BLOWUP RATE ESTIMATES FOR THE HEAT EQUATION WITH A NONLINEAR GRADIENT SOURCE TERM

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Abstract. The gradient blowup rate of the equation $u_t = \Delta u + |\nabla u|^p$, where $p > 2$, is studied. It is shown that the blowup rate will never match that of the self-similar variables. In the one space dimensional case when assumptions are made on the initial data so that the solution is monotonically increasing in time, the exact blowup rate is found.

1. Introduction

In this paper, we study the following initial boundary value problem for the heat equation with a nonlinear gradient source term:

$$u_t = \Delta u + |\nabla u|^p \quad \text{for} \quad x \in \Omega, \ t > 0, \tag{1.1}$$

$$u(x,t) = g(x) \quad \text{for} \quad x \in \partial\Omega, \ t > 0, \tag{1.2}$$

$$u(x,0) = u_0(x) \quad \text{for} \quad x \in \overline{\Omega}, \tag{1.3}$$

where $p > 2$, g is a bounded smooth function, Ω is a smooth domain (not necessarily bounded) in $\mathbb{R}^n$, and

$$u_0 \in BC^2(\overline{\Omega}), \quad u_0(x) = g(x) \quad \text{for} \quad x \in \partial\Omega. \tag{1.4}$$

The space BC^2 is the set of all functions with bounded and continuous derivatives up to order 2. Here p is assumed to be > 2, since $|\nabla u|$ is always bounded for any finite time interval when $0 \leq p \leq 2$ (cf. [24]). Note that by the maximum principle u is uniformly bounded.

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When $\Omega \neq \mathbb{R}^n$ (which implies that $\partial\Omega \neq \emptyset$), it is known that under certain conditions $|\nabla u|$ blows up in a finite time $t = T$, i.e.,

$$\limsup_{t \to T^-} \{\sup_{x \in \overline{\Omega}} |\nabla u(x,t)|\} = \infty.$$

For example, in the one space dimensional case with $\Omega = (0,1)$, if $u_0(x) \geq 0$, $g(0) = 0$, and $g(1) = M$ with $M > M_c := (p-2)^{-1}(p-1)^{(p-2)/(p-1)}$, then all solutions are uniformly bounded, but their gradients blow up in a finite time (cf. [1]). We shall call such phenomenon as *gradient blowup*. This is different from the usual *blowup* in which the L^∞ norm of the solution tends to infinity as $t \to T^-$.

Finite time blowup phenomena have attracted a lot of attention in the past years. Many studies have concentrated on the blowup of solution itself. These include blowup criteria, blowup locations, blowup rates, and blowup profiles. In particular, for the classical semilinear heat equation

$$u_t = \Delta u + u^p, \tag{1.5}$$

we refer the reader to, e.g., [9, 11, 23, 8, 15, 16, 17, 18, 26].

For the gradient blowup, most previous works were about the blowup criteria, i.e., when the gradient blowup occurs. See, for example, [1, 6, 10, 2, 4, 27, 3, 28, 7]. *Little is known* about the blowup rates. The main purpose of this paper is to investigate the blowup rate of the gradient for the model problem (1.1)-(1.3).

It has been known for years that the blowup rates are usually determined by the so-called self-similar rates. For example, it is $(T-t)^{-1/(p-1)}$ for the equation (1.5) for $1 < p < (n+2)/(n-2)_+$. This self-similar rate is indeed related to the scaling invariance of the equation.

For our problem, assuming $|\nabla u(0, T^-)| = \infty$, we have the following self-similar transformations:

$$u(x,t) := (T-t)^\alpha w(y,s), \quad \alpha := \frac{p-2}{2(p-1)},$$
$$y := \frac{x}{\sqrt{T-t}}, \quad s := -\ln(T-t).$$

Then the equation (1.1) is transformed into the equation

$$w_s = \Delta_y w - \frac{1}{2} y \cdot \nabla_y w + \alpha w + |\nabla_y w|^p. \tag{1.6}$$

In this case, *assuming that* $|\nabla_y w(0,s)|$ *is bounded from above and below*, then the self-similar rate of the gradient blowup is given by

$$|\nabla u(0,t)| \sim (T-t)^{-(1/2-\alpha)} \quad \text{as} \quad t \to T^-. \tag{1.7}$$

It is quite surprising that this *will never be the case*, since we shall show in the next section that there exists a positive constant c_0 such that

$$\sup_{x\in\overline{\Omega},\, 0\le\tau\le t} |\nabla u(x,\tau)| \ge c_0(T-t)^{-1/(p-2)}. \tag{1.8}$$

Note that $1/(p-2) > 1/[2(p-1)] = 1/2-\alpha$ for all $p>2$.

For the equation (1.5) with

$$n \ge 11 \quad \text{and} \quad p > p_* := 1 + \frac{4}{n-4-2\sqrt{n-1}},$$

Herrero and Velazquez [19, 20] have constructed blowup radial solutions with blowup rates different from the self-similar rate. We call such type of blowup as *Type II* blowup; see also [25]. Indeed, this type II singularity is also found in the dead-core problem (cf. [12]). An important feature of Type II singularity is that the rates are not unique. This can be seen in the construction of solutions with different rates in [19, 20, 25, 13].

We note that (1.8) has been proved by Conner and Grant [5] in one space dimensional case with $\Omega = (0,1)$, under the assumption $u_0' > 0$ in $(0,1)$ so that the solution is monotonically increasing in x. It should be noted that the estimate in [5] is more precise than (1.8) (only t level is used, not the supremum over $[0,t]$); in the case the solution is monotone increasing in t, then these estimates coincide (cf. Lemma 3.2 and Theorem 2.1). Our proof here is completely different and much simpler. It works for any space dimensions. Also, we do not need the additional assumption $u_0' > 0$ in $(0,1)$. On the other hand, based on some numerical simulations, Conner and Grant [5] observed that $(T-t)^{-1/(p-2)}$ should be the exact gradient blowup rate for the one dimensional problem. In this paper, we shall provide a rigorous proof of this result under the assumption that the solution is monotonically increasing in t. It should be very interesting to see if, without the assumption on the initial data so that the solution is monotonically increasing in t, this is the only gradient blowup rate. We leave it here as an open problem. Also, for the upper bound estimate for gradient blowup rate in higher space dimensional cases, it is still open.

In this paper, we shall establish:

(i) the lower bound of the blowup rate is always $(T-t)^{-1/(p-2)}$ for any space dimensions;

(ii) in the one space dimensional case, the upper bound of the blowup rate is also $(T-t)^{-1/(p-2)}$ under the additional assumption on the initial data so that the solution is monotonically increasing in t, therefore the blowup rate is proportional to $(T-t)^{-1/(p-2)}$ in this case.

From the proof we can also find that, rather than the usual $y = x/\sqrt{T-t}$, the appropriate change of variables is $y = x/(T-t)^{(p-1)/(p-2)}$. In fact, letting

$$u(x,t) = (T-t)v(y,s), \quad s = \frac{p-2}{p}(T-t)^{-p/(p-2)},$$

the equation (1.1) is transformed into

$$v_s = \Delta_y v + |\nabla_y v|^p - \frac{p-2}{p}s^{-1}\Big(-v + \frac{p-1}{p-2} y \cdot \nabla_y v \Big).$$

Note that in the one dimensional case when the solution is monotone in t, this is the correct scale of transformation, since, by our estimates $v_y(y,s)$ is bounded from above and below. This means that the equation in the new variable is nearly invariant, modulo some small terms as $t \to T^-$ (or $s \to \infty$).

This paper is organized as follows. In Section 2, we derive the gradient blowup rate lower bound estimate. In Section 3, we first quote a result from [29]. This implies that gradient blowup must occur on the boundary. Indeed, more general results than Theorem 3.1 (below) are well-known already. More results can be found in, e.g., [29]. Then we show that, for t close to T, $\sup_{x\in\overline{\Omega}} |\nabla_x u(x,t)|$ is monotonically increasing if $u(x,t)$ is monotonically increasing in t. For the one space dimensional case, the blowup rate upper bound estimate is proven in Section 4, with an additional assumption on the initial data so that the solution is monotonically increasing in time. Finally, in Section 5 we describe the asymptotic behavior in the one space dimensional case as an application of our blowup rate estimates. See also Theorem 1.1 of [5]. It should be noticed that we do not need the assumption $u_0' > 0$ in $(0,1)$ throughout this paper.

2. Gradient blowup rate lower bound: non-self-similar blowup

Assuming that the gradient blowup occurs at a finite time $t = T$, we shall derive the following lower bound estimate which exclude the self-similar blowup rate. If we compare with the equation $u_t = \Delta u + u^p$, where self-similar blowup is referred as type I, this case could be called a type II blowup. However, we shall resist the temptation from doing so, since, the type I blowup never exists, by the following theorem.

Theorem 2.1. *Under the assumption (1.4), if the gradient blowup occurs at a finite time $t = T$, then there exists $c_0 > 0$ such that*

$$\sup_{x\in\overline{\Omega},\, 0\le\tau\le t} |\nabla_x u(x,\tau)| \ge c_0(T-t)^{-1/(p-2)}. \tag{2.1}$$

Proof. Under the assumption (1.4), we can easily establish

$$|u(x,t)| \le \|u_0\|_{L^\infty(\Omega)} \ \text{ for } \ x\in\overline{\Omega},\ 0<t<T, \tag{2.2}$$

$$|u_t(x,t)| \le C_1 \ \text{ for } \ x\in\overline{\Omega},\ 0<t<T, \tag{2.3}$$

where $C_1 := \sup_{x\in\overline{\Omega}} |\Delta u_0(x) + |\nabla_x u_0(x)|^p|$. Let

$$m(t) := \sup_{x\in\overline{\Omega},\, 0\le\tau\le t} |\nabla_x u(x,\tau)|. \tag{2.4}$$

It is clear that $v := u_t$ satisfies the equations

$$v_t = \Delta_x v + p|\nabla_x u|^{p-2}\nabla_x u\cdot\nabla_x v \ \text{ for } \ x\in\Omega,\ 0<t<T, \tag{2.5}$$

$$v(x,t) = 0, \quad x\in\partial\Omega,\ 0<t<T. \tag{2.6}$$

Given $t^* \in (0,T)$. Let

$$\phi(y,s) := v(\lambda y, \lambda^2 s + t^*), \quad \lambda := \frac{1}{m^{p-1}(t^*)}.$$

Then ϕ satisfies

$$\phi_s - \Delta_y\phi = \vec{b}(y,s)\cdot\nabla_y\phi \ \text{ for } \ y\in\Omega_\lambda := \{y;\ \lambda y\in\Omega\},\ \frac{-t^*}{\lambda^2} < s \le 0,$$

where $\vec{b}(y,s) = \lambda p|\nabla_x u|^{p-2}\nabla_x u$ is clearly bounded by p, and by (2.3), (2.6),

$$|\phi| \le C_1 \ \text{ for } \ y\in\Omega_\lambda,\ \frac{-t^*}{\lambda^2} < s \le 0,$$

$$\phi = 0 \ \text{ for } \ \partial\Omega_\lambda,\ \frac{-t^*}{\lambda^2} < s \le 0.$$

Although Ω_λ becomes unbounded as $\lambda\to 0$, its boundary is approaching to a hyperplane as $\lambda\to 0$, and therefore $\partial\Omega_\lambda$ is uniformly in C^2, independently of λ as $\lambda\to 0$. Denote $B_R(y^*)$ the ball centered at y^* with radius R.

If $y^*\in\partial\Omega_\lambda$, we can apply the $W_p^{2,1}$ interior and interior-boundary estimates [24] in the domain $(\Omega_\lambda\cap B_1(y^*))\times[-1,0]$ to obtain

$$\|\phi\|_{W_p^{2,1}(\Omega_\lambda\cap B_1(y^*))\times[-1,0])} \le C, \qquad C \text{ is independent of } y^*,\ \lambda.$$

By Sobolev's embedding theorem [24], if we choose $p > n+2$, then

$$\|\nabla_y \phi\|_{L^\infty(\Omega_\lambda \cap B_1(y^*))\times[-1,0])} \le C, \qquad C \text{ is independent of } y^*,\ \lambda.$$

If $B_1(y^*) \subset \Omega_\lambda$, then we can repeat the above procedure by applying the interior $W_p^{2,1}$ estimates and Sobolev's embedding theorem to obtain

$$\|\nabla_y \phi\|_{L^\infty(\Omega_\lambda \cap B_{1/2}(y^*))\times[-1,0])} \le C, \qquad C \text{ is independent of } y^*,\ \lambda.$$

Since the balls $\{B_1(y^*);\ y^* \in \partial\Omega_\lambda\}$ and $\{B_{1/2}(y^*);\ B_1(y^*) \subset \Omega_\lambda\}$ cover the entire Ω_λ, we conclude that

$$\sup_{y\in\overline{\Omega}_\lambda} |\nabla_y \phi(y,0)| \le C, \tag{2.7}$$

where the constant C is independent of t^* and λ.

Since the supremum of a family of uniformly Lipschitz functions is Lipschitz continuous (by the inequality $|\sup_\beta a_\beta - \sup_\beta b_\beta| \le \sup_\beta |a_\beta - b_\beta|$), the function $m(t)$ is Lipschitz continuous for $\delta \le t \le T-\delta$ for any small $\delta > 0$. It follows that m is absolutely continuous, m' exists almost everywhere, and satisfies the Fundamental Theorem of Calculus.

Let $T/2 \le t < s < T$. For any $\tau \in [t,s]$ and $x \in \Omega$, it follows from the mean-value inequality and (2.7) that

$$|\nabla_x u(x,\tau) - \nabla_x u(x,t)| \le (\tau - t) \sup_{\Omega\times[t,\tau]} |\nabla_x u_t| \le C(\tau - t) m^{p-1}(\tau);$$

hence

$$|\nabla_x u(x,\tau)| \le |\nabla u_x(x,t)| + C(\tau-t)m^{p-1}(\tau) \le m(t) + C(\tau-t)m^{p-1}(\tau).$$

Taking supremum for (x,τ) over $\Omega\times[t,s]$, we get

$$0 \le m(s) - m(t) \le C(s-t)m^{p-1}(s),$$

which implies

$$m'(t) \le Cm^{p-1}(t) \qquad a.e. \quad t \in [T/2, T). \tag{2.8}$$

Dividing (2.8) by $m^{p-1}(t)$ and integrating (2.8) over $t^* \in [t, T_j]$ on a sequence $T_j \to T^-$ such that $m(T_j) \to \infty$, we obtain the conclusion. □

Remark 2.1. In general, the lower bound estimate is a result of regularity of the parabolic equations. This is the case for the equation (1.5). It is also the case for equations with boundary heat source, cf. [22, 21, 14]. In our case, the lower bound (2.1) is also derived by parabolic estimates.

3. Blowup set: no interior blowup

The gradient blowup cannot occur in the interior of the domain, by the regularity estimates:

Theorem 3.1 ([29], Theorem 3.2). *For any compact subset $K \subset \Omega$,*

$$\sup_{0<t<T} \|\nabla_x u\|_{L^\infty(K)} < C \operatorname{dist}(K, \partial\Omega)^{-1/(p-1)}.$$

Thus the gradient blowup must occur on the boundary. We next show that, for t close to T, $\sup_{x\in\overline{\Omega}} |\nabla_x u(x,t)|$ is actually monotonically increasing under the additional assumption on the initial data so that $u(x,t)$ is monotonically increasing in t. Assume

$$\Delta_x u_0(x) + |\nabla_x u_0(x)|^p \geq 0 \quad \text{for} \quad x \in \Omega. \tag{3.1}$$

In this case we can easily establish, by the maximum principle, that

$$0 < u_t(x,t) < \sup_{x\in\overline{\Omega}} \{\Delta_x u_0(x) + |\nabla_x u_0(x)|^p\} \quad \text{for} \quad x \in \Omega,\ 0 < t < T. \tag{3.2}$$

Lemma 3.2. *Let Ω be a bounded domain. If $0 < T - t \ll 1$, then*

$$\sup_{x\in\overline{\Omega}} |\nabla_x u(x,t)|$$

is monotonically increasing, and the maximum in (2.4) is actually taken on the boundary at time t, i.e.,

$$m(t) = \sup_{x\in\partial\Omega} |\nabla_x u(x,t)|.$$

Proof. It is clear that $v = |\nabla_x u|^2/2$ satisfies the equation

$$v_t = \Delta_x v + p|\nabla_x u|^{p-2}\nabla_x u \cdot \nabla_x v - \sum_{i,j} (u_{x_i x_j})^2 \quad \text{for} \quad x \in \Omega,\ 0 < t < T. \tag{3.3}$$

Therefore, the maximum of v can only be achieved at the parabolic boundary of the domain. Thus if we take $0 < T - t \ll 1$ such that $m(t) > \max_{x\in\overline{\Omega}} |\nabla_x u_0(x)|$, then the maximum of v must be achieved at the boundary $\partial\Omega$, and hence

$$m(t) = \sup_{x\in\partial\Omega,\ 0\leq\tau\leq t} |\nabla_x u(x,\tau)|. \tag{3.4}$$

Since $u(x,t) \geq u_0(x)$ for $x \in \Omega$ and by the compatibility condition (1.4), $u(x,t) = g(x) = u_0(x)$ for $x \in \partial\Omega$, we have

$$\nabla_x u(x,t) \cdot \vec{n} \leq \nabla_x u_0(x) \cdot \vec{n} \quad \text{for} \quad x \in \partial\Omega, \tag{3.5}$$

where we use $\vec{n}$ to denote the exterior unit normal vector on the boundary $\partial\Omega$. Therefore, at any point $x^* \in \partial\Omega, t^* \in (0,T)$ where $\nabla_x u(x^*,t^*)\cdot\vec{n} \ge 0$, we have

$$\begin{aligned} |\nabla_x u(x^*,t^*)| &= \sqrt{|\nabla_x u(x^*,t^*)\cdot\vec{n}|^2 + |\nabla_x u(x^*,t^*)\cdot\vec{\tau}|^2} \\ &= \sqrt{|\nabla_x u(x^*,t^*)\cdot\vec{n}|^2 + |\nabla_x u_0(x^*)\cdot\vec{\tau}|^2} \\ &\le \sqrt{|\nabla_x u_0(x^*)\cdot\vec{n}|^2 + |\nabla_x u_0(x^*)\cdot\vec{\tau}|^2} = |\nabla_x u_0(x^*)|, \end{aligned} \tag{3.6}$$

where we use $\vec{\tau}$ to denote the unit tangential vector on the boundary $\partial\Omega$. Since we have assumed $m(t) > \sup_{x\in\overline{\Omega}} |\nabla_x u_0(x)|$, the maximum in (3.4) must be achieved at a point (x^*,t^*) where $\nabla_x u(x^*,t^*)\cdot\vec{n} < 0$, $x^* \in \partial\Omega$.

Since $u_t = 0$ on $\partial\Omega$ and $u_t > 0$ in Ω, we have, by the strong maximum principle,

$$\nabla_x u_t(x,t)\cdot\vec{n} < 0 \quad \text{for } \ x \in \partial\Omega,\ 0 < t < T. \tag{3.7}$$

Since $u_t(x,t) \equiv 0$ for $x \in \partial\Omega$, its tangential derivative vanishes on $\partial\Omega$. Hence

$$\nabla_x u_t(x,t) = [\nabla_x u_t(x,t)\cdot\vec{n}]\vec{n} \qquad \text{for } \ x \in \partial\Omega.$$

Therefore, at any point (x^*,t^*), $x^* \in \partial\Omega$, where the maximum in (3.4) is reached, we have

$$\frac{\partial}{\partial t}|\nabla_x u|^2\Big|_{(x^*,t^*)} = 2\nabla_x u\cdot\nabla_x u_t\Big|_{(x^*,t^*)} = 2(\nabla_x u\cdot\vec{n})(\nabla_x u_t\cdot\vec{n})\Big|_{(x^*,t^*)} > 0. \tag{3.8}$$

As noted in the proof of Theorem 2.1, the function $m(t)$ is Lipschitz continuous and therefore m' exists almost everywhere. This implies that $m'(t) > 0$ a.e. for $0 < T - t \ll 1$. The lemma follows. $\square$

4. Gradient blowup rate upper bound: One dimensional case

We now consider the special one space dimensional case with $\Omega = (0,1)$. In this case the equations can be written as

$$u_t = u_{xx} + |u_x|^p \ \text{ for } \ 0 < x < 1,\ t > 0, \tag{4.1}$$

$$u(0,t) = 0, \quad u(1,t) = M \ \text{ for } \ t > 0, \tag{4.2}$$

$$u(x,0) = u_0(x) \ \text{ for } \ 0 \le x \le 1, \tag{4.3}$$

where $p > 2$ and M is a constant. Throughout this section we assume that

$$u_0 \in C^2[0,1], \quad u_0(0) = 0,\ u_0(1) = M, \tag{4.4}$$

$$u_0''(x) + |u_0'(x)|^p \ge 0 \ \text{ for } \ 0 \le x \le 1. \tag{4.5}$$

In this case we can easily establish, by the maximum principle, that

$$|u(x,t)| \le \|u_0\|_{L^\infty(0,1)} \quad \text{for } \ 0<x<1,\ 0<t<T, \tag{4.6}$$

$$0 < u_t(x,t) < \sup_{0\le x\le 1}\{u_0''(x) + |u_0'(x)|^p\} \quad \text{for } \ 0<x<1,\ 0<t<T. \tag{4.7}$$

In this section we shall establish

Theorem 4.1. *Let the assumptions (4.4)–(4.5) be in force and $2<p<\infty$. If the gradient blowup occurs at a finite time $t=T$, then there exists a constant $C_0>0$ such that*

$$\max_{0\le x\le 1} |u_x(x,t)| \le C_0 (T-t)^{-1/(p-2)}. \tag{4.8}$$

Remark 4.1. First note that there always exist initial data satisfying (4.4)–(4.5) (e.g., $u_0(x) = Mx$). In the case (4.4) and (4.5) are satisfied (i.e., we only look for solutions increasing in time), the necessary and sufficient condition for a finite time gradient blowup is

$$|M| > M_c := (p-2)^{-1}(p-1)^{(p-2)/(p-1)}.$$

Note that if $|M| > M_c$, then a finite time gradient blowup occurs (cf. [1], [3]).

On the other hand, the condition is necessary. Indeed, the arguments in Section 5 show that, if the blowup occurs at $x=0$, then $u(\cdot, T^-)$ is bigger than the stationary solution with boundary condition $f(0)=0$ and $f(1)=M_c$, and thus $M = u(1,t) > f(1) = M_c$ for $0 < T-t \ll 1$. Similarly, one must have $M < -M_c$ if the blowup occurs at $x=1$.

Proof of Theorem 4.1. For convenience, we define

$$\mathcal{L}[\phi] := \phi_t - \phi_{xx} - p|u_x|^{p-2}u_x\phi_x.$$

Then it is clear that

$$\mathcal{L}[u_t] \equiv 0, \quad \mathcal{L}[u_x] \equiv 0. \tag{4.9}$$

By Theorem 3.1,

$$\sup_{1/3<x<2/3,\ 0<t<T} |u_x(x,t)| < \infty.$$

Moreover, the gradient blowup will only occur at $x=0$, or $x=1$, or both.

If the gradient blowup occurs at $x=0$, we let

$$m(t) = \max_{0\le x\le 1/2,\ 0\le\tau\le t} |u_x(x,\tau)|. \tag{4.10}$$

Take $0<\sigma<1$ and set

$$w=\Big(1+\frac{1}{m^\sigma(t)}\Big)\Big(1-\frac{u_x}{m(t)}\Big),\quad x\in\Omega:=(0,1/2).$$

Since $u_0(x)<u(x,t)$, we deduce that $u_0'(0)\le u_x(0,t)$, so that Lemma 3.2 (the same proof is valid for this truncated domain, since u_x is bounded on $x=1/2$) implies that, for $0<T-t\ll 1$,

$$m(t)=\max_{x\in\overline{\Omega},\,0\le\tau\le t}|u_x(x,\tau)|=\max_{x\in\partial\Omega}|u_x(x,t)|=u_x(0,t). \tag{4.11}$$

Take $0<T-t\ll 1$ so that $m(t)\ge 1$. Then a direct computation shows that

$$\mathcal{L}[w]=-\frac{\sigma m'}{m^{\sigma+1}}\Big(1-\frac{u_x}{m}\Big)+\Big(1+\frac{1}{m^\sigma}\Big)\,\frac{u_x m'}{m^2}. \tag{4.12}$$

Recalling that $m'>0$ a.e. by the proof of Lemma 3.2, we have, in case $|u_x(x,t)|<\frac{\sigma}{\sigma+2}m^{1-\sigma}(t)$,

$$\begin{aligned}\mathcal{L}[w] &= \frac{m'}{m^{\sigma+1}}\Big(-\sigma+(\sigma+1)\frac{u_x}{m}+\frac{u_x}{m^{1-\sigma}}\Big)\\ &\le \frac{m'}{m^{\sigma+1}}\Big(-\sigma+(\sigma+2)\frac{|u_x|}{m^{1-\sigma}}\Big)\\ &\le 0 \ \text{ for } \ 0<T-t\ll 1.\end{aligned} \tag{4.13}$$

On the other hand, if $|u_x(x,t)|\ge\frac{\sigma}{\sigma+2}m^{1-\sigma}(t)$, then by (2.8), (4.12), and Lemma 3.2, we have

$$\begin{aligned}\mathcal{L}[w] &\le \Big(1+\frac{1}{m^\sigma}\Big)\,\frac{|u_x|m'}{m^2}\ \le\ 2|u_x|\frac{\widehat{C}m^{p-1}}{m^2}\\ &\le \frac{2\widehat{C}}{m}\Big(\frac{\sigma+2}{\sigma}\Big)^{(p-2)/(1-\sigma)}|u_x|^{(p-1-\sigma)/(1-\sigma)},\end{aligned} \tag{4.14}$$

where we use $\widehat{C}$ to denote the constant from (2.8) (note that the proof of (2.8) is still valid for $\Omega=(0,1/2)$ since the solution u and its derivative u_x are uniformly bounded in a neighborhood of $x=1/2$, which imply that x and t derivatives of any order are bounded near $x=1/2$, by the standard parabolic estimates). Combining the inequalities (4.13) with (4.14) we find that (4.14) is actually valid in both cases.

We now choose $\sigma=1/(p-1)$ so that

$$\frac{p-1-\sigma}{1-\sigma}=p.$$

Thus, for $0 < T - t \ll 1$ and $C^* = 2\widehat{C}\left(\frac{\sigma+2}{\sigma}\right)^{(p-2)/(1-\sigma)}$, we have

$$\mathcal{L}[w+u] \le \frac{C^*}{m}|u_x|^p - (p-1)|u_x|^p = |u_x|^p\Big(\frac{C^*}{m} - (p-1)\Big) < 0. \tag{4.15}$$

We take t_0 such that $0 < T - t_0 \ll 1$ and (4.15) holds for $t \in [t_0, T)$. Clearly, for any constant $C > 0$,

$$w(0,t) + u(0,t) = 0 = Cu_t \ \text{ for } \ t_0 < t < T.$$

By the strong maximum principle, for any $x_0 \in \Omega$,

$$\inf_{t_0 \le t < T} u_t(x_0, t) > 0, \quad u_{xt}(0, t_0) > 0.$$

By parabolic estimates, u and u_x are uniformly bounded on Γ, where

$$\Gamma := \{x = x_0,\ t_0 \le t < T\} \cup \{0 \le x \le x_0, t = t_0\}.$$

We can take a positive constant C such that $(w+u)(x,t) \le Cu_t(x,t)$ on Γ. Thus we can apply the maximum principle to $w + u - Cu_t$ in the region $(0, x_0) \times (t_0, T)$ to obtain $w + u < Cu_t$ for $0 < x < x_0,\ t_0 < t < T$. This implies that, for $t_0 < t < T$,

$$\begin{aligned} w_x(0,t) &= \lim_{x\to 0+} \frac{w(x,t)}{x} \le \lim_{x \to 0+} \frac{w(x,t) + u(x,t)}{x} \\ &\le \lim_{x\to 0+} C\frac{u_t(x,t)}{x} = Cu_{xt}(0,t). \end{aligned} \tag{4.16}$$

A direct computation shows

$$\begin{aligned} w_x(0,t) &= -\Big(1 + \frac{1}{m^\sigma(t)}\Big)\frac{u_{xx}(0,t)}{m(t)} \\ &= \Big(1 + \frac{1}{m^\sigma(t)}\Big)\frac{|u_x(0,t)|^p}{m(t)} \\ &\ge m^{p-1}(t). \end{aligned}$$

Thus (4.16) implies

$$m^{p-1}(t) \le Cm'(t) \ \text{ for } \ t_0 < t < T.$$

This inequality implies $|u_x(0,t)| \le C_0(T-t)^{-1/(p-2)}$. The proof is the same if the gradient blowup up occurs at $x = 1$. ∎

5. Asymptotic behavior

It is established in [1] (see also [3]) the following estimate

Lemma 5.1. *Let* $C_1 = \max_{0\le x\le 1}\{u_0''(x)+|u_0'(x)|^p\}$ *and the assumptions (4.4)–(4.5) be in force. If the gradient blowup occurs at* $x = 0$*, then for* $0 < T-t \ll 1$*,*

$$0 < u_x(x,t) - \{[u_x(0,t)]^{1-p} + (p-1)x\}^{-1/(p-1)} < C_1 x, \quad 0 < x < 1. \tag{5.1}$$

Proof. For reader's convenience we include the simplified proof from [3] here. From (4.7), for each fixed $t \in (0,T)$,

$$0 < u_{xx}(x,t) + |u_x(x,t)|^p < C_1 \text{ for } 0 < x < 1.$$

By the strong maximum principle, $u_{xt}(0,t) > 0$. Furthermore, since $|u_x(0,t)| \to \infty$ as $t \to T^-$, we must have $u_x(0,t) \ge 1$ for $0 < T-t \ll 1$.

Clearly, $f(x) = \{[u_x(0,t)]^{1-p} + (p-1)x\}^{-1/(p-1)}$ satisfies

$$f_x + f^p = 0 \text{ for } 0 < x < 1,\ 0 < T-t \ll 1.$$

Since $u_x(0,t) = f(0)$, by comparison principle, we derive that $u_x(x,t) > f(x)$ for $0 < x < 1$ and $0 < T-t \ll 1$.

Similarly, for $0 < T-t \ll 1$, $\widetilde{f}(x) = f(x) + C_1 x$ satisfies

$$\widetilde{f}_x + \widetilde{f}^p > C_1 + f_x + f^p \ge C_1 \text{ for } 0 < x < 1, \qquad \widetilde{f}(0) = u_x(0,t),$$

so that $u_x(x,t) < \widetilde{f}(x)$ for $0 < x < 1$. This proves (5.1). □

As a corollary it is clear that in this case $u_x(x,t) > 0$ for $0 < x < 1$, $0 < T-t \ll 1$. In particular, $0 < u_x(1,t) \le u_0'(1)$, since $u(x,t) > u_0(x)$. Thus $x = 1$ cannot be a blowup point. Letting $t \to T^-$ in (5.1), we also have $u_x(x,T^-) \ge \{(p-1)x\}^{-1/(p-1)}$, and the strict inequality holds for $0 < x < 1$ by the strong maximum principle. Integrating over $(0,1)$, we find that $M > M_c := (p-2)^{-1}(p-1)^{(p-2)/(p-1)}$ in this case. As indicated in the introduction, the finite time gradient blowup must occur in this case ([1, 3]). Similar argument can be made in the case gradient blowup occurs at $x = 1$. Therefore, under the assumptions (4.4)–(4.5), *(i) the gradient* u_x *blows up at* $x = 0$ *if and only if* $M > M_c$*, in this case* $x = 0$ *is the only point where* $u_x(\cdot,t)$ *is unbounded; (ii) the gradient* u_x *blows up at* $x = 1$ *if and only if* $M < -M_c$*, in this case* $x = 1$ *is the only point where* $u_x(\cdot,t)$ *is unbounded.*

The estimate in Lemma 5.1, together with our blowup rate estimates, is enough for us to establish the asymptotic behavior of the solution. We consider the function

$$g(y,s) := [s+(p-1)y]^{-1/(p-1)}, \quad 0 \le s < \infty,\ 0 \le y < \infty. \tag{5.2}$$

Note that, for any $s \ge 0$, g satisfies the equation

$$g_{yy}(y,s) + [g_y(y,s)]^p = 0 \ \text{ for } \ 0 < y < \infty,$$

Set

$$y := \frac{x}{(T-t)^{(p-1)/(p-2)}}, \quad c(t) := [u_x(0,t)(T-t)^{1/(p-2)}]^{1-p}. \tag{5.3}$$

Then it follows from (5.1) that

$$0 < u_x(x,t)(T-t)^{1/(p-2)} - g(y,c(t)) < C_1 x(T-t)^{1/(p-2)}, \tag{5.4}$$

for all $x \in (0,1)$ and $t \in (0,T)$. Thus we obtain the uniform convergence of $u_x(x,t)(T-t)^{1/(p-2)}$ to $g(y,c(t))$ over $x \in [0,1]$ as $t \to T^-$.

Note that, by (2.1) and (4.8), $c(t)$ is bounded from above and below by positive constants.

We state it as a theorem:

Theorem 5.2. *Let the assumptions (4.4)-(4.5) be in force. If the gradient blowup occurs at $x=0$, then for all $0 < x < 1$ and $0 < T-t \ll 1$*

$$0 < u_x(x,t)(T-t)^{1/(p-2)} - g\Big(\frac{x}{(T-t)^{(p-1)/(p-2)}}, c(t)\Big) < C_1 x(T-t)^{1/(p-2)},$$

where $g(y,s)$ and $c(t)$ are given in (5.2) and (5.3). Moreover, the function $c(t)$ satisfies, for some constants $C_0 > c_0 > 0$,

$$0 < C_0^{1-p} \le c(t) \le c_0^{1-p} \ \textit{ for } \ 0 < T-t \ll 1.$$

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