BLOW-UP FOR A REACTION-DIFFUSION EQUATION WITH VARIABLE COEFFICIENT

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Abstract. We study the blow-up behavior for positive solutions of a reaction-diffusion equation with nonnegative variable coefficient. When there is no stationary solution, we show that the solution blows up in finite time. Under certain conditions, we then show that any point with zero source cannot be a blow-up point.

1. Introduction

In this paper, we study the positive solution of the following initial boundary value problem (P):

$$u_t = \Delta u + V(x)f(u), \quad x \in \Omega,\ t > 0, \tag{1.1}$$

$$u(x,0) = u_0(x), \quad x \in \bar{\Omega},, \tag{1.2}$$

$$u(x,t) = 0, \quad x \in \partial\Omega,\ t > 0, \tag{1.3}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 3$, $f(u)$ is a nonnegative increasing smooth function defined on $[0,\infty)$, V is a nonnegative smooth function defined in $\bar{\Omega}$, and u_0 is a nonnegative bounded smooth function defined in $\bar{\Omega}$. Throughout this paper, we assume that $V \not\equiv 0$. The solution u of (P) is said to blow up (in finite time), if

$$\limsup_{t\to T^-}\{\sup_{x\in\Omega} u(x,t)\} = \infty$$

for some $T < \infty$. A point $a \in \bar{\Omega}$ is called a blow-up point if there exists a sequence $\{(x_n,t_n)\}$ in $\Omega \times (0,T)$ such that $x_n \to a$, $t_n \uparrow T$ and $u(x_n,t_n) \to \infty$ as $n \to \infty$.

The phenomena of blow-up have attracted a lot of attention for past years. Most literature are concerned with spatially homogeneous equations, i.e., equations with constant V. Interesting questions, for example, are about criteria of blow-up, locations of blow-up points, blow-up rate, spatial blow-up profile and so on. See, for example, [20, 7, 23, 21, 22, 28, 3, 6, 8, 9, 10, 4, 14, 15, 16, 17, 24, 25] for spatially homogeneous equations.

Recently, there are many interesting works on the problem (P) in which V is not a constant function (see, e.g., [5, 26, 12, 13]). In particular, when the function V takes zero value at a point in Ω, there is no source at this point locally. Therefore, it is interesting to see whether such a point can be a blow-up point. Intuitively it

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Keywords: blow-up, variable coefficient, stationary solution, blow-up point.

The first author of this work was partially supported by the National Science Council of the Republic of China under the grant NSC 99-2115-M-032-006-MY3. We also thank the anonymous referee for the helpful comments.

seems that the answer to this question is negative. Surprisingly, the answer can be positive or negative depending on the situation. By constructing some self-similar solutions, it is shown in [5] that the origin is a blow-up point for the solution of the Cauchy problem (1.1)-(1.2) with $N \geq 3$, $V(x) = |x|^\sigma$, $\sigma > 0$, and $f(u) = u^p$ for certain range of p. On the other hand, when $f(u) = u^p$ with $p > 1$, in [12, 13] we have established the following results:

(1) Let $\Omega = B_R$, u be radially symmetric, and $V(x) = |x|^\sigma$, $\sigma > 0$. Suppose that u blows up in finite time. Then the origin is not a blow-up point, if either $N = 3$, $p = 1 + \sigma$; or $N \geq 3$, $1 < p < 1 + 2\sigma/(N-1)$.
(2) Let $\Omega = B_R$, u be radially symmetric, and $V(x) = |x|^\sigma$, $\sigma > 0$. If $N = 3$ and $p > 5 + 2\sigma$, then there are certain radial solutions which blow up in finite time such that the origin is a blow-up point.
(3) Let Ω be a bounded smooth domain in $\mathbb{R}^N$. Suppose that u is monotone increasing in time. Then any zero of V cannot be a blow-up point, if all zeros of V are contained in Ω.

It is also interesting to see whether a function V which vanishing somewhere prevents the blow-up of the solution of (P). Indeed, in [12] we show that the solution u of (P) blows up in a finite time when the solution u is strictly monotone increasing in time for any V not identically zero. For the case with constant positive V, we refer the reader to [27].

All of the above mentioned results share the property that $f(0) = 0$. When $f(0) > 0$, e.g., $f(u) = \lambda e^u$ or $f(u) = \lambda(1+u)^p$ with $\lambda > 0, p > 1$, the situation is quite different. Note that in this case zero is no longer a stationary solution and the equation (1.1) has a positive source near the boundary. The purpose of this paper is to study the problem (P) when $f(0) > 0$. We shall take $f(u) = \lambda e^u$ as our typical example. Our results can be easily extended to the case when $f(u) = \lambda(1+u)^p$ with $p > 1$. Henceforth we assume throughout this paper that $f(u) = \lambda e^u$.

For $f(u) = \lambda e^u$, if λ is sufficiently large so that there do not exist any regular stationary solutions, then any solution u of the problem (P) blows up in finite time. For such a result, we also refer the reader to [1, 21, 2]. Furthermore, for general nonnegative function $V(x)$, we prove that any zero of V cannot be a blow-up point, if u is strictly increasing in time and all blow-up points are included in a compact subset of Ω.

This paper is organized as follows. In Section 2, in the case without stationary solutions, we prove that all solutions of (P) blow up in finite time. In Section 3, a sufficient condition that blow-up cannot occur at zeros of V is given.

2. Finite time blow-up for general V

In this section, we shall study the problem (P) with $f(u) = \lambda e^u$ and general nonnegative V. First we define

$$\lambda^* := \sup\{\lambda > 0 : \text{a (regular) stationary solution of (1.1)-(1.3) exists}\}$$

Using the results obtained in [18, 19] for constant V, it is easy to see that the quantity λ^* is well-defined as long as $V \not\equiv 0$. Indeed, using the monotone iteration method we can easily see that a stationary solution of (1.1)-(1.3) for all $\lambda < \lambda_0$

exists, if a stationary solution of (1.1)-(1.3) exists for a certain $\lambda = \lambda_0 > 0$. This simply follows from the fact that the latter solution is a supersolution of the stationary problem for all $\lambda < \lambda_0$. Note that 0 is always a subsolution. Also, the existence of stationary solution for small positive λ can be derived by a contraction mapping principle. Hence the quantity λ^* is well-defined. The finiteness of λ^* can be shown as that in [18, 19]. See also [11].

We shall prove the finite time blow-up of solutions to the problem (1.1)-(1.3) with $\lambda > \lambda^*$ for any nonnegative bounded smooth initial data u_0. In [21], the author has obtained the same blow-up result for the case $V \equiv 1$, $2 < N < 10$ and $\lambda > \lambda^*$ (see also [1, 2]). Based on the idea of [2], we can prove the following blow-up result. Although the proof is the same as the one in [2], we provide the details here for the reader's convenience.

Theorem 1. *For $\lambda > \lambda^*$, the solution u of* (1.1)-(1.3) *with nonnegative bounded smooth initial data u_0 blows up in finite time.*

Proof. Assume that $\lambda > \lambda^*$. Choosing $\varepsilon \in (0,1)$ sufficiently small such that $\lambda(1-\varepsilon) > \lambda^*$. We define

$$g(w) = e^w, \quad h(w) := \int_0^w \frac{ds}{g(s)},$$
$$h_\varepsilon(w) := (1-\varepsilon)^{-1} h(w), \quad \eta_\varepsilon(w) = h_\varepsilon^{-1}(h(w)).$$

Indeed, we have

$$h(w) = 1 - e^{-w}, \quad \eta_\varepsilon(w) = -\ln[1 - (1-\varepsilon)(1 - e^{-w})].$$

Note that $h(\infty) = 1$ and $\eta_\varepsilon(\infty) = -\ln \varepsilon < \infty$. We can easily check that $\eta_\varepsilon(0) = 0$, $0 \le \eta_\varepsilon(w) < w$, and

$$\eta_\varepsilon'(w) = (1-\varepsilon) g(\eta_\varepsilon(w))/g(w) > 0, \tag{2.1}$$

$$\eta_\varepsilon''(w) = -\varepsilon(1-\varepsilon) g^2(\eta_\varepsilon(w))/g(w) \le 0. \tag{2.2}$$

By the comparison principle, we only need to prove that the case with initial value $u_0 \equiv 0$. Assume on the contrary that the solution u of (P) exists globally and we define

$$v_\varepsilon(x,t) := \eta_\varepsilon(u)(x,t).$$

This is equivalent to $h(v_\varepsilon) = (1-\varepsilon) h(u)$. Then (2.1) and (2.2) yield

$$-\Delta v_\varepsilon = -\eta_\varepsilon'(u) \Delta u - \eta_\varepsilon''(u) |\nabla u|^2 \ge \eta_\varepsilon'(u) \{\lambda V(x) e^u - u_t\} = \lambda(1-\varepsilon) V(x) g(v_\varepsilon) - (v_\varepsilon)_t.$$

This means that v_ε is a supersolution of (1.1)-(1.3) with λ replaced by $\lambda(1-\varepsilon)$. Note that v_ε is uniformly bounded and 0 is a subsolution. The iteration sequence $\{v_k\}$, $k \ge 0$, defined by

$$\begin{cases} (v_{k+1})_t = \Delta v_{k+1} + \lambda(1-\varepsilon) V(x) e^{v_k}, & x \in \Omega,\ t > 0, \\ v_{k+1}(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ v_{k+1}(x,0) = 0, & x \in \bar{\Omega} \end{cases}$$

with $v_0 = v_\varepsilon$, is monotone decreasing in k and it converges to a bounded classical solution u_ε of (1.1)-(1.3) with λ replaced by $\lambda(1-\varepsilon)$ and $u_0 \equiv 0$. Furthermore,

$(u_\varepsilon)_t \geq 0$ by the comparison principle. Therefore, u_ε converges to a classical stationary solution of (1.1)-(1.3) with λ replaced by $\lambda(1-\varepsilon)$ as $t \to \infty$. This contradicts the definition of λ^*. Thus the theorem follows. □

3. Blow-up points

In this section, we give a sufficient condition such that any zero of V cannot be a (finite time) blow-up point for the problem (1.1)-(1.3). Without loss of generality, we may assume that $\lambda = 1$ so that $f(u) = e^u$.

Theorem 2. *Suppose that a solution $u(x,t)$ of* (1.1)-(1.3) *blows up in the finite time T such that $u_t > 0$ in $\Omega \times [0,T)$. If all blow-up points are included in a compact subset of Ω, then any zero of V cannot be a blow-up point.*

Proof. By assumption, we may take a domain Ω' such that $\bar{\Omega}' \subset \Omega$ and u is bounded in $[\bar{\Omega} \setminus \Omega'] \times [0,T)$. Since $u_t > 0$ in $\Omega \times [0,T)$, u_t is bounded below by a positive constant in the region $\bar{\Omega}' \times [0,T)$. Hence we can choose a positive constant ε small enough so that the function (cf. [6])

$$J := u_t - \varepsilon e^u$$

is nonnegative on the parabolic boundary of $\Omega' \times [0,T)$. By a simple calculation, we have

$$J_t - \Delta J = V(x)e^u J + \varepsilon e^u |\nabla u|^2 \geq V(x)e^u J.$$

It follows from the maximum principle that $J \geq 0$ in $\Omega' \times [0,T)$. Consequently, we have

$$e^{-u}u_t \geq \varepsilon \quad \text{in} \quad \Omega' \times [0,T).$$

By integrating this inequality from $t \in (0,T)$ to $\tau \in (t,T)$ and letting $\tau \to T$, we obtain

$$u(x,t) \leq C - \ln(T-t), \quad x \in \Omega',\ t \in (0,T) \tag{3.1}$$

for some positive constant C.

Let x_0 be any zero point of $V(x)$ in Ω'. We may assume that $\{x : |x-x_0| \leq 2r_0\} \subset \Omega'$ for some $r_0 > 0$. Then we define

$$\begin{aligned} w(x,t) &:= A - \ln[v(x) + (T-t)], \\ v(x) &:= \delta \cos^2\Big(\frac{\pi|x-x_0|}{2r_0}\Big), \\ B_0 &:= \{x : |x-x_0| \leq r_0\}, \end{aligned}$$

where δ, A are positive constants to be specified later. Note that $w(x,t) \geq u(x,t)$ for $x \in \partial B_0$ and $t \in (0,T)$, by (3.1), if we choose $A > C$. Also,

$$w(x,0) = A - \ln[v(x) + T] \geq u_0(x), \quad |x-x_0| \leq r_0,$$

if we take A sufficiently large.

For w to be a super-solution, we need the following inequality

$$w_t - \Delta w - V(x)e^w \geq 0$$

which is equivalent to

$$1 - e^A V(x) + \Delta v(x) - \frac{|\nabla v(x)|^2}{v(x) + (T-t)} \geq 0$$

for all $(x,t) \in B_0 \times (0,T)$. We have this inequality if

$$1 - e^A V(x) + \Delta v(x) - \frac{|\nabla v(x)|^2}{v(x)} \geq 0 \tag{3.2}$$

for all $x \in B_0$. It is easy to see that Δv and $[|\nabla v|^2/v]$ are bounded in B_0 by $M\delta$ for a constant M independent of δ for any positive constant δ. Furthermore, by fixing A and taking r_0 sufficiently small, we have $e^A V(x) < 1/3$ for all $x \in B_0$. For these fixed A and r_0, we can take $\delta > 0$ sufficiently small so that the last two terms in the inequality (3.2) are bounded by $1/3$ in B_0. Hence (3.2) holds in B_0 and, by the comparison principle, we conclude that

$$w(x,t) \geq u(x,t), \quad |x - x_0| \leq r_0,\ t \in (0,T).$$

This implies that x_0 cannot be a blow-up point. The proof is completed. $\square$

We make some remarks on the assumptions of Theorem 2. First, the condition that $u_t > 0$ can be realized if we assume that

$$\Delta u_0 + V f(u_0) \geq 0 \quad \text{in } \Omega.$$

The assumption that the blow-up set is compact made in Theorem 2 can be verified in the case of homogeneous nonlinearity (i.e., when V is constant) by the moving plane argument as in [6]. Assume that Ω is convex and the function V is decreasing in the direction normal to the boundary in a neighborhood of the boundary of the domain, the moving plane argument also works well so that we have the compactness of the blow-up set.

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