SELF-SIMILAR SOLUTIONS FOR A QUENCHING PROBLEM WITH SPATIALLY DEPENDENT NONLINEARITY

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Abstract. This paper is devoted to the study of the self-similar solutions for a semilinear parabolic equation with spatially dependent nonlinearity which arises in the model of micro-electro mechanical system. We first provide a result on the non-existence of slow orbit for a certain range of parameters. Next, we prove the existence of backward solutions with the desired polynomial growth condition at infinity to the associated equation by a fixed point argument. Then we give a detailed analysis of the behavior of global solutions at the origin. Finally, as an application of the above results, we prove a uniqueness theorem.

Keywords: self-similar solution, slow orbit, Pohozaev's type identity, backward solution

1. Introduction

In this paper, we study the Cauchy problem for the following semilinear parabolic equation with spatially dependent nonlinearity:

$$u_t = u_{xx} - |x|^q u^{-p}, \quad x \in \mathbb{R},\ t > 0, \tag{1.1}$$

supplemented with a positive initial datum, where $p > 1$ and $q > 0$. This problem arises in the study of the dynamic deflection of an elastic membrane inside a micro-electro mechanical system (MEMS). The function u represents the distance between the membrane and the fixed bottom plate. The function $|x|^q$ is a particular choice of the permittivity profile (the dielectric property of the membrane).

Due to the wide applications of MEMS, there have been a lot of research on the related problems to (1.1) recently. We refer the reader to the works [16, 17, 2] for more details on the background and derivation of the MEMS model. Depending on whether the device is embedded in an electrical circuit with a capacitor, the model can be of nonlocal nature in the sense that an integral term appears in the partial differential equation. For works on this nonlocal problem, we refer the reader to a recent survey paper [6] and the references

Date: August 28, 2016. Corresponding author: J.-S. Guo.

2000 Mathematics Subject Classification. 34B15, 34A34, 35K55, 74H35.

The first author was supported in part by the Ministry of Science and Technology of Taiwan under the grant 102-2115-M-032-003-MY3. The second author was supported in part by JSPS KAKENHI Grant Number 26287025, 26247013. The third author was supported in part by JSPS KAKENHI Grant Number 15K04972. Part of this work was carried out during a visit of the first author to Ryukoku University. We would like to thank the anonymous referee for the valuable suggestions on the revision of this paper.

cited therein. The equation (1.1) is the case when there is no capacitor embedded in the circuit. This case has been studied extensively for past years, see, e.g., the works [5, 12, 3, 4, 10, 11, 13, 15, 19].

Physically, the system breaks down when the function u reaches zero, i.e., the membrane touches the bottom plate. This is the so-called touchdown phenomenon. An important question is to determine the locations of touchdown (the touchdown points). Since there is no source at $x = 0$ for (1.1), it is very interesting to see whether $x = 0$ is a touchdown point when touchdown occurs. For the equation (1.1) in a bounded domain, it was conjectured numerically in [12] that $x = 0$ is not a touchdown point. Then this conjecture is recently verified rigorously by the authors in [9]. Indeed, replacing the permittivity profile by a general nonnegative nontrivial function, it is shown in [9] that any interior zero point of the permittivity profile cannot be a touchdown point. Moreover, this is true not only for the one-dimensional case but also for the higher space dimension.

For the Cauchy problem of equation (1.1) with a positive initial datum, by the standard parabolic theory, it is trivial that there is a unique local (in time) solution. A local (in time) positive solution may reach zero in a finite time. This (touchdown) phenomenon is also called *quenching* mathematically. More precisely, if there is a finite time T such that $u(\cdot, t) > 0$ for all $t < T$ and

$$\liminf_{t\to T^-}\{\inf_{x\in(-\infty,\infty)} u(x,t)\} = 0,$$

then we say that u quenches in finite time. A point x_0 is a (finite) quenching point, if there is a sequence $\{(x_n, t_n)\}$ such that $x_n \to x_0$, $t_n \to T^-$ and $u(x_n, t_n) \to 0$ as $n \to \infty$. One interesting question is to determine whether $x = 0$ is a quenching point. Furthermore, how fast the solution quenches, if $x = 0$ is a quenching point.

Motivated by the work of Filippas and Tertikas [1] for blow-up problems, we study the self-similar solution of (1.1) in the form

$$u(x,t) = (T-t)^\gamma w(y), \quad y = \frac{|x|}{\sqrt{T-t}}, \tag{1.2}$$

where $T > 0$ is given and

$$\gamma := \frac{1+q/2}{1+p}.$$

Then w satisfies the equation

$$w'' - \frac{y}{2}w' + \gamma w - y^q w^{-p} = 0, \quad y > 0, \tag{1.3}$$

with the initial condition $w'(0) = 0$ and $w(0) > 0$. Hereafter $w' := dw/dy$. It is trivial that a local solution w of (1.3) with $w'(0) = 0$ and $w(0) > 0$ exists.

If there is a global solution w of (1.3) such that $w'(0) = 0$, $w > 0$ on $[0, \infty)$ and the limit

$$\ell := \lim_{y\to\infty} [y^{-2\gamma} w(y)] \tag{1.4}$$

exists for some finite positive constant ℓ, then the corresponding self-similar solution u defined by (1.2) quenches at the finite time T with the self-similar rate $(T-t)^\gamma$. Note that u is positive for all $t < T$ and $u(0, T^-) = 0$. Moreover, by (1.4),

$$u(x, T^-) := \lim_{t \uparrow T} u(x,t) = |x|^{2\gamma} \cdot \lim_{y \to \infty} [y^{-2\gamma} w(y)] = \ell |x|^{2\gamma} > 0$$

for all $x \neq 0$. Hence $x = 0$ is the (only) quenching point.

In this paper, we call a positive global solution w of (1.3) such that $w'(0) = 0$, $w > 0$ on $[0, \infty)$ and (1.4) holds for a finite positive ℓ a *slow orbit*. The existence of a slow orbit renders a solution of (1.1) that quenches in finite time T with the quenching point $x = 0$ and the self-similar quenching rate $(T-t)^\gamma$.

For a given global positive solution w of (1.3), if we set

$$\hat{w}(y) := w(y) y^{-2\gamma},$$

then it is easy to check that $\hat{w}$ satisfies

$$\hat{w}'' + \left(\frac{4\gamma}{y} - \frac{y}{2}\right)\hat{w}' = \frac{1}{y^2}[\hat{w}^{-p} + 2\gamma(1-2\gamma)\hat{w}], \quad y > 0. \tag{1.5}$$

Note that a constant solution of (1.5) exists if and only if $\gamma > 1/2$. In fact, the constant solution of (1.5) renders the function

$$w_s(y) := \kappa y^{2\gamma}, \quad y \geq 0, \quad \kappa := [2\gamma(2\gamma - 1)]^{-1/(1+p)},$$

a solution of (1.3) with $w(0) = w'(0) = 0$ and $\ell = \kappa$ in (1.4). Note that w_s is not a slow orbit, since it is singular in the sense that $w(0) = 0$.

The rest of this paper is organized as follows. In §2, we prove the non-existence of slow orbits for $\gamma \leq 1/4$, by using a Pohozaev type identity. Also, some comments on the range $\gamma \in (1/4, 1/2]$ are given. Then, in §3, we derive the existence of backward solutions to (1.3) and (1.4) for $\gamma > 1/2$. This is done by utilizing a fixed-point argument and the finiteness of an associated energy. Section 4 is devoted to the behavior of global solutions to (1.3) at $y = 0$ for $\gamma > 1/2$. The analysis of the behavior at $y = 0$ is much more involved. We first derive the monotonicity of the transformed solution by analyzing the associated energy. Then, with the help of three invariant sets by the phase plane analysis, we are able to derive the exact two possible asymptotic behaviors. More details on the desired behavior towards slow orbit is carried out. Also, some comments on the non-existence of slow orbits for $\gamma > 1/2$ are given in §4. Finally, we apply the results obtained in §4 to prove a uniqueness result in §5. This type of uniqueness results can be found, e.g., in [8, 7, 18]. However, it seems that the method developed in [8] (see also [7, 18]) does not work here. Note that, instead of the monotonicity condition (imposed in [8, 7, 18]), we impose the condition $w(0) = 0$ here. Due to this singular condition, the standard uniqueness theory cannot be applied.

2. Non-existence of slow orbits for $\gamma \leq 1/4$

In this section, we prove that there are no slow orbits when $\gamma \leq 1/4$.

Theorem 1. *There is no slow orbit, if $\gamma \leq 1/4$.*

Proof. Suppose that there exists a slow orbit w of (1.3). Using (1.3), $w'(0) = 0$ and (1.4), we can derive the following Pohozaev's type identity

$$\left(\frac{1}{2} - \frac{1+q}{p-1}\right) \int_0^\infty \rho(y)[w'(y)]^2 dy + \frac{1+p}{4(p-1)} \int_0^\infty y^2 \rho(y)[w'(y)]^2 dy \\ + \frac{(\gamma+1/4)q}{p-1} \int_0^\infty \rho(y)[w(y)]^2 dy + \frac{q}{8(p-1)} \int_0^\infty y^2 \rho(y)[w(y)]^2 dy = 0, \tag{2.1}$$

where $\rho(y) = \exp\{-y^2/4\}$. In fact, re-writing (1.3) as

$$(\rho w')' + \gamma \rho w - y^q \rho w^{-p} = 0, \tag{2.2}$$

multiplying (2.2) by w, $y^2 w$ and yw' respectively, integrating these equations over $[0, \infty)$ using integration by parts, the identity (2.1) can be deduced.

Note that $(1 + q)/(p - 1) \leq 1/2$, if $\gamma \leq 1/4$. In this case, all coefficients in (2.1) are nonnegative. This gives a contradiction and the theorem follows. □

Now, we consider the range $\gamma \in (1/4, 1/2]$. It is easy to see from (1.5) that any critical point of $\hat{w}$ must be a strictly minimal point, if $\gamma \leq 1/2$. Suppose that there is a slow orbit w. Note that we always have $w''(0) < 0$ so that w is decreasing near $y = 0$. Hence $\hat{w}'(y) < 0$ for any small $y > 0$. This implies that either $\hat{w}' < 0$ in $(0, \infty)$, or there exists a unique $y_0 > 0$ such that $\hat{w}' < 0$ in $(0, y_0)$ and $\hat{w}' > 0$ in (y_0, ∞).

Suppose that $\hat{w}'(y) > 0$ for $y > y_0$ for some positive constant y_0. Then $\hat{w}''(y) > 0$ for all $y \gg 1$, due to (1.5) and $\gamma \leq 1/2$. This implies that $\hat{w}(y) \to \infty$ as $y \to \infty$, a contradiction to the fact that w is a slow orbit. Therefore, we conclude that $\hat{w}' < 0$ in $(0, \infty)$ for any slow orbit w (if it exists), when $\gamma \leq 1/2$. However, it may happen that $\hat{w}(y) \to 0$ as $y \to \infty$. We were unable to exclude this case and so leave it open here.

3. Existence of backward solutions to (1.3) and (1.4) for $\gamma > 1/2$

From now on, we shall always assume that $\gamma > 1/2$. Set

$$v(\xi) := w(y) y^{-2\gamma}, \quad y = e^\xi. \tag{3.1}$$

Then it is easy to check that w satisfies the equation (1.3) if and only if v satisfies

$$\frac{d^2 v}{d\xi^2} - (e^{2\xi}/2 + 1 - 4\gamma)\frac{dv}{d\xi} + g(v) = 0, \quad \xi \in (-\infty, \infty), \tag{3.2}$$

where

$$g(v) := \beta v - v^{-p}, \qquad \beta := 2\gamma(2\gamma - 1) > 0.$$

Note that $v \equiv \kappa$ is the only constant solution of (3.2).

Let v be a nonconstant solution of (3.2). Introduce the following energy functional

$$E(\xi) := \frac{1}{2}v_\xi^2(\xi) + G(v(\xi)), \quad G(v) := \int_\kappa^v g(s)ds.$$

It is easy to see that

$$\frac{dE}{d\xi}(\xi) = (e^{2\xi}/2 + 1 - 4\gamma)v_\xi^2(\xi)$$

so that E is monotone non-decreasing (non-increasing, resp.) for $\xi > \bar{\xi}$ ($\xi < \bar{\xi}$, resp.), where $\bar{\xi} := [\ln(8\gamma - 2)]/2$. In fact, it is strictly increasing (decreasing, resp.) for $\xi > \bar{\xi}$ ($\xi < \bar{\xi}$, resp.). Thus the limits

$$L_\pm := \lim_{\xi \to \pm\infty} E(\xi)$$

exist. Moreover, we have $L_\pm > 0$, since $v \not\equiv \kappa$ and $G(v) > 0$ for all $v \in (0, \infty) \setminus \{\kappa\}$.

For notational convenience, hereafter we let $\delta := 2\gamma - 1/2$. We first prove the local existence of positive solutions to (3.2) at $\xi = +\infty$ as follows.

Proposition 3.1. *For any $\ell > 0$ with $\ell \neq \kappa$, the equation* (3.2) *has a solution v defined in a neighborhood of $+\infty$ such that*

$$v(\xi) = \ell - g(\ell)e^{-2\xi} + O(e^{-4\xi}) \quad \textit{as} \quad \xi \to \infty. \tag{3.3}$$

Proof. We write the equation (3.2) as the following system

$$\frac{dv}{d\xi} = W,$$
$$\frac{dW}{d\xi} = [e^{2\xi}/2 - (4\gamma - 1)]W - g(v).$$

Then we introduce the new independent variable $\tau := e^{2\xi}$ to transform the system into

$$\frac{dv}{d\tau} = \frac{W}{2\tau}, \tag{3.4}$$

$$\frac{dW}{d\tau} = \left(\frac{1}{4} - \frac{\delta}{\tau}\right)W - \frac{g(v)}{2\tau}. \tag{3.5}$$

First, using $v(\infty) = \ell$, we write (3.4) in the integral form

$$v(\tau) = \ell - \int_\tau^\infty \frac{W(s)}{2s}ds. \tag{3.6}$$

Similarly, assuming $\lim_{\tau \to \infty} W(\tau) = 0$, we write (3.5) in the integral form as

$$W(\tau) = \frac{1}{2}\left\{\int_\tau^\infty e^{\frac{1}{4}(\tau - s)}\left(\frac{s}{\tau}\right)^\delta \frac{g(v(s))}{s}\right\}ds. \tag{3.7}$$

Combining (3.6) and (3.7) yields

$$W(\tau) = \frac{1}{2}\int_\tau^\infty e^{\frac{1}{4}(\tau - s)}\left(\frac{s}{\tau}\right)^\delta \frac{1}{s}g\left(\ell - \int_s^\infty \frac{W(z)}{2z}dz\right)ds. \tag{3.8}$$

We will find a solution $W(\tau)$, defined on $[a, \infty)$ for some $a > 0$, of (3.8) satisfying

$$\lim_{\tau \to \infty} \tau|W(\tau) - 2g(\ell)/\tau| = 0. \tag{3.9}$$

For this, we put

$$\phi(\tau) = \tau[W(\tau) - 2g(\ell)/\tau].$$

Then

$$\int_s^\infty \frac{W(z)}{2z} dz = \frac{g(\ell)}{s} + \int_s^\infty \frac{\phi(z)}{2z^2} dz$$

and we define

$$\mathcal{F}[\phi](\tau) := \frac{\tau}{2} \int_\tau^\infty e^{\frac{1}{4}(\tau-s)} \left(\frac{s}{\tau}\right)^\delta \frac{1}{s} g\left(\ell - \frac{g(\ell)}{s} - \int_s^\infty \frac{\phi(z)}{2z^2} dz\right) ds - 2g(\ell)$$

for $\tau \in [a, \infty)$ for $\phi \in X$, where

$$X := \{\phi \in C[a, \infty) : \|\phi\|_1 := \sup_{a \le s < \infty} s|\phi(s)| \le K\},$$

with positive constants a, K to be determined later. The space X is a Banach space with the norm $\|\phi\|_1$.

We shall prove in Lemma 3.1 below that $\mathcal{F}$ is a contraction mapping on X. Hence $\mathcal{F}$ has a fixed point, say ϕ^*. Then the function

$$w^*(\tau) := \frac{2g(\ell)}{\tau} + \frac{\phi^*(\tau)}{\tau},\ \tau \ge a, \tag{3.10}$$

gives a solution of (3.8) on $[a, \infty)$. Consequently,

$$v^*(\tau) := \ell - \frac{g(\ell)}{\tau} - \int_\tau^\infty \frac{\phi^*(z)}{2z^2} dz,\ \tau \ge a, \tag{3.11}$$

is a desired solution, since there is an $M_1 > 0$ such that

$$\left| v^*(\tau) - \ell + \frac{g(\ell)}{\tau} \right| \le \int_\tau^\infty \frac{|\phi^*(z)|}{2z^2} dz \le \int_\tau^\infty \frac{M_1}{2z^3} dz = \frac{M_1}{4\tau^2} \quad \text{for all } \tau \ge a.$$

This proves the proposition. □

Lemma 3.1. *The mapping $\mathcal{F}$ is a contraction mapping from X to X.*

Proof. To show that $\mathcal{F}$ is a contraction mapping, we may rewrite $\mathcal{F}$ as

$$\mathcal{F}[\phi](\tau) = \frac{g(\ell)}{2} \left[e^{\frac{\tau}{4}} \left(\frac{1}{\tau}\right)^{\delta-1} \int_\tau^\infty \left\{ e^{-\frac{s}{4}} s^{\delta-1} - g'(\ell) e^{-\frac{s}{4}} s^{\delta-2} \right\} ds - 4 \right]$$
$$+ \frac{1}{2} \int_\tau^\infty e^{\frac{1}{4}(\tau-s)} \left(\frac{s}{\tau}\right)^{\delta-1} R(\phi, s) ds,$$

where

$$R(\phi, s) := g\left(\ell - \frac{g(\ell)}{s} - \int_s^\infty \frac{\phi(z)}{2z^2} dz\right) - g(\ell) + \frac{g'(\ell) g(\ell)}{s}.$$

Integration by parts shows that

$$\int_\tau^\infty e^{-\frac{s}{4}} s^{\delta-1} ds \quad = \quad 4e^{-\frac{\tau}{4}} \tau^{\delta-1} + 4(\delta - 1) \int_\tau^\infty e^{-\frac{s}{4}} s^{\delta-2} ds.$$

By this, we obtain that

$$\mathcal{F}[\phi](\tau) = A g(\ell) \int_\tau^\infty e^{\frac{\tau-s}{4}} \left(\frac{s}{\tau}\right)^{\delta-1} s^{-1} ds + \frac{1}{2} \int_\tau^\infty e^{\frac{\tau-s}{4}} \left(\frac{s}{\tau}\right)^{\delta-1} R(\phi, s) ds, \tag{3.12}$$

where $A := [-g'(\ell) + 4(\delta - 1)]/2$.

We define the constant

$$C_a := \int_0^\infty e^{-y/4}\left(\frac{y}{a}+1\right)^\delta dy.$$

Setting $s = y+\tau$, it is clear that

$$\int_\tau^\infty e^{\frac{\tau-s}{4}}\left(\frac{s}{\tau}\right)^{\delta-1} ds = \int_0^\infty e^{-y/4}\left(\frac{y+\tau}{\tau}\right)^{\delta-1} dy \le \int_0^\infty e^{-y/4}\left(\frac{y}{\tau}+1\right)^\delta dy \le C_a$$

for all $\tau \ge a$. Hence we see that

$$\left|\int_\tau^\infty e^{\frac{\tau-s}{4}}\left(\frac{s}{\tau}\right)^{\delta-1} s^{-1}ds\right| \le \frac{C_a}{\tau}, \quad \forall\tau\in[a,\infty). \tag{3.13}$$

Notice that $C_a \to 4$ as $a\to\infty$.

On the other hand, rewrite

$$\begin{aligned} R(\phi,s) &= g\left(\ell - g(\ell)/s - \int_s^\infty \frac{\phi(z)}{2z^2}dz\right) - g(\ell-g(\ell)/s) \\ &\quad + g(\ell-g(\ell)/s) - g(\ell) + \frac{g'(\ell)g(\ell)}{s}. \end{aligned}$$

Then for $s\ge a$ there exists $\theta\in(0,1)$ such that

$$\begin{aligned} R(\phi,s) &= -\int_0^1 g'\left(\ell - g(\ell)/s - h\int_s^\infty \frac{\phi(z)}{2z^2}dz\right)dh\left\{\int_s^\infty \frac{\phi(z)}{2z^2}dz\right\} \\ &\quad + \frac{1}{2}g''(\ell-\theta g(\ell)/s)(g(\ell)/s)^2. \end{aligned} \tag{3.14}$$

Now we choose a positive constant K such that

$$K>1,\ K\ge \max_{a\ge 1}\{2Ag(\ell)C_a\},\ K>\max_{a\ge1}\{4C_a\}.$$

For this constant K, we choose a sufficiently large constant a such that

$$a\ge\beta(1+p)K,\ a\ge 2p(1+p)\kappa^{-2-p}g(\ell)^2,\ \ell-\frac{g(\ell)}{a}-\frac{K}{4a^2}\ge\kappa. \tag{3.15}$$

For $\phi\in X$, we have

$$\int_s^\infty\frac{\phi(z)}{2z^2}dz \le \int_s^\infty \frac{K}{2z^3}dz = \frac{K}{4s^2},\ \forall s\ge a.$$

It follows that

$$\begin{aligned} \ell-\theta g(\ell)/s &\ge \ell - g(\ell)/s \ge \ell-g(\ell)/s - h\int_s^\infty\frac{\phi(z)}{2z^2}dz \\ &\ge \ell-g(\ell)/s - \int_s^\infty \frac{\phi(z)}{2z^2}dz \ge \ell-\frac{g(\ell)}{a}-\frac{K}{4a^2}\ge\kappa \end{aligned}$$

for any $h,\theta\in[0,1]$ for all $s\ge a$. Note that we have

$$\begin{aligned} g'(v) &= \beta + pv^{-1-p} \le \beta(1+p), \\ |g''(v)| &= |p(1+p)v^{-2-p}| \le p(1+p)\kappa^{-2-p} \end{aligned}$$

for all $v\ge\kappa$. It then follows from (3.14) and (3.15) that

$$|R(\phi,s)| \le \beta(1+p)K/(4s^2) + \frac{1}{2}p(1+p)\kappa^{-2-p}g(\ell)^2/s^2 \le \frac{1}{2s} \tag{3.16}$$

for all $s \geq a$. Hence, we obtain that

$$\left| \int_\tau^\infty e^{\frac{\tau - s}{4}} \left(\frac{s}{\tau}\right)^{\delta - 1} R(\phi, s) ds \right| \leq \frac{C_a}{2\tau}, \quad \forall \tau \in [a, \infty). \tag{3.17}$$

Combining (3.13) and (3.17), it follows from (3.12) that

$$|\mathcal{F}[\phi](\tau)| \leq [Ag(\ell)C_a + C_a/4]/\tau \leq K/\tau \quad \text{for all } \tau \geq a.$$

Hence $\mathcal{F}[\phi] \in X$.

Finally, we show that $\mathcal{F}$ is a contraction mapping. For this, we first observe from (3.12) that

$$\mathcal{F}[\phi_1](\tau) - \mathcal{F}[\phi_2](\tau) = \frac{1}{2} \int_\tau^\infty e^{\frac{1}{4}(\tau - s)} \left(\frac{s}{\tau}\right)^{\delta - 1} [R(\phi_1, s) - R(\phi_2, s)] ds.$$

Also, by the mean value theorem, for each $s \geq a$ there exists $h \in [0, 1]$ such that

$$R(\phi_1, s) - R(\phi_2, s) = -g'\left(\ell - g(\ell)/s - h \int_s^\infty \frac{\phi(z)}{2z^2} dz\right) \int_s^\infty \frac{\phi_1(z) - \phi_2(z)}{2z^2} dz.$$

It follows that

$$|R(\phi_1, s) - R(\phi_2, s)| \leq \beta(1 + p)\|\phi_1 - \phi_2\|_1 \int_s^\infty \frac{dz}{2z^3} \leq \beta(1 + p)\|\phi_1 - \phi_2\|_1/(4s^2)$$

for all $s \geq \tau$. Hence we obtain

$$\begin{aligned} \tau|\mathcal{F}[\phi_1](\tau) - \mathcal{F}[\phi_2](\tau)| &\leq \frac{\tau}{2} \int_\tau^\infty e^{\frac{1}{4}(\tau - s)} \left(\frac{s}{\tau}\right)^{\delta - 1} |R(\phi_1, s) - R(\phi_2, s))| ds \\ &\leq \frac{\tau}{2} \frac{C_a}{4\tau^2} \beta(1 + p)\|\phi_1 - \phi_2\|_1 \\ &\leq \frac{1}{8a} \beta(1 + p) C_a \|\phi_1 - \phi_2\|_1 \leq \frac{1}{2}\|\phi_1 - \phi_2\|_1 \end{aligned}$$

for all $\tau \geq a$. Therefore, $\mathcal{F}$ is a contraction mapping on X. This concludes the proof of the lemma and the proof of Proposition 3.1. □

The following lemma gives the global existence of positive solutions to (3.2) in $\mathbb{R}$.

Lemma 3.2. *Let v be a solution of (3.2) in an infinite interval $[\xi_0, \infty)$ for some ξ_0, constructed in Proposition 3.1 with $v(\infty) = \ell > 0$ and $\ell \neq \kappa$. Then $v(\xi)$ can be extended backwards to $\xi = -\infty$ and $v(\xi) > 0$ for all $\xi \in (-\infty, \infty)$.*

Proof. Let $\tilde{v}(\tau) = v(-\tau)$. Then $\tilde{v}$ is a solution to

$$\frac{d^2\tilde{v}}{d\tau^2} + (e^{-2\tau}/2 + 1 - 4\gamma)\frac{d\tilde{v}}{d\tau} + g(\tilde{v}) = 0 \tag{3.18}$$

for $\tau \in (-\infty, -\xi_0]$. Then the energy

$$\tilde{E}(\tau) := \frac{1}{2}\tilde{v}_\tau^2 + G(\tilde{v}), \quad G(v) := \int_\kappa^v g(s) ds, \tag{3.19}$$

satisfies

$$\begin{aligned} \frac{d}{d\tau}\tilde{E}(\tau) &= (4\gamma - 1 - e^{-2\tau}/2)\tilde{v}_\tau(\tau)^2 \\ &\leq 2(4\gamma - 1)\left(\frac{1}{2}\tilde{v}_\tau(\tau)^2 + G(\tilde{v}(\tau))\right) = 2(4\gamma - 1)\tilde{E}(\tau). \end{aligned}$$

This implies that $\tilde{E}(\tau)$ is finite for any τ finite and grows at most exponentially. Hence we can continue the solution up to $\tau = \infty$. This yields the desired result. $\square$

By returning to the original variable, i.e., letting $w(y) = y^{2\gamma} v(\ln y)$, Lemma 3.2 implies that a positive solution $w(y)$ to the equation (1.3) exists for $y \in (0, \infty)$ such that $w(y)y^{-2\gamma} \to \ell$ as $y \to \infty$. We have proved

Theorem 2. *For each positive constant $\ell \neq \kappa$, there is a positive solution w of* (1.3) *in* $(0, \infty)$ *such that $w(y)y^{-2\gamma} \to \ell$ as $y \to \infty$.*

Since $v(\infty) = \ell \neq \kappa$, it is easy to see from (3.2) that $v_{\xi\xi}$ has a fixed sign for any critical point ξ of v as long as $\xi \gg 1$. It follows that v is monotone ultimately at $\xi = \infty$.

4. Behavior of global solutions of (1.3) at $y = 0$ for $\gamma > 1/2$

Throughout this section, we always assume that $\gamma > 1/2$. Let w be a global positive solution of (1.3) in $(0, \infty)$ such that $w \neq w_s$. Then the corresponding function v defined by (3.1) satisfies the equation (3.2) in $(-\infty, \infty)$. Note that v is non-constant. For a slow orbit, we note that conditions $w'(0^+) = 0$ and $w(0^+) \in (0, \infty)$ are equivalent to the following conditions:

$$\lim_{\xi \to -\infty} \{e^{(2\gamma - 1)\xi} [v_\xi(\xi) + 2\gamma v(\xi)]\} = 0, \tag{4.1}$$

(4.2) the limit $\lim_{\xi \to -\infty} [e^{2\gamma\xi} v(\xi)]$ exists and is a finite positive constant.

To study the asymptotic behavior of solutions v of (3.2) at $\xi = -\infty$, we set $V(\tau) = \tilde{v}(\tau) = v(-\tau)$, $\tau = -\xi$, and introduce $W = \tilde{v}_\tau$. Then the equation (3.18) is converted to the system

$$\begin{cases} V' := \frac{dV}{d\tau} = W, \\ W' := \frac{dW}{d\tau} = (4\gamma - 1 - e^{-2\tau}/2)W - g(V). \end{cases} \tag{4.3}$$

Therefore, to study the behavior of w at $y = 0$, we are reduced to study the asymptotic behavior of V at $\tau = \infty$.

First, we have the following lemma on the divergence of the energy $\tilde{E}$ as $\tau \to \infty$.

Lemma 4.1. *Let (V, W) be a solution of* (4.3) *for $\tau \in (-\infty, \infty)$ such that $V \not\equiv \kappa$. Then*

$$\lim_{\tau \to \infty} \tilde{E}(\tau) = \infty,$$

where $\tilde{E}$ is defined by (3.19).

Proof. We show the assertion of the lemma by the contradiction argument. Suppose that

$$\lim_{\tau \to \infty} \tilde{E}(\tau) = e_\infty < \infty. \tag{4.4}$$

Then there are three cases: (a) there is a T_1 such that $W(\tau) > 0$ for all $\tau > T_1$; (b) there is a T_2 such that $W(\tau) < 0$ for all $\tau > T_2$; (c) $W(\tau)$ has infinitely many zeros.

First, we consider the case (a). Then $V(\tau)$ is monotone for $\tau > T_1$ and it has the limit $V_\infty (\neq \kappa)$ as $\tau \to \infty$. On the other hand the assumption (4.4) implies $G(V_\infty) = e_\infty$, otherwise $\lim_{\tau\to\infty} V'(\tau) > 0$, which is a contradiction. Thus we have $\lim_{\tau\to\infty} W(\tau) = 0$. However, by the equation for W

$$\lim_{\tau\to\infty} W'(\tau) = -g(V_\infty) \neq 0$$

which contradicts the fact that $\lim_{\tau\to\infty} W(\tau) = 0$. Hence (a) cannot occur.

In a similar way we can exclude the case (b).

We next consider (c). Notice that the function $F(V,W) := W^2/2 + G(V)$ has a unique minimum $F(\kappa,0)$ and every level set of $F(V,W)$ is a simple closed curve if $F(V,W) > F(\kappa,0) = G(\kappa)$. Moreover, every curve $\{(V,W) : F(V,W) = s\}$ takes the maximum $W = \sqrt{2(s - G(\kappa))}$ for $s > G(\kappa)$. Then

$$(V(\tau),W(\tau)) \in D := \{(V,W) :\ W^2/2 + G(V) \le e_\infty\} \quad (\forall \tau > T)$$

since $\tilde{E}(\tau)$ is nondecreasing for $\tau > -\frac{1}{2}\log(2(4\gamma-1))$. We note $(\kappa,0) \in D$. Set the closed curve $\mathcal{C} := \{(V,W) :\ W^2/2 + G(V) = e_\infty\}$. The trajectory of the solution goes around $(\kappa,0)$ clockwise infinitely many times and it approaches $\mathcal{C}$. Thus there are sequences $\{\tau_k^-\}$ and $\{\tau_k^+\}$ such that

$$\begin{aligned} &\tau_1^- < \tau_1^+ < \cdots < \tau_k^- < \tau_k^+ < \tau_{k+1}^- < \tau_{k+1}^+ < \cdots, \quad \lim_{k\to\infty} \tau_k^{\pm} = \infty, \\ &W(\tau_k^{\pm}) = b/2 \quad (k = 1,2,\ldots), \qquad W(\tau) > b/2 \quad (\tau_k^- < \tau < \tau_k^+), \end{aligned}$$

where $b := \sqrt{2(e_\infty - G(\kappa))}$. We may assume that there is a positive number c_1 such that

$$\tau_k^+ - \tau_k^- \ge c_1. \tag{4.5}$$

Indeed, take $V_1 < V_2$ so that

$$\sqrt{2(e_\infty - G(V_j))} = \frac{b}{2} \quad (j = 1,2),$$

and put $V_k^{\pm} = V(\tau_k^{\pm})$. Then for the solution in $D \cap \{(V,W) : W \ge b/2\}$, we see $W(\tau) \le b$ and

$$\frac{d\tau}{dV} = \frac{1}{W}$$

yields

$$\tau_k^+ - \tau_k^- = \int_{V_k^-}^{V_k^+} \frac{1}{W} dV \ge \frac{V_k^+ - V_k^-}{b} \ge \frac{V_2 - V_1}{2b}$$

for sufficiently large k. If necessary, by relabeling of k, we have (4.5).

We estimate

$$\begin{aligned} \tilde{E}(\tau_k^+) - \tilde{E}(\tau_k^-) &= \int_{\tau_k^-}^{\tau_k^+} (4\gamma - 1 - e^{-2\tau}/2) W^2\, d\tau \\ &\ge (\tau_k^+ - \tau_k^-)(4\gamma - 1 - \varepsilon)(b/2)^2 \\ &\ge (V_2 - V_1)(4\gamma - 1 - \varepsilon) b^2/8. \end{aligned}$$

The left-hand side, however, goes to 0 as $k \to \infty$. This is again a contradiction. As a result any case of (a), (b) and (c) never occurs. This leads to a contradiction of (4.4), so $\tilde{E}(\tau)$ goes to infinity as $\tau \to \infty$. □

With Lemma 4.1, we now prove that any non-constant solution v of (3.2) must be monotone ultimately at $\xi = -\infty$ as follows.

Lemma 4.2. *Let $v(\xi)$ be a non-constant global solution of* (3.2) *in* $(-\infty, \infty)$*. Then $v_\xi(\xi)$ has at most finitely many zeros in* $(-\infty, 0)$*.*

Proof. Consider $\tilde{v}(\tau) = v(-\tau)$ and (3.18). Then to show that v_ξ has at most finitely many zeros in $(-\infty, 0)$, it suffices to prove that there are only finitely many zeros of W in a half infinite interval $[T, \infty)$ for some large T.

We show by a contradiction and suppose that W has infinitely many zeros in $[T, \infty)$ for some large T. Take $c \in (2\gamma - 1, 2\gamma)$ and $T \gg 1$ so that

$$c^2 - (4\gamma - 1 - \varepsilon)c + \beta < 0, \qquad \varepsilon := e^{-2T}/2 \tag{4.6}$$

holds. Indeed, since $\beta = 2\gamma(2\gamma - 1)$, it is possible to choose such an ε (and so T) for any $c \in (2\gamma - 1, 2\gamma)$.

We consider the orbit of the solution in the phase plane (V, W). Define

$$\Sigma_c := \{(V, W) : V > 0, \quad W \geq cV\}.$$

Assume $\tau > T$. Then in $\{(V, W) : V > 0,\ W > 0\}$

$$\begin{aligned} \frac{dW}{dV} &= 4\gamma - 1 - \frac{1}{2}e^{-2\tau} - \beta(V/W) + V^{-p}/W \\ &> 4\gamma - 1 - \varepsilon - \beta(V/W). \end{aligned}$$

Thus on the half line $\{W = cV,\ V > 0\}$

$$\frac{dW}{dV} > 4\gamma - 1 - \varepsilon - \beta/c > c$$

holds by (4.6). This implies that Σ_c is a positively invariant region for the flow defined by (4.3) for $\tau \geq T$.

There are four possible cases: (i) there is a $T_1 > T$ such that $(V(\tau), W(\tau)) \in \Sigma_c$ for all $\tau > T_1$; (ii) there is a $T_2 > T$ such that $(V(\tau), W(\tau)) \notin \Sigma_c$ and $W(\tau) > 0$ for all $\tau > T_2$; (iii) there is a $T_3 > T$ such that $W(\tau) < 0$ for all $\tau > T_3$; (iv) $(V(\tau), W(\tau)) \notin \Sigma_c$ for any $t > T$ and $W(\tau)$ has infinitely many zeros. It suffices for the assertion of the lemma to show that (iv) is excluded. To carry it out, we will use the fact that $\lim_{\tau\to\infty} \tilde{E}(\tau) = \infty$.

Recall that κ is the zero of $g(V)$. Then $W' > 0$ and $W' < 0$ hold on $\{(V, W) : W = 0,\ 0 < V < \kappa\}$ and $\{(V, W) : W = 0,\ \kappa < V\}$ respectively. Suppose that there are infinite number of zeros of W. We can take an infinite sequence $\{\tau_k\}$ such that $\tau_k > T$, $W(\tau_k) = 0$, $\kappa > V(\tau_k) > V(\tau_{k+1}) > 0$ for all $k \in \mathbb{N}$, and

$$\lim_{k\to\infty} \tau_k = \infty, \quad \lim_{k\to\infty} V(\tau_k) = 0.$$

Note that the fact that $\lim_{k\to\infty} V(\tau_k) = 0$ follows from $\lim_{\tau\to\infty} \tilde{E}(\tau) = \infty$.

Put $\alpha_k := V(\tau_k)$ and consider

$$\frac{dV}{dW} = \frac{W}{(4\gamma - 1 - e^{-2\tau}/2)W - g(V)}$$

for V in the region $\alpha_k \leq V \leq V_1 := \kappa/2$ and $\tau \geq \tau_k$. Since $V(\tau_k) < V(\tau)$ for $\tau_k < \tau$ as long as $W > 0$, we have $g(\alpha_k) \leq g(V(\tau)) \leq g(V_1) < 0$, thus

$$\frac{dV}{dW} \leq \frac{W}{(4\gamma - 1 - \varepsilon)W + K}, \qquad K := -g(V_1) > 0.$$

Let (V_1, W_1) be a point which the orbit of the solution starting from $(\alpha_k, 0)$ goes through. Namely, W_1 is given by $W_1 = W(\tau^\dagger)$ where $\tau^\dagger$ first achieves $V(\tau^\dagger) = V_1$ from τ_k. We see that the $V(\tau)$ and $W(\tau)$ is monotone increasing in the region under consideration. Integrating the both side of the above inequality yields

$$\begin{aligned} V_1 - \alpha_k &< \int_0^{W_1} \frac{W}{(4\gamma - 1 - \varepsilon)W + K}\, dW \\ &= \frac{1}{4\gamma - 1 - \varepsilon} \left\{ W_1 - \frac{K}{4\gamma - 1 - \varepsilon} \log \frac{(4\gamma - 1 - \varepsilon)W_1 + K}{K} \right\}. \end{aligned}$$

As a result

$$V_1 - \frac{1}{4\gamma - 1 - \varepsilon} W_1 < \alpha_k - \frac{K}{(4\gamma - 1 - \varepsilon)^2} \log \frac{(4\gamma - 1 - \varepsilon)W_1 + K}{K}. \tag{4.7}$$

On the other hand if $\alpha_k < \kappa/4$, we have a lower estimate for W_1 by

$$\kappa/4 = V_1/2 < \frac{1}{4\gamma - 1 - \varepsilon} W_1.$$

Take k sufficiently large in (4.7) so that the right-hand side is negative. Then invoking of (4.6) we obtain

$$cV_1 - W_1 < (c + \beta/c)V_1 - W_1 < (4\gamma - 1 - \varepsilon)V_1 - W_1 < 0.$$

This implies that (V_1, W_1) exists in Σ_c. However, since Σ_c is positively invariant, it contradicts the choices of $\{\tau_k\}$. Therefore, we have the desired conclusion. □

Next, we prove that v tends to infinity as $\xi \to -\infty$ as follows.

Lemma 4.3. *Let v be a non-constant solution of* (3.2) *defined in a neighborhood of $\xi = -\infty$. Then $v(\xi) \to \infty$ as $\xi \to -\infty$.*

Proof. Suppose that v is a non-constant solution of (3.2) defined in $(-\infty, \xi_0]$ for some ξ_0. Recall from Lemma 4.2 that v is monotone near $\xi = -\infty$. Hence the limit $l_- := \lim_{\xi \to -\infty} v(\xi)$ exists.

By contradiction, we assume that l_- is finite. Then we can find a sequence $\xi_n \to -\infty$ such that $v_\xi(\xi_n) \to 0$ as $n \to \infty$. Moreover, by Lemma 4.1, we have $l_- = 0$ due to the fact that $G(v) = \infty$ only when $v = 0, \infty$.

Now, integrating (3.2) from ξ_n to ξ_1 for any $n > 1$, we obtain

$$v_\xi(\xi_1) - v_\xi(\xi_n) + \int_{\xi_n}^{\xi_1} \left[(4\gamma - 1) - \frac{1}{2} e^{2\xi} \right] v_\xi(\xi) d\xi = -\int_{\xi_n}^{\xi_1} g(v(\xi)) d\xi. \tag{4.8}$$

We compute that

$$\int_{\xi_n}^{\xi_1} \left[(4\gamma - 1) - \frac{1}{2} e^{2\xi} \right] v_\xi(\xi) d\xi = (4\gamma - 1)[v(\xi_1) - v(\xi_n)] - \frac{1}{2} \int_{\xi_n}^{\xi_1} e^{2\xi} v_\xi(\xi) d\xi.$$

Furthermore,

$$\int_{\xi_n}^{\xi_1} e^{2\xi} v_\xi(\xi) d\xi = [e^{2\xi_1} v(\xi_1) - e^{2\xi_n} v(\xi_n)] - 2\int_{\xi_n}^{\xi_1} e^{2\xi} v(\xi) d\xi.$$

Then, due to the boundedness of v, the integral

$$\int_{\xi_n}^{\xi_1} e^{2\xi} v(\xi) d\xi$$

is uniformly bounded for all n. Hence the left-hand side of (4.8) is uniformly bounded for all n. However, the right-hand side of (4.8) is unbounded as $n \to \infty$. This contradiction leads to the conclusion of the lemma. $\square$

Recall from the proof of Lemma 4.2 that the set Σ_c is positively invariant for the system (4.3) for any $c \in (2\gamma - 1, 2\gamma)$. In fact, we have the following two more types of positively invariant regions.

Lemma 4.4. *There is a sufficiently large $T > 0$ such that the sets*

$$\Sigma_c^- := \{(V, W) \mid V > 0,\ 0 < W \le cV\}, \quad c \in (0, 2\gamma - 1),$$
$$\Sigma_{c_1,c_2} := \{(V, W) \mid V > 0,\ c_1 V \le W \le c_2 V\}, \quad c_1 \in (2\gamma - 1, 2\gamma),\ c_2 > 2\gamma,$$

are positively invariant regions for the system (4.3) *for $\tau \ge T$.*

Proof. Take an arbitrary $c \in (0, 2\gamma - 1) \cup (2\gamma, \infty)$ and fix it. Then there is an $\varepsilon > 0$ such that

$$c^2 - (4\gamma - 1)c + \beta > \varepsilon.$$

At $W = cV$, we have

$$\begin{aligned} \frac{dW}{dV} &= 4\gamma - 1 - \frac{1}{2}e^{-2\tau} - \beta(V/W) + V^{-p}/W \\ &\le 4\gamma - 1 - \beta/c + V^{-p}/W \\ &< \frac{c^2 - \varepsilon}{c} + V^{-p}/W = c - \frac{\varepsilon}{c} + \frac{1}{cV^{p+1}} < c \end{aligned}$$

for all sufficiently large τ such that

$$\frac{1}{V^{p+1}(\tau)} < \varepsilon,$$

due to Lemma 4.3.

Combining this with the fact that Σ_c is positively invariant region for any $c \in (2\gamma - 1, 2\gamma)$ (proved in Lemma 4.2), the lemma follows. $\square$

Now, we are ready to prove the following lemma by using the above invariant sets.

Lemma 4.5. *Let (V, W) be a global solution of* (4.3) *in $(-\infty, \infty)$ such that $V \not\equiv \kappa$. Then the limit*

$$\mu := \lim_{\tau \to \infty} \frac{W(\tau)}{V(\tau)} \tag{4.9}$$

exists and $\mu \in \{2\gamma - 1, 2\gamma\}$.

Proof. By contradiction, we assume that

$$A := \liminf_{\tau\to\infty} \frac{W(\tau)}{V(\tau)} < \limsup_{\tau\to\infty} \frac{W(\tau)}{V(\tau)} := B.$$

Note that due to Lemma 4.3 we have $W(\tau) > 0$ for all τ sufficiently large. Hence we have $0 \le A < B \le \infty$.

We divide our discussion into the following cases.

Case 1. $A < 2\gamma - 1$. In this case, we choose $c \in (A, \min\{2\gamma - 1, B\})$ and consider the positively invariant region Σ_c^-. By the definition of A, there is a sequence $\tau_n \to \infty$ such that $W(\tau_n)/V(\tau_n) \to A$ as $n \to \infty$. Let $T \gg 1$ such that $(V, W)(\tau) \in \Sigma_c^-$ for all $\tau \ge \tau_0$ as long as $(V, W)(\tau_0) \in \Sigma_c^-$ for some $\tau_0 \ge T$. Choose $N \gg 1$ such that $\tau_N \ge T$ and $W(\tau_n)/V(\tau_n) < c$ for all $n \ge N$. Without loss of generality, we may assume that $W(\tau) > 0$ for all $\tau \ge T$. Then $(V, W)(\tau_N) \in \Sigma_c^-$ and so $(V, W)(\tau) \in \Sigma_c^-$ for all $\tau \ge \tau_N$. This contradicts the definition of B.

Case 2. $A \ge 2\gamma$. In this case, we consider the invariant region Σ_{c_1,c_2} for $c_1 \in (2\gamma - 1, 2\gamma)$ and $c_2 = (A + B)/2$. Then the flow defined by (4.3) would enter the region Σ_{c_1,c_2} for all sufficiently large τ by the same argument as Case 1, a contradiction.

Case 3. $A \in [2\gamma - 1, 2\gamma)$. In this case, we consider the invariant region Σ_c for $c \in (A, \min\{2\gamma, B\})$. By taking a sequence $\tau_n \to \infty$ such that $W(\tau_n)/V(\tau_n) \to B$ as $n \to \infty$, it also leads to a contradiction to the definition of A by the a similar argument to that of Case 1.

Hence we have proved that the limit μ exists such that $0 \le \mu \le \infty$.

To proceed further, in the sequel we set $\phi(\tau) := W(\tau)/V(\tau)$. Then ϕ satisfies the equation

$$\phi'(\tau) = -[\phi(\tau) - 2\gamma][\phi(\tau) - (2\gamma - 1)] + V^{-p-1}(\tau) - \frac{1}{2} e^{-2\tau} \phi(\tau). \tag{4.10}$$

Suppose that $\mu = 0$. Then, due to Lemma 4.3, we can choose τ_0 large enough such that

$$\phi(\tau) < \gamma - 1/2, \quad V^{-p-1}(\tau) < \delta := \frac{1}{2}(\gamma^2 - 1/4), \quad V(\tau) > \kappa$$

for all $\tau \ge \tau_0$. Hence we obtain from (4.10) that

$$\phi'(\tau) < -\delta \quad \text{for all } \tau \ge \tau_0.$$

This implies that $\phi(\tau)$ reaches zero at a finite τ, say $\phi(\tau_1) = 0$ for some $\tau_1 > \tau_0$. However, from (4.3) we see that $W'(\tau_1) < 0$, since $V(\tau_1) > \kappa$. Moreover, by the phase plane analysis, $W(\tau) < 0$ for all $\tau > \tau_1$. This contradicts the fact that $V(\infty) = \infty$. Hence we must have $\mu > 0$.

Since $\mu > 0$, it follows from Lemma 4.3 that $W(\tau) \to \infty$ as $\tau \to \infty$. Applying the l'Hôpital's rule and using the system (4.3), we compute

$$\lim_{\tau\to\infty} \frac{W(\tau)}{V(\tau)} = \lim_{\tau\to\infty} \frac{W'(\tau)}{V'(\tau)} = 4\gamma - 1 - \beta \lim_{\tau\to\infty} \frac{V(\tau)}{W(\tau)}.$$

We conclude that $\mu < \infty$ and $\mu \in \{2\gamma - 1, 2\gamma\}$. $\square$

To derive the exact asymptotic behavior of V, we shall focus on the case $\mu = 2\gamma$, since the other case ($\mu = 2\gamma - 1$) does not produce any slow orbits. For this, we first prove the following lemma.

Lemma 4.6. *Assume that $\mu = 2\gamma$ for a global solution (V, W) of (4.3). Then there is a constant $\lambda \in (0, 1)$ such that*

$$\lim_{\tau\to\infty} \{e^{\lambda\tau}[\phi(\tau) - 2\gamma]\} = 0. \tag{4.11}$$

Proof. Set $\psi := \phi - 2\gamma$. Then ψ satisfies the equation

$$\psi' = -\psi - \psi^2 - \frac{1}{2}\psi e^{-2\tau} + V^{-p-1}(\tau) - \gamma e^{-2\tau}. \tag{4.12}$$

Since $\phi(\tau) \to 2\gamma$ as $\tau \to \infty$, for a given $\epsilon \in (0, q/(1 + p))$ there is a τ_0 large enough such that

$$\phi(\tau) \geq 2\gamma - \epsilon \quad \text{for all } \tau \geq \tau_0.$$

An integration gives that

$$V(\tau) \geq C_0 e^{(2\gamma-\epsilon)\tau} \quad \text{for all } \tau \geq \tau_0 \tag{4.13}$$

for some positive constant C_0. Since $(2\gamma - \epsilon)(p+1) = (2+q) - (1+p)\epsilon > 2$, by the choice of ϵ, there is a $\tau_1 > \tau_0$ such that $V^{-p-1}(\tau) - \gamma e^{-2\tau} < 0$ for all $\tau \geq \tau_1$. This implies that $\psi'(\tau) < 0$ when $\psi(\tau) = 0$ for some $\tau > \tau_1$. Hence it is easy to see that either $\psi(\tau) > 0$ for all $\tau \gg 1$, or $\psi(\tau) < 0$ for all $\tau \gg 1$.

Suppose that $\psi(\tau) > 0$ for all $\tau \gg 1$. It follows from (4.12) that $(\psi' + \psi)(\tau) < 0$ for all $\tau \gg 1$. This gives

$$0 < \psi(\tau) \leq C_1 e^{-\tau} \tag{4.14}$$

for $\tau \gg 1$ for some positive constant C_1. Then for any $\lambda \in (0, 1)$ we have

$$0 \leq \lim_{\tau\to\infty} \{e^{\lambda\tau}[\phi(\tau) - 2\gamma]\} \leq C_1 \lim_{\tau\to\infty} e^{-(1-\lambda)\tau} = 0,$$

and (4.11) follows.

On the other hand, for the case $\psi(\tau) < 0$ for all $\tau \gg 1$, taking an $\eta \in (0, 1/2)$, there exists τ_2 large enough such that $\psi(\tau) > -\eta$ for all $\tau \geq \tau_2$. Hence we have

$$\psi'(\tau) \geq -(1 - \eta)\psi - \gamma e^{-2\tau}.$$

This implies that

$$-C_2 e^{-(1-\eta)\tau} < \psi(\tau) < 0 \quad \text{for } \tau \gg 1 \text{ for some positive constant } C_2 \tag{4.15}$$

for the case $\psi(\tau) < 0$ for all $\tau \gg 1$. Now, taking $\lambda = (1 - \eta)/2 \in (0, 1)$, it follows from (4.15) that

$$0 \geq \lim_{\tau\to\infty} \{e^{\lambda\tau}[\phi(\tau) - 2\gamma]\} \geq -C_2 \lim_{\tau\to\infty} e^{-(1-\eta)\tau/2} = 0,$$

and (4.11) follows. This proves the lemma. □

Applying Lemma 4.6, we have the following lemma on the asymptotic behavior of V at $\tau = \infty$ when $\mu = 2\gamma$.

Lemma 4.7. *Suppose that* $\mu = 2\gamma$ *for a global solution* (V, W) *of* (4.3). *Then the limit*

$$\theta := \lim_{\tau\to\infty} \{e^{-2\gamma\tau} V(\tau)\}$$

exists and is positive.

Proof. For the $\lambda \in (0, 1)$ defined in Lemma 4.6, it follows from (4.11) that

$$\frac{V'(\tau)}{V(\tau)} = 2\gamma + e^{-\lambda\tau} o(1),$$

where $o(1) \to 0$ as $\tau \to \infty$. An integration gives that

$$e^{-2\gamma\tau} V(\tau) = V(\tau_0) e^{-2\gamma\tau_0} \exp\left\{\int_{\tau_0}^{\tau} e^{-\lambda s} o(1) ds\right\}$$

for $\tau > \tau_0 \gg 1$. Hence the lemma follows. □

The following lemma gives a dichotomy for the asymptotic behavior of V at $\tau = \infty$ when $\mu = 2\gamma$.

Lemma 4.8. *Suppose that* $\mu = 2\gamma$. *Set* $\psi = \phi - 2\gamma$ *and recall* (4.12). *Then we have the following two alternatives, namely, either* (I) *the limit*

$$\chi := \lim_{\tau\to\infty} [e^{\tau} \psi(\tau)] \tag{4.16}$$

exists and is a nonzero finite number, or, (II) $\lim_{\tau\to\infty}[e^{2\tau}\psi(\tau)] = \gamma$.

Proof. Due to Lemma 4.7, we have $V^{-p-1}(\tau) - \gamma e^{-2\tau} < 0$ for all $\tau \geq \tau_0$ for some $\tau_0 \gg 1$. Then $\psi'(\tau) < 0$ when $\psi(\tau) = 0$ for some $\tau \gg 1$. Hence, as before, either $\psi(\tau) > 0$ for all $\tau \gg 1$, or $\psi(\tau) < 0$ for all $\tau \gg 1$.

First, we deal with the case when $\psi(\tau) < 0$ for all $\tau \geq \tau_0$ for some $\tau_0 \gg 1$. in this case, using (4.12) we have

$$\psi'(\tau) \leq -\left(1 + \frac{1}{2}e^{-2\tau}\right)\psi(\tau),$$

since $V^{-p-1}(\tau) - \gamma e^{-2\tau} < 0$ for all $\tau \gg 1$. From this, we easily deduce that

$$\psi(\tau) \leq -C_4 e^{-(1+\eta)\tau} \tag{4.17}$$

for all $\tau \gg 1$ for some positive constant C_4. Hereafter the constant η is the same as in (4.15).

Now, we re-write (4.12) as

$$\frac{\psi'(\tau)}{\psi(\tau)} = -1 - \psi(\tau) - \frac{1}{2}e^{-2\tau} + \frac{V^{-p-1}(\tau) - \gamma e^{-2\tau}}{\psi(\tau)}.$$

Using (4.13), (4.15) and (4.17), we obtain

$$\frac{\psi'(\tau)}{\psi(\tau)} = -1 + O(e^{-(1-\eta)\tau}) \quad \text{for all } \tau \gg 1,$$

where

$$|O(e^{-(1-\eta)\tau})| \leq Ce^{-(1-\eta)\tau} \quad \text{for all } \tau \gg 1$$

for some positive constant C. An integration gives that

$$e^{\tau}\psi(\tau) = e^{\tau_0}\psi(\tau_0)\exp\left\{\int_{\tau_0}^{\tau} O(e^{-(1-\eta)s})ds\right\} \quad \text{for all } \tau \gg 1.$$

Thus the limit (4.16) exists and is a finite nonzero number. The alternative (I) holds for the case $\psi(\tau) < 0$ for all $\tau \gg 1$.

Now, we consider the case $\psi(\tau) > 0$ for all $\tau \gg 1$. Then it follows from (4.12) and (4.14) that

$$\psi'(\tau) \geq -\psi\left\{1 + Ce^{-(1-\eta)\tau} + \frac{1}{2}e^{-2\tau}\right\} - \gamma e^{-2\tau} \quad \text{for all } \tau \gg 1.$$

For the same constant η, we end up with

$$\psi'(\tau) \geq -(1+\eta)\psi(\tau) - \gamma e^{-2\tau} \quad \text{for all } \tau \gg 1.$$

This implies that

$$\psi(\tau) \geq e^{-(1+\eta)\tau}e^{(1+\eta)\tau_0}\left\{\psi(\tau_0) - \frac{\gamma}{1-\eta}e^{-2\tau_0}\right\} \tag{4.18}$$

for all $\tau \geq \tau_0$ for some $\tau_0 \gg 1$.

If

$$\psi(\tau_0) > \frac{\gamma}{1-\eta}e^{-2\tau_0} \quad \text{for some } \tau_0 \gg 1, \tag{4.19}$$

then we have

$$\psi(\tau) \geq C_3 e^{-(1+\eta)\tau} \tag{4.20}$$

for all $\tau \gg 1$ for some positive constant C_3. Using (4.13), (4.14) and (4.20), by the same argument as in the negative case, we easily prove that the limit (4.16) exists and is a finite nonzero number.

Finally, it remains to consider the case when (4.19) is violated, namely,

$$\psi(\tau_0) \leq \frac{\gamma}{1-\eta}e^{-2\tau_0} \quad \text{for all } \tau_0 \gg 1. \tag{4.21}$$

Under this assumption, we claim that

$$\lim_{\tau\to\infty}[e^{2\tau}\psi(\tau)] = \gamma. \tag{4.22}$$

For this, we set $h(\tau) := e^{2\tau}\psi(\tau)$. Then h satisfies

$$h'(\tau) - h(\tau) = -\gamma - e^{-2\tau}h^2(\tau) - \frac{1}{2}e^{-2\tau}h(\tau) + O(e^{-q\tau}), \tag{4.23}$$

where we have used Lemma 4.7. Note that h is uniformly bounded for all $\tau \gg 1$ due to (4.21).

Suppose that h is monotone. Then the limit $h(\infty)$ exists and there is a sequence $\{\tau_n\}$ tending to infinity such that $h'(\tau_n) \to 0$ as $n \to \infty$. It follows from (4.23) that $h(\infty) = \gamma$. On the other hand, suppose that h is oscillatory at $\tau = \infty$. Then along its extremal sequence $\{\tau_n\}$, $\tau_n \to \infty$ as $n \to \infty$, we have $h(\tau_n) \to \gamma$ as $n \to \infty$. Hence we also have $h(\infty) = \gamma$. Thus the claim (4.22) is proved.

The proof of the lemma is then completed. $\square$

We remark that there are no slow orbits for $\gamma > 1/2$, if the alternative (II) in Lemma 4.8 never happen. To see this, we claim that there are no global solutions (V, W) of (4.3) for $\tau \in (-\infty, \infty)$ such that (4.1) and (4.2) hold simultaneously. Let (V, W) be a global solution of (4.3) in $(-\infty, \infty)$. Recall from Lemma 4.5 that $\mu \in \{2\gamma - 1, 2\gamma\}$.

It is trivial that condition (4.2) is not valid for a global solution (V, W) of (4.3) with $\mu = 2\gamma - 1$. For $\mu = 2\gamma$, we write

$$e^{-(2\gamma-1)\tau}[V'(\tau) - 2\gamma V(\tau)] = [e^{-2\gamma\tau}V(\tau)]\{e^{\tau}[\phi(\tau) - 2\gamma]\}.$$

Then, for a global solution (V, W) of (4.3) with $\mu = 2\gamma$, Lemmas 4.7 and 4.8(I) imply that

$$\lim_{\tau\to\infty}\{e^{-(2\gamma-1)\tau}[V'(\tau) - 2\gamma V(\tau)]\} = \theta\chi \neq 0.$$

Hence the condition (4.1) does not hold for a global solution (V, W) of (4.3) with $\mu = 2\gamma$ such that the alternative (I) holds in Lemma 4.8. This proves the claim.

In the original variable y, the alternative (II) corresponds to the condition $w'(0) = 0$. It is trivial that a unique solution w of (1.3) with $w'(0) = 0$ and $w(0) > 0$ exists locally. It remains open whether the alternative (II) occurs for a backward solution of (1.3) such that (1.4) holds for some positive constant ℓ.

5. An application: a uniqueness theorem

In this section, we prove the following uniqueness theorem.

Theorem 3. *Assume that $q > p+1$. Let w be a global positive solution of* (1.3) *in* $(0,\infty)$ *such that $w(0) = 0$, $w'(0) = 0$ and* (1.4) *holds for some positive constant ℓ. Then $w = w_s$.*

Proof. Note that $\gamma > 1/2$ due to $q > p + 1$. By contradiction, we suppose that there is a global positive solution of (1.3) in $(0, \infty)$ such that $w(0) = 0$, $w'(0) = 0$ and (1.4) holds for some positive constant $\ell \neq \kappa$. Let v be the corresponding function defined by (3.1). Then v is a global solution of (3.2) and its corresponding function pair (V, W) is a global solution of (4.3) such that $V \not\equiv \kappa$. Moreover, $\mu \in \{2\gamma - 1, 2\gamma\}$ for the limit μ defined by (4.9).

Suppose that $\mu = 2\gamma - 1$. Set $\phi = 2\gamma - 1 + \omega$. Then ω satisfies the equation

$$\omega'(\tau) = \omega(\tau) - \omega^2(\tau) - \frac{1}{2}e^{-2\tau}\omega + V^{-p-1} - (\gamma - 1/2)e^{-2\tau}. \tag{5.1}$$

Due to $q > p + 1$, we can choose a positive constant ϵ small enough such that

$$(2\gamma - 1 - \epsilon)(p + 1) = 2 + q - (1 + \epsilon)(p + 1) > 2.$$

Since $\phi(\tau) \to 2\gamma - 1$ as $\tau \to \infty$, there exists a τ_0 such that

$$\frac{V'(\tau)}{V(\tau)} = \phi(\tau) > 2\gamma - 1 - \epsilon \quad \text{for all } \tau \geq \tau_0.$$

An integration gives that

$$V(\tau) \geq Ce^{(2\gamma-1-\epsilon)\tau} \quad \text{for all } \tau \geq \tau_0. \tag{5.2}$$

By taking τ_0 larger (if necessary), it follows from (5.2) that

$$V^{-p-1}(\tau) < (\gamma - 1/2)e^{-2\tau}$$

for all $\tau \geq \tau_0$. Hence $\omega'(\tau) < 0$, if $\omega(\tau) = 0$ for some $\tau \geq \tau_0$. This implies that either $\omega(\tau) > 0$ for all $\tau \gg 1$, or $\omega(\tau) < 0$ for all $\tau \gg 1$.

Suppose that $\omega(\tau) < 0$ for all $\tau \gg 1$. Then it follows from (5.1) that

$$\omega'(\tau) \leq \frac{1}{2}\omega(\tau) \quad \text{for all } \tau \gg 1,$$

due to $(1 - e^{-2\tau}/2) > 1/2$ for $\tau > 0$. Then an integration gives

$$\omega(\tau) \leq e^{\tau/2}e^{-\tau_1/2}\omega(\tau_1) \tag{5.3}$$

for all $\tau \geq \tau_1$ for some $\tau_1 \gg 1$. This leads to a contradiction, since the right-hand side of (5.3) tends to $-\infty$ as $\tau \to \infty$. We conclude that $\omega(\tau) > 0$ for all $\tau \gg 1$, i.e., $\phi(\tau) > 2\gamma - 1$ for all $\tau \gg 1$. An integration gives that $V(\tau) \geq Ke^{(2\gamma-1)\tau}$ for all $\tau \gg 1$ for some positive constant K. Then we obtain

$$\begin{aligned} & \limsup_{\tau\to\infty}\{e^{-(2\gamma-1)\tau}[V'(\tau) - 2\gamma V(\tau)]\} \\ = \ & \limsup_{\tau\to\infty}\{e^{-(2\gamma-1)\tau}V(\tau)[\phi(\tau) - 2\gamma]\} \leq -K < 0, \end{aligned}$$

a contradiction to $w'(0) = 0$.

On the other hand, if $\mu = 2\gamma$, then Lemma 4.7 implies that $w(0) > 0$. We also have reached a contradiction. Hence we must have $\ell = \kappa$. Using the energy defined by (3.19), we conclude that $V \equiv \kappa$, i.e., $w = w_s$. Therefore, the theorem is proved. $\square$

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