THE EXISTENCE OF TRAVELING WAVE SOLUTIONS FOR A BISTABLE THREE-COMPONENT LATTICE DYNAMICAL SYSTEM

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Abstract. We study the traveling wave solutions for a three-component lattice dynamical system. This problem arises in the modeling of three species competing two food resources in an environment with migration in which the habitat is of one-dimensional and is divided into countable niches. We are concerned with the case when two species have different preferences of foods and the third species has both preferences of foods. To understand which species win the competition under the bistable condition, the existence of a traveling wave solution for this lattice dynamical system is proven.

1. Introduction

In this paper, we study the following three-component lattice dynamical system (LDS):

$$u_j'(t) = d_1\mathcal{D}_2[u_j](t) + r_1u_j(t)[1-u_j(t)-b_2v_j(t)],\ j\in\mathbb{Z},\ t\in\mathbb{R}, \tag{1.1}$$

$$v_j'(t) = d_2\mathcal{D}_2[v_j](t) + r_2v_j(t)[1-b_1u_j(t)-v_j(t)-b_3w_j(t)],\ j\in\mathbb{Z},\ t\in\mathbb{R}, \tag{1.2}$$

$$w_j'(t) = d_3\mathcal{D}_2[w_j](t) + r_3w_j(t)[1-b_2v_j(t)-w_j(t)],\ j\in\mathbb{Z},\ t\in\mathbb{R}, \tag{1.3}$$

where $\mathcal{D}_2[z_j] := (z_{j+1}-z_j)+(z_{j-1}-z_j)$ and d_i, r_i, b_i, $i=1,2,3$, are positive constants.

The system (1.1)-(1.3) models three species competing the food resources in an environment with migration (or diffusion), when the habitat is of one-dimensional and is divided into countable niches. Here $\{u_j\},\{v_j\},\{w_j\}$, as a function of time t, denotes the population density of each species at position j, d_i is the diffusion coefficient and r_i is the net growth rate of species i, $i=1,2,3$, respectively, and b_1,b_2,b_3 are the interspecific competition coefficients. We are interested in the case when there are only two different food resources, A and B, such that the species 1 prefers food A only, the species 3 prefers food B only, and the species 2 has both preferences of food A and B. Under this assumption, there is no competition between species 1 and 3. Also, with a certain normalization, the carrying capacity of each species is taken to be 1. By taking the competition coefficients of species 2 to species 1 and 3 to be equal, we arrive at the system (1.1)-(1.3).

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In fact, system (1.1)-(1.3) is a spatially discrete model of the following continuous model:

$$u_t = d_1 u_{xx} + r_1 u(1 - u - b_2 v), \quad x, t \in \mathbb{R}, \tag{1.4}$$

$$v_t = d_2 v_{xx} + r_2 v(1 - b_1 u - v - b_3 w), \quad x, t \in \mathbb{R}, \tag{1.5}$$

$$w_t = d_3 w_{xx} + r_3 w(1 - b_2 v - w), \quad x, t \in \mathbb{R}. \tag{1.6}$$

However, for the aggregated dispersion, LDS model (1.1)-(1.3) is more suitable than the continuous model (1.4)-(1.6) to describe the phenomenon of competition between species. Therefore, we shall focus on the LDS model (1.1)-(1.3) in this work. Indeed, lattice dynamic systems have been extensively used to model biological problems, see, for example, the books [6, 15, 13] and the survey paper [5].

In the competition system, it is very interesting to see, under what conditions, whether one species will survive or die out eventually. For this purpose, traveling wave solutions serve an important object to understand the competition mechanism. The aim of this paper is to study the existence of traveling wave for the three species competition system (1.1)-(1.3). Here a traveling wave solution of (1.1)-(1.3) is a solution in the form

$$(u_j(t), v_j(t), w_j(t)) = (\bar{U}(\xi), \bar{V}(\xi), \bar{W}(\xi)), \quad \xi = j + ct,$$

that connecting two constant equilibria, where c is the wave speed and $\{\bar{U}, \bar{V}, \bar{W}\}$ are the wave profiles.

It is trivial that $(0,0,0), (1,0,0), (0,1,0), (0,0,1), (1,0,1)$ are constant equilibria of the system (1.1)-(1.3). In this paper, we shall always assume the following **(BS)** condition:

$$0 < b_1, b_3 < 1 < b_2, \quad b_1 + b_3 > 1.$$

Under the assumption **(BS)**, there is the positive equilibrium (u^*, v^*, w^*) given by

$$(u^*, v^*, w^*) := \frac{1}{b_2(b_1 + b_3) - 1}(b_2 - 1, b_1 + b_3 - 1, b_2 - 1),$$

and no other constant equilibria of (1.1)-(1.3) in the cube $[0,1]^3$. Moreover, the equilibria $(1,0,1), (0,1,0)$ are stable nodes and the others are saddle points.

Biologically, the first condition in **(BS)** means that species 2 is a strong competitor and species 1 and 3 are weaker competitors. Intuitively, species 2 should win the competition. However, putting species 1 and 3 together with the second condition in **(BS)** may kill the species 2. This makes the problem of determining which species win the competition more interesting. Therefore, we are interesting in finding the traveling wave solution of (1.1)-(1.3) connecting $(1,0,1)$ and $(0,1,0)$. This is equivalent to find $(c, \bar{U}, \bar{V}, \bar{W}) \in \mathbb{R} \times [C^1(\mathbb{R})]^3$ such that

$$
\begin{cases}
c\bar{U}' = d_1 D_2[\bar{U}] + r_1 \bar{U}(1 - \bar{U} - b_2\bar{V}), \quad \xi \in \mathbb{R}, \\
c\bar{V}' = d_2 D_2[\bar{V}] + r_2 \bar{V}(1 - b_1\bar{U} - \bar{V} - b_3\bar{W}), \quad \xi \in \mathbb{R}, \\
c\bar{W}' = d_3 D_2[\bar{W}] + r_3 \bar{W}(1 - b_2\bar{V} - \bar{W}), \quad \xi \in \mathbb{R}, \\
(\bar{U}, \bar{V}, \bar{W})(-\infty) = (1, 0, 1), \quad (\bar{U}, \bar{V}, \bar{W})(+\infty) = (0, 1, 0), \\
0 \le \bar{U}, \bar{V}, \bar{W} \le 1,
\end{cases}
\tag{1.7}
$$

where $D_2[Z](\xi) := Z(\xi+1) + Z(\xi-1) - 2Z(\xi)$ for a function $Z = Z(\xi)$, $\xi \in \mathbb{R}$. The main result of this paper is to derive the existence of a traveling wave solution of (1.1)-(1.3) connecting $(1,0,1)$ and $(0,1,0)$. Note that the sign of the wave speed determines which species win the competition.

The study of traveling waves for LDS has attracted a lot attention in past years. For this, we refer the reader to [12, 18, 19, 20, 11, 1, 2, 3, 4, 17, 8, 9] and the references cited therein. For the case of three species competition, we also refer the reader to the recent work [7]. It is important to note that the monotone property does not hold for a general three species competition model. However, system (1.1)-(1.3) (or (1.4)-(1.6)) is a tridiagonal system with sign symmetric in the sense that all partial derivatives of the nonlinearities on off-diagonal of the Jacobian matrix are negative. Such a system generates a monotone dynamical system (cf. [14, 16]). The study of monotone dynamical system has attracted a lot of attention since the pioneer work of Hirsch [10]. Mathematically, it is interesting to extend the existing works (as mentioned above) on traveling waves for one and two species competition systems to this special three species competition system.

The rest of this paper is organized as follows. In §2, we shall introduce the truncated problems as in [4] and recall some known results from [4]. Then we give a proof of the existence of traveling wave solutions of (1.1)-(1.3) connecting $(1,0,1)$ and $(0,1,0)$ in §3. Although our approach is the same as that in [4], it is not obvious to derive a solution for the problem (1.7). The main difficulty, in applying the method of [4] (where actually is applicable for any finite number of equations), is to exclude the possibility of limits with a trivial component (i.e., a component which is identically equal to either 0 or 1), when we pass the limit from the solutions of truncated problems. We remark that this cannot happen in [4], due to the special structure of the model there. Indeed, the existence of one nontrivial component can be derived easily by applying an idea of [4]. However, to derive that all the components are nontrivial needs a delicate analysis with a careful choice of subsequence of truncated solutions. Indeed, the key idea here is by a contradiction argument using the sign of wave speed for 1-component or 2-component system. This is done in §3 and it is the main contribution of this work. As a simple application of this idea, in particular, we can exclude the second possibility in [17, Theorem 1.1] to obtain a nontrivial traveling wave solution for two species strong competition lattice dynamical system.

2. Truncated problems

In this section, following [4], we first introduce the truncated problems for the problem (1.7). Then we recall some important results for these truncated problems.

Set $(U,V,W) = (1-\bar{U}, \bar{V}, 1-\bar{W})$. Then $0 \le U,V,W \le 1$ and the problem (1.7) is reduced to the system

$$\begin{cases} cU' = d_1 D_2[U] + f(U,V,W), & \xi \in \mathbb{R}, \\ cV' = d_2 D_2[V] + g(U,V,W), & \xi \in \mathbb{R}, \\ cW' = d_3 D_2[W] + h(U,V,W), & \xi \in \mathbb{R}, \end{cases} \tag{2.1}$$

where

$$\begin{cases} f(U,V,W) := r_1(1-U)(b_2 V - U), \\ g(U,V,W) := r_2 V[1 - b_1(1-U) - V - b_3(1-W)], \\ h(U,V,W) := r_3(1-W)(b_2 V - W), \end{cases}$$

supplemented with the asymptotic boundary conditions

$$(U,V,W)(-\infty) = (0,0,0), \quad (U,V,W)(+\infty) = (1,1,1). \tag{2.2}$$

Note that $(U_0, V_0, W_0) := (1-u^*, v^*, 1-w^*)$ is the positive constant equilibrium of (2.1).

For a positive constant μ, we define

$$\begin{cases} H_1(U,V,W)(\xi) = \mu U(\xi) + d_1 D_2[U](\xi) + f(U,V,W)(\xi), \\ H_2(U,V,W)(\xi) = \mu V(\xi) + d_2 D_2[V](\xi) + g(U,V,W)(\xi), \\ H_3(U,V,W)(\xi) = \mu W(\xi) + d_3 D_2[W](\xi) + h(U,V,W)(\xi). \end{cases}$$

Then it is easy to check that, for $c \neq 0$, the differential system (2.1) is equivalent to the following integral system (IS):

$$\begin{aligned} U(\xi) = T_1(c,U,V,W)(\xi) &:= \int_{-\infty}^0 e^{\mu s} H_1(U,V,W)(\xi + cs)\,ds, \\ V(\xi) = T_2(c,U,V,W)(\xi) &:= \int_{-\infty}^0 e^{\mu s} H_2(U,V,W)(\xi + cs)\,ds, \\ W(\xi) = T_3(c,U,V,W)(\xi) &:= \int_{-\infty}^0 e^{\mu s} H_3(U,V,W)(\xi + cs)\,ds. \end{aligned}$$

In fact, this is also true for the case $c = 0$. Moreover, if the constant μ is chosen sufficiently large, then the following monotonic property holds, namely,

$$\begin{aligned} & 0 \le U_1(\cdot) \le U_2(\cdot) \le 1,\ 0 \le V_1(\cdot) \le V_2(\cdot) \le 1,\ 0 \le W_1(\cdot) \le W_2(\cdot) \le 1 \ \text{ in } \mathbb{R} \\ \Rightarrow\ & T_i(c,U_1,V_1,W_1)(\cdot) \le T_i(c,U_2,V_2,W_2)(\cdot) \ \text{ in } \mathbb{R},\ i = 1,2,3. \end{aligned}$$

Note that the integral system (IS) is only equivalent to the differential system (2.1), without the boundary condition (2.2). It is the main task of this work to choose a suitable solution of the integral system (IS) so that the boundary condition (2.2) is satisfied.

Following [4], for each $n \in \mathbb{N}$, we consider the following truncated problem:

$$\begin{aligned} cU' &= d_1 D_2[U] + f(U,V,W) \quad \text{in } (-n,n),\\ cV' &= d_2 D_2[V] + g(U,V,W) \quad \text{in } (-n,n),\\ cW' &= d_3 D_3[W] + h(U,V,W) \quad \text{in } (-n,n) \end{aligned}$$

with the exterior conditions:

$$\begin{aligned} U(\xi) = V(\xi) = W(\xi) = 1, \, \forall\, \xi \in (n,+\infty),\\ U(\xi) = V(\xi) = W(\xi) = 0, \, \forall\, \xi \in (-\infty,-n). \end{aligned}$$

To solve this truncated system, it is more convenient to consider the following system of integral equations

$$U(\xi) = T_1^n(c,U,V,W)(\xi),\; V(\xi) = T_2^n(c,U,V,W)(\xi),\; W(\xi) = T_3^n(c,U,V,W)(\xi), \tag{2.3}$$

where $T_i^n(c,U,V,W)(\xi) := P_n T_i(c,U,V,W)(\xi)$ and

$$P_n T_i(\xi) := \begin{cases} 0 & \text{if } \xi < -n,\\ T_i(\xi) & \text{if } \xi \in [-n,n],\\ 1 & \text{if } \xi > n, \end{cases}$$

for $i = 1,2,3$.

First, recall from [4, Lemma 13] that we have the following existence result for the truncated problem.

Lemma 2.1. *For each $c \neq 0$ and each $n \in \mathbb{N}$, there exists a unique solution $(U^{n,c}, V^{n,c}, W^{n,c})$ of (2.3) such that $0 < U^{n,c}, V^{n,c}, W^{n,c} < 1$ on $[-n,n]$ and $(U^{n,c})', (V^{n,c})', (W^{n,c})' > 0$ in $(-n,n)$. For $c = 0$, (2.3) has a minimal solution (U_*^n, V_*^n, W_*^n) and a maximal solution (U^{*n}, V^{*n}, W^{*n}). Moreover, (U_*^n, V_*^n, W_*^n) and (U^{*n}, V^{*n}, W^{*n}) are nondecreasing and are constant in $(l, l+1)$ for each $l \in \mathbb{Z}$.*

Also, the following monotonicity in c of solutions of (2.3) can be found in [4, lemma 14].

Lemma 2.2. *Fix $n \in \mathbb{N}$. Let $c_1 < c_2$ and $(U^{n,c_i}, V^{n,c_i}, W^{n,c_i})$ be the solution of (2.3) with $c = c_i$, $i = 1,2$. Then $(U^{n,c_2}, V^{n,c_2}, W^{n,c_2})(\xi) < (U^{n,c_1}, V^{n,c_1}, W^{n,c_1})(\xi)$ for all $\xi \in [-n,n]$. Let (U_*^n, V_*^n, W_*^n) and (U^{*n}, V^{*n}, W^{*n}) be the minimal solution and maximal solution to (2.3) with $c = 0$ respectively. Then*

$$\begin{aligned} \lim_{c \nearrow 0} (U^{n,c}, V^{n,c}, W^{n,c})(\xi) &= (U^{*n}, V^{*n}, W^{*n})(\xi),\\ \lim_{c \searrow 0} (U^{n,c}, V^{n,c}, W^{n,c})(\xi) &= (U_*^n, V_*^n, W_*^n)(\xi) \end{aligned}$$

for all $\xi \in \mathbb{R} \setminus \mathbb{Z}$.

Finally, we have the following useful bounds for the later purpose. Hereafter $(U_1, V_1, W_1) \leq (U_2, V_2, W_2)$ means $U_1 \leq U_2, V_1 \leq V_2, W_1 \leq W_2$, similar for the strict inequality.

Lemma 2.3. *Let* $c^* = (d_1 + d_2 + d_3)(e + e^{-1}) + r_1 b_2 + r_2(1 + b_1 + b_3) + r_3 b_2$. *Then for any* $\xi \in \mathbb{R}$ *we have*

(i) *For any* $c \geq c^*$, $(U^{n,c}, V^{n,c}, W^{n,c})(\xi) \leq (e^{\xi-n}, e^{\xi-n}, e^{\xi-n})$;

(ii) *For any* $c \leq -c^*$, $(U^{n,c}, V^{n,c}, W^{n,c})(\xi) \geq (1 - e^{-\xi-n}, 1 - e^{-\xi-n}, 1 - e^{-\xi-n})$.

Since the proof of this lemma is almost the same as the one given in [4], we safely omit it.

3. Existence of traveling waves

In this section, we first prove that (2.1) has a solution (c, U, V, W) such that

$$(0,0,0) < (U,V,W)(x) \leq (U,V,W)(y) < (1,1,1) \quad \text{for } x \leq y. \tag{3.1}$$

Then we derive the condition (2.2) to obtain the existence of a traveling wave solution of (1.1)-(1.3) connecting $(1, 0, 1)$ and $(0, 1, 0)$.

To begin with, we note that for a fixed $a \in (0, 1)$ there are only the following three possibilities:

$$\text{(i)} \liminf_{n\to+\infty} V^{*n}(a) \leq \frac{1}{2}; \quad \text{(ii)} \limsup_{n\to+\infty} V_*^n(a) \geq \frac{1}{2};$$
$$\text{(iii)} \limsup_{n\to+\infty} V_*^n(a) < \frac{1}{2} < \liminf_{n\to+\infty} V^{*n}(a).$$

Case (i). $\liminf_{n\to+\infty} V^{*n}(a) \leq 1/2$. In this case, there is a sequence $\{n_l\} \in \mathbb{N}$ with $n_l \to +\infty$ as $l \to +\infty$ such that $V^{*n_l}(a) < 1/2 + 1/l$ for each l. For each $l \in \mathbb{N}$ sufficiently large, it follows from Lemmas 2.2 and 2.3 that $V^{n_l,c_l}(a) = 1/2 + 1/l$ for some $c_l \in (-c^*, 0)$. Due to the monotonicity of $\{(U^{n_l,c_l}, V^{n_l,c_l}, W^{n_l,c_l})\}$ and the boundedness of $\{c_l\}$, there is a subsequence of $\{(n_l, c_l)\}$ (still denoted by $\{(n_l, c_l)\}$) such that $c_l \to c$ as $l \to +\infty$ and

$$\lim_{l\to+\infty} (U^{n_l,c_l}, V^{n_l,c_l}, W^{n_l,c_l})(\xi) = (U,V,W)(\xi),\ \forall\, \xi \in \mathbb{R},$$

for some $c \in [-c^*, 0]$ and monotone non-decreasing functions U, V, W defined in $\mathbb{R}$ such that $0 \leq U, V, W \leq 1$ in $\mathbb{R}$ and $V(a) = 1/2$. By Lebesgue's dominated convergence theorem, (c, U, V, W) satisfies the integral system (IS) and so is a solution of (2.1).

We claim that $U(a), W(a) \in (0, 1)$. Suppose that $U(a) = 0$. Then $U \equiv 0$. It follows from the first equation of (2.1) that $V \equiv 0$, a contradiction. Hence $U(a) > 0$. Similar, we have $W(a) > 0$.

On the other hand, suppose that $U(a) = 1$. Then $U \equiv 1$ and (2.1) is reduced to the system

$$cV' = d_2 D_2[V] + r_2 V[1 - V - b_3(1 - W)],$$
$$cW' = d_3 D_2[W] + r_3(1 - W)(b_2 V - W).$$

By taking the limits as $\xi \to \pm\infty$, $(V,W)(\pm\infty)$ satisfies

$$r_2V[1-V-b_3(1-W)](\pm\infty)=0, \quad r_3(1-W)(b_2V-W)(\pm\infty)=0. \tag{3.2}$$

If $W\equiv 1$, then we can deduce that $V(-\infty)=0$ and $V(+\infty)=1$ due to $V(a)=1/2$. Moreover, V is a nontrivial solution of the equation

$$cV'=d_2D_2[V]+r_2V(1-V) \quad \text{in } \mathbb{R}.$$

This implies that $c>0$ (cf. [20, 1, 2]), a contradiction. Hence $W(a)\in(0,1)$ and, by (3.2), we must have $(V,W)(-\infty)=(0,0)$ and $(V,W)(+\infty)=(1,1)$. Since $b_2>1>b_3$, we have $c>0$ (cf. [9]), a contradiction. This leads that $U(a)<1$. Similarly, we can show that $W(a)<1$. We conclude that $U(a),W(a)\in(0,1)$ and so (3.1) holds.

Case (ii). $\limsup_{n\to+\infty} V_*^n(a)\ge 1/2$. We divide this case into the following three subcases:

(a) $\displaystyle\limsup_{n\to+\infty} U_*^n(a)\ge\frac{1}{2}$ and $\displaystyle\limsup_{n\to+\infty} W_*^n(a)\ge\frac{1}{2}$;

(b) $\displaystyle\limsup_{n\to+\infty} U_*^n(a)<\frac{1}{2}$ and $\displaystyle\limsup_{n\to+\infty} W_*^n(a)<\frac{1}{2}$;

(c) $\displaystyle\limsup_{n\to+\infty} U_*^n(a)<\frac{1}{2}\le\limsup_{n\to+\infty} W_*^n(a)$ or $\displaystyle\limsup_{n\to+\infty} W_*^n(a)<\frac{1}{2}\le\limsup_{n\to+\infty} U_*^n(a)$.

For subcase (a), there is a sequence $\{n_l\}$ in $\mathbb{N}$ with $n_l\to+\infty$ as $l\to+\infty$ such that

$$\max\{U_*^{n_l}(a),W_*^{n_l}(a)\}\ge 1/2-1/l$$

for each l. For each $l\in\mathbb{N}$ sufficiently large, it follows from Lemmas 2.2 and 2.3 that there exists a $c_l\in(0,c^*)$ such that

$$\max\{U^{n_l,c_l}(a),W^{n_l,c_l}(a)\}=1/2-1/l.$$

Using the monotonicity of $\{(U^{n_l,c_l},V^{n_l,c_l},W^{n_l,c_l})\}$ and the boundedness of $\{c_l\}$, there is a subsequence of $\{(n_l,c_l)\}$ (still denoted by $\{(n_l,c_l)\}$) such that $c_l\to c$ as $l\to+\infty$ and

$$\lim_{l\to+\infty}(U^{n_l,c_l},V^{n_l,c_l},W^{n_l,c_l})(\xi)=(U,V,W)(\xi),\ \forall\,\xi\in\mathbb{R},$$

for some $c\in[0,c^*]$ and monotone non-decreasing functions U,V,W defined in $\mathbb{R}$ such that either $U(a)=1/2$ or $W(a)=1/2$. Note that $U(a)\le 1/2$, $W(a)\le 1/2$, $0\le U,V,W\le 1$ in $\mathbb{R}$ and (c,U,V,W) satisfies (2.1).

Without loss of generality, we may assume that $U(a)=1/2$. Note that $W(a)\le 1/2$. We claim that $V(a),W(a)\in(0,1)$. We argue it by contradiction.

First, suppose that $V(a)=0$. Then $V\equiv 0$. Then, by (2.1), U satisfies the equation

$$cU'=d_1D_2[U]-r_1U(1-U) \quad \text{in } \mathbb{R}$$

such that $U(-\infty)=0$ and $U(+\infty)=1$, due to

$$U(\pm\infty)[1-U(\pm\infty)]=0, \quad 0\le U(-\infty)\le U(a)=\frac{1}{2}\le U(+\infty)\le 1.$$

This implies that $c<0$, a contradiction. On the other hand, if $V(a)=1$, then $V\equiv 1$. It follows from (2.1) that

$$0=0-r_2[b_1(1-U)+b_3(1-W)]<0 \quad \text{at} \;\; \xi=a,$$

a contradiction. This proves that $V(a)\in(0,1)$.

Next, suppose that $W(a)=0$. Then $W\equiv 0$. It follows from the third equation of (2.1) that $V\equiv 0$, a contradiction. We conclude that $W(a)\in(0,1)$ and so (3.1) holds.

For subcase (b), by a similar argument as in case (i), there is a sequence $\{(n_l,c_l)\}$ with $c_l\in(0,c^*)$ for each l such that $c_l\to c$ as $l\to+\infty$ and

$$\lim_{l\to+\infty}(U^{n_l,c_l},V^{n_l,c_l},W^{n_l,c_l})(\xi)=(U,V,W)(\xi),\ \forall\,\xi\in\mathbb{R},$$

for some $c\in[0,c^*]$ and monotone non-decreasing functions U,V,W defined in $\mathbb{R}$ with $V(a)=1/2$. Note that (c,U,V,W) satisfies (2.1). Since

$$U^{n_l,c_l}(a)\le U_*^{n_l}(a), \quad W^{n_l,c_l}(a)\le W_*^{n_l}(a),$$

by assumption, we have $U(a)\le 1/2$ and $W(a)\le 1/2$.

If $U(a)=0$, then $U\equiv 0$. This implies that $V\equiv 0$ by (2.1), a contradiction. Similarly, if $W(a)=0$, then $W\equiv 0$. Again, by (2.1), we have $V\equiv 0$, a contradiction. We conclude that $U(a),W(a)\in(0,1)$ and so (3.1) holds.

For subcase (c), we only consider the case when

$$\limsup_{n\to+\infty}U_*^n(a)<\frac{1}{2}\le\limsup_{n\to+\infty}W_*^n(a).$$

The other case can be treated similarly. In this case, for $\{W_*^n\}$, there is a sequence $\{(n_l,c_l)\}$ with $c_l\in(0,c^*)$ and $W^{n_l,c_l}(a)=1/2-1/l$ such that $c_l\to c$ as $l\to+\infty$ and

$$\lim_{l\to+\infty}(U^{n_l,c_l},V^{n_l,c_l},W^{n_l,c_l})(\xi)=(U,V,W)(\xi),\ \forall\,\xi\in\mathbb{R},$$

for some $c\in[0,c^*]$ and monotone non-decreasing functions U,V,W defined in $\mathbb{R}$ with $W(a)=1/2$. Note that $U(a)\le 1/2$ and (c,U,V,W) satisfies (2.1).

If $U(a)=0$, then $U\equiv 0$. It follows from (2.1) that $V\equiv 0$ and W satisfies

$$cW'=d_3D_2[W]-r_3W(1-W) \quad \text{in} \;\; \mathbb{R}.$$

This leads that $c<0$, a contradiction. Hence $U(a)\in(0,1)$. Similar argument as before, we have $V(a)\in(0,1)$. Hence (3.1) holds.

Case (iii). $\limsup_{n\to+\infty} V_*^n(a) < 1/2 < \liminf_{n\to+\infty} V^{*n}(a)$. By assumption, we have $V_*^n(a) < 1/2$ and $V^{*n}(a) > 1/2$ for all $n \geq n_0$ for some n_0 large. Set

$$\begin{aligned}\alpha_n &:= \sup\{\xi \mid (U^{*n}, V^{*n}, W^{*n})(\xi) \leq (1/2, 1/2, 1/2)\},\\ \beta_n &:= \inf\{\xi \mid (U_*^n, V_*^n, W_*^n)(\xi) \geq (1/2, 1/2, 1/2)\}.\end{aligned}$$

Then $\alpha_n \in [-n, a]$ and $\beta_n \in [a, n]$ for all $n \geq n_0$. Let

$$\begin{aligned}(U_1^n, V_1^n, W_1^n)(\xi) &:= (U^{*n}, V^{*n}, W^{*n})(\xi + \alpha_n),\\ (U_2^n, V_2^n, W_2^n)(\xi) &:= (U_*^n, V_*^n, W_*^n)(\xi + \beta_n).\end{aligned}$$

Then

$$\begin{aligned}&(0,0,0) < (U_1^n, V_1^n, W_1^n)(\xi) < (1,1,1) \text{ for all } \xi \in [-n-\alpha_n, n-\alpha_n],\\ &(0,0,0) < (U_2^n, V_2^n, W_2^n)(\xi) < (1,1,1) \text{ for all } \xi \in [-n-\beta_n, n-\beta_n],\\ &(U_1^n, V_1^n, W_1^n)(\xi) = (0,0,0),\ \forall\, \xi < -n-\alpha_n,\ (U_1^n, V_1^n, W_1^n)(\xi) = (1,1,1),\ \forall\, \xi > n-\alpha_n,\\ &(U_2^n, V_2^n, W_2^n)(\xi) = (0,0,0),\ \forall\, \xi < -n-\beta_n,\ (U_2^n, V_2^n, W_2^n)(\xi) = (1,1,1),\ \forall\, \xi > n-\beta_n.\end{aligned}$$

Notice that $n - \alpha_n \to +\infty$ and $-n - \beta_n \to -\infty$ as $n \to +\infty$. Therefore, there are three possibilities, namely,

$$\begin{aligned}&\text{(I)} \limsup_{n\to+\infty}(n+\alpha_n) = +\infty; \quad \text{(II)} \limsup_{n\to+\infty}(n-\beta_n) = +\infty;\\ &\text{(III)} \limsup_{n\to+\infty}(n+\alpha_n) < +\infty \quad \text{and} \quad \limsup_{n\to+\infty}(n-\beta_n) < +\infty.\end{aligned}$$

For case (I), we can choose a suitable subsequence of (U^{*n}, V^{*n}, W^{*n}) and pass to the limit to obtain a non-decreasing solution (U, V, W) of (IS) with $c = 0$ such that

$$\max\{U(0), V(0), W(0)\} = 1/2, \tag{3.3}$$

by using the definition of α_n. This gives us a non-decreasing solution $(0, U, V, W)$ of (2.1) satisfying (3.3). To proceed further, we first assume that $U(0) = 1/2$. Then $0 < U(x) < 1$ for all $x \in \mathbb{R}$. We claim that $V(0) > 0$ and $W(0) > 0$. If $V(0) = 0$, then $V \equiv 0$. It follows from (2.1) that

$$d_1 D_2[U] - r_1 U(1-U) = 0 \quad \text{in } \mathbb{R},$$

which is impossible. Hence $V(0) \in (0,1)$. On the other hand, if $W(0) = 0$, then $W \equiv 0$. This implies that $V \equiv 0$ by (2.1), a contradiction again. Hence $W(0) \in (0,1)$. Therefore, (3.1) holds. The other cases can be treated similarly and we omit it. The case (II) is similar.

Finally, for case (III), we use the idea of super-solution and sub-solution as introduced in [4] to obtain a solution (U, V, W) of (2.1) with $c = 0$ such that (3.1) holds. Since the proof is similar to that in [4], we omit it here.

To derive the asymptotic boundary conditions (2.2), we first set

$$
\begin{aligned}
\alpha &:= \min\left\{\frac{U_0}{4}, \frac{(b_2-1)(1-U_0)}{4(b_2+1)}, \frac{1-V_0}{4}, \frac{(b_1+b_3-1)V_0}{4(b_1+b_3+1)}, \frac{(b_2-1)(1-W_0)}{4(b_2+1)}, \frac{W_0}{4}\right\},\\
\xi_1^{n,c} &:= \min\{\xi \mid (U^{n,c}, V^{n,c}, W^{n,c})(\xi) \geq (U_0-\alpha, V_0-\alpha, W_0-\alpha)\},\\
\xi_2^{n,c} &:= \max\{\xi \mid (U^{n,c}, V^{n,c}, W^{n,c})(\xi) \leq (U_0+\alpha, V_0+\alpha, W_0+\alpha)\}.
\end{aligned}
$$

Hereafter we use $(U^{n,0}, V^{n,0}, W^{n,0})$ to denote either (U_*^n, V_*^n, W_*^n) or (U^{*n}, V^{*n}, W^{*n}). Note that $\xi_1^{n,c}$ and $\xi_2^{n,c}$ are well-defined numbers in $(-n, n)$, due to the choice of α.

Following the proof of [4, Lemma 15], we have

$$
\eta := \limsup_{n\to+\infty} \sup\{(\xi_2^{n,c} - \xi_1^{n,c}) \mid c \in [-c^*, c^*]\} < +\infty. \tag{3.4}
$$

Since the proof is almost the same as that of [4, Lemma 15], we omit it here.

With this property, we are ready to derive the asymptotic boundary conditions (2.2) as follows. For reader's convenience, we give a proof of $(U,V,W)(+\infty) = (1,1,1)$. The other is similar.

Let $\{(U^{n_l,c_l}, V^{n_l,c_l}, W^{n_l,c_l})\}$ be the sequence such that

$$
\lim_{l\to+\infty} (U^{n_l,c_l}, V^{n_l,c_l}, W^{n_l,c_l})(\xi) = (U,V,W)(\xi) \quad \text{for all } \ \xi \in \mathbb{R}. \tag{3.5}
$$

Since

$$
(0,0,0) < (U,V,W)(\xi) < (1,1,1) \quad \text{for all } \ \xi \in \mathbb{R},
$$

$(U,V,W)(+\infty)$ is either $(1,1,1)$ or (U_0, V_0, W_0). We argue by a contradiction and assume that $(U,V,W)(+\infty) = (U_0, V_0, W_0)$. Then there exists a ξ_0 such that

$$
(U,V,W)(\xi) \geq (U_0 - \frac{\alpha}{2}, V_0 - \frac{\alpha}{2}, W_0 - \frac{\alpha}{2}) \ \text{ for all } \ \xi \geq \xi_0.
$$

For ξ_0, it follows from (3.5) that

$$
(U^{n_l,c_l}, V^{n_l,c_l}, W^{n_l,c_l})(\xi_0) \geq (U(\xi_0) - \frac{\alpha}{2}, V(\xi_0) - \frac{\alpha}{2}, W(\xi_0) - \frac{\alpha}{2}) \quad \text{for all } \ l \geq l_0
$$

for some l_0 large enough. This implies, using the monotonicity of (U,V,W), that

$$
(U^{n_l,c_l}, V^{n_l,c_l}, W^{n_l,c_l})(\xi) \geq (U_0-\alpha, V_0-\alpha, W_0-\alpha) \text{ for all } \xi \geq \xi_0 \text{ and } l \geq l_0.
$$

Hence we obtain that $\xi_1^{n_l,c_l} \leq \xi_0$. On the other hand, for each ξ, due to (3.5), we have

$$
(U^{n_l,c_l}, V^{n_l,c_l}, W^{n_l,c_l})(\xi) \leq (U_0+\alpha, V_0+\alpha, W_0+\alpha)
$$

for some l large enough. Thus $\xi_2^{n_l,c_l} \to +\infty$ as $l \to +\infty$. It follows that $\xi_2^{n_l,c_l} - \xi_1^{n_l,c_l} \to +\infty$ as $l \to +\infty$, which contradicts (3.4). We conclude that $(U,V,W)(+\infty) = (1,1,1)$.

We summarize the above discussions as the following main theorem of this paper.

Theorem 1. *Problem* (1.7) *has a solution* $(c, \bar{U}, \bar{V}, \bar{W})$ *such that*

$$
(0,0,0) < (1-\bar{U}, \bar{V}, 1-\bar{W})(x) \leq (1-\bar{U}, \bar{V}, 1-\bar{W})(y) < (1,1,1) \quad \textit{for } \ x \leq y.
$$

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