NON-SELF-SIMILAR DEAD-CORE RATE FOR THE FAST DIFFUSION EQUATION WITH STRONG ABSORPTION

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Abstract. We study the dead-core problem for the fast diffusion equation with strong absorption. Unlike in many other related problems of singularity formation, we show that the temporal rate of formation of the dead-core is **not self-similar**. We moreover obtain precise estimates on rescaled solutions and on the single-point final dead-core profile. Results of this type were up to now known only for problems with linear diffusion. The proofs rely on self-similar variables and require a delicate use of the Zelenyak method.

1. Introduction

We study the following initial boundary value problem:

$$\begin{cases} u_t = (u^m)_{xx} - u^p, & x \in (-1,1), \quad t > 0, \\ u(\pm 1, t) = k, & t > 0, \\ u(x,0) = u_0(x), & x \in [-1,1], \end{cases} \tag{1.1}$$

in the parameter range

$$0 < p < m < 1. \tag{1.2}$$

It is assumed that $k > 0$ and that the initial data u_0 satisfies

$$u_0 \in C([-1,1]), \quad u_0 > 0 \ \text{ in } [-1,1], \quad u_0(\pm 1) = k. \tag{1.3}$$

Also, throughout this paper, we denote

$$\beta = \frac{1}{1-p}.$$

Problem (1.1) admits a unique, global classical solution $u \geq 0$. We set

$$\Lambda(t) := \min_{|x| \leq 1} u(x,t)$$

and denote

$$T = T(u_0) := \inf \{ t > 0; \ \Lambda(t) = 0 \} > 0.$$

For suitable initial data, we shall show that $T(u_0) < \infty$ (see Theorem 1 below). We say that the solution develops a dead-core in finite time, and T is called the dead-core time. The

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main goal of this paper is to study the asymptotic behavior of the solution as $t \to T^-$ when $T < \infty$.

In the semilinear case $0 < p < m = 1$, the question of temporal dead-core rates has been studied in [23, 24, 36]. We refer to [3, 38, 5] for earlier work on the semilinear dead-core problem and to [42, 4, 13, 39, 6] for studies of the regularity and behavior of interfaces for this problem. It was shown in [23] that the rate is *not* self-similar, i.e. its order is not the same as for the corresponding ODE $y' = -y^p$. Namely, it was found that

$$\lim_{t \to T} (T-t)^{-\beta} \Lambda(t) = 0. \tag{1.4}$$

It is clear that such phenomenon is due to a tight interaction between absorption and reaction near the level $u = 0$, since diffusion dominance would prevent the appearance of a dead-core and absorption dominance would lead to an ODE rate. The corresponding Cauchy problem was further investigated in [24], in which they constructed some special solutions with different dead-core rates following the idea of Herrero and Velázquez [27, 28]. Recently, this construction for general higher spatial dimension was carried out by Seki [36].

It was also observed in [23] that the non-self-similar behavior (1.4) strongly departs from the related extinction problem for the same equation on the whole real line: starting from compactly supported initial data, the solutions becomes identically zero after a finite time and the rate of extinction is self-similar (see [10, 33, 25, 26]). Other singularity formation mechanisms in related reaction-diffusion equations, such as blow-up and quenching, also exhibit self-similar behaviors, at least in one space dimension (see [40, 11, 16, 17, 32] for blow-up and [18, 9, 19, 30] for quenching). However, another exception is the phenomenon of boundary gradient blow-up for the equation $u_t - u_{xx} = |u_x|^p$ $(p > 2)$ under Dirichlet boundary conditions, for which the rate was recently found to be non-self-similar (see [21] and cf. also [29, 31] for recent related results).

The goal of this paper is to study the dead-core rate in the presence of fast diffusion $(m < 1)$. In view of the above observation concerning the interaction of diffusion and absorption, this question is of interest since the effect of fast diffusion, as compared with linear diffusion, is much stronger near the level $u = 0$. It will turn out that again the rate is non-self-similar. Although our strategy of proof is close to that in [23], the proof is technically much more difficult due to the presence of a nonlinear diffusion operator.

The fast diffusion equation with strong absorption in (1.1) was studied before from the point of view of decay and extinction [34, 7, 8, 35, 41], and of regularity and behavior of interfaces [13, 12, 1]. We note that the porous medium or slow diffusion case $m > 1 > p$ was also studied from the latter point of view [4, 13, 14, 15, 12] and that the existence of finite-time dead-core was proved in [2]. However, our methods do not seem apply for $m > 1$ and the dead-core rate remains an open question in that case.

Our first result gives sufficient conditions under which the solution of problem (1.1) develops a dead-core in finite time. To formulate this, let us first recall some well-known facts: (1.1) admits a unique steady state $U_k \in C^2([-1,1])$ for each given $k > 0$. Moreover, U_k is an even and nondecreasing function of $|x|$ and it is a nondecreasing function of k. Furthermore, there exists $k_0 = k_0(m,p) > 0$ such that: if $k \in (0, k_0)$ then U_k vanishes on an interval of positive length, if $k = k_0$ then U_k vanishes only at $x = 0$, and if $k > k_0$ then U_k is positive.

Theorem 1. *Assume* (1.2) *and* (1.3).

(i) Let $0 < k < k_0$. *Then* $T(u_0) < \infty$ *for any* u_0.

(ii) Let $k \geq k_0$. *For any* $\eta, M > 0$ *there exists* $\delta = \delta(\eta, M) > 0$ *such that* $T(u_0) < \infty$ *whenever* $\|u_0\|_\infty \leq M$ *and* $u_0 \leq \delta$ *on a subinterval of* $[-1,1]$ *of length* η.

Remark 1.1. We note that the assumption $k < k_0$ in Theorem 1(i) is essentially optimal due to existence of a positive steady state for $k > k_0$. The assumption (1.2) is also necessary throughout this paper, since the dead-core phenomenon never occurs if $p \geq m$ (indeed $\underline{u}(x) = [\varepsilon(1+x^2)]^{1/m}$ is then a positive subsolution to (1.1) for sufficiently small $\varepsilon > 0$).

For our main results on the asymptotic dead-core behavior, we shall assume that u_0 satisfies the conditions

$$u_0 \in C^2([-1,1]), \qquad (u_0^m)'' \leq u_0^p \quad in \quad [-1,1], \tag{1.5}$$

and

$$u_0 \ \ is \ \ even \ \ and \ \ nondecreasing \ \ in \ \ |x| \ \ and \ \ T(u_0) < \infty. \tag{1.6}$$

It then follows from the strong maximum principle that $u_t < 0$ in $Q_T := (-1,1) \times (0,T)$, $u(-x,t) = u(x,t)$ for $(x,t) \in Q_T$ and $u_x > 0$ in $(0,1) \times (0,T)$.

Theorem 2. *Let* $k > 0$ *and assume* (1.2), (1.3), (1.5) *and* (1.6). *Then*

$$\lim_{t \to T^-} (T-t)^{-\beta} \Lambda(t) = 0.$$

As a basic ingredient in the proof of Theorem 2, we use the following self-similar variables

$$y = \frac{x}{(T-t)^\alpha}, \quad s = -\ln(T-t), \quad z(y,s) := \left[\frac{u(x,t)}{(T-t)^\beta}\right]^m, \tag{1.7}$$

where the exponent α is given by

$$\alpha = \frac{m-p}{2(1-p)}.$$

Then z satisfies the equation

$$\gamma z^{\gamma-1} z_s = z_{yy} - \alpha\gamma y z^{\gamma-1} z_y + \beta z^\gamma - z^q \quad \text{in} \ \ \Omega, \tag{1.8}$$

where $\gamma := 1/m$, $q := p/m$, and

$$\Omega := \{(y,s);\ |y| < e^{\alpha s},\ -\ln(T) =: s_0 < s < \infty\}\,.$$

The boundary and initial conditions are transformed into

$$z(e^{\alpha s}, s) = k^m e^{\beta m s}, \quad s > s_0, \tag{1.9}$$

$$z(y, s_0) = z_0(y) := T^{-\beta m} u_0^m(yT^\alpha), \quad y \in [-T^{-\alpha}, T^{-\alpha}]. \tag{1.10}$$

Theorem 2 will actually be a consequence of the following more precise result:

Theorem 3. *Under the assumptions of Theorem 2, the corresponding global solution z of* (1.8)-(1.10) *satisfies*

$$\lim_{s\to\infty} z(y,s) = V_1(y) := k_{p,m}|y|^{\frac{2m}{m-p}},$$

where $k_{p,m} = (\frac{(m-p)^2}{2m(m+p)})^{\frac{m}{m-p}}$, uniformly on $\{|y| < R\}$, for any $R > 0$ fixed.

The key ingredients to prove Theorem 3 are the following:

(1) derive some a priori estimates of z from above and below;
(2) construct a Lyapunov function by the method of Zelenyak [43] (cf. also [22]). Note that, unlike for $m = 1$, the standard energy argument does not work here;
(3) classify the possible steady states for z on the whole real line.

In the course of the proof, we shall obtain the following precise estimates on the single-point final dead-core profile near $x = 0$.

Theorem 4. *Under the assumptions of Theorem 2, there exist $c_1, c_2 > 0$ (depending on u) such that*

$$c_1|x|^{2/(m-p)} \le u(x,T) \le c_2|x|^{2/(m-p)}, \qquad |x| \le 1.$$

This paper is organized as follows. Section 2 is devoted to the proof of Theorem 1. Some a priori estimates will be derived in section 3. As a by-product, this also gives a proof of Theorem 4. In section 4, we study the stationary equation associated with (1.8), from the point of view of uniqueness of global solutions (with suitable growth at infinity) and of backward continuation of local solutions. This is needed, respectively, in the study of omega limits and in the construction of a Lyapunov function. Finally, the proof of Theorem 3 is given in section 5.

2. Proof of Theorem 1

Step 1. We look for a supersolution $\overline{u}$ of $u_t - (u^m)_{xx} + u^p = 0$ in $Q_T := (-1,1)\times(0,T)$, which develops a dead-core at time T. Namely, motivated by an idea from [37], for any $T \in (0,T_0)$ we shall construct $\overline{u}$ under the following self-similar form:

$$\overline{u}(x,t) = \varepsilon(T-t)^{\beta} V(y), \qquad y = x(T-t)^{-\gamma}, \qquad V(y) = (1+y^2)^{\nu},$$

where

$$0 < \gamma < \frac{\beta(m-p)}{2} = \frac{m-p}{2(1-p)}$$

and $\nu, \varepsilon, T_0 > 0$ will be determined. Note that $\overline{u}(0,T) = 0$. Simple computations yield

$$\begin{aligned} P\overline{u} &:= \overline{u}_t - (\overline{u}^m)_{xx} + \overline{u}^p \\ &= \varepsilon(T-t)^{\beta-1}(-\beta V + \gamma y V') - \varepsilon^m (T-t)^{\beta m - 2\gamma}(V^m)'' + \varepsilon^p (T-t)^{\beta p} V^p \\ &= \varepsilon(T-t)^{\beta p}\big\{\varepsilon^{p-1}V^p - \beta V + \gamma y V' - \varepsilon^{m-1}(T-t)^{\theta}(V^m)''\big\} \end{aligned}$$

for $(x,t) \in Q_T$, where $\theta = \beta(m-p) - 2\gamma > 0$. Assuming $T \le T_0(\varepsilon) := \varepsilon^{(1-m)/\theta}$, we see that

$$P\overline{u} \ge \varepsilon(T-t)^{\beta p}\big\{\varepsilon^{p-1} - h(y)\big\}, \quad \text{where } h(y) = \beta V - \gamma y V' + |(V^m)''|.$$

Next taking $\nu > \beta/(2\gamma)$ and using $|(V^m)''| \sim C|y|^{2m\nu-2}$ as $|y| \to \infty$, we observe that

$$h(y) \sim (\beta - 2\gamma\nu)|y|^{2\nu} \to -\infty, \quad \text{as } |y| \to \infty.$$

It follows that $\sup_{y\in\mathbb{R}} h(y) < \infty$ and choosing $\varepsilon = \varepsilon(m,p,\gamma,\nu) > 0$ sufficiently small, we conclude that $P\overline{u} \ge 0$ in Q_T. For further reference we also note that

$$\overline{u}(x,t) \ge \varepsilon|x|^{2\nu} T^{-\mu} \ \text{ in } Q_T, \quad \text{where } \mu = 2\gamma\nu - \beta > 0. \tag{2.1}$$

Step 2. We prove assertion (ii). Fix $\eta, M > 0$ and $x_0 \in [-1+\eta/2, 1-\eta/2]$. Let $\overline{u}$, T_0 be as in Step 1 and set $\overline{v}(x,t) = \overline{u}(x-x_0,t)$. Taking $T \le \min(T_0, T_1)$, where $T_1 = T_1(\eta, M) > 0$ is sufficiently small, and using (2.1), we see that $\overline{v}(x,t) \ge M$ for $|x-x_0| \ge \eta/2$ and $t \in [0,T)$, hence in particular $\overline{v}(\pm 1, t) \ge k$. Next put $\delta := \min_{|x-x_0|\le\eta/2} \overline{v}(x,0)$. Then assuming $\|u_0\|_\infty \le M$ and $u_0 \le \delta$ for $|x-x_0| < \eta/2$, we get $u_0 \le \overline{v}(\cdot,0)$ and it follows from the comparison principle that $u \le \overline{v}$ in Q_T, hence $T(u_0) \le T < \infty$. This proves assertion (ii).

Step 3. We prove assertion (i). First observe that assertion (ii) is actually true for any $k > 0$ in view of Step 2. On the other hand, by standard energy arguments, one can show that $u(\cdot,t)$ converges to U_k in $L^\infty(-1,1)$ as $t \to \infty$. Since $U_k = 0$ on $[-\eta/2, \eta/2]$ for some $\eta > 0$, it follows that for t_0 large, the new initial data $\tilde{u}_0 := u(\cdot, t_0)$ satisfies the assumptions of part (ii) with $M = k+1$. The conclusion follows. □

3. Some a priori estimates

In this section, we shall derive some a priori estimates for solutions of (1.1). Let u be a solution of (1.1). Since $u_t < 0$ in Q_T and $u_x > 0$ in $(0,1)\times(0,T)$, we have $(u^m)_{xx} < u^p$. Let $v = u^m$. Then from $v_x = mu^{m-1}u_x$, $0 < u \le k$ in Q_T and

$$v_{xx}v_x < v^q v_x \quad \text{in } (0,1)\times(0,T),$$

it follows that v_x (and so u_x) is bounded in Q_T. We have the following two lemmas (which in particular imply Theorem 4).

Lemma 3.1. *Let the assumptions of Theorem 2 be in force and fix $t_0 \in (0,T)$. Then there exists $\varepsilon > 0$ such that the auxiliary function*

$$J := (u^m)_x - \varepsilon x u^p$$

satisfies $J \ge 0$ in $[0,1]\times(t_0,T)$. In particular, there exists $c > 0$ such that

$$u(x,t) \ge c|x|^{2/(m-p)}, \quad x \in (-1,1),\ t_0 < t < T. \tag{3.1}$$

Proof. Notice that the differential equation in (1.1) can be written under the form

$$u_t - au_{xx} = m(m-1)u^{m-2}(u_x)^2 - u^p,$$

with

$$a = mu^{m-1}.$$

For $(x,t) \in (0,1)\times(0,T)$, we compute

$$\begin{aligned}
(xu^p)_t &= pxu^{p-1}u_t,\\
(xu^p)_x &= u^p + pxu^{p-1}u_x,\\
(xu^p)_{xx} &= 2pu^{p-1}u_x + p(p-1)xu^{p-2}(u_x)^2 + pxu^{p-1}u_{xx}.
\end{aligned}$$

Therefore

$$\begin{aligned}
(xu^p)_t - a(xu^p)_{xx} &= pxu^{p-1}\big(u_t - au_{xx}\big) - a\big(2pu^{p-1}u_x + p(p-1)xu^{p-2}(u_x)^2\big)\\
&= -pxu^{2p-1} - 2mpu^{p+m-2}u_x - mp(p-m)xu^{p+m-3}(u_x)^2\\
&= -pxu^{2p-1} - 2pu^{p-1}(u^m)_x - m^{-1}p(p-m)xu^{p-m-1}((u^m)_x)^2.
\end{aligned}$$

Using

$$(u^m)_x = J + \varepsilon x u^p,$$

we deduce that

$$(xu^p)_t - a(xu^p)_{xx} = b_1 J - pxu^{2p-1}\big\{1 + 2\varepsilon + m^{-1}(p-m)\varepsilon^2 x^2 u^{p-m}\big\}. \tag{3.2}$$

Here and below, $b_1, b_2, \dots$ denote functions which are smooth in $[-1,1]\times(0,T)$. On the other hand, we have

$$\begin{aligned}(u^m)_{xt} - a(u^m)_{xxx} &= (au_x)_t - a(u^m)_{xxx} = a(u_t - (u^m)_{xx})_x + a_t u_x\\ &= -pau^{p-1}u_x + m(m-1)u^{m-2}u_t u_x\\ &= [-p + (m-1)u^{-p}u_t]u^{p-1}(u^m)_x\\ &= b_2 J + \varepsilon x u^{2p-1}[-p + (m-1)u^{-p}u_t]\end{aligned}$$

hence

$$(u^m)_{xt} - a(u^m)_{xxx} = b_2 J + \varepsilon x u^{2p-1}[1 - m - p + (m-1)u^{-p}(u^m)_{xx}].$$

Since

$$\begin{aligned}(u^m)_{xx} &= (J + \varepsilon x u^p)_x = J_x + \varepsilon(u^p + pxu^{p-1}u_x)\\ &= J_x + \varepsilon u^p[1 + pm^{-1}xu^{-m}(u^m)_x]\\ &= J_x + b_3 J + \varepsilon u^p[1 + pm^{-1}\varepsilon x^2 u^{p-m}],\end{aligned}$$

it follows that

$$(u^m)_{xt} - a(u^m)_{xxx} = b_4 J + b_5 J_x + \varepsilon x u^{2p-1}\big\{1 - m - p + (m-1)\varepsilon[1 + pm^{-1}\varepsilon x^2 u^{p-m}]\big\}. \tag{3.3}$$

Combining (3.2) and (3.3), we obtain

$$\begin{aligned}J_t - aJ_{xx} - b_6 J - b_7 J_x &= \varepsilon x u^{2p-1}\big\{1 - m - p + (m-1)\varepsilon[1 + pm^{-1}\varepsilon x^2 u^{p-m}]\big\}\\ &\quad + p\varepsilon x u^{2p-1}\big\{1 + 2\varepsilon + m^{-1}(p-m)\varepsilon^2 x^2 u^{p-m}\big\}\\ &= \varepsilon x u^{2p-1}\big\{1 - m + (m-1)\varepsilon[1 + pm^{-1}\varepsilon x^2 u^{p-m}]\\ &\quad + p\varepsilon[2 + m^{-1}(p-m)\varepsilon x^2 u^{p-m}]\big\}.\end{aligned}$$

Since $m < 1$, by choosing $0 < \varepsilon \le \varepsilon_0$ with $\varepsilon_0 = \varepsilon_0(m,p) > 0$ small enough, it follows that

$$\begin{aligned}J_t - aJ_{xx} - b_6 J - b_7 J_x &\ge \varepsilon x u^{2p-1}\Big\{\frac{1-m}{2} - \frac{p(1-p)}{m}\varepsilon^2 x^2 u^{p-m}\Big\}\\ &= \frac{1-m}{2}\varepsilon x u^{3p-m-1}\big\{u^{m-p} - \lambda\varepsilon^2 x^2\big\},\end{aligned}$$

where $\lambda := 2p(1-p)/[m(1-m)] > 0$. Now observe that

$$\begin{aligned}\Big[u^{m-p} - \frac{m-p}{2m}\varepsilon x^2\Big]_x &= (m-p)[u^{m-p-1}u_x - m^{-1}\varepsilon x]\\ &= \frac{m-p}{m}u^{-p}\big[(u^m)_x - \varepsilon x u^p\big] = \frac{m-p}{m}u^{-p}J,\end{aligned}$$

hence

$$u^{m-p} - \frac{m-p}{2m}\varepsilon x^2 \ge \frac{m-p}{m}\int_0^x u^{-p}J(y,t)\,dy.$$

Thus, taking ε_0 possibly smaller, we get

$$J_t - aJ_{xx} - b_6 J - b_7 J_x \ge \frac{1-m}{2}\varepsilon x u^{3p-m-1}\big\{u^{m-p} - \frac{m-p}{2m}\varepsilon x^2\big\}$$

hence

$$J_t - aJ_{xx} - b_6 J - b_7 J_x \geq C\varepsilon x u^{3p-m-1} \int_0^x u^{-p} J(y,t)\, dy,$$

with $C = (1-m)(m-p)/2m > 0$. Now, for any $0 < t_0 < t_1 < T$, it follows from the maximum principle that J attains its minimum in $Q = [0,1] \times [t_0, t_1]$ on the parabolic boundary of Q (see p.659 of [23] for details in a similar situation).

It is thus sufficient to check that $J \geq 0$ on the parabolic boundary of Q for ε small. Clearly $J = 0$ for $x = 0$. Since u_x is bounded on Q_T, $u(x,t) \geq \eta > 0$ in $[1-\delta, 1] \times (t_0, T)$ for some small constant $\delta > 0$. Therefore u extends to a classical solution on $[1-\delta, 1] \times (t_0, T]$ and Hopf's Lemma implies that $u_x(1,t) \geq \tilde{\delta} > 0$ for $t_0 < t < T$, hence $J(1,t) \geq 0$ for $t_0 < t < T$ if ε is chosen small enough. Moreover, also as a consequence of Hopf's Lemma, we have $u_x(x, t_0) \geq cx$ in $[0,1]$ for some $c > 0$. Again decreasing ε if necessary, we deduce that $J(x, t_0) \geq 0$ in $[0,1]$. The lemma follows. □

From (3.1) and using similarity variables, we have for any fixed $t_0 \in (0, T)$,

$$z(y,s) \geq D^* |y|^\delta \quad \text{in } \Omega_0 := \{(y,s);\ |y| < e^{\alpha s},\ -\ln(T-t_0) < s < \infty\}, \tag{3.4}$$

where $\delta := \frac{2m}{m-p} = \frac{2}{1-q}$ and $D^* > 0$. Next we shall prove a local uniform bound for z.

Lemma 3.2. *Under the assumptions of Theorem 2, the corresponding global solution z of* (1.8) *satisfies*

$$z(y,s) \leq c(1+|y|)^{\frac{2}{1-q}}, \tag{3.5}$$

$$|z_y(y,s)| \leq c|y|^{\frac{1+q}{1-q}}, \quad \text{if } |y| \geq 1, \tag{3.6}$$

$$|z_y(y,s)| \leq c|y|, \quad \text{if } |y| \leq 1, \tag{3.7}$$

for some constant $c > 0$ in $\{(y,s); |y| < e^{\alpha s},\ -\ln(T-t_0) < s < \infty\}$.

Proof. Without loss of generality, we may consider the case $y > 0$. Since $x = 0$ is the unique minimum point for each t, we have $u_t(0,t) \geq -u^p(0,t)$. By an integration we obtain

$$z(0,s) \leq \kappa := (1-p)^{\frac{m}{1-p}} \quad \text{for any } s \geq s_0.$$

Since $u_t < 0$ in Q_T and (1.7), we obtain $\gamma z^{\gamma-1} z_s + \alpha\gamma y z^{\gamma-1} z_y - \beta z^\gamma < 0$ in Ω and so

$$z_{yy} - z^q < 0 \quad \text{in } \Omega. \tag{3.8}$$

Multiplying (3.8) by z_y and integrating it over $[0, y]$ for each s, we obtain

$$\frac{1}{2} z_y^2(y,s) \leq \frac{1}{q+1}(z^{q+1}(y,s) - z^{q+1}(0,s)) < \frac{1}{q+1} z^{q+1}(y,s). \tag{3.9}$$

Then (3.5) follows by an integration of (3.9) from 0 to y. Clearly, (3.6) follows from (3.5) and (3.9), whereas (3.7) follows from (3.5), (3.8) and $z_y(0,s) = 0$. □

4. The associated ordinary differential equation

This section is devoted to the study of the ordinary differential equation associated with (1.8). We first consider the equation on the whole real line $\mathbb{R}$.

$$V'' - \alpha\gamma y V^{\gamma-1}V' + \beta V^\gamma - V^q = 0, \quad y \in \mathbb{R}. \tag{4.1}$$

Recall that

$$\alpha = \frac{m-p}{2(1-p)}, \quad \beta = \frac{1}{1-p}, \quad \gamma = \frac{1}{m}, \quad q = \frac{p}{m}, \quad 0 < p < m < 1.$$

Proposition 4.1. *Let $V \in C^2(\mathbb{R})$ be a solution of (4.1) such that*

$$V = V(|y|), \quad \text{with} \quad V' \geq 0, \quad V > 0 \quad \text{for all } y > 0,$$

and such that V is polynomially bounded. Then

$$V = V_1 := k_{p,m}|y|^{\frac{2m}{m-p}} \quad \text{or} \quad V = V_2 := \kappa$$

where $k_{p,m} = (\frac{(m-p)^2}{2m(m+p)})^{\frac{m}{m-p}}$ and $\kappa = (1-p)^{\frac{m}{1-p}}$.

Proof. Let $W := V^{\frac{m-p}{m}}$ and denote $' = d/dy$. At any point $y \in \mathbb{R}$ such that $W(y) > 0$, the equation for W is:

$$W'' - \frac{m-p}{2m(1-p)} y W^{\frac{1-m}{m-p}} W' + \frac{p}{m-p}\frac{(W')^2}{W} + \frac{m-p}{m(1-p)} W^{\frac{1-p}{m-p}} = \frac{m-p}{m}. \tag{4.2}$$

By differentiating (4.2), we obtain that

$$\begin{aligned} W''' - \frac{m-p}{2m(1-p)} y W^{\frac{1-m}{m-p}} W'' + \frac{2-m-p}{2m(1-p)} W^{\frac{1-m}{m-p}} W' \\ - \frac{1-m}{2m(1-p)} y W^{\frac{1-2m+p}{m-p}} (W')^2 = -\frac{p}{m-p}\frac{W'(2W''W - (W')^2)}{W^2}. \end{aligned} \tag{4.3}$$

Set $H := W - yW'/2$. Then

$$H' = \frac{1}{2}W' - \frac{y}{2}W'', \quad H'' = -\frac{y}{2}W'''.$$

Define

$$D := \big\{ y \in \mathbb{R};\ W(y) > 0 \text{ and } H(y) \neq 0 \big\}.$$

The function $Z := |H|^{\frac{m}{m-p}}$ is smooth in D and we can compute

$$Z' = \frac{m}{m-p}|H|^{\frac{2p-m}{m-p}} HH', \quad Z'' = \frac{m}{m-p}|H|^{\frac{2p-m}{m-p}} \Big(HH'' + \frac{p}{m-p} H'^2 \Big).$$

Hence

$$Z'' - \frac{m-p}{2m(1-p)} y W^{\frac{1-m}{m-p}} Z' = \frac{m}{m-p}|H|^{\frac{2p-m}{m-p}} F, \quad y \in D,$$

where

$$F := \frac{p}{m-p} H'^2 + H\Big(H'' - \frac{m-p}{2m(1-p)} y W^{\frac{1-m}{m-p}} H' \Big).$$

From the definition of H, we have

$$F=\frac{p}{m-p}\Big(\frac{1}{2}W'-\frac{y}{2}W''\Big)^2-\frac{y}{2}(W-\frac{y}{2}W')\Big[W'''+\frac{m-p}{2m(1-p)}W^{\frac{1-m}{m-p}}(W'-yW'')\Big].$$

Using (4.3), it follows that

$$\begin{aligned}
F &= \frac{p}{m-p}\Big(\frac{1}{2}W'-\frac{y}{2}W''\Big)^2-\frac{y}{2}(W-\frac{y}{2}W')\Big[\frac{-2+2m}{2m(1-p)}W^{\frac{1-m}{m-p}}W'\\
&\quad+\frac{1-m}{2m(1-p)}yW^{\frac{1-2m+p}{m-p}}W'^2-\frac{p}{m-p}\frac{W'(2W''W-W'^2)}{W^2}\Big]\\
&= \frac{p}{4(m-p)}\Big\{(W'-yW'')^2-y(2W-yW')\Big[-\frac{(1-m)(m-p)}{mp(1-p)}W^{\frac{1-m}{m-p}}W'\\
&\quad+\frac{(1-m)(m-p)}{2mp(1-p)}yW^{\frac{1-2m+p}{m-p}}W'^2-\frac{W'(2W''W-W'^2)}{W^2}\Big]\Big\}\\
&= \frac{p}{4(m-p)}\Big\{\Big[(W'-yW'')^2+y(2W-yW')\frac{W'(2W''W-W'^2)}{W^2}\Big]\\
&\quad+y(2W-yW')\frac{(1-m)(m-p)}{mp(1-p)}W^{\frac{1-2m+p}{m-p}}\Big(WW'-\frac{y}{2}W'^2\Big)\Big\}.
\end{aligned}$$

Then we have

$$\begin{aligned}
F &= \frac{p}{4(m-p)}\Big\{\Big[W'+y\Big(\frac{WW''-W'^2}{W}\Big)\Big]^2\\
&\quad+y(2W-yW')\frac{(1-m)(m-p)}{mp(1-p)}W^{\frac{1-2m+p}{m-p}}\Big(WW'-\frac{y}{2}W'^2\Big)\Big\}\\
&= \frac{p}{4(m-p)}\Big\{\Big[W'+y\Big(\frac{WW''-W'^2}{W}\Big)\Big]^2\\
&\quad+\frac{(1-m)(m-p)}{mp(1-p)}yW^{\frac{1-2m+p}{m-p}}(2W-yW')W'\Big(W-\frac{y}{2}W'\Big)\Big\}\\
&= \frac{p}{4(m-p)}\Big\{\Big[W'+y\Big(\frac{WW''-W'^2}{W}\Big)\Big]^2\\
&\quad+\frac{2(1-m)(m-p)}{mp(1-p)}yW^{\frac{1-2m+p}{m-p}}\Big(W-\frac{y}{2}W'\Big)^2W'\Big\}\geq 0
\end{aligned}$$

for all $y\in D$, and hence

$$(e^{-\rho(y)}Z')'\geq 0 \ \text{ in } \ D, \tag{4.4}$$

where

$$\rho(y):=\int_0^y\frac{m-p}{2m(1-p)}sW^{\frac{1-m}{m-p}}(s)ds.$$

Next, we claim that the function Z is nonincreasing in $[0,\infty)$. Otherwise, we can find a $y_0>0$ such that $Z(y_0)>0$ and $Z'(y_0)>0$. Hence $e^{-\rho(y)}Z'(y)\geq e^{-\rho(y_0)}Z'(y_0)$ for $y\geq y_0$ by

(4.4). Noting that $W(y_0) > 0$ and $W' \geq 0$ for all $y > 0$, it follows that for $y \geq y_0$

$$\rho(y) \geq \frac{m-p}{2m(1-p)} \int_{y_0}^{y} sW(y_0)^{\frac{1-m}{m-p}} ds \geq cy^2$$

for some $c > 0$. Therefore, we get $Z'(y) \geq e^{-\rho(y_0)} Z'(y_0) e^{cy^2} = \widehat{C} e^{cy^2}$ for $y \geq y_0$, where $\widehat{C} = e^{-\rho(y_0)} Z'(y_0) > 0$. From $|(y^{-2}W)'| = 2y^{-3} Z^{1-(p/m)}$, we would get $W \geq e^{\hat{\eta} y^2}$ as $y \to \infty$ for some $\hat{\eta} > 0$, contradicting the polynomially bounded assumption on V. Hence Z is nonincreasing in $[0, \infty)$.

Now, we claim that Z is constant on $(0, \infty)$. For contradiction, we assume that there is $R > 0$ such that $Z(0) > Z(R)$. Then we can choose $\epsilon > 0$ small enough such that $f(0) > Z(0) > f(R)$, where $f := Z + \epsilon e^{2\rho(y)}$. Hence f has a local maximum at some point y_1 in $(-R, R)$. Note that $Z(0) > 0$. Hence $W(0) > 0$ and so we have $D = \{y \in \mathbb{R};\ H(y) \neq 0\}$. Also, it follows from

$$\begin{aligned} &Z(y_1) + \epsilon e^{2\rho(R)} \geq Z(y_1) + \epsilon e^{2\rho(y_1)} = f(y_1) \geq f(0) > Z(0), \\ &Z(0) > f(R) = Z(R) + \epsilon e^{2\rho(R)}, \end{aligned}$$

that $Z(y_1) > Z(R)$. Hence $Z(y_1) > 0$ and so $y_1 \in D$.

Denoting $a = \frac{1-m}{m-p} > 0$ and $c = \frac{m-p}{2m(1-p)} > 0$, we compute, for $y \in D$,

$$\begin{aligned} \rho'' &= c\big[W^a + y(W^a)'\big] \geq cW^a(0) > 0, \\ f' &= Z' + 2\epsilon\rho' e^{2\rho(y)}, \\ f'' &= Z'' + \epsilon(2\rho')^2 e^{2\rho(y)} + 2\epsilon\rho'' e^{2\rho(y)} > Z'' + 4\epsilon\rho'^2 e^{2\rho(y)}. \end{aligned}$$

Therefore, at $y = y_1$, using the fact that f has a local maximum and (4.4), we obtain

$$0 \leq \rho' f' - f'' < \rho'(Z' + 2\varepsilon\rho' e^{2\rho}) - (Z'' + 4\varepsilon\rho'^2 e^{2\rho}) = (\rho' Z' - Z'') - 2\varepsilon\rho'^2 e^{2\rho} \leq 0,$$

a contradiction. We conclude that $W - yW'/2 = C$ on $(0, \infty)$ for some constant C. By an integration, we get $W = A + By^2$ for some constants A and B. Putting this into (4.2), the conclusion follows. □

Remark 4.1. Proposition 4.1 generalizes [23, Proposition 3.3] (for $m = 1$). Actually there was a small inaccuracy in the proof of [23, Proposition 3.3], since D was defined there as $\{y > 0;\ H(y) \neq 0\}$ and we cannot rule out the possibility that $y_1 = 0$. This is easily overcome by the current definition of D (the present proof works for $m = 1$ as well).

Next, in order to construct a Lyapunov function, we need to study the backward continuation of solution to the ODE associated with (1.8). For this, we take a smooth and nonincreasing function ζ on $\mathbb{R}$ such that

$$\zeta(\varrho) = 0,\ \varrho \geq 2, \quad \zeta(\varrho) = 1,\ \ \varrho \leq 1, \quad 0 \leq \zeta(\varrho) \leq 1, \quad \varrho \in (1, 2).$$

Let $g(v) := \beta v^\gamma - v^q$. Following [22], we define

$$\begin{aligned}\widehat{g}(\xi, v) &= g(v)\left[1 - \zeta\left(\frac{2v}{D^*\xi^\delta\zeta(\xi) + D^*[1-\zeta(\xi)]}\right)\right] \\ &\quad -v^{-1}\zeta\left(\frac{2v}{D^*\xi^\delta\zeta(\xi) + D^*[1-\zeta(\xi)]}\right).\end{aligned} \tag{4.5}$$

Without loss of generality, we may assume that $D^* < \kappa/\left(2^\delta\right)$. Note that $\widehat{g}(\xi, v) = g(v)$ for all ξ whenever $v \geq \kappa$.

Let $\psi(\xi; y, v, w)$ be the solution of the problem:

$$\psi_{\xi\xi} - \alpha\gamma\xi\psi^{\gamma-1}\psi_\xi + \widehat{g}(\xi, \psi) = 0, \quad \xi < y, \tag{4.6}$$

$$\psi(y; y, v, w) = v, \quad \psi_\xi(y; y, v, w) = w, \tag{4.7}$$

where $v > 0$, $w \in \mathbb{R}$, and the subscript ξ denotes the derivative with respect to ξ. By the standard ODE theory, $\psi(\xi; y, v, w)$ is defined in a neighborhood of $\xi = y$. Clearly, the solution $\psi(\xi; y, v, w)$ can be extended as long as $\psi > 0$ and ψ, ψ_ξ remain bounded.

Now, we assume that $(\hat{y}, y]$, $0 \leq \hat{y} < y$, is the maximal existence interval for the solution ψ of (4.6)-(4.7) in $[0, y]$. We shall prove that $\hat{y} = 0$, i.e., the solution ψ can be continued backward to $\xi = 0$. For this, multiplying (4.6) by ψ_ξ and integrating in ξ, we have

$$\begin{aligned}&\frac{1}{2}\psi_\xi^2(\xi; y, v, w) + \alpha\gamma\int_\xi^y \tau\psi^{\gamma-1}(\tau; y, v, w)\psi_\xi^2(\tau; y, v, w)d\tau + \int_\kappa^{\psi(\xi;y,v,w)} \widehat{g}(\xi, \mu)d\mu \\ &\quad = \frac{1}{2}w^2 + \int_\kappa^v \widehat{g}(y, \mu)d\mu - \int_\xi^y \int_\kappa^{\psi(\tau;y,v,w)} \widehat{g}_\xi(\tau, \mu)d\mu d\tau.\end{aligned} \tag{4.8}$$

Next, we have to estimate the last term in the above equation. Clearly, we have

$$\widehat{g}_\xi(\xi, v) = 2v[g(v) + v^{-1}]\zeta'\left(\frac{2v}{D^*\xi^\delta\zeta(\xi) + D^*[1-\zeta(\xi)]}\right)\frac{\left\{D^*\xi^\delta\zeta(\xi) + D^*[1-\zeta(\xi)]\right\}'}{\left\{D^*\xi^\delta\zeta(\xi) + D^*[1-\zeta(\xi)]\right\}^2}.$$

Hence

$$\widehat{g}_\xi(\xi, v) = 0 \ \text{ in } \{v > D^*\xi^\delta\zeta(\xi) + D^*[1-\zeta(\xi)]\} \cup \{v < \{D^*\xi^\delta\zeta(\xi) + D^*[1-\zeta(\xi)]\}/2\}, \tag{4.9}$$

and so $|\widehat{g}_\xi(\xi, v)|$ is uniformly bounded for $\xi > 1$ for all $v > 0$. Therefore, the integral

$$-\int_\xi^y \int_\kappa^{\psi(\tau;y,v,w)} \widehat{g}_\xi(\tau, \mu)d\mu d\tau$$

is bounded if $1 < \xi < y$. This bound may depend on y, but it is independent of ξ, v, w.

If $\xi \leq 1$, then we have

$$\widehat{g}_\xi(\xi, v) = 2v[g(v) + v^{-1}]\zeta'\left(\frac{2v}{D^*\xi^\delta}\right)\frac{\delta}{D^*\xi^{\delta+1}}.$$

Note that $D^* < \kappa$. Hence

$$g(v) + v^{-1} \geq 0 \quad \text{for} \quad 0 < v < D^*\xi^\delta, \quad 0 < \xi \leq 1.$$

Since $\zeta' \le 0$, we obtain

$$\widehat{g}_\xi(\xi, v) \le 0 \quad \text{for} \quad 0 < v < D^*\xi^\delta, \quad 0 < \xi \le 1. \tag{4.10}$$

For $0 \le \xi \le 1$, we write

$$\begin{aligned} -\int_\xi^y \int_\kappa^{\psi(\tau;y,v,w)} \widehat{g}_\xi(\tau,\mu)d\mu d\tau &= -\int_\xi^{\min\{1,y\}} \int_\kappa^{\psi(\tau;y,v,w)} \widehat{g}_\xi(\tau,\mu)d\mu d\tau \\ &\quad -\int_{\min\{1,y\}}^y \int_\kappa^{\psi(\tau;y,v,w)} \widehat{g}_\xi(\tau,\mu)d\mu d\tau. \end{aligned}$$

Since the last integral in the above equation is bounded, from (4.9) and (4.10) it follows that

$$-\int_\xi^y \int_\kappa^{\psi(\tau;y,v,w)} \widehat{g}_\xi(\tau,\mu)d\mu d\tau \le C + \int_\xi^{\min\{1,y\}} \int_{\psi(\tau;y,v,w)}^\kappa \widehat{g}_\xi(\tau,\mu)d\mu d\tau \le C, \tag{4.11}$$

where the constant C is independent of w, v, and ξ (it may depend on y). Hence, by (4.8), we obtain $|\psi_\xi|$ and ψ are bounded from above. On the other hand, we have $\int_0^\kappa \widehat{g}(\xi,\mu)d\mu = -\infty$ for $\xi > 0$. By (4.8) again, we also have $\psi(\xi; y, v, w) > 0$ if $\xi > 0$. Thus the solution ψ can be extended beyond $\hat{y}$ if $\hat{y} > 0$, and therefore we must have $\hat{y} = 0$.

Finally, in this section, we shall derive the following identity

$$\psi_y(\xi; y, v, w) = -w\psi_v(\xi; y, v, w) - \left[\alpha\gamma y v^{\gamma-1} w - \widehat{g}(y, v)\right] \psi_w(y; y, v, w). \tag{4.12}$$

Differentiating (4.7) in y, we obtain

$$\psi_y(y; y, v, w) = -\psi_\xi(y; y, v, w) = -w; \quad \psi_{\xi\xi}(y; y, v, w) + \psi_{\xi y}(y; y, v, w) = 0.$$

Then we have

$$\begin{aligned} \psi_{\xi y}(y; y, v, w) &= -\psi_{\xi\xi}(y; y, v, w) \\ &= -\alpha\gamma y\psi^{\gamma-1}(y; y, v, w)\psi_\xi(y; y, v, w) + \widehat{g}(y, \psi(y; y, v, w)) \\ &= -\alpha\gamma y v^{\gamma-1} w + \widehat{g}(y, v). \end{aligned}$$

Furthermore, differentiating (4.7) in v and w respectively, we obtain

$$\psi_v(y; y, v, w) = 1, \quad \psi_{v\xi}(y; y, v, w) = 0, \quad \psi_w(y; y, v, w) = 0, \quad \psi_{w\xi}(y; y, v, w) = 1.$$

Hence

$$\psi_y(y; y, v, w) = -w\psi_v(y; y, v, w) - \left[\alpha\gamma y v^{\gamma-1} w - \widehat{g}(y, v)\right] \psi_w(y; y, v, w), \tag{4.13}$$

$$\psi_{y\xi}(y; y, v, w) = -w\psi_{v\xi}(y; y, v, w) - \left[\alpha\gamma y v^{\gamma-1} w - \widehat{g}(y, v)\right] \psi_{w\xi}(y; y, v, w). \tag{4.14}$$

Differentiating (4.6) in y, v, and w respectively, we obtain that both functions

$$\psi_y(\xi; y, v, w), \quad -w\psi_v(\xi; y, v, w) - \left[\alpha\gamma y v^{\gamma-1} w - \widehat{g}(y, v)\right] \psi_w(y; y, v, w)$$

satisfy the following ODE:

$$\theta_{\xi\xi} - \alpha\gamma\xi\psi^{\gamma-1}\theta_\xi + [\widehat{g}_\psi(\xi, \psi) - \alpha\gamma(\gamma-1)\xi\psi^{\gamma-2}\psi_\xi]\theta = 0$$

for unknown θ. Then, using (4.13)-(4.14) and the uniqueness of the solution to the initial value problem, we obtain (4.12).

5. Proof of Theorem 3

This section is devoted to the proof of the main result, Theorem 3. We first construct a suitable Lyapunov function using a method of Zelenyak [43] (see also [22]). Define

$$E_{R(s)}[z](s) = \int_0^{R(s)} \Phi(y, z(y,s), z_y(y,s))dy \tag{5.1}$$

where $R(s) = e^{\alpha s}$ and $\Phi = \Phi(y,v,w)$ will be chosen below. Since $z(y,s) \geq D^*|y|^\delta$ in Ω_0, z also satisfies the equation

$$\gamma z^{\gamma-1} z_s = z_{yy} - \alpha\gamma y z^{\gamma-1} z_y + \widehat{g}(y,z) \quad \text{in } \Omega_0. \tag{5.2}$$

Then, integrating by parts and using (5.2), we have

$$\frac{d}{ds} E_{R(s)}[z](s) = J_0 + J_1 + J_2,$$

where

$$\begin{aligned}
J_0 &= -\int_0^{R(s)} \gamma \Phi_{ww}(y, z(y,s), z_y(y,s)) z^{\gamma-1} |z_s|^2 dy,\\
J_1 &= \Phi_w(R(s), z(R(s),s), z_y(R(s),s)) z_s(R(s),s) - \Phi_w(0, z(0,s), 0) z_s(0,s)\\
&\quad + \Phi(R(s), z(R(s),s), z_y(R(s),s)) R'(s),\\
J_2 &= \int_0^{R(s)} K(y, z(y,s), z_y(y,s)) z_s(y,s) dy,\\
K(y,v,w) &:= \Phi_v - \Phi_{wy} - \Phi_{wv} w - \Phi_{ww}\left[\alpha\gamma y v^{\gamma-1} w - \widehat{g}(y,v)\right],
\end{aligned}$$

where $\widehat{g}$ is defined in (4.5).

Let

$$\Phi(y,v,w) := \int_0^w (w-\sigma) P(y,v,\sigma) d\sigma - \int_\kappa^v \widehat{g}(y,\mu) P(y,\mu,0) d\mu.$$

Then we have

$$\begin{aligned}
\Phi_w(y,v,w) &= \int_0^w P(y,v,\sigma) d\sigma,\\
\Phi_w(y,v,0) &= 0, \quad \Phi_{ww}(y,v,w) = P(y,v,w),\\
K(y,v,w) &= \int_0^w \Big\{ -\sigma \frac{\partial P}{\partial v}(y,v,\sigma) - \frac{\partial P}{\partial y}(y,v,\sigma)\\
&\qquad + \frac{\partial}{\partial \sigma}\left[P(y,v,\sigma)\left(\widehat{g}(y,v) - \alpha\gamma y v^{\gamma-1}\sigma\right)\right]\Big\} d\sigma.
\end{aligned}$$

Moreover, let

$$P(y,v,w) := \exp\left\{-\alpha\gamma \int_0^y \xi \psi(\xi; y, v, w)^{\gamma-1} d\xi\right\},$$

where ψ is defined in (4.6)-(4.7). Then using (4.12) and by a simple computation we obtain that $K(y,v,w)\equiv 0$, and therefore $J_2=0$.

Next, we derive some estimates for large v. Recall that $\widehat{g}(y,v)=g(v)$ for $v\geq\kappa$. From (4.8) it follows that

$$|\psi_\xi(\bar\xi;y,v,w)|^2\leq w^2+2\int_\kappa^v g(\mu)d\mu:=A(v,w)\quad\text{ for all }\bar\xi\in[\xi,y].$$

Obviously, we have

$$\psi(\xi;y,v,w)\geq v-\max_{\xi\leq\bar\xi\leq y}|\psi_\xi(\bar\xi;y,v,w)|(y-\xi)\geq v-A(v,w)^{1/2}(y-\xi)\geq v/2$$

for any ξ satisfying $0<y-\xi\leq vA(v,w)^{-1/2}/2$. Let

$$\widehat{\xi}:=\max\left\{y/2,\ y-vA(v,w)^{-1/2}/2\right\}.$$

Then

$$P(y,v,w)\leq\exp\left\{-\alpha\gamma\Big(\frac{v}{2}\Big)^{\gamma-1}\int_{\widehat{\xi}}^y\xi d\xi\right\}\leq\exp\{-B(y,v,w)\},\tag{5.3}$$

where

$$B(y,v,w):=\min\left\{\frac{\alpha\gamma}{4}\left(\frac{v}{2}\right)^{\gamma-1}y^2,\ \ \frac{\alpha\gamma}{2}\left(\frac{v}{2}\right)^{\gamma}yA(v,w)^{-1/2}\right\}.$$

Therefore, we have

$$\Phi(y,v,w)\leq\int_0^w(w-\sigma)P(y,v,\sigma)d\sigma\leq w^2\exp\left\{-B(y,v,w)\right\},\tag{5.4}$$

$$|\Phi_w(y,v,w)|=\Big|\int_0^w P(y,v,\sigma)d\sigma\Big|\leq|w|\exp\left\{-B(y,v,w)\right\}.\tag{5.5}$$

We shall estimate J_1 from above. Here we substitute $y=R(s)(=e^{\alpha s})$, $z(R(s),s)=v$, and $w=z_y(R(s),s)$ into the above estimates. Since $z(R(s),s)=k^me^{\beta ms}$, $z(R(s),s)\to\infty$ as $s\to\infty$. Note that

$$\int_\kappa^v g(\mu)d\mu\sim\frac{\beta}{\gamma+1}v^{\gamma+1}\quad\text{for}\quad v\gg1.$$

Hence it follows from (5.4) and (5.5) that

$$\begin{aligned}&\Phi(y,z(R(s),s),z_y(R(s),s))\leq|z_y(R(s),s)|^2\exp\left\{-c^*z(R(s),s)^{(\gamma-1)/2}R(s)\right\},\\&|\Phi_w(y,z(R(s),s),z_y(R(s),s))|\leq|z_y(R(s),s)|\exp\left\{-c^*z(R(s),s)^{(\gamma-1)/2}R(s)\right\}\end{aligned}$$

for large s for some constant $c^*>0$.

Next, we shall follow an idea from [20, p.54] to obtain an estimate of $u_x(1,t)$. Since $u_t<0$ in Q_T and $u_x>0$ in $(0,1]\times(0,T)$, we have

$$(u^m)_{xx}<u^p\quad\text{ in }\ Q_T,\quad(u^m)_{xx}(u^m)_x<u^p(u^m)_x\quad\text{ in }\ (0,1]\times(0,T).$$

An integrating of the last inequality from 0 to $x \in (0,1]$ gives

$$0 \le m(u^{m-1}u_x)(x,t) = (u^m)_x(x,t) \le \sqrt{\frac{2m}{p+m}}u^{(p+m)/2}(x,t)$$

for $(x,t) \in (0,1] \times (0,T)$. Therefore, by (1.1), we have $0 \le u_x(1,t) \le c^{**}$ for all $t \in (0,T)$ for some positive constant c^{**}. Then

$$0 \le z_y(R(s),s) \le c^{**}mk^{m-1}\exp[(\beta m - \alpha)s]. \tag{5.6}$$

This implies that

$$\begin{aligned} &\Phi(y, z(R(s),s), z_y(R(s),s))R'(s) \\ &\quad \le \alpha(c^{**})^2 m^2 k^{2(m-1)} \exp\left[(2\beta m - \alpha)s\right] \exp\left\{-c^* k^{(1-m)/2} e^{s/2}\right\}. \end{aligned} \tag{5.7}$$

Similarly, using $z(R(s),s) = k^m e^{\beta m s}$ and (5.6), we obtain

$$(\beta m k^m - \alpha m c^{**} k^{m-1}) e^{\beta m s} \le z_s(R(s),s) \le \beta m k^m e^{\beta m s}.$$

Thus we have

$$\begin{aligned} &|\Phi_w(y, z(R(s),s), z_y(R(s),s))||z_s(R(s),s)| \\ &\quad \le \overline{c}\exp\left[(2\beta m - \alpha)s\right] \exp\left\{-c^* k^{(1-m)/2} e^{s/2}\right\} \end{aligned} \tag{5.8}$$

for some positive constant $\overline{c}$. Since $\Phi_w(0, z(0,s), 0) = 0$, together with (5.7) and (5.8), we obtain that J_1 is bounded from above by a function that decays exponentially fast.

Next, we prove that J_1 is bounded below. By (5.3) and noting that $\widehat{g}(y,\mu) = g(\mu)$ for $\mu \ge \kappa$, we have

$$P(y,v,0) \le \exp\left\{-c^* v^{(\gamma-1)/2} y\right\} \quad for \quad v \ge \kappa.$$

Furthermore, we have

$$\begin{aligned} &\Phi(y, z(R(s),s), z_y(R(s),s))R'(s) \\ \ge\ & -R'(s)\int_\kappa^{z(R(s),s)} g(\mu)P(R(s),\mu,0)d\mu \\ \ge\ & -\frac{\beta}{\gamma+1}R'(s)z(R(s),s)^{\gamma+1}\exp\left\{-c^*\kappa^{(\gamma-1)/2}R(s)\right\} \\ \ge\ & -\frac{\alpha\beta}{\gamma+1}k^{m(\gamma+1)}\exp\left\{\frac{(3m-p+2)s}{2(1-p)}\right\}\exp\left\{-c^*\kappa^{(\gamma-1)/2}\exp(\alpha s)\right\}. \end{aligned} \tag{5.9}$$

Hence by (5.8) and (5.9) we obtain that J_1 is bounded from below by a function that decays exponentially fast. Thus we have proved:

Lemma 5.1. *We have*

$$\frac{d}{ds}E_{R(s)}[z](s) = -\gamma\int_0^{R(s)} P(y, z(y,s), z_y(y,s))z^{\gamma-1}(y,s)|z_s(y,s)|^2 dy + J_1(s),$$

where J_1 satisfies the property $\int_{s_0}^\infty |J_1(s)|ds < \infty$.

From this lemma, we obtain

$$\frac{d}{ds}\left[E_{R(s)}[z](s) - \int_{s_0}^{s} J_1(\tau)d\tau\right] \le 0,$$

and therefore for any $s > s_0$,

$$E_{R(s)}[z](s) \le E_{R(s)}[z](s_0) + \int_{s_0}^{s} J_1(\tau)d\tau \le E_{R(s)}[z](s_0) + \int_{s_0}^{\infty} |J_1(\tau)d\tau| \equiv \overline{C}.$$

Proof of Theorem 3. Let s_j be a sequence with $s_j \to \infty$ as $j \to \infty$. We define $z_j(y,s) = z(y, s+s_j)$ for all $j \in \mathbb{N}$ and $(y,s) \in \Omega$. Define $\tilde{\Omega} := \{(y,s) \in \mathbb{R}^2;\, y \neq 0\}$ and observe that, for any given $\omega \subset\subset \tilde{\Omega}$, z_j is defined on ω for j large enough. Using (3.4)-(3.7), applying parabolic L^p estimates to (1.8) and to the equation satisfied by z_y, and then parabolic Hölder estimates, it follows that the sequence $\{z_j\}_{j\in\mathbb{N}}$ is compact in $C^{2,1}(\omega)$ for any $\omega \subset\subset \tilde{\Omega}$. Therefore, using a diagonal process, there exists a subsequence $\{j_l\}$ and a function $z_\infty \in C^{2,1}(\tilde{\Omega})$, such that $z_{j_l}(y,s) \to z_\infty(y,s)$ as $l \to \infty$, locally uniformly in $C^{2,1}(\tilde{\Omega})$. Moreover, z_∞ satisfies (1.8) in $\tilde{\Omega}$ and, due to (3.7),

$$|z_y(y,s)| \le c|y|, \quad 0 < |y| \le 1,\ s \in \mathbb{R}. \tag{5.10}$$

Consider the Lyapunov function $E_{R(s)}[z](s)$ defined in (5.1). Then we have

$$E_{R(s)}[z](s) \le E_{R(s)}[z](s_0) + \int_{s_0}^{\infty} |J_1(\tau)|d\tau \equiv \overline{C} < \infty \quad \text{ for } s > s_0.$$

On the other hand, since

$$P(y,v,0) \le \exp\left\{-c^* v^{(\gamma-1)/2} y\right\}$$

for $v \ge \kappa$ and $z(y,s) \ge D^*|y|^\delta \ge D^*$ for any $y \ge 1$, we have

$$\begin{aligned} &\int_0^{R(s)} \int_\kappa^{z(y,s)} \widehat{g}(y,\mu)P(y,\mu,0)d\mu dy \\ &\quad \le \int_0^1 \int_0^{c^*} |g(\mu)|P(y,\mu,0)d\mu dy + \int_1^\infty \int_{D^*}^\infty |g(\mu)|P(y,\mu,0)d\mu dy \le \widetilde{C} \end{aligned} \tag{5.11}$$

where c^* is the upper bound for $z(y,s)$ over $y \in [0,1]$. From (5.11) we obtain

$$E_{R(s)}[z](s) \ge -\int_0^{R(s)} \int_\kappa^{z(y,s)} \widehat{g}(y,\mu)P(y,\mu,0)d\mu dy \ge -\widetilde{C},$$

and hence by using Lemma 5.1, we have

$$\gamma \int_{s_0}^\infty \int_0^{R(s)} P(y,z(y,s),z_y(y,s))z^{\gamma-1}(y,s)|z_s(y,s)|^2 dyds \le \widetilde{C}. \tag{5.12}$$

Note that $P(y,v,w)$ is bounded below away from 0 for y,v,w in bounded sets. Also, z and z_y are bounded for y bounded, $z(y,s) \ge D^*|y|^\delta$ in $\Omega_{\overline{s}}$ and $\gamma > 1$. For each $0 < \varepsilon < 1$

and $M > e^{\alpha s_0}$, putting $s_M = \alpha^{-1} \ln M$, it follows from (5.12) that

$$\int_{s_M}^{\infty} \int_{\varepsilon}^{M} |z_s(y,s)|^2 dyds < \infty.$$

For each $S > 0$, we thus deduce

$$\int_{-S}^{S} \int_{\varepsilon}^{M} |\partial_s z_\infty(y,s)|^2 dyds \le \liminf_{j\to\infty} \int_{s_j - S}^{\infty} \int_{\varepsilon}^{M} |z_s(y,s)|^2 dyds = 0,$$

hence $\partial_s z_\infty(y,s) = 0$ and $z_\infty = z_\infty(y)$ satisfies

$$z'' - \alpha\gamma y z^{\gamma-1} z' + \beta z^\gamma - z^q = 0, \quad y > 0. \tag{5.13}$$

Using (5.13) and (5.10), it follows that z_∞ can be extended to a C^2 solution of (5.13) on $[0,\infty)$ with $\partial_y z_\infty(0) = 0$, hence to a symmetric C^2 solution on $\mathbb{R}$, in view of the symmetry of z in y. Note that z_∞ is monotone in $y > 0$. Therefore, the conclusion follows from (3.4)-(3.5) and Proposition 4.1. This completes the proof of the theorem. □

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