EXISTENCE AND UNIQUENESS OF STABILIZED PROPAGATING WAVE SEGMENTS IN WAVE FRONT INTERACTION MODEL

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Abstract. Recent experimental studies of photosensitive Belousov-Zhabotinskii reaction has revealed the existence of propagating wave segments. The propagating wave segments are unstable, but can be stabilized by using a feedback control to continually adjust the excitability of the medium. Experimental studies also indicate that the locus of the size of a stabilized wave segment as a function of the excitability of the medium gives the excitability boundary for the existence of 2D wave patterns with free ends in excitable media. To study the properties of this boundary curve, we use the wave front interaction model proposed by Zykov and Showalter. This is equivalent to study a first order system of three ordinary differential equations which includes a singular nonlinearity. Using two different reduced first order systems of two ordinary differential equations, we first show the existence of wave segments for any given propagating velocity. Then the wave profiles can be classified into two types, namely, convex and non-convex types. More precisely, when the normalized propagating velocity is small, we show that the wave profile is of convex type, while the wave profile is of non-convex type when the normalized velocity is close to 1.

Keywords: Stabilized propagating wave segment, wave front, wave back, wave profile

1. Introduction

Wave propagation in excitable media has been investigated in depth, both by theoreticians and experimentalists, and has a variety of applications in physical model, chemical reaction, and biological system. Among them, spiral waves have been recognized as a fascinating and important spatio-temporal pattern, such as waves of oxidation in the Belousov-Zhabotinskii reaction [13], waves of cyclic-AMP signaling in the social amoeba colonies of *Dioctyostelium discoideum* [5], and waves of neuromuscular in the heart muscle [9].

A fundamental problem on spiral waves is to understand what is the region of the excitability of the medium for which spiral waves can exist. For this, we note

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that the existence of spiral waves is closely related to the existence of 1D pulse [14]. It has been shown in [8] that there exists an excitability limit, which we denote by ζ_0, below which the propagation of 1D waves is not possible. Therefore, below such a excitability limit ζ_0, the underlying medium cannot support the propagation of spiral waves [12, 14]. On the other hand, it is known that spiral waves can exist in the medium with sufficiently high excitability [12, 14]. Moreover, the spiral wavelength increases as the excitability of the medium decreases. When the excitability of the medium decreases from a high value to a critical value, the corresponding wavelength tends to infinity, and the associated wave pattern becomes an unbounded (nearly) planar wave with one free end, which propagates in the normal direction without sprouting and retracting at its free end (tip), and is known as a critical finger [8, 3]. It is worthy to note that this critical excitability does not coincide with ζ_0. This suggests that there should exist a second excitability limit for 2D wave pattern with free ends.

Recently, Mihaliuk, Sakurai, Chirila and Showalter [6, 7] have used the photosensitive Belousov-Zhabotinskii reaction to find another wave pattern: a wave segment which has two free ends, and moves with a constant velocity and fixed shape (see Fig. 1 for three different wave segments). These wave segments are unstable, but can be stabilized by using a feedback control to continually adjust the excitability of the medium. Specifically, when the observed wave segment increases (resp. decreases) in size, by adjusting the incident light one can decrease (resp. increase) the size of the wave segment, thereby stabilizing the size of the wave segment. Their experimental study and numerical simulations [6, 7] based on the Oregonator model [4] also showed that there is a unique stabilized wave segment for each given (admissible) excitability of the medium determined by the light intensity. Therefore, the size S of a stabilized wave segment can be viewed as a function, $S = h(\zeta)$, of the medium excitability ζ (determined by the light intensity) for any ζ larger than a critical value ζ_∞. Note that the size $h(\zeta)$ of a stabilized wave segment approaches infinity as $\zeta \searrow \zeta_\infty$. Also, this critical value ζ_∞ defines the second excitability limit for the existence of 2D wave patterns [6, 7]. More precisely, the numerical results in [6, 7] show that the medium with excitability lying between ζ_∞ and ζ_0 can only support the propagation of 2D wave patterns without free ends (e.g., planar waves and circular waves) [11]. Hence spiral waves cannot exist for the medium with excitability lying between ζ_∞ and ζ_0. Furthermore, this second excitability limit ζ_∞ is exactly the same excitability limit suggested by the critical fingers [6, 7].

Motivated by the above discussion, we see that a good understanding of stabilized propagating wave segments is necessary. Although the experimental and numerical study in [6, 7] makes a significant step in the understanding of stabilized propagating wave segments, it does not provide a satisfactory description of these waves. For example, what is the relationship between the width of the wave segment and the excitability of the medium? Another possible approach to attack the stabilized propagating wave segments is to resort to the two-component reaction-diffusion systems (e.g., the typical FitzHugh-Nagumo equation) analytically, which

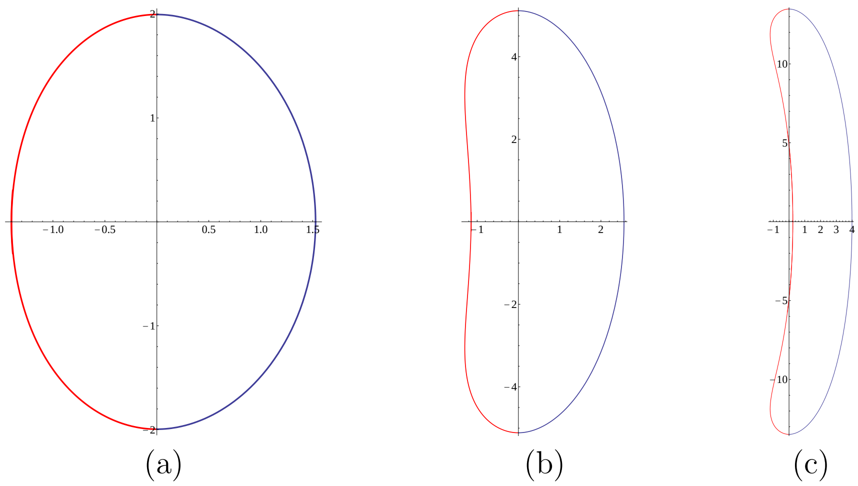

FIGURE 1. Profiles of the wave segments (with different scales): (a) $(\sigma, b) = (0.6, 0.44)$, (b) $(\sigma, b) = (0.9, 0.5247)$, (c) $(\sigma, b) = (0.98, 0.5334145)$.

is quite difficult to perform. To summarize, in order to get a deeper understanding of the stabilized propagating wave segments, one needs an analytical approach, and a further reduction of the reaction-diffusion system to a simpler system is necessary.

To do this, Zykov and Showalter [15] have used the so-called free-boundary approach [1, 12, 3, 10] (also known as wave front interaction model) to reduce the reaction-diffusion system to two first order systems of three ordinary differential equations, which still can capture the essential behavior of stabilized propagating wave segments. Within this frame, Zykov and Showalter [15] have obtained a relationship between the size of the wave segment and the excitability of the medium when the excitability is very close to the one corresponding to the critical finger (see [16] for the higher excitability case). In this paper, we shall use the model of Zykov and Showalter [15] to establish that (i) for each given size of the wave segment, there exists a unique excitability such that the stabilized wave segment with the given size can propagate in the corresponding medium, which, together with the properties of ζ for small sizes of stabilized wave segments, shows that the locus $S = h(\zeta)$ of stabilized wave segments in the parameter space (ζ, S) is monotone decreasing in the excitability ζ; and (ii) there are two types of the profiles of stabilized propagating wave segments.

To describe our results, we need to introduce some notation and the governing equations. In the wave front interaction model for stabilized propagating wave segments, a wave segment propagating in a two-dimensional medium can be characterized by its wave front and wave back which form the boundary of the wave segment. Here the front and the back are realized as dividing the boundary of the wave segment by its top and bottom points (see Fig. 2). Suppose that *the wave segment is symmetric with respect to the horizontal line through the midpoint of the front.* Let this horizontal line be the x-axis, the y-coordinate of the midpoint of the

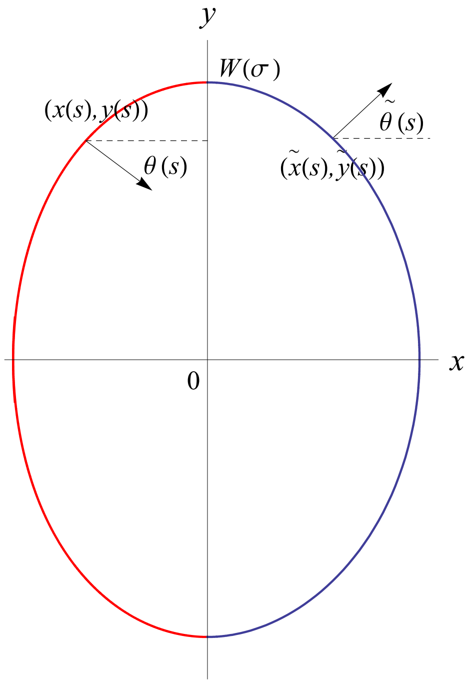

FIGURE 2. A traveling wave segment and its variables.

wave front be zero and the x-coordinate of the top point of the wave segment be also zero. Then, the wave front/back is described by the relation

$$(x(s), y(s), \theta(s)) = (x_\pm(s), y_\pm(s), \theta_\pm(s)),$$

where s the arc-length measured from the top point, (x, y) the position, and θ is the angle of the outward/inward normal measured from the x-axis.

Then in the dimensionless form, $(\tilde{x}, \tilde{y}, \tilde{\theta}) = (x_+, y_+, \theta_+)$ is governed by the following system

$$\left\{ \begin{array}{rcl} \tilde{x}' & = & \sin\tilde{\theta}, \\ \tilde{y}' & = & -\cos\tilde{\theta}, \\ \tilde{\theta}' & = & -1 + \sigma\cos\tilde{\theta} \end{array} \right. \tag{1.1}$$

with $(\tilde{x}(0), \tilde{y}(0), \tilde{\theta}(0)) = (0, W(\sigma), \pi/2)$, where $W(\sigma)$ is the half-width of the wave segment and σ (the normalized propagation velocity of the wave, see the equation (2.9)) is a given constant such that $\sigma \in (0, 1)$. Note that $\tilde{\theta} \in [0, \pi/2]$. Since the wave segment is symmetric with respect to the x-axis, we require that $\tilde{y}(s) = 0$ when $\tilde{\theta}(s) = 0$. Using this condition, we can solve system (1.1) to obtain the expression for the wave front $(\tilde{x}, \tilde{y}) = (x_+, y_+)$ in terms of $\tilde{\theta}$ (see [15, 6] for detail).

Precisely, we have

$$\begin{aligned} x_+ &= \frac{1}{\sigma}\log\frac{1}{1-\sigma\cos\tilde\theta}, \\ y_+ &= -\frac{\tilde\theta}{\sigma}+\frac{2}{\sigma\sqrt{1-\sigma^2}}\tan^{-1}\Big(\frac{(1+\sigma)\tan(\tilde\theta/2)}{\sqrt{1-\sigma^2}}\Big) \end{aligned}$$

for $\tilde\theta \in [0,\pi/2]$. In particular, the half-width of the wave segment $W(\sigma)$ is the evaluation of y_+ at $\tilde\theta = \pi/2$, i.e.,

$$W = W(\sigma) := -\frac{\pi}{2\sigma}+\frac{2}{\sigma\sqrt{1-\sigma^2}}\tan^{-1}\Big(\frac{1+\sigma}{\sqrt{1-\sigma^2}}\Big).$$

Note that $W(0^+) = 1$ and $W(1^-) = \infty$. Hereafter we use the notation $W(0^+) = \lim_{\sigma\searrow 0} W(\sigma), W(1^-) = \lim_{\sigma\nearrow 1} W(\sigma)$ and so on for simplicity. This orbit (x_+, y_+) gives the relation $x_+ = f_\sigma(y_+)$. Note that f_σ (depending on σ) is a decreasing function defined only on $[0, W(\sigma)]$ such that

$$\begin{aligned} &f_\sigma(0) = -\frac{\log(1-\sigma)}{\sigma},\ f_\sigma(W(\sigma)) = 0, \\ &f'_\sigma(0) = 0,\ f'_\sigma(W^-(\sigma)) = -\infty,\ f''_\sigma(0) = \sigma - 1 < 0. \end{aligned}$$

Hence we have the complete description for the wave front of a stabilized wave segment for each normalized propagation velocity σ of the wave segment.

It remains to describe the wave back. Indeed, in the dimensionless form, the wave back $(x, y, \theta) = (x_-, y_-, \theta_-)$ satisfies the following initial value problem $(P_{\sigma,b})$:

$$x' := \frac{dx}{ds} = \sin\theta, \tag{1.2}$$

$$y' := \frac{dy}{ds} = -\cos\theta, \tag{1.3}$$

$$\theta' := \frac{d\theta}{ds} = 1+\sigma\cos\theta - b[f_\sigma(y) - x], \tag{1.4}$$

with the initial conditions

$$x(0) = 0,\ \ y(0) = W(\sigma),\ \ \theta(0) = -\pi/2. \tag{1.5}$$

Due to the symmetric assumption of the wave back with respect to the x-axis, the positive constant b associated with the wave back is to be determined so that the following boundary condition holds

$$y(s^*) = 0,\ \theta(s^*) = 0\ \text{ and }\ y' < 0 \text{ in } (0, s^*) \tag{1.6}$$

for some $s^* > 0$. Here the parameter b is decreasing in the excitability of the medium (see Section 2). Indeed, the parameter b can be used as an excitability measure by (2.11).

We recall that for each normalized propagation velocity σ of the wave segment, we have the complete description for the wave front which is independent of b. On the other hand, the b dependence of the problem $(P_{\sigma,b})$ implies that the normalized propagation velocity σ of the wave segment alone cannot characterize the wave back. Indeed, to describe the wave back of a stabilized wave segment for a given normalized propagation velocity σ of the wave segment, we need to look for a particular excitability (a positive constant b) so that the problem $(P_{\sigma,b})$ with the boundary condition (1.6) admits a solution.

For the reader's convenience and physical implication, in Section 2, we shall follow [15] to give a brief description on deriving these governing equations for the wave front and back. We also refer to [15] for more details of the background.

We now state our main existence and uniqueness result as follows.

Theorem 1. *For each $\sigma \in (0,1)$, there exists a unique $b^* = b^*(\sigma) > 0$ and $s^* = s^*(\sigma) > 0$ such that the solution (x, y, θ) of (P_{σ,b^*}) defined on $[0, s^*]$ satisfies $y' < 0$ on $(0, s^*)$, $y(s^*) = 0$ and $\theta(s^*) = 0$. Moreover, $|\theta| < \pi/2$ and $x < f_\sigma(y)$ on $(0, s^*]$.*

We note that the half-width $W(\sigma)$ of the wave segment is increasing in σ (see Lemma 3.4), and that the parameter b is decreasing in the excitability ζ of the medium (see the equation (2.11)). From this and Theorem 1, we may conclude that for each given size of the wave segment, there exists a unique excitability such that the stabilized wave segment with the given size can propagate in the corresponding medium. This shows that the locus $S = h(\zeta)$ of stabilized wave segments in the parameter space (ζ, S) is monotone in the excitability ζ. Furthermore, from Section 6 we see that $b^*(\sigma)$ is increasing in σ for small σ. Together with (2.11) and the fact that the parameter I_s is decreasing in the excitability ζ of the medium, we have that the locus $S = h(\zeta)$ is decreasing in ζ provided S is small. Hence the monotonicity of the locus $S = h(\zeta)$ in ζ yields that $S = h(\zeta)$ is decreasing in ζ, which is consistent with experimental studies [6, 7].

We also remark that, as will be shown in Lemma 6.2, the stationary wave segment (i.e., when $\sigma = 0$) is precisely the unit circle which corresponds to $b = 0$. On the other hand, when $b = 0$, by Theorem 1, the problem $(P_{\sigma,b})$ with (1.6) can be solved only if $\sigma = 0$.

Note that $y' < 0$ on $(0, s^*)$ if and only if $\theta \in (-\pi/2, \pi/2)$. Also, θ may be positive and θ' may change sign in $(0, s^*)$. We can roughly classify the wave segments into the following two types:

(I) Convex type : $\theta' > 0$ on $[0, s^*)$.
(II) Non-convex type : θ' can change its sign in $(0, s^*)$.

By using numerical simulation, we find that θ' can change sign at most once. Furthermore, the wave segment is of convex type for $\sigma \in (0, \hat{\sigma}]$, and of non-convex type for $\sigma \in (\hat{\sigma}, 1)$. Here $\hat{\sigma}$ is roughly between 0.83 and 0.85. Indeed, we can prove the following result on the types of wave profiles.

Theorem 2. *The propagating wave segment obtained in Theorem 1 is convex when σ is small, while it becomes non-convex when σ is close to* 1.

Finally, the plan of this paper is as follows. In Section 2, following [15], we present a brief derivation of the mathematical model of stabilized wave segments. Section 3 is concerned with some mathematical preparations. In particular, the local existence and uniqueness of solution to the problem $(P_{\sigma,b})$ is established. Then in Section 4 we prove that there is at most one stabilized wave segment for a given $\sigma \in (0,1)$ via the comparison principle. The existence of stabilized wave segments is established in Section 5. In Section 6, we classified the profiles of stabilized wave segments. Finally, a summary and discussion of the results is given in Section 7.

2. Derivation of the model

In this section, we follow [15] to derive the problem $(P_{\sigma,b})$ for the wave back, and the governing equations (1.1) for the wave front. We first consider the two-component reaction-diffusion model of the form

$$\frac{\partial u}{\partial t} = D\nabla^2 u + F(u,v), \tag{2.1}$$

$$\frac{\partial v}{\partial t} = \epsilon[G(u,v) + I(t)], \tag{2.2}$$

where $u = u(x,y,t)$ and $v = v(x,y,t)$ represent, respectively, the activator and the inhibitor in a two-dimensional medium with the diffusion coefficient D, and the parameter ϵ is sufficiently small. Here the function $I(t)$ corresponds to the light intensity in the photosensitive Belousov-Zhabotinskii reaction. It controls the excitability of the medium. Precisely, the excitability of the medium decreases with increasing light intensity [6, 7]. When the wave segment tends to a stead-state, $I(t)$ will approach a constant value I_s. Therefore, we can view I_s as an decreasing function of the excitability of the medium for which the corresponding stabilized wave segment exists. For the reaction functions F and G, we use the typical FitzHugh-Nagumo nonlinearity, i.e.,

$$F(u,v) = 3u - u^3 - v, \quad G(u,v) = u - \delta$$

with the constant δ satisfying $0 < \delta + \sqrt{3} \ll 1$. With this choice of δ, the system (2.1)-(2.2) with $I(t) \equiv 0$ has a unique uniform resting state $(u_0, v_0) = (\delta, 3\delta - \delta^3)$. Note that the quantity $\mathcal{E} := -v_0$ measures the excitability of the system (2.1)-(2.2) with $I(t) \equiv 0$ (see [3]). In the remaining of this section, we assume $|\mathcal{E}| \ll 1$.

A wave segment propagating in the x direction can be described by its two boundaries: the wave front and the wave back. These two boundaries separate the enclosed domain Ω of excitation from the refractory region. Due to the small ϵ, the wave front can be viewed as a sharp interface which connects the resting state (u_0, v_0) and the excited state $(u_e(v_0), v_0)$, and propagates with the velocity $c_p(v_0)$. Here $u = u_e(v)$ is the largest root of the equation $F(u,v) = 0$, and the propagation velocity $c_p(v)$ of a planar interface with the slow variable v at the moving boundary is given by

$$c_p(v) = -\alpha\sqrt{D}v, \tag{2.3}$$

provided v is sufficiently small. Here α is a constant determined by $F(u,0)$ (see [12, 3]). We also note that the slow variable v is approximately equal to v_0 on the wave front. In the excited domain Ω, the inhibitor v evolves slowly according to (2.2) with u replaced by $u = u_e(v)$, while the activator u approximately equals $u_e(v_0) \approx \sqrt{3}$. Now we let v_- be the value of v evaluated on the wave back. After the excited region Ω, the activator u abruptly changes from $u_e(v_-) \approx \sqrt{3}$ to $u_r(v_-) \approx -\sqrt{3}$, where $u = u_r(v)$ is the smallest root of the equation $F(u,v) = 0$. As before, the wave back can be viewed as a sharp interface which connects the excited state $(u_e(v_-), v_-)$ and the state $(u_r(v_-), v_-)$, and propagates with the velocity $c_p(v_-)$.

It is well known [12] that the normal velocity c_n of the wave front (back) obeys the following linear eikonal equation

$$c_n(v) = c_p(v) - D\kappa. \tag{2.4}$$

Also, the normal direction of the stabilized wave segment at the midpoint ($y = 0$) coincides with the propagation direction of the full wave segment along the x axis. Then the free-boundary problem for the stabilized wave segment propagating along the x axis with the constant velocity c_s reads as

$$\begin{aligned} c_p(v_0) - D\kappa_+ &= c_s \cos(\Theta_+), && (2.5)\\ -c_p(v_-) - D\kappa_- &= c_s \cos(\Theta_-), && (2.6)\\ c_s \frac{dv}{dx} &= -\epsilon[G(u_e(v), v) + I_s] \quad \text{in } \Omega, && (2.7) \end{aligned}$$

where $I_s := I(+\infty)$, Θ_+ (resp. Θ_-) denotes the angle between the x axis and the outward (resp. inward) normal on the front (resp. back), s_+ (resp. s_-) measures the arc length from the top of the stabilized wave segment, and $\kappa_\pm = -d\Theta_\pm/ds_\pm$ is the local curvature on the front (resp. back). According to (2.4), $c_s = c_p(v_0) - D\kappa_m$, where κ_m is the local curvature at the midpoint of the wave front.

The equation (2.5) (resp. (2.6)-(2.7)) describes the shape $x_+ = x_+(y)$ (resp. $x_- = x_-(y)$) of the wave front (resp. the wave back). Moreover, if we let (x_+, y_+) (resp. (x_-, y_-)) denote the Cartesian coordinates of the wave front (back), then we have

$$\frac{dx_\pm}{ds_\pm} = \sin\Theta_\pm, \quad \frac{dy_\pm}{ds_\pm} = -\cos\Theta_\pm. \tag{2.8}$$

Now set $s = c_p(v_0)s_+/D$, $\tilde{x} = c_p(v_0)x_+/D$, $\tilde{y} = c_p(v_0)y_+/D$, $\tilde{\theta} = \Theta_+$, and

$$\sigma = 1 - \frac{D\kappa_m}{c_p(v_0)}. \tag{2.9}$$

Then the equation (2.8) for (x_+, y_+) together with the equation (2.5) for Θ_+ can be transformed into the governing equations (1.1) for $(\tilde{x}, \tilde{y}, \tilde{\theta})$ (see [15, 6, 7]).

To characterize the shape $x_- = x_-(y)$ of the wave back, we need to take (2.7) into account due to v_-. Note that $G(u_e(v), v) + I_s \approx G^* := G(u_e(0), 0) + I_s$ in the excited region Ω. Hence (2.7) gives

$$v_- = v_0 + \frac{G^*\epsilon}{c_s}[x_+(y) - x_-(y)],$$

which, together with (2.6) and (2.3), yields

$$D\frac{d\Theta_-}{ds_-} = c_p(v_0) - \frac{\alpha\sqrt{D}G^*\epsilon}{c_s}[x_+(y) - x_-(y)] + c_s\cos\Theta_-. \tag{2.10}$$

Then, by setting $s = c_p(v_0)s_-/D$, $x = c_p(v_0)x_-/D$, $y = c_p(v_0)y_-/D$, $\theta = \Theta_-$,

$$\sigma = 1 - \frac{D\kappa_m}{c_p(v_0)} \quad \text{and} \quad b = \frac{G^*\epsilon}{\alpha^2\mathcal{E}^3\sigma} = \frac{\epsilon}{\alpha^2\mathcal{E}^3\sigma}[G(u_e(0),0) + I_s], \tag{2.11}$$

the equation (2.8) for (x_-, y_-) together with the equation (2.10) for Θ_- can be transformed into the problem $(P_{\sigma,b})$ for (x, y, θ). Note that the parameter b is decreasing in the excitability of the medium by the definition of I_s.

3. Preliminaries

From now on we assume that $\sigma \in (0,1)$ and $b \geq 0$. We first investigate the local existence and uniqueness of solutions to the problem $(P_{\sigma,b})$. Indeed, f_σ is continuous and bounded on $[0, W(\sigma)]$, the local existence of a solution to the problem $(P_{\sigma,b})$ follows from the standard existence theorem for differential equations (see [2]).

Now we turn to the uniqueness. Note that the uniqueness of the solution to the problem $(P_{\sigma,b})$ with $\sigma \in (0,1)$ and $b = 0$ is trivial. When $b > 0$, since f_σ is not Lipschitz continuous at $y = W(\sigma)$, the usual theory of uniqueness cannot be applied to the problem $(P_{\sigma,b})$. Nevertheless, we have the following uniqueness and asymptotic expansion for solutions of $(P_{\sigma,b})$ near $s = 0$.

Lemma 3.1. *Suppose that $b > 0$ and $\sigma \in (0,1)$. Then there exists a unique solution (x, y, θ) of $(P_{\sigma,b})$ such that*

$$\begin{aligned} x(s) &= -s + O(s^3), \\ y(s) &= W(\sigma) - \frac{1}{2}s^2 + O(s^3), \\ \theta(s) &= -\frac{\pi}{2} + s + \frac{\sigma - 2b}{2}s^2 + O(s^3) \end{aligned}$$

for small $s \geq 0$.

Proof. First, we study the behavior of f_σ. By (1.1), we have

$$\frac{\partial\tilde{x}}{\partial\tilde{\theta}} = \frac{\sin\tilde{\theta}}{-1 + \sigma\cos\tilde{\theta}}, \quad \frac{\partial\tilde{y}}{\partial\tilde{\theta}} = \frac{\cos\tilde{\theta}}{1 - \sigma\cos\tilde{\theta}}, \quad \tilde{\theta} \in [0, \pi/2],$$

which implies

$$\frac{\partial\tilde{x}}{\partial\tilde{\theta}}(\pi/2) = -1,\ \frac{\partial\tilde{y}}{\partial\tilde{\theta}}(\pi/2) = 0,\ \frac{\partial^2\tilde{x}}{\partial\tilde{\theta}^2}(\pi/2) = \sigma,\ \frac{\partial^2\tilde{y}}{\partial\tilde{\theta}^2}(\pi/2) = -1.$$

It follows from the implicit function theorem that

$$f_\sigma(y) = \sqrt{2[W(\sigma) - y]} + O([W(\sigma) - y]).$$

Set

$$x = -s + \xi, \quad y = W(\sigma) - \frac{1}{2}s^2 - \eta, \quad \theta = -\frac{\pi}{2} + s + \varphi.$$

Then (ξ, η, φ) satisfies

$$\begin{aligned} \xi' &= \sin(-\pi/2 + s + \varphi) + 1 = 1 - \cos(s + \varphi), \\ \eta' &= \cos(-\pi/2 + s + \varphi) - s = \sin(s + \varphi) - s, \\ \varphi' &= \sigma \cos(-\pi/2 + s + \varphi) - b[f_\sigma(W(\sigma) - s^2/2 - \eta) + s - \xi] \end{aligned}$$

with $(\xi(0), \eta(0), \varphi(0)) = (0, 0, 0)$.

Now, we consider the metric space

$$\begin{aligned} \mathcal{M} := \{(\xi, \eta, \varphi) \quad | \quad & \xi, \eta, \varphi \in C^0[0, s_0], \\ & 0 \le \xi(s) \le As^2,\ 0 \le \eta(s) \le Bs^3,\ 0 \le \varphi(s) \le Cs^2\} \end{aligned}$$

with the metric

$$\|(\xi, \eta, \varphi)\|_{\mathcal{M}} := \sup_{0 \le s \le s_0} (s^{-2}|\xi(s)| + s^{-3}|\eta(s)| + s^{-2}|\varphi(s)|)$$

and the mapping $\mathcal{F}$ on $\mathcal{M}$ given by

$$\mathcal{F}(\xi, \eta, \varphi)(s) = (\mathcal{F}_1(\xi, \eta, \varphi)(s), \mathcal{F}_2(\xi, \eta, \varphi)(s), \mathcal{F}_3(\xi, \eta, \varphi)(s)),$$

where

$$\begin{aligned} \mathcal{F}_1(\xi, \eta, \varphi)(s) &:= \int_0^s \Big(1 - \cos(\tau + \varphi(\tau))\Big) d\tau, \\ \mathcal{F}_2(\xi, \eta, \varphi)(s) &:= \int_0^s \Big(\sin(\tau + \varphi(\tau)) - \tau\Big) d\tau, \\ \mathcal{F}_3(\xi, \eta, \varphi)(s) &:= \int_0^s \Big(\sigma \sin(\tau + \varphi(\tau)) - b[f_\sigma(W(\sigma) - \tau^2/2 - \eta(\tau)) + \tau - \xi(\tau)]\Big) d\tau. \end{aligned}$$

We can easily check that

$$\begin{aligned} |\mathcal{F}_1(\xi, \eta, \varphi)| &\le (1 + Cs_0)^2 s_0 s^2 \\ |\mathcal{F}_2(\xi, \eta, \varphi)| &\le \frac{C}{3} s^3 \\ |\mathcal{F}_3(\xi, \eta, \varphi)| &\le \left(\frac{|\sigma - 2b|}{2} + \frac{\sigma C s_0}{3} + \frac{K_1 b B}{4} s_0^2 + \frac{bAs_0}{3}\right) s^2 \end{aligned}$$

where K_1 is a constant depending only on f_σ. An appropriate choice of positive constants A, B, C and s_0 assures that $\mathcal{F}$ maps from $\mathcal{M}$ into $\mathcal{M}$. Similarly we see that $\mathcal{F}$ is a contraction mapping on $\mathcal{M}$ and that the solution uniquely exists for $0 \le s \le s_0$. This calculation also implies the asymptotic expansion of x, y, θ for $s \ge 0$ small. $\square$

This lemma also implies that the solution (x, y, θ) depends on (σ, b) continuously for $(\sigma, b) \in [0, 1) \times [0, \infty)$.

For a given $\sigma \in (0, 1)$ and $b \geq 0$, we let $(x(s; \sigma, b), y(s; \sigma, b), \theta(s; \sigma, b))$ denote the solution of $(P_{\sigma,b})$ and let $[0, S_{\sigma,b})$ be the corresponding maximal existence interval such that $y(s; \sigma, b) > 0$. Note that $y(S_{\sigma,b}^{-}; \sigma, b) = 0$ if $S_{\sigma,b} < +\infty$. If there is no ambiguity, we shall suppress the subindex.

Lemma 3.2. *Let (x, y, θ) be the solution of $(P_{\sigma,b})$.*

(i) *If $-\pi/2 \leq \theta(s; \sigma, b) < 0$ and $\theta'(s; \sigma, b) = 0$, then*
$$\theta''(s; \sigma, b) < 0.$$
(ii) *If $\theta(s; \sigma, b) = \pi/2$ and $\theta'(s; \sigma, b) = 0$, then*
$$\theta''(s; \sigma, b) > 0.$$
(iii) *If $y(s; \sigma, b) = 0$, $\theta(s; \sigma, b) = 0$ and $\theta'(s; \sigma, b) = 0$, then*
$$\theta''(s; \sigma, b) = 0, \quad \theta'''(s; \sigma, b) = -bf_\sigma''(0) > 0.$$

Proof. Differentiating Eq. (1.4) with respect to s, we have

$$\theta'' = -\sigma\theta' \sin\theta - b\Big(-f_\sigma'(y)\cos\theta - \sin\theta\Big) = b\Big(f_\sigma'(y)\cos\theta + \sin\theta\Big), \tag{3.1}$$

if $\theta' = 0$. The first statement follows from $f_\sigma' \leq 0$, $\cos\theta \geq 0$ and $\sin\theta < 0$. The equation (3.1) also implies the second statement.

For the last statement, (3.1) and the fact that $f_\sigma'(0) = 0$ implies
$$\theta''(s; \sigma, b) = 0$$
at (s, σ, b) as in the lemma. We also have
$$\theta''' = -\sigma\theta'' \sin\theta - \sigma(\theta')^2\cos\theta - b\Big(f_\sigma''(y)\cos^2\theta + f_\sigma'(y)\theta'\sin\theta - \theta'\cos\theta\Big),$$
which implies
$$\theta'''(s; \sigma, b) = -bf_\sigma''(0).$$
Thus we have completed the proof. □

Next, we recall the following comparison principle from p.28 of Hartman [2].

Lemma 3.3. (Comparison Principle)

(a) *Let $\mathbf{f}(t, \mathbf{y}) = (f^1(t, \mathbf{y}), ..., f^d(t, \mathbf{y}))$, $\mathbf{g}(t, \mathbf{y}) = (g^1(t, \mathbf{y}), ..., g^d(t, \mathbf{y}))$ be continuous on the strip $S = \{(t, \mathbf{y}) \mid a \leq t \leq b,\ \mathbf{y} \in R^d\}$ such that $f^k(t, \mathbf{y}) < g^k(t, \mathbf{y})$ for $k = 1, \cdots, d$, and that, for each $k = 1, \cdots, d$, either $f^k(t, \mathbf{y})$ or $g^k(t, \mathbf{y})$ is nondecreasing with respect to y^i with $i \neq k$. On $a \leq t \leq b$, let $\mathbf{y} = \mathbf{y}(t)$ be the solution of $\mathbf{y}' = \mathbf{f}(t, \mathbf{y})$ with the initial condition $\mathbf{y}(a) = \mathbf{y}_0$; and $\mathbf{z} = \mathbf{z}(t)$ be the solution of $\mathbf{z}' = \mathbf{g}(t, \mathbf{z})$ with the initial condition $\mathbf{z}(a) = \mathbf{z}_0$, where $y_0^k \leq z_0^k$ for $k = 1, \cdots, d$. Then $y^k(t) \leq z^k(t)$ for $a \leq t \leq b$ and $k = 1, \cdots, d$.*

(b) *If, in part* (a)*, all initial value problems associated with* $\mathbf{y}' = \mathbf{f}(t, \mathbf{y})$ *and* $\mathbf{z}' = \mathbf{g}(t, \mathbf{z})$ *have unique solutions,* $f^k(t, \mathbf{y})$ *and* $g^k(t, \mathbf{y})$ *are increasing with respect to* $y^i, i \neq k$ *and* $k = 1, 2, ..., d$*, and* $y_0^j < z_0^j$ *for at least one index* j*, then* $y^k(t) < z^k(t)$ *for* $a < t \leq b$ *and* $k = 1, ..., d$.

(c) *If, in addition to the assumption of* (a)*, there is an index* h *such that either* $g^h(t, \mathbf{y})$ *or* $f^h(t, \mathbf{y})$ *is nondecreasing with respect to* y^h*, then* $z^h(t) - y^h(t)$ *is nondecreasing on* $a \leq t \leq b$.

Finally, in this section, we show that $W(\sigma)$ is strictly increasing in $\sigma \in (0, 1)$ as the following lemma.

Lemma 3.4. *The following holds:*

$$\frac{dW}{d\sigma}(\sigma) > 0 \quad \text{for } \sigma \in (0, 1).$$

Proof. Since

$$\frac{d}{d\sigma}\left(\tan^{-1}\frac{1+\sigma}{\sqrt{1-\sigma^2}}\right) = \frac{1}{2\sqrt{1-\sigma^2}},$$

we have

$$\frac{dW}{d\sigma} = \frac{\pi(1-\sigma^2)^{3/2} + 4(2\sigma^2-1)\tan^{-1}\dfrac{1+\sigma}{\sqrt{1-\sigma^2}} + 2\sigma\sqrt{1-\sigma^2}}{2\sigma^2(1-\sigma^2)^{3/2}},$$

$$\frac{d^2W}{d\sigma^2} = \frac{-[2\sigma - 5\sigma^3 + \pi(1-\sigma^2)^2]\sqrt{1-\sigma^2} + 2(2-5\sigma^2+6\sigma^4)\tan^{-1}\dfrac{1+\sigma}{\sqrt{1-\sigma^2}}}{\sigma^3(1-\sigma^2)^{5/2}}.$$

Note that

$$2 - 5\sigma^2 + 6\sigma^4 \geq -5\sigma^2 + 2\sqrt{2\cdot 6\sigma^4} = (\sqrt{48} - 5)\sigma^2 > 0.$$

Set

$$\xi(\sigma) := -\frac{[2\sigma - 5\sigma^3 + \pi(1-\sigma^2)^2]\sqrt{1-\sigma^2}}{(2-5\sigma^2+6\sigma^4)} + 2\tan^{-1}\frac{1+\sigma}{\sqrt{1-\sigma^2}}.$$

Then we have

$$\frac{d\xi}{d\sigma}(\sigma) = \frac{\sigma^2(1-\sigma^2)^{3/2}(8 + 9\pi\sigma + 36\sigma^2 + 6\pi\sigma^3)}{(2-5\sigma^2+6\sigma^4)^2} > 0$$

for $\sigma > 0$. It follows from $\xi(0) = 0$ that $\xi(\sigma) > 0$ for $0 < \sigma < 1$. Hence

$$\frac{d^2W}{d\sigma^2}(\sigma) > 0 \quad \text{for } 0 < \sigma < 1.$$

Also, the l'Hôpital rule implies that

$$\lim_{\sigma\to 0^+}\frac{dW}{d\sigma}(\sigma) = \frac{\pi}{4}.$$

Thus the lemma follows. ∎

4. Uniqueness

This section is devoted to the proof of uniqueness. We first introduce the following definition.

Definition 1. *Let (x, y, θ) be the solution of $(P_{\sigma,b})$ defined on $[0, S_{\sigma,b})$. Let $\hat{s}_\pm = \hat{s}_\pm(\sigma, b)$ be the smallest positive s such that $\theta(s) = \pm\pi/2$, if it exists. Otherwise, set $\hat{s}_\pm = S_{\sigma,b}$. We also set $\hat{\theta}_\pm = \hat{\theta}_\pm(\sigma, b) := \theta(\hat{s}_\pm; \sigma, b)$, $\hat{y}_\pm = \hat{y}_\pm(\sigma, b) := y(\hat{s}_\pm; \sigma, b)$, $\hat{z}_\pm = \hat{z}_\pm(\sigma, b) := W(\sigma) - \hat{y}_\pm$, $\hat{s} = \min\{\hat{s}_+, \hat{s}_-\}$, $\hat{\theta} = \hat{\theta}(\sigma, b) := \theta(\hat{s}; \sigma, b)$, $\hat{y} = \hat{y}(\sigma, b) := y(\hat{s}; \sigma, b)$ and $\hat{z} = \hat{z}(\sigma, b) := W(\sigma) - \hat{y}$.*

Let (x, y, θ) be the solution of $(P_{\sigma,b})$. Since $y' < 0$ on $(0, \hat{s})$, we can express x and θ as functions of $z := W(\sigma) - y$. Let $\mathbf{X}(z) = x(s(z))$ and $\Theta(z) = \theta(s(z))$ for $z \in (0, W(\sigma) - \hat{y})$. Also set $g(z) = f_\sigma(W(\sigma) - z)$. Note that g depends on σ. Then it follows from (1.2)-(1.4) that $(\mathbf{X}, \Theta)$ satisfies the following problem $(R_{\sigma,b})$:

$$\mathbf{X}' := \frac{d\mathbf{X}}{dz} = \tan\Theta, \tag{4.1}$$

$$\Theta' := \frac{d\Theta}{dz} = \frac{1 + \sigma\cos\Theta - b(g(z) - \mathbf{X})}{\cos\Theta} \tag{4.2}$$

for $z \in (0, W(\sigma) - \hat{y})$ with the initial conditions

$$\mathbf{X}(0) = 0, \quad \Theta(0) = -\pi/2. \tag{4.3}$$

Lemma 4.1. *Let $(\mathbf{X}_i(z), \Theta_i(z)) = (\mathbf{X}(z; \sigma, b_i), \Theta(z; \sigma, b_i))$ be the solution of (R_{σ,b_i}) defined on $[0, \hat{z}(\sigma, b_i))$, $i = 1, 2$. If $0 \le b_1 < b_2$ and $\mathbf{X}_i(z) < g(z)$ for each $z \in (0, \hat{z}(\sigma, b_i))$, $i = 1, 2$, then $\mathbf{X}_2 < \mathbf{X}_1$ and $\Theta_2 < \Theta_1$ on $(0, \min\{\hat{z}(\sigma, b_1), \hat{z}(\sigma, b_2)\})$.*

Proof. Since $\cos\Theta(0; \sigma, b_i) = 0$, $i = 1, 2$, we cannot apply the comparison principle near $z = 0$. However, Lemma 3.1 guarantees that $\Theta_2 < \Theta_1$ and $\mathbf{X}_2 < \mathbf{X}_1$ when z is sufficiently small.

Note that the function of the right hand side of (4.1) is an increasing function of Θ for $\Theta \in (-\pi/2, \pi/2)$. Also, the function of the right hand side of (4.2) is an increasing and decreasing function of $\mathbf{X}$ and b respectively, as long as $b \ge 0$ and $\Theta \in (-\pi/2, \pi/2)$. By Definition 1, we always have $-\pi/2 < \Theta_i(z) < \pi/2$ for each $z \in (0, \hat{z}(\sigma, b_i))$, $i = 1, 2$. Therefore, by Lemma 3.3, we have

$$\mathbf{X}_2 < \mathbf{X}_1 \quad \text{and} \quad \Theta_2 < \Theta_1$$

for all $z \in (0, \min\{\hat{z}(\sigma, b_1), \hat{z}(\sigma, b_2)\})$. □

Similarly, we have the following lemma.

Lemma 4.2. *Let $(\mathbf{X}_i(z), \Theta_i(z)) = (\mathbf{X}(z; \sigma_i, b), \Theta(z; \sigma_i, b))$ be the solution of $(R_{\sigma_i,b})$ defined on $[0, \hat{z}(\sigma_i, b))$, $i = 1, 2$. If $0 \le \sigma_1 < \sigma_2$ and $\mathbf{X}_i(z) < g(z)$ for each $z \in (0, \hat{z}(\sigma_i, b))$, $i = 1, 2$, then $\mathbf{X}_1 < \mathbf{X}_2$ and $\Theta_1 < \Theta_2$ on $(0, \min\{\hat{z}(\sigma_1, b), \hat{z}(\sigma_2, b)\})$.*

Proof. Lemma 3.1 guarantees that $\Theta_1 < \Theta_2$ and $\mathbf{X}_1 < \mathbf{X}_2$ when z is sufficiently small. Note that the function of the right hand side of (4.1) is an increasing function of Θ for $\Theta \in (-\pi/2, \pi/2)$. Since the function of the right hand side of (4.2) is an increasing function of $\mathbf{X}$ and σ as long as $b \geq 0$ and $\Theta \in (-\pi/2, \pi/2)$, Lemma 3.3 shows $\mathbf{X}_1 < \mathbf{X}_2$ and $\Theta_1 < \Theta_2$ for all $z \in (0, \min\{\hat{z}(\sigma_1, b), \hat{z}(\sigma_2, b)\})$. □

The following corollary follows easily from Lemma 4.1.

Corollary 4.3. *For each fixed $\sigma \in (0,1)$, the (b^*, s^*) in Theorem 1 is unique.*

Similarly, we define the following.

Definition 2. *Let (x, y, θ) be the solution of $(P_{\sigma,b})$ defined on $[0, S_{\sigma,b})$. Let $\bar{s} = \bar{s}(\sigma, b)$ be the first zero of θ' if it exists. Otherwise, set $\bar{s} = S_{\sigma,b}$. We also set $\bar{\theta} = \bar{\theta}(\sigma, b) := \theta(\bar{s}; \sigma, b)$, $\bar{y} = \bar{y}(\sigma, b) := y(\bar{s}; \sigma, b)$ and $\bar{z} = \bar{z}(\sigma, b) := W(\sigma) - \bar{y}$.*

Let (x, y, θ) be the solution of $(P_{\sigma,b})$. Since $\theta' > 0$ on $[0, \bar{s})$, we can express x and y as functions of θ for $\theta \in [-\pi/2, \bar{\theta})$. Let $X(\theta) = x(s(\theta))$ and $Y(\theta) = y(s(\theta))$ for $\theta \in [-\pi/2, \bar{\theta})$. Then X and Y satisfy the following problem $(Q_{\sigma,b})$:

$$X' := \frac{dX}{d\theta} = \frac{\sin\theta}{1 + \sigma\cos\theta - b(f_\sigma(Y) - X)}, \tag{4.4}$$

$$Y' := \frac{dY}{d\theta} = -\frac{\cos\theta}{1 + \sigma\cos\theta - b(f_\sigma(Y) - X)}, \tag{4.5}$$

for $\theta \in [-\pi/2, \bar{\theta})$ with the initial conditions

$$X(-\pi/2) = 0, \quad Y(-\pi/2) = W(\sigma).$$

Note that $X'(\theta) < 0$ for $\theta \in [-\pi/2, \min\{\bar{\theta}, 0\})$ and $X(\theta) < 0$ for $\theta \in (-\pi/2, \min\{\bar{\theta}, 0\}]$.

From Lemma 3.3, we also have the following lemma.

Lemma 4.4. *Let (X_i, Y_i) be the solution of (Q_{σ,b_i}) defined on $[-\pi/2, \bar{\theta}(\sigma, b_i))$, $i = 1, 2$. If $0 < b_1 < b_2$, then we have $X_2 < X_1 < 0$, $Y_2 < Y_1$, $X_2' < X_1'$ and $Y_2' < Y_1'$ on $(-\pi/2, \min\{\bar{\theta}(\sigma, b_1), \bar{\theta}(\sigma, b_2), 0\})$. Moreover, $X_1 - X_2$ and $Y_1 - Y_2$ are nondecreasing on $(-\pi/2, \min\{\bar{\theta}(\sigma, b_1), \bar{\theta}(\sigma, b_2), 0\}]$.*

Proof. Note that the function of the right hand side of (4.4) is an increasing function of X, Y and decreasing function of b, respectively, as long as $b > 0$, $\theta \in (-\pi/2, 0)$, and $1 + \sigma\cos\theta - b(f_\sigma(Y) - X) > 0$. The same conclusion holds for the function of the right hand side of (4.5). Then by applying Lemma 3.3 to (Q_{σ,b_i}), $i = 1, 2$, we reach the desired conclusion. □

5. Existence

In this section, we shall prove Theorem 1 by analyzing the behavior of $y(s;\sigma,b)$ and $\theta(s;\sigma,b)$ for $(\sigma,b)\in(0,1)\times[0,\infty)$.

First we consider the case when $b=0$.

Lemma 5.1. *Assume $b=0$. Let $(\mathbf{X}(z;\sigma,0),\Theta(z;\sigma,0))$ be the solution of the problem $(R_{\sigma,0})$. Then there exists $\sigma^*\in(0,1)$ such that*

(i) *for $0<\sigma<\sigma^*$, $\hat{z}(\sigma,0)=W(\sigma)$, $-\pi/2<\Theta(z;\sigma,0)<\pi/2$ for $0<z\leq W(\sigma)$ and $\Theta(W(\sigma)^-;\sigma,0):=\lim\limits_{z\to W(\sigma)^-}\Theta(z;\sigma,0)\in(0,\pi/2)$;*
(ii) *for $\sigma=\sigma^*$, $\hat{z}(\sigma,0)=W(\sigma)$, $-\pi/2<\Theta(z;\sigma,0)<\pi/2$ for $0<z<W(\sigma)$ and $\Theta(W(\sigma)^-;\sigma,0)=\pi/2$;*
(iii) *for $\sigma^*<\sigma<1$, $0<\hat{z}(\sigma,0)<W(\sigma)$, $-\pi/2<\Theta(z;\sigma,0)<\pi/2$ for $0<z<\hat{z}(\sigma,0)=\hat{z}_+(\sigma,0)$ and $\Theta(\hat{z}(\sigma,0)^-;\sigma,0)=\pi/2$.*

Proof. First, we note that, by the definition of $S_{\sigma,b}$, $\hat{z}(\sigma,0)\leq W(\sigma)$ for any $\sigma\in(0,1)$. Let $b=0$. By (4.2), we have

$$z=\int_{-\pi/2}^{\Theta(z;\sigma,0)}\frac{\cos\Theta}{1+\sigma\cos\Theta}d\Theta.$$

The function

$$H(\Theta,\sigma):=W(\sigma)-\int_{-\pi/2}^{\Theta}\frac{\cos\theta}{1+\sigma\cos\theta}d\theta$$

is decreasing in $\Theta\in(-\pi/2,\pi/2]$ and increasing in $\sigma\in[0,1)$. We also have

$$H(\Theta,0^+)=-\sin\Theta,\quad H(\Theta,1^-)=\infty$$

for $\Theta\in(-\pi/2,\pi/2]$. Let σ^* be the unique root of

$$H(\pi/2,\sigma)=0,\quad \sigma\in(0,1).$$

Then $H(\pi/2,\sigma)<0$ for $\sigma\in(0,\sigma^*)$ and $H(\pi/2,\sigma)>0$ for $\sigma\in(\sigma^*,1)$. The lemma follows. □

To derive the information of $\bar{\theta}$, we consider the orbit by the function of θ. Namely, let $(X(\theta;\sigma,0),Y(\theta;\sigma,0))$ be a solution of $(Q_{\sigma,0})$ with $\sigma\in(0,1)$. We claim that $\bar{\theta}(\sigma,0)>0$. Indeed, proceeding as in the proof of Lemma 5.1, it is sufficient to prove that the function

$$K(\sigma):=W(\sigma)-\int_{-\pi/2}^{0}\frac{\cos\theta}{1+\sigma\cos\theta}d\theta$$

is positive for $\sigma\in(0,1)$. As before, we can show that K is increasing in $\sigma\in(0,1)$, $K(0^+)=0$ and $K(1^-)=+\infty$. Hence we have $\bar{\theta}(\sigma,0)>0$ for each $\sigma\in(0,1)$. Therefore, $(X(\theta;\sigma,0),Y(\theta;\sigma,0))$ is defined at least on $[-\pi/2,0]$.

Clearly, $Y'(\theta;\sigma,0) < 0$ on $(-\pi/2,\pi/2)$. Integrating (4.5) over $[-\pi/2,0]$, we obtain

$$Y(0;\sigma,0) = -\frac{\pi}{\sigma} + \frac{2}{\sigma\sqrt{1-\sigma^2}}\Big[\tan^{-1}\Big(\frac{1+\sigma}{\sqrt{1-\sigma^2}}\Big) + \tan^{-1}\Big(\frac{1-\sigma}{\sqrt{1-\sigma^2}}\Big)\Big].$$

We see that $\lim_{\sigma\to 0^+} Y(0;\sigma,0) = 0$ and $Y(0;\sigma,0)$ is an increasing function in σ. Thus we have that $0 < Y(0;\sigma,0)$ for all $\sigma \in (0,1)$. Moreover, by a similar argument as the proof of Lemma 3.4, one can show that $Y(0;\sigma,0) < W(\sigma)$, and hence

$$0 < Y(0;\sigma,0) < W(\sigma) \ \text{ for all } \ \sigma \in (0,1).$$

Next we consider the case when b is large.

Lemma 5.2. *Assume that $\sigma \in (0,1)$ and $\eta \in (-\pi/2,0]$. If $b \geq (1+\sigma)/f_\sigma(Y(\eta;\sigma,0))$, then*

$$-\frac{\pi}{2} < \bar\theta(\sigma,b) \leq \eta.$$

In particular, the following equalities hold:

$$\lim_{b\to+\infty} \bar\theta(\sigma,b) = -\frac{\pi}{2}, \quad \lim_{b\to+\infty} \bar y(\sigma,b) = W(\sigma). \tag{5.1}$$

Proof. We show this lemma by a contradiction argument. Assume that $\bar\theta(\sigma,b) > \eta$ for some $b \geq (1+\sigma)/f_\sigma(Y(\eta;\sigma,0))$. Then the solution (x,y,θ) of $(P_{\sigma,b})$ satisfies that $\theta' > 0$ for $0 \leq s < \bar s$. Since $\eta \in (-\pi/2,0]$, there is a constant $s_1 \in (0,\bar s)$ such that $\theta' > 0$ on $[0,s_1]$, $x(s_1;\sigma,b) < 0$, $y(s_1;\sigma,b) > 0$ and $\theta(s_1;\sigma,b) = \eta$. It follows from Lemma 4.4 that

$$Y(\eta;\sigma,0) > y(s_1;\sigma,b).$$

Using (1.4) and noting that f is decreasing, we have

$$\begin{aligned}
\theta'(s_1;\sigma,b) &= 1+\sigma\cos\eta - b[f_\sigma(y(s_1;\sigma,b)) - x(s_1;\sigma,b)] \\
&< 1+\sigma - bf_\sigma(y(s_1;\sigma,b)) \\
&< 1+\sigma - bf_\sigma(Y(\eta;\sigma,0)) \leq 0,
\end{aligned}$$

which is a contradiction. Hence we have $\bar\theta(\sigma,b) \in (-\pi/2,\eta]$. Furthermore, since

$$\lim_{\eta\to-\pi/2} Y(\eta;\sigma,0) = W(\sigma),$$

we have

$$\lim_{\eta\to-\pi/2} \frac{1+\sigma}{f_\sigma(Y(\eta;\sigma,0))} = \infty,$$

which gives the first equality in (5.1) and then the second one. This completes the proof. □

Using this lemma, we can derive the following behavior of the solution $(\mathbf{X},\Theta)$ of $(R_{\sigma,b})$ for large b.

Lemma 5.3. *Let $(\mathbf{X}(z;\sigma,b),\Theta(z;\sigma,b))$ be the solution of the problem $(R_{\sigma,b})$. Then there exists a positive number $b^\#$ such that the following statements hold for $b \geq b^\#$:*

(i) *$0 < \bar z < \hat z < W(\sigma)$ and $\Theta' > 0$ on $(0,\bar z)$ and $\Theta' < 0$ on $(\bar z,\hat z)$, where $\bar z = \bar z(\sigma,b)$, $\hat z = \hat z(\sigma,b)$ and $\Theta = \Theta(z;\sigma,b)$.*

(ii) $\Theta(z;\sigma,b)\in(-\pi/2,0)$ *for all* $z\in(0,\hat{z})$ *and* $\Theta(\hat{z}^-;\sigma,b)=-\pi/2$.

Proof. Lemma 5.2 assures us that there is a sufficiently large constant b_1 such that $\bar{\theta}(\sigma,b)\in(-\pi/2,-\pi/4)$ and $\bar{y}(\sigma,b)\in(7W(\sigma)/8,W(\sigma))$ for each $b>b_1$. Recall that $\bar{z}(\sigma,b)=W(\sigma)-\bar{y}(\sigma,b)$. Then, for $b>b_1$, we have $\bar{z}<W(\sigma)/8$ and $\Theta'>0$ on $(0,\bar{z})$. At $z=\bar{z}$, we have $\Theta'=0$, $\Theta=\bar{\theta}\in(-\pi/2,-\pi/4)$, and so

$$\frac{d^2\Theta}{dz^2}\bigg|_{\Theta'=0}=\frac{-b(g'(z)-\tan\Theta)}{\cos\Theta}<0. \tag{5.2}$$

Thus we see that Θ' becomes negative just after $z=\bar{z}$. The equation (5.2) also implies that $\Theta'<0$ as long as $\Theta(z;\sigma,b)\in(-\pi/2,0)$.

Now set

$$b^{\#}:=\max\left\{b_1,\ \frac{\dfrac{\sqrt{2}\pi}{W(\sigma)}+(1+\sigma)}{\min\limits_{z\in[W(\sigma)/4,W(\sigma)/2]}g(z)}\right\}.$$

Fix a $b\geq b^{\#}$. We claim that $\hat{z}\in(\bar{z},W(\sigma))$ such that $\Theta'<0$ on $(\bar{z},\hat{z})$ and $\Theta(\hat{z}^-;\sigma,b)=-\pi/2$. Suppose not. Then Θ is defined on $[0,W(\sigma))$, $\Theta'<0$ on $(\bar{z},W(\sigma))$ and $\Theta(W(\sigma)^-)\in[-\pi/2,-\pi/4)$. Since $\mathbf{X}'<0$ for $\Theta\in[-\pi/2,-\pi/4)$, $\mathbf{X}<0$. Thus we have

$$\frac{d\Theta}{dz}\ =\ \frac{1+\sigma\cos\Theta-b(g(z)-\mathbf{X})}{\cos\Theta}<\frac{1+\sigma-bg(z)}{\cos\Theta}$$

for $z\in[W(\sigma)/4,W(\sigma)/2]$. By the choice of b, for $z\in[W(\sigma)/4,W(\sigma)/2]$, we obtain

$$\frac{d\Theta}{dz}\ <\ -\frac{\sqrt{2}\pi}{W(\sigma)}\frac{1}{\cos\Theta}\leq-\frac{2\pi}{W(\sigma)}.$$

Integrating this inequality over $[W(\sigma)/4,W(\sigma)/2]$ yields that

$$\Theta(W(\sigma)/2)-\Theta(W(\sigma)/4)<-\frac{\pi}{2}.$$

This contradicts that $\Theta(W(\sigma)^-)\in[-\pi/2,-\pi/4)$, and hence the claim is established. The proof is thus completed. $\square$

To prove the existence part of Theorem 1, it is equivalent to find solutions (x,y,θ) of $(P_{\sigma,b})$ satisfying

$$\begin{cases} -\dfrac{\pi}{2}<\theta(s;\sigma,b)<\dfrac{\pi}{2}, \quad y(s;\sigma,b)>0 \ \text{ for } \ 0<s<s^*, \\ f_\sigma(y(s;\sigma,b))-x(s;\sigma,b)>0 \ \text{ for } \ 0<s\leq s^*, \\ (\theta(s^*;\sigma,b),y(s^*;\sigma,b))=(0,0) \end{cases} \tag{5.3}$$

with some $b=b^*(\sigma)>0$ and $s^*=s^*(\sigma)>0$ for any $\sigma\in(0,1)$. For this, we first prove the following lemma.

Lemma 5.4. *For each $b>0$, we have that*
$$-\frac{\pi}{2} < \Theta(z;\sigma,b) < \frac{\pi}{2}, \qquad g(z) > \mathbf{X}(z;\sigma,b)$$
for $0<z<\hat{z}(\sigma,b)$.

Proof. The assertion
$$-\frac{\pi}{2} < \Theta(z;\sigma,b) < \frac{\pi}{2}$$
for $0<z<\hat{z}(\sigma,b)$ follows from the definition of $\hat{z}$.

For the second assertion, we first note that, by Lemma 5.3, there exists a sufficiently large b_0 such that $\mathbf{X}(z;\sigma,b)<0$ for all $z\in(0,\hat{z}]$ and $b\ge b_0$. Hence $g(z)>\mathbf{X}(z;\sigma,b)$ for all $z\in(0,\hat{z}]$ and $b\ge b_0$.

Next, recall from (1.1) that
$$\begin{cases} \dfrac{d\tilde{x}}{d\tilde{y}} &= -\tan\tilde{\theta}, \\[2mm] \dfrac{d\tilde{\theta}}{d\tilde{y}} &= \dfrac{1-\sigma\cos\tilde{\theta}}{\cos\tilde{\theta}} \end{cases}$$
and $\tilde{\theta}\in[0,\pi/2)$. This orbit corresponds to $f_\sigma(\tilde{y})=\tilde{x}$. Similarly the back is represented by the solution $(x(y),\theta(y))$ of
$$\begin{cases} \dfrac{dx}{dy} &= -\tan\theta, \\[2mm] \dfrac{d\theta}{dy} &= -\dfrac{1+\sigma\cos\theta-b(f_\sigma(y)-x)}{\cos\theta} \end{cases}$$
with $\theta\in(-\pi/2,\pi/2)$.

Now we show that $g(z)>\mathbf{X}(z;\sigma,b)$ for all $z\in(0,\hat{z})$ for all $b>0$, by a contradiction argument. Indeed, by the choice of b_0, we assume that there is a largest $b<b_0$ and $y_2\in[0,W(\sigma))$ such that
$$\begin{aligned} &\tilde{x}(y)>x(y), \quad \tilde{\theta}(y)<\pi/2, \quad -\pi/2<\theta(y) \quad \text{for } 0<y<y_2, \\ &\tilde{\theta}(y_2)=\theta(y_2), \quad \tilde{x}(y_2)=x(y_2), \\ &\frac{d}{dy}(\tilde{x}-x)|_{y=y_2}=0, \quad \frac{d^2}{dy^2}(\tilde{x}-x)|_{y=y_2}\ge 0. \end{aligned}$$
We have
$$\begin{aligned} \frac{d}{dy}(\tilde{x}-x) &= \tan\theta-\tan\tilde{\theta}, \\ \frac{d^2}{dy^2}(\tilde{x}-x) &= \frac{\theta_y}{\cos^2\theta}-\frac{\tilde{\theta}_y}{\cos^2\tilde{\theta}} \\ &= -\frac{1+\sigma\cos\theta-b(f_\sigma(y)-x)}{\cos^3\theta}-\frac{1-\sigma\cos\tilde{\theta}}{\cos^3\tilde{\theta}}. \end{aligned}$$

This implies

$$\frac{d^2}{dy^2}(\tilde{x}-x)|_{y=y_2} = -\frac{2}{\cos^3\theta} < 0.$$

This contradicts the choice of y_2. Hence the proof is completed. $\square$

Finally, we prove the following lemma, which completes the proof of Theorem 1.

Lemma 5.5. *For each $\sigma \in (0,1)$, there is a positive constant b^* such that*

$$\hat{z}(\sigma, b^*) = W(\sigma), \quad \Theta(W(\sigma)^-; \sigma, b^*) = 0.$$

Proof. First, we claim that there exists a $b_1 > 0$ such that

$$\hat{z}(\sigma, b_1) = \min\{\hat{z}_+(\sigma, b_1), \hat{z}_-(\sigma, b_1)\} = W(\sigma).$$

Indeed, the assertion holds for $\sigma \in (0, \sigma^*]$ by Lemma 5.1 and the continuous dependence on the parameter b. Therefore, we focus on the case $\sigma \in (\sigma^*, 1)$. For contradiction, we assume that $\hat{z}(\sigma, b) < W(\sigma)$ for any $b > 0$. It follows from Lemma 5.1 that $\hat{z}_+(\sigma, b) < W(\sigma)$ for small b. By Lemma 5.3 (ii), $\hat{z}_-(\sigma, b) < \hat{z}_+(\sigma, b)$ for sufficiently large b. Hence we can take the smallest $b = b_*$ such that $\hat{z}_-(\sigma, b_*) \le \hat{z}_+(\sigma, b_*)$. Actually, we have $\hat{z}_-(\sigma, b_*) < \hat{z}_+(\sigma, b_*)$. Then

$$\Theta(\hat{z}_-(\sigma, b_*); \sigma, b_*) = -\frac{\pi}{2}, \quad -\frac{\pi}{2} < \Theta(z; \sigma, b_*) < \frac{\pi}{2}$$

for $z \in (0, \hat{z}_-(\sigma, b_*)) \cup (\hat{z}_-(\sigma, b_*), \hat{z}_+(\sigma, b_*))$, which implies that there is s_1 such that

$$\theta(s_1; \sigma, b_*) = -\frac{\pi}{2}, \quad \theta'(s_1; \sigma, b_*) = 0, \quad \theta''(s_1; \sigma, b_*) \ge 0.$$

This contradicts Lemma 3.2 (i). Therefore, we conclude that there exists a $b_1 > 0$ such that $\hat{z}(\sigma, b_1) = W(\sigma)$.

Next, we define the set

$$\mathcal{A}(\sigma) := \Big\{ b > 0 \ \Big| \ \Theta(z; \sigma, b) \text{ is defined on } [0, W(\sigma)),\ \Theta(z; \sigma, b) \in (-\pi/2, \pi/2) \\ \text{for all } z \in (0, W(\sigma)), \text{ and } \Theta(W(\sigma)^-; \sigma, b) \in (-\pi/2, \pi/2) \Big\}.$$

We shall show that the set $\mathcal{A}(\sigma)$ is nonempty. If $\Theta(W(\sigma)^-; \sigma, b_1) \in (-\pi/2, \pi/2)$, then $b_1 \in \mathcal{A}$ and we are done.

Suppose that $\Theta(W(\sigma)^-; \sigma, b_1) = \pi/2$. Let $(x(s; \sigma, b_1), y(s; \sigma, b_1), \theta(s; \sigma, b_1))$ be the solution of the problem (P_{σ, b_1}). We claim that $\theta'(\hat{s}^-; \sigma, b_1) > 0$, where $\hat{s} = \hat{s}(\sigma, b_1)$. We first note that $\theta'(\hat{s}^-; \sigma, b_1) \ge 0$ by the definition of b_1. Thus, for contradiction, we assume that $\theta'(\hat{s}^-; \sigma, b_1) = 0$. By differentiating (1.4), we can compute that $\theta''(\hat{s}^-; \sigma, b_1) = b_1 > 0$. This is a contradiction, and so establishing the assertion of this claim.

By the continuous dependence on the parameter b and Lemma 4.1, we can choose a sufficiently small positive number $\delta > 0$ and $z_1 \in (0, W(\sigma))$ such that the solution $(\mathbf{X}(z;\sigma,b),\Theta(z;\sigma,b))$ of $(R_{\sigma,b})$ satisfies

$$\begin{aligned}&\mathbf{X}(z;\sigma,b)<\mathbf{X}(z;\sigma,b_1),\quad \Theta(z;\sigma,b)<\Theta(z;\sigma,b_1),\\ &\Theta(z;\sigma,b)\in(-\pi/2,\pi/2)\quad\text{for all } z\in[0,z_1],\\ &\Theta(z_1;\sigma,b)\in(0,\pi/2),\quad \Theta'(z_1;\sigma,b)>0\end{aligned}$$

for each $b\in(b_1,b_1+\delta)$. Recall that $g'(W(\sigma))=f'_\sigma(0)=0$ and $\tan\theta\to+\infty$ as $\theta\to(\pi/2)^-$. Therefore, we can choose δ small enough and z_1 sufficiently close to $W(\sigma)$ such that $\tan\theta-g'(z)>0$ for all $(z,\theta)\in[z_1,W(\sigma))\times[\Theta(z_1;\sigma,b),\pi/2)$ for any $b\in(b_1,b_1+\delta)$.

Fix a $b\in(b_1,b_1+\delta)$. We claim that $\hat{z}(\sigma,b)=W(\sigma)$ and $\Theta(W(\sigma)^-;\sigma,b)\in(0,\pi/2)$. We first show that $\hat{z}(\sigma,b)=W(\sigma)$. Suppose not. Then from Lemmas 4.1 and 5.4 it follows that $\Theta(z;\sigma,b)<\Theta(z;\sigma,b_1)$ for all $z\in[0,\hat{z}(\sigma,b))$. Hence there exists a $z_0\in(z_1,\hat{z}(\sigma,b))$ such that

$$\Theta(z_0;\sigma,b)\in(\Theta(z_1;\sigma,b),\pi/2),\ \Theta'(z;\sigma,b)>0 \text{ on } [z_1,z_0) \text{ and } \Theta'(z_0;\sigma,b)=0.$$

By the choice of z_1, we have

$$\tan\Theta(z_0;\sigma,b)-g'(z_0)>0.$$

On the other hand, by using (4.2), we can compute

$$\Theta''(z_0;\sigma,b)=b\cdot\frac{\tan\Theta(z_0;\sigma,b)-g'(z_0)}{\cos\Theta(z_0;\sigma,b)}>0,$$

a contradiction. Therefore, we have $\hat{z}(\sigma,b)=W(\sigma)$.

Moreover, from the above proof it yields that $\Theta'(z;\sigma,b)>0$ for all $z\in[z_1,W(\sigma))$, which implies $\Theta(W(\sigma)^-;\sigma,b)\in(0,\pi/2]$. Since $\theta'(\hat{s}^-;\sigma,b_1)>0$, by (1.4) and (4.2), we have $1-b_1(g(W(\sigma))-\mathbf{X}(W(\sigma^-);\sigma,b_1))>0$. Hence by choosing a smaller δ, we may assume that $1-b(g(z)-\mathbf{X}(z;\sigma,b))>0$ for all $(z,b)\in[z_1,W(\sigma))\times[b_1,b_1+\delta)$. By Lemma 4.1, we have that $\mathbf{X}(z;\sigma,b)<\mathbf{X}(z;\sigma,b_1)$, $\Theta(z;\sigma,b)<\Theta(z;\sigma,b_1)$ for all $z\in[z_1,W(\sigma))$. Together with the choice of δ, we have

$$\begin{aligned}\Theta'(z;\sigma,b_1)&=\sigma+\frac{1-b_1\Big(g(z)-\mathbf{X}(z;\sigma,b_1)\Big)}{\cos\Theta(z;\sigma,b_1)}\\ &>\sigma+\frac{1-b\Big(g(z)-\mathbf{X}(z;\sigma,b)\Big)}{\cos\Theta(z;\sigma,b)}\\ &=\Theta'(z;\sigma,b).\end{aligned}$$

for all $(z,b)\in[z_1,W(\sigma))\times(b_1,b_1+\delta)$, where we have used the fact that $g(z)-\mathbf{X}(z;\sigma,b)>0$ for all $(z,b)\in[z_1,W(\sigma))\times[b_1,b_1+\delta)$ (see Lemma 5.4). Therefore, we have $\Theta(W(\sigma)^-;\sigma,b)\in(0,\pi/2)$ for all $b\in(b_1,b_1+\delta)$.

The case when $\Theta(W(\sigma)^-;\sigma,b_1)=-\pi/2$ can be treated similarly. This establishes the nonempty of the set $\mathcal{A}(\sigma)$.

For each $b \in \mathcal{A}(\sigma)$, we see that the curve $\{(z, \Theta(z; \sigma, b))\}$ intersects the line $\{z = W(\sigma)\}$ transversely (i.e., $\Theta'(W(\sigma)^-; \sigma, b) \neq \pm\infty$). Hence, with the use of the continuous dependence on the parameter b, we can conclude that $\mathcal{A}(\sigma)$ is open.

Now we choose a nonempty open interval $\hat{\mathcal{A}}(\sigma)$ contained in $\mathcal{A}(\sigma)$. Consider the following two quantities:

$$b_- = b_-(\sigma) := \inf \hat{\mathcal{A}}(\sigma), \quad b_+ = b_+(\sigma) := \sup \hat{\mathcal{A}}(\sigma).$$

Since $\hat{\mathcal{A}}(\sigma)$ is nonempty, b_- is well-defined. From Lemma 5.3, we have that b_+ is also well-defined. Since $\hat{\mathcal{A}}(\sigma)$ is open, the function $\Theta(\cdot\ ; \sigma, b_+)$ and $\Theta(\cdot\ ; \sigma, b_-)$ are defined on $[0, W(\sigma))$ such that $\Theta(W(\sigma)^-; \sigma, b_+) = -\pi/2$ and $\Theta(W(\sigma)^-; \sigma, b_-) = \pi/2$, if $b_- > 0$; $\Theta(W(\sigma)^-; \sigma, b_-) \in (0, \pi/2]$, if $b_- = 0$ (by Lemma 5.1). Furthermore, we have $\Theta'(W(\sigma)^-; \sigma, b_+) = -\infty$ and $\Theta'(W(\sigma)^-; \sigma, b_-) = +\infty$, if $\Theta(W(\sigma)^-; \sigma, b_-) = \pi/2$. Therefore, together with the theory of continuous dependence on the parameter b, we can conclude that the function

$$G : \hat{\mathcal{A}}(\sigma) \longmapsto (-\frac{\pi}{2}, \Theta(W(\sigma)^-; \sigma, b_-))$$

given by $G(b) = \Theta(W(\sigma)^-; \sigma, b)$ is continuous and onto. Hence there is a $b^* \in \hat{\mathcal{A}}(\sigma) \subseteq \mathcal{A}(\sigma)$ such that $\Theta(W(\sigma)^-; \sigma, b^*) = 0$. This completes the proof of this lemma, thereby completing the proof of Theorem 1. □

6. Profiles of wave segments

6.1. **Classification of the wave profile.** Although we have shown the existence and uniqueness of wave segments in the previous sections, we do not have too much information about the profile of a wave segment. Let (x, y, θ) be a wave segment defined on $[0, s^*]$. We classify the wave segments into the following two types:

(I) Convex type : $\theta' > 0$ on $[0, s^*)$.
(II) Non-convex type : θ' can change its sign in $(0, s^*)$.

6.2. **Existence of convex wave segments.** First, we show the continuity of $b^*(\sigma)$, where the function $b^*(\sigma)$ is obtained in Theorem 1.

Lemma 6.1. *The function $b^*(\sigma)$ is continuous in $\sigma \in (0, 1)$.*

Proof. Suppose that b^* is not continuous at some σ_0. Then there exist a positive constant ε and a sequence $\{\sigma_j\}$ such that

$$\lim_{j\to\infty} \sigma_j = \sigma_0, \quad |b^*(\sigma_0) - b^*(\sigma_j)| > \varepsilon.$$

By Lemma 5.2, $b^*(\sigma)$ is bounded for σ close to σ_0. Hence we can take a subsequence $\{\sigma_{j_n}\}$ of $\{\sigma_j\}$ such that $b^*_{j_n} := b^*(\sigma_{j_n})$ converges to $b^{**} (\neq b^*(\sigma_0))$ as $n \to \infty$. The facts that $(\mathbf{X}(z; \sigma, b), \Theta(z; \sigma, b))$ is continuously depending on z, σ, b and that

$$\Theta(W(\sigma_{j_n}); \sigma_{j_n}, b^*_{j_n}) = 0 \quad \text{for all } n \in \mathbb{N},$$

imply

$$\Theta(W(\sigma_0); \sigma_0, b^{**}) = 0.$$

This contradicts the uniqueness of b^* in Theorem 1. Hence the lemma follows. □

For small σ, we show that the propagating wave segment is of convex type by the following lemma.

Lemma 6.2. *There holds*

$$\lim_{\sigma \to 0^+} b^*(\sigma) = 0.$$

Proof. When $\sigma = b = 0$, the solution is given by

$$(x(s;0,0), y(s;0,0), \theta(s;0,0)) = (-\cos(-\pi/2+s), -\sin(-\pi/2+s), -\pi/2+s)$$

for $s \in [0, \pi/2]$. Indeed, it is the unit circle.

Suppose that $b^*(0^+) > 0$. Note that $W(\sigma)$ is increasing in σ and $W(0^+) = 1$. Then we can fix a b_0 sufficiently close to $b^*(0^+)$ and $\sigma_0 \in (0,1)$ such that $\Theta(1; 0, b_0)$ is defined and $b_0 < b^*(\sigma)$ for all $\sigma \in (0, \sigma_0)$. Together with the fact that $\Theta(W(\sigma); \sigma, b^*(\sigma)) = 0$ for $\sigma \in (0, \sigma_0)$, it follows from Lemma 4.1 that $\Theta(1; 0, b_0) > 0$.

On other hand, since $\Theta(1; 0, 0) = 0$, it follows from Lemma 4.1 that $\Theta(1; 0, b_0) < 0$, a contradiction. This completes the proof. □

Now with the help of the above lemma, we can show that for small σ, the propagating wave segment is of convex type. Indeed, since $\theta'(s; 0, 0) = 1$ for all $s \in [0, \pi/2]$, the continuity of θ' with respect to $s, \sigma, b^*(\sigma)$ implies that $\theta'(s; \sigma, b^*(\sigma)) > 0$ for all $s \in [0, s^*(\sigma)]$, provided that σ is small enough. Therefore, the wave segment is of convex type for small σ.

6.3. **Existence of non-convex wave segments.** In this subsection, we will show that when σ is close to 1, the wave segment must be non-convex.

To begin with, we need the following lemma which shows that if the wave segment is of convex type for $\sigma = \sigma_1, \sigma_2$ with $\sigma_1 < \sigma_2$, then $b^*(\sigma_1) < b^*(\sigma_2)$.

Lemma 6.3. *If $0 < \sigma_1 < \sigma_2 < 1$ and $\theta'(\cdot; \sigma_j, b^*(\sigma_j)) > 0$ on $[0, s^*(\sigma_j))$, $j = 1, 2$, then $b^*(\sigma_1) < b^*(\sigma_2)$.*

Proof. Note that $\Theta(W(\sigma_j); \sigma_j, b^*(\sigma_j)) = 0$ and $\Theta(z; \sigma_j, b^*(\sigma_j)) \in (-\pi/2, 0)$ for all $z \in (0, W(\sigma_j))$, $j = 1, 2$. Recall $W(\sigma_1) < W(\sigma_2)$, by Lemma 3.4.

For contradiction, we suppose that $b^*(\sigma_1) \geq b^*(\sigma_2)$. Then, by Lemma 4.1,

$$\Theta(z; \sigma_2, b^*(\sigma_1)) \leq \Theta(z; \sigma_2, b^*(\sigma_2)) < 0 \tag{6.1}$$

for all $z \in (0, \min\{\hat{z}(\sigma_2, b^*(\sigma_1)), W(\sigma_2)\})$. Also, by Lemma 4.2, we have

$$\Theta(z; \sigma_2, b^*(\sigma_1)) \geq \Theta(z; \sigma_1, b^*(\sigma_1)) > -\pi/2 \tag{6.2}$$

for all $z \in (0, \min\{W(\sigma_1), \hat{z}(\sigma_2, b^*(\sigma_1))\})$.

Set $\hat{Z} := \hat{z}(\sigma_2, b^*(\sigma_1))$. We claim that $\hat{Z} \geq W(\sigma_1)$. Otherwise, if $\hat{Z} < W(\sigma_1)$, then it follows from (6.1) and (6.2) that $\Theta(\hat{Z}; \sigma_2, b^*(\sigma_1)) \in (-\pi/2, 0)$, a contradiction to the definition of $\hat{Z}$. This establishes the assertion of the claim. Now from (6.1) we have $\Theta(W(\sigma_1); \sigma_2, b^*(\sigma_1)) < 0$. However, (6.2) implies

$$\Theta(W(\sigma_1); \sigma_2, b^*(\sigma_1)) \geq \Theta(W(\sigma_1)^-; \sigma_1, b^*(\sigma_1)) = 0,$$

a contradiction. This completes the proof. □

Now we are in a position to establish the existence of non-convex wave segments for σ close to 1. Recall that the wave segment is of convex type for $\sigma \in (0, \sigma_0]$ for some $\sigma_0 \in (0, 1)$, by the conclusion of Section 6.2. Since $f_\sigma(0)$ converges to $+\infty$ as $\sigma \to 1^-$, we can choose a $\sigma_1 \in (\sigma_0, 1)$ such that $f_\sigma(0) > 2/b^*(\sigma_0)$ for all $\sigma \in [\sigma_1, 1)$.

Now we claim that the wave profile is of non-convex type for $\sigma \in [\sigma_1, 1)$. Otherwise, there exists a $\sigma \in [\sigma_1, 1)$ such that $\Theta(z; \sigma, b^*(\sigma)) \in (-\pi/2, 0)$ for any $z \in (0, W(\sigma))$. This implies that $\theta(s; \sigma, b^*(\sigma)) \in (-\pi/2, 0)$ and $x(s; \sigma, b^*(\sigma)) < 0$ for all $s \in (0, s^*(\sigma))$. By Lemma 6.3, $b^*(\sigma) > b^*(\sigma_0)$. Thus we have

$$1 + \sigma \cos\theta - b^*(\sigma)(f_\sigma(y) - x) \leq 1 + \sigma \cos\theta - b^*(\sigma) f_\sigma(y) \leq 2 - b^*(\sigma_0) f_\sigma(y),$$

for $s \in (0, s^*(\sigma))$. This, together with the choice of σ_1, implies that $\theta'(\cdot; \sigma, b^*(\sigma))$ becomes negative near $s = s^*(\sigma)$, a contradiction. Thus we have shown that the wave profile is non-convex when σ is close to 1.

7. Discussion

In this paper, we use the model of Zykov and Showalter [15] to study the existence of stabilized propagating wave segments. In this model, a stabilized propagating wave segment can be described by two systems of ordinary differential equations for the wave front, (1.1), and for the wave back (i.e., the problem $(P_{\sigma,b})$ and (1.6)). Although this model is a reduction of two-component reaction-diffusion system, it can still reflect the essential behavior of stabilized propagating wave segments. Moreover, it is easier to handle analytically, and only the key parameters are involved in this model.

Within this frame, we have established that for each given size of the wave segment, there exists a unique excitability such that the stabilized wave segment with the given size can propagate in the corresponding medium. The physical implication of this is that the excitability limit $S = h(\zeta)$ is monotone decreasing in the excitability, which agrees with the experimental study. Our analysis also shows that there are two types of the profiles of stabilized wave segments (see Fig. 2), namely, convex and non-convex types. In particular, the wave profile is of convex type when the normalized propagating velocity is small. However, it is of non-convex type when the normalized propagating velocity is close to 1.

References

[1] P.C. Fife, *Understanding the patterns in the BZ reagent*, J. Statist. Phys. **39** (1985), 687–703.
[2] P. Hartman, "Ordinary Differential Equations", SIAM, Philadelphia, 2002.
[3] A. Karma, *Universal limit of spiral wave propagation in excitable media*, Phys. Review Letters **66** (1991), 2274–2277.
[4] J.P. Keener and J.J. Tyson, *Spiral waves in the Belousov-Zhabotinskii reaction*, Physical D **21** (1986), 307–324.
[5] W.F. Loomis, "The Development of Dioctyostelium Discoideum", Academic Press, New York, 1982.
[6] E. Mihaliuk, T. Sakurai, F. Chirila and K. Showalter, *Experimental and theoretical studies of feedback stabilization of propagating wave segments*, Faraday Discuss **120** (2001), 383–394.
[7] E. Mihaliuk, T. Sakurai, F. Chirila and K. Showalter, *Feedback stabilization of unstable propagating waves*, Phys. Review E. **65** (2002), 065602.
[8] A.S. Mikhailov and V.S. Zykov, *Kinematical theory of spiral waves in excitable media: comparison with numerical simulations*, Physica D **52** (1991), 379–397.
[9] J.D. Murray, "Mathematical biology. I: An introduction", Springer-Verlag, New York, 2004.
[10] P. Pelce and J. Sun, *On the stability of steadily rotating waves in the free boundary formulation*, Physica D **63** (1993), 273–281.
[11] Á. Tóth, V. Gaspar, and K. Showalter, *Signal transmission in chemical systems: propagation of chemical waves through capillary tubes*, J. Phys. Chem. **98** (1994), 522-531.
[12] J.J. Tyson and J.P. Keener, *Singular perturbation theory of traveling waves in excitable media (a review)*, Physica D **32** (1988), 327–361.
[13] W.F. Winfree, "When Time Breaks Down", Princeton Univ. Press, Princeton, 1987.
[14] V.S. Zykov, "Simulation of wave process in excitable media", Manchester University Press, 1984.
[15] V.S. Zykov and K. Showalter, *Wave front interaction model of stabilized propagating wave segments*, Phys. Review Letters **94** (2005), 068302.
[16] A. Kothe, V.S. Zykov, and H. Engel, *Second universal limit of wave segment propagation in excitable media*, Phys. Review Letters **103** (2005), 154102.

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