (with $\phi_3(\zeta) = \phi_2(-\zeta)$), one can show that $(\underline{U},\underline{W})$ is a subsolution for all $-t \gg 1$ (with suitable ℓ_0).

Step 3. Complete the proof.

Now we can take $T_1 \gg 1$ such that $(\overline{U},\overline{W})(x,t)$ is a (weak) supersolution and $(\underline{U},\underline{W})$ is a subsolution for all $t \le -T_1$. Next, following the argument used in [5, Theorem1.3] with minor changes, we can obtain

$$(\overline{U},\overline{W})(x,t) \ge (\underline{U},\underline{W})(x,t), \quad x \in \mathbb{R},\ t \le -T_2$$

for some $T_2 \ge T_1$.

Denote the solution of (2.1) with initial data (u_0,w_0) by $(u,w)(x,t;u_0,w_0)$. For any given $T > T_2$, define $(u^T,w^T)(x,t) := (u,w)(x,t+T;\overline{U}(\cdot,-T),\overline{W}(\cdot,-T))$. By comparison,

$$(\underline{U},\underline{W})(x,t) \le (u^T,w^T)(x,t) \le (\overline{U},\overline{W})(x,t), \quad x \in \mathbb{R},\ t \le -T.$$

Note that by comparison, $\{u^T\}$ and $\{w^T\}$ is decreasing in T. It follows that the limit function $(u^\infty,w^\infty) := \lim_{T\to\infty}(u^T,w^T)$ is well-defined in the whole space and time such that

$$(\underline{U},\underline{W})(x,t) \le (u^\infty,w^\infty)(x,t) \le (\overline{U},\overline{W})(x,t), \quad x \in \mathbb{R},\ t \in \mathbb{R}.$$

Moreover, since $(\overline{U},\overline{W})$ is even in x, $(u^T,w^T)(x,t)$ is also even in x for $x \in \mathbb{R}$ and $t \le -T$ and so does $(u^\infty,w^\infty)(x,t)$ for $x,t \in \mathbb{R}$.

To finish the proof, we need to show that (u^∞,w^∞) satisfies the desired asymptotic behavior as $t \to \pm\infty$. For this, we first use a similar procedure as that in [5, Theorem 1.3] to derive the estimate (3.5) for all t with $-t \gg 1$. Then the asymptotic behavior of (u^∞,w^∞) can be done by using the same process as used in Theorem 1. We leave the details for the reader. Hence, we complete the proof of Theorem 3. □

Proof of Theorem 4. Set $w(x,t) := 1 - v(x,t)$ and $\overline{W} := 1 - \overline{V}$. Then we see that an entire solution (u,w) originates from N fronts if and only if there exists $(c_j,\overline{U}_j,\overline{W}_j)$ for $j = 1,\cdots,N$ satisfying (1.9), (1.10) and (1.11) with v and $\overline{V}$ replaced by w and $\overline{W}$, respectively, where $(c_j,\overline{U}_j,\overline{W}_j)$ satisfies

$$\begin{cases} c_j\overline{U}_j' = \overline{U}_j'' + f(\overline{U}_j,\overline{W}_j), \quad \xi \in \mathbb{R}, \\ c_j\overline{W}_j' = D\overline{W}_j'' + g(\overline{U}_j,\overline{W}_j), \quad \xi \in \mathbb{R}, \\ (\overline{U}_j,\overline{W}_j)(-\infty) := L_i, \quad (\overline{U}_j,\overline{W}_j)(+\infty) := R_i, \quad i = 1,\cdots,N, \end{cases}$$

where $L_i, R_i \in \{\mathbf{1},\mathbf{0},\mathbf{E_I},\mathbf{E_B}\}$ and $\mathbf{E_B} := (0,1)$. Reducing to this form allows us to use the argument in [5] to prove desired result.

Given an entire solution (u,w) originating from N fronts, one can define the sequence of boundary conditions

$$A_N := \{L_1,R_1,L_2,R_2,\cdots,L_N,R_N\}$$

with $2N$ terms describing $(\overline{U}_j,\overline{W}_j)(\pm\infty)$, $1 \le j \le N$. Since entire solutions are continuous and satisfy (1.10) and (1.11) (with v and $\overline{V}$ replaced by w and $\overline{W}$), we must have

$$R_j = L_{j+1} \quad \text{for all } j = 1,\cdots,N-1. \tag{3.19}$$

Under the above setting, we see that the corresponding sequences to Theorem 1, Theorem 2 and Theorem 3 are

$$\{\mathbf{1},\mathbf{0},\mathbf{0},\mathbf{E_I},\mathbf{E_I},\mathbf{1}\}, \quad \{\mathbf{1},\mathbf{0},\mathbf{0},\mathbf{E_I},\mathbf{E_I},\mathbf{0}\}, \quad \{\mathbf{1},\mathbf{0},\mathbf{0},\mathbf{E_I},\mathbf{E_I},\mathbf{0},\mathbf{0},\mathbf{1}\},$$

respectively.

We say that A_N is non-extendable if there is no entire solution (u, w) originating from $N+1$ fronts with the corresponding sequence A_{N+1} such that the first $2N$ terms of A_{N+1} are exactly the same as A_N. Then one can observe that A_N is non-extendable if

$$(L_N, R_N) = (\mathbf{0}, \mathbf{1}) \text{ or } (\mathbf{E_I}, \mathbf{1}) \tag{3.20}$$

Indeed, if (3.20) occurs, by (3.19), we must have $L_{N+1} = \mathbf{1}$. Then $R_{N+1} = \mathbf{0}$ or $\mathbf{E_I}$. Since both $(\mathbf{1}, \mathbf{0})$-front and $(\mathbf{1}, \mathbf{E_I})$-front move to the right, we have $c_{N+1} < 0 < c_N$, which reaches a contradiction with (1.9).

Note that system (1.2) may admit an $(\mathbf{E_B}, \mathbf{E_I})$-front as we have mentioned in the introduction section. The sign of the wave speed plays an important role to determine all possible non-extendable sequences. Suppose that an $(\mathbf{E_B}, \mathbf{E_I})$-front (c, U, V) exists. Then

$$\begin{cases} cU'(\xi) = U''(\xi) + U(\xi)(1 - U - kV)(\xi), \quad \xi \in \mathbb{R}, \\ cV'(\xi) = DV''(\xi) + rV(\xi)(1 - V - hU)(\xi), \quad \xi \in \mathbb{R}, \\ (U, V)(-\infty) = (0, 0), \quad (U, V)(+\infty) = (u^*, v^*). \end{cases} \tag{3.21}$$

We now show that $c > 0$. Note that the monotonicity of wave profiles is not known. However, by a similar argument as in section 2.2, we see that both U and V decay exponentially near $-\infty$. Thus, there exists $\xi_0 \gg 1$ such that $U'(\xi) > 0$ and $V'(\xi) > 0$ for all $\xi \leq -\xi_0$. Together with the boundary condition at $\xi = -\infty$, we may further assume that

$$U(\xi)(1 - U - kV)(\xi) > \frac{1}{2} U(\xi), \quad \xi \leq -\xi_0.$$

Integrating the first equation of (3.21) over $(-\infty, \xi)$ with $\xi < -\xi_0$, we have

$$cU(\xi) = U'(\xi) + \int_{-\infty}^{\xi} U(s)(1 - U - kV)(s) > \frac{1}{2} U(\xi) > 0.$$

Thus, $c > 0$ if an $(\mathbf{E_B}, \mathbf{E_I})$-front (c, U, V) exists. Together with (3.20), one can list all possible non-extendable sequences corresponding to entire solutions originating from N fronts:

When $L_1 = \mathbf{E_B}$, the possible longest non-extendable sequence is

$$\{\mathbf{E_B}, \mathbf{E_I}, \mathbf{E_I}, \mathbf{0}, \mathbf{0}, \mathbf{1}\},$$

which corresponds to 3 fronts. When $L_1 = \mathbf{E_I}$, the possible longest non-extendable sequence is

$$\{\mathbf{E_I}, \mathbf{E_B}, \mathbf{E_B}, \mathbf{E_I}, \mathbf{E_I}, \mathbf{0}, \mathbf{0}, \mathbf{1}\},$$

which corresponds to 4 fronts. When $L_1 = \mathbf{0}$, the possible longest non-extendable sequence is

$$\{\mathbf{0}, \mathbf{E_I}, \mathbf{E_I}, \mathbf{E_B}, \mathbf{E_B}, \mathbf{E_I}, \mathbf{E_I}, \mathbf{0}, \mathbf{0}, \mathbf{1}\},$$

which corresponds to 5 fronts. When $L_1 = \mathbf{1}$, the possible longest non-extendable sequence is

$$\{\mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{E_I}, \mathbf{E_I}, \mathbf{E_B}, \mathbf{E_B}, \mathbf{E_I}, \mathbf{E_I}, \mathbf{0}, \mathbf{0}, \mathbf{1}\},$$

which corresponds to 6 fronts. Consequently, there does not exist a sequence corresponding to entire solutions originating from N fronts for $N \geq 7$. This completes the proof. □

Remark 3.1. *In* [5], *the authors showed that for the Allan-Cahn equation, there is no entire solution originating from N fronts if $N \geq 5$. The dynamics of competition systems may become more complicated due to the increment of the number of equilibria. Theorem 4 suggests that the bistable competition system* (1.2) *may support entire solutions originating from* 5 *fronts and* 6 *fronts. It will be of interest to show that* 5 *fronts and* 6 *fronts entire solutions do exist. We leave this issue to a future work.*

4. Appendix

We provide a proof of Lemma 2.9 as follows:

Proof of Lemma 2.9. We first deal with (i). From Lemma 2.8, we see that

$$Q_{1z}(1,z,\eta) = Q_{1\eta}(1,z,\eta) = Q_i(1,z,\eta) = 0, \quad i = 1,2. \tag{4.1}$$

Let us write $\mathbf{x}_i = (x_{i1}, x_{i2})$ for $i = 1,2,3$. It follows that

$$\begin{aligned} H_1(\mathbf{1}, \mathbf{x}_2, \mathbf{x}_3) &= f(Q_1(1,x_{21},x_{31}), Q_2(1,x_{22},x_{23})) - Q_{1y}(1,x_{21},x_{31})f(1,1) \\ &\quad -Q_{1z}(1,x_{21},x_{31})f(x_{21},x_{31}) - Q_{1\eta}(1,z,\eta)f(x_{31},x_{32}) \\ &= 0, \end{aligned}$$

where we have used (4.1) and the fact $f(1,1) = 0$. The other cases in (i) can be shown in a similar way. We omit the details here.

We next prove (ii) by using the mean value theorem several times. Given $i \in \{1,2\}$. Since $H_i(\mathbf{1}, \mathbf{x}_2, \mathbf{x}_3) = 0$ (from (i)), the mean value theorem implies that

$$\begin{aligned} H_i(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) &= \Big(\frac{\partial H_i}{\partial x_{i1}}, \frac{\partial H_i}{\partial x_{i2}}\Big) \cdot (\mathbf{x_1} - \mathbf{1}) \\ &= \|\mathbf{x_1} - \mathbf{1}\| \Big[\Big(\frac{\partial H_i}{\partial x_{i1}}, \frac{\partial H_i}{\partial x_{i2}}\Big) \cdot \frac{\mathbf{x_1} - \mathbf{1}}{\|\mathbf{x_1} - \mathbf{1}\|}\Big] \end{aligned} \tag{4.2}$$

where $\mathbf{x}_i := (x_{i1}, x_{i2})$. Define

$$\nu_{i1}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) := \Big(\frac{\partial H_i}{\partial x_{i1}}, \frac{\partial H_i}{\partial x_{i2}}\Big) \cdot \frac{\mathbf{x_1} - \mathbf{1}}{\|\mathbf{x_1} - \mathbf{1}\|}.$$

Clearly, ν_{i1} is a smooth function defined on D_H.

Next, since $H_i(\mathbf{x}_1, \mathbf{0}, \mathbf{x}_3) = 0$, we have $\nu_{i1}(\mathbf{x}_1, \mathbf{0}, \mathbf{x}_3) = 0$. By the mean value theorem as above, there exists a smooth function ν_{i2} such that

$$\nu_{i1}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \|\mathbf{x}_2\| \nu_{i2}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3).$$

From $H_i(\mathbf{x}_1, \mathbf{E}_I, \mathbf{x}_3) = 0$, we see that $\nu_{i2}(\mathbf{x}_1, \mathbf{E}_I, \mathbf{x}_3) = 0$. Again, by the mean value theorem as above, there exists a smooth function ν_{i3} such that

$$\nu_{i2}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \|\mathbf{E}_I - \mathbf{x}_2\| \nu_{i3}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3).$$

From $H_i(\mathbf{x}_1, \mathbf{x}_2, \mathbf{1}) = 0$, we see that $\nu_{i3}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{1}) = 0$. By the mean value theorem as above, there exists a smooth function ν_{i4} such that

$$\nu_{i3}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \|\mathbf{1} - \mathbf{x}_3\| \nu_{i4}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3).$$

Putting ν_{ij}, $j = 1,2,3,4$, into (4.2), we obtain

$$H_i(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \|\mathbf{1} - \mathbf{x}_1\| \times \|\mathbf{x}_2\| \times \|\mathbf{E_I} - \mathbf{x}_2\| \times \|\mathbf{1} - \mathbf{x}_3\| \times \widehat{H}_i(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$$

with $\widehat{H}_i := \nu_{i4}$.

Finally, from $H_i(\mathbf{0}, \mathbf{x}_2, \mathbf{E}_I) = 0$ we see that $\widehat{H}_i(\mathbf{0}, \mathbf{x}_2, \mathbf{E}_I) = 0$. Then the mean value theorem gives (2.27). This completes the proof. ∎

Lemma A. ([12]) *Consider*

$$p'(t) = s + Le^{\kappa p(t)} \quad \textit{for } t < 0, \quad p(0) := p_0$$

for some constants $s > 0$ and $\kappa > 0$. Then

$$p(t) = p_0 + st - \frac{1}{\kappa}\log\Big[1 + \frac{L}{s}e^{\kappa p_0}(1 - e^{s\kappa t})\Big],\ t \le 0.$$

Moreover, the following hold:

(i) *If $L > 0$, then p is increasing in $(-\infty, 0]$ and*

$$0 < p(t) - st - \omega \le R_0 e^{s\kappa t}, \quad t \le 0,$$

for some constant $R_0 > 0$, where

$$\omega := p_0 - \frac{1}{\kappa}\log\Big[1 + \frac{L}{s}e^{\kappa p_0}\Big]. \tag{4.3}$$

(ii) *If $L < 0$ and $p_0 < (1/\kappa)\log(-s/L)$, then p is increasing in $(-\infty, 0]$ and*

$$-R_0 e^{s\kappa t} < p(t) - st - \omega < 0, \quad t \le 0,$$

for some constant $R_0 > 0$, where ω is given in (4.3).

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References

[1] C. Carrere, *Spreading speeds for a two-species competition-diffusion system.* J. Diff. Equ. **264** (2018), 2133–2156.

[2] X. Chen, J.-S. Guo, *Existence and uniqueness of entire solutions for a reaction-diffusion equation*, J. Diff. Equ. **212** (2005), 62–84.

[3] X. Chen, J.-S. Guo, H. Ninomiya, *Entire solutions of reaction-diffusion equations with balanced bistable nonlinearity*, Proc. Roy. Soc. Edinburgh Sect. A **136** (2006), 1207–1237.

[4] Y.-Y. Chen, *Entire solution originating from three fronts for a discrete diffusive equation*, Tamkang J. Math. **48** (2017), 215–226.

[5] Y.-Y. Chen, J.-S. Guo, H. Ninomiya and C.-H. Yao, *Entire solutions originating from monotone fronts to the Allen-Cahn equation*, Physica D **378-379** (2018), 1–19.

[6] C. Conley, R. Gardner, *An application of the generalized Morse index to traveling wave solutions of a competitive reaction diffusion model*, Indiana Univ. Math. J. **33** (1984), 319–343.

[7] J. Fang, X.-Q. Zhao, *Monotone wavefronts for partially degenerate reactiondiffusion systems*, J. Dyn. Diff. Equ. **21** (2009), 663–680.

[8] Y. Fukao, Y. Morita, H. Ninomiya, *Some entire solutions of the Allen-Cahn equation*, Taiwanese J. Math. **8** (2004), 15-32.

[9] R.A. Gardner, *Existence and stability of traveling wave solutions of competition models: a degree theoretic*, J. Diff. Equ. **44** (1982), 343–364.

[10] L. Girardin, G. Nadin, *Travelling waves for diffusive and strongly competitive systems: relative motility and invasion speed*, European J. Appl. Math. **26** (2015), 521–534.

[11] J.-S. Guo, Y.-C. Lin, *The sign of the wave speed for the Lotka-Volterra competition-diffusion system*, Comm. Pure Appl. Anal. **12** (2013), 2083–2090.

[12] J.-S. Guo, Y. Morita, *Entire solutions of reaction-diffusion equations and an application to discrete diffusive equations*, Discrete Cont. Dyn. Syst. **12** (2005), 193–212.

[13] J.-S. Guo, C.-H. Wu, *Entire solutions for a two-component competition system in a lattice*, Tohoku Math. J. **62** (2010), 17–28.

[14] F. Hamel, N. Nadirashvili, *Entire solutions of the KPP equation*, Comm. Pure Appl. Math. **52** (1999), 1255-1276.

[15] F. Hamel, N. Nadirashvili, *Travelling fronts and entire solutions of the Fisher-KPP equation in* $\mathbb{R}^N$, Arch. Ration. Mech. Anal. **157** (2001), 91–163.

[16] Y. Hosono, *Singular perturbation analysis of travelling waves for diffusive Lotka-Volterra competitive models*, Numerical and Applied Mathematics, Part II (Paris, 1988), Baltzer, Basel, (1989) 687–692.

[17] Y. Hosono, *The minimal speed of traveling fronts for a diffusive Lotka-Volterra competition model*, Bull. Math. Biology **60** (1998), 435–448.

[18] J.I. Kanel, L. Zhou, *Existence of wave front solutions and estimates of wave speed for a competition-diffusion system*, Nonlinear Anal. **27** (1996), 579–587.

[19] Y. Kan-on, *Parameter dependence of propagation speed of travelling waves for competition-diffusion equations*, SIAM J. Math. Anal. **26** (1995), 340–363.

[20] Y. Kan-on, *Existence of standing waves for competition-diffusion equations*, Japan J. Indust. Appl. Math. **13** (1996), 117–133.

[21] Y. Kan-on, *Fisher wave fronts for the Lotka-Volterra competition model with diffusion*, Nonlinear Anal. **28** (1997), 145–164.

[22] Y. Kan-on, *Instability of stationary solutions for a Lotka-Volterra competition model with diffusion*, J. Math. Anal. Appl. **208** (1997), 158-170.

[23] Y. Kan-on, Q. Fang, *Stability of monotone travelling waves for competition-diffusion equations*, Japan J. Indust. Appl. Math. **13** (1996), 343–349.

[24] M.A. Lewis, B. Li, H.F. Weinberger, *Spreading speed and linear determinacy for two-species competition models*, J. Math. Biol. **45** (2002), 219–233.

[25] B. Li, H. Weinberger, M. Lewis, *Spreading speeds as slowest wave speeds for cooperative systems*, Math. Biosci. **196** (2005), 82–98.

[26] W.-T. Li, Z.-C. Wang, J. Wu, *Entire solutions in monostable reaction-diffusion equations with delayed nonlinearity*, J. Diff. Equ. **245** (2008), 102–129.

[27] Y. Morita, H. Ninomiya, *Entire solutions with merging fronts to reaction-diffusion equations*, J. Dynam. Diff. Equ. **18** (2006), 841–861.

[28] Y. Morita, K. Tachibana, *An entire solution to the Lotka-Volterra competition-diffusion equations*, SIAM J. Math. Anal. **40** (2009), 2217–2240.

[29] J. D. Murray, *Mathematical Biology*, Springer-Verlag, Berlin, 1993.

[30] A. Okubo, P.K. Maini, M.H. Williamson, J.D. Murray, *On the spatial spread of the grey squirrel in Britain*, Proc. R. Soc. Lond. B **238** (1989), 113–125.

[31] M. Rodrigo, M. Mimura, *Exact solutions of a competition-diffusion system*, Hiroshima Math. J. **30** (2000). 257–270.

[32] M.M. Tang, P.C. Fife, *Propagating fronts for competing species equations with diffusion*, Arch. Ration. Mech. Anal. **73** (1980) 69–77.

[33] M. Wang, G. Lv, *Entire solutions of a diffusion and competitive Lotka-Volterra type system with nonlocal delayed*, Nonlinearity **23** (2010), 1609–1630.

[34] A. I. Volpert, V. A. Volpert, V. A. Volpert, Traveling wave solutions of parabolic systems, Amer. Math. Soc., Providence, 1994.

[35] Z.-C. Wang, W.-T. Li, S. Ruan, *Traveling fronts in monostable equations with nonlocal delayed effects*, J. Dyn. Diff. Equ. **20** (2008), 563–607.

[36] Z.-C. Wang, W.-T. Li, S. Ruan, *Entire solutions in bistable reaction-diffusion equations with nonlocal delayed nonlinearity*, Trans. Amer. Math. Soc. **361** (2009), 2047–2084.

[37] Z.-C. Wang, W.-T. Li, J. Wu, *Entire solutions in delayed lattice differential equations with monostable nonlinearity*, SIAM J. Math. Anal. **40** (2009), 2392–2420.

[38] C.-H. Wu, *A general approach to the asymptotic behavior of traveling waves in a class of three-component lattice dynamical systems*, J. Dyn. Diff. Equ. **28** (2016) 317–338.

[39] S.-L. Wu, C.-H. Hsu, *Entire solutions of nonlinear cellular neural networks with distributed time delays*, Nonlinearity **25** (2012), 1–17.

[40] S.-L. Wu, H. Wang, *Front-like entire solutions for monostable reaction-diffusion systems*, J. Dyn. Diff. Equ. **25** (2013), 505–533.

[41] H.F. Weinberger, M.A. Lewis, B. Li, *Analysis of the linear conjecture for spread in cooperative models*, J. Math. Biol. **45** (2002), 183–218.

[42] H. Yagisita, *Backward global solutions characterizing annihilation dynamics of travelling fronts*, Publ. Res. Inst. Math. Sci. **39** (2003), 117-164.

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