# DYNAMICS FOR A COMPLEX-VALUED HEAT EQUATION WITH AN INVERSE NONLINEARITY

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ABSTRACT. We study the Cauchy problem for a parabolic system which is derived from a complex-valued heat equation with an inverse nonlinearity. First, we provide some criteria for the global existence of solutions. Then we consider the case when the initial data are asymptotically constants and obtain that, depending on the asymptotic limits, the solution quenches at space infinity or exists globally in time.

## 1. INTRODUCTION

In this paper, we study the following equation

$$z_t = z_{xx} - \frac{1}{z}, \tag{1.1}$$

where $z = z(x,t)$ is a complex-valued function of the spatial variable $x \in \mathbb{R}$ and the time variable $t \geq 0$. If we set $z(x,t) = u(x,t) + iv(x,t)$, where $i = \sqrt{-1}$ and $u(x,t)$, $v(x,t) \in \mathbb{R}$, then (1.1) can be written as a system of parabolic equations

$$\begin{cases} u_t = u_{xx} - u/(u^2+v^2), \\ v_t = v_{xx} + v/(u^2+v^2). \end{cases} \tag{1.2}$$

If $z(x,t)$ is real-valued (i.e., $v \equiv 0$), then the system is reduced to the equation

$$u_t = u_{xx} - \frac{1}{u}.$$

An initial boundary value problem for the above equation was first studied by Kawarada [7] in 1975. For more general negative power nonlinearity, we refer the reader to, e.g., [4, 6, 8] and the references cited therein. The goal of this paper is to study the dynamics of solutions of the system (1.2) with $v \not\equiv 0$.

First of all, we consider a spatially homogeneous solution of (1.2), namely, $(u,v) = (U(t),V(t))$. We obtain that $(U(t),V(t))$ satisfies the following ODE system:

$$\begin{cases} U_t = -U/(U^2+V^2), \\ V_t = V/(U^2+V^2). \end{cases} \tag{1.3}$$

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Given $(U(0), V(0)) \in \mathbb{R}^2 \setminus \{(0,0)\}$. By a simple computation, we obtain that

$$U(t)V(t) = U(0)V(0) := C, \quad \forall t \geq 0. \tag{1.4}$$

for some constant $C \in \mathbb{R}$.

If $U(0) = 0$, then the trajectory stays on the the $V$-axis, exists globally and tends to $\pm\infty$ as $t \to \infty$. On the other hand, if $V(0) = 0$, then $V(t) \equiv 0$ and $U$ tends to zero in finite time. When $C \neq 0$, by (1.3) and (1.4) we have $(U(t), V(t)) \to (0, \pm\infty)$ as $t \to \infty$.

In this paper, we consider the initial value problem (P) for (1.2) with the initial condition

$$(u(\cdot, 0), v(\cdot, 0)) = (u_0, v_0). \tag{1.5}$$

In the sequel, we shall always assume that

$$u_0 > 0, \quad v_0 \geq 0, \quad u_0, v_0 \in L^\infty(\mathbb{R}) \cap C(\mathbb{R}), \quad \inf_{\mathbb{R}} u_0 + \inf_{\mathbb{R}} v_0 > 0.$$

Then the problem (P) has a unique solution $(u, v) \in (C([0, T); L^\infty(\mathbb{R})))^2$, where $T = T(u_0, v_0) \in (0, \infty]$ is the maximal existence time of the solution. Furthermore, we have either $T = \infty$, or

$$T < \infty \ \text{ and } \ \liminf_{t \to T} \{\inf_{x \in \mathbb{R}} u(x,t) + \inf_{x \in \mathbb{R}} v(x,t)\} = 0.$$

In the first case, we have the *global* existence. For the second case, we say that the solution of (P) *quenches* in a finite time $T$ in which $T$ is called the *quenching time*. Moreover, we say that $x_Q \in \mathbb{R}$ is a (finite) *quenching point* for $(u, v)$ if there exists a sequence $\{(x_j, t_j)\}$ such that $x_j \to x_Q$, $t_j \uparrow T$ and $u(x_j, t_j) + v(x_j, t_j) \to 0$ as $j \to \infty$. We shall investigate the global and non-global existence of solutions of (P).

The first result is about the global existence and (time) asymptotic behavior of solution of the problem (P).

**Theorem 1.** *Suppose that the initial data satisfy*

$$\begin{cases} u_0(x) > 0, \ v_0(x) > 0, \ \forall x \in \mathbb{R}, \ u_0 \textit{ and } v_0 \textit{ are bounded in } \mathbb{R}, \\ u_0(x)v_0(x) \geq K, \ \forall x \in \mathbb{R}, \ \textit{for some constant } K > 0. \end{cases} \tag{1.6}$$

*Then the solution of* (1.2) *with* (1.5) *exists globally in time and* $(u, v)$ *converges to* $(0, \infty)$ *as* $t \to \infty$ *uniformly in* $\mathbb{R}$.

For $t \geq 0$, we set

$$\mathcal{R}(t) := \left\{(u(x,t), v(x,t)) \in \mathbb{R}^2; \ x \in \mathbb{R}\right\}$$

to be the image of the solution on $(u, v)$-plane. We remark that, under the hypothesis of Theorem 1, the closure of the convex hull of $\mathcal{R}(0)$ lies in the first quadrant of $(u, v)$-plane. Indeed, under the condition (1.6), we shall see that $\mathcal{R}(t)$ stays in the first quadrant for all $t > 0$. This implies the global existence of solutions.

On the other hand, if the initial data do not satisfy (1.6), in view of the dynamics of (1.3), it is interesting to see what happen. One question is to see under what conditions the quenching occurs. From (1.2) it is easy to see that both $u$ and $v$ quench simultaneously whenever quenching occurs. On the contrary, there might be *non-simultaneous quenching* in which just one component quenches and the other remains bounded away from zero. For this, we refer the reader to, e.g., [1, 9, 11, 13, 16].

To find solutions quenching in finite time, we consider the case when the initial data are *asymptotically constants*. Namely, we impose the following conditions on initial data:

$$u_0, v_0 \in C^1(\mathbb{R}), \quad u_0 \geq M, \quad u_0 \not\equiv M, \quad v_0 \geq 0, \quad v_0 \not\equiv 0, \tag{1.7}$$

$$\lim_{|x|\to\infty} u_0(x) = M, \quad \lim_{|x|\to\infty} v_0(x) = N \tag{1.8}$$

for some constants $M > 0$ and $N \geq 0$.

The following theorem shows that the solution of (1.2) with initial data satisfying (1.7) and (1.8) with $N > 0$ behaves like the solution the ODE system (1.3) with $(U(0), V(0)) = (M, N)$.

**Theorem 2.** *Let $(u, v)$ be a solution of (1.2) with initial data $(u_0, v_0)$ satisfying (1.7) and (1.8). If $N > 0$, then the solution of (1.2) with (1.5) exists globally for all $t \geq 0$ and $(u, v)$ converges to $(0, \infty)$ as $t \to \infty$ uniformly in $\mathbb{R}$.*

On the other hand, if the initial data of (1.2) satisfy (1.7) and (1.8) with $N = 0$, then the solution of (1.2) and (1.5) quenches only at space infinity. Namely, there are no (finite) quenching points, while there exists a sequence $\{(x_j, t_j)\}$ such that $|x_j| \to \infty$, $t_j \uparrow T$ and $u(x_j, t_j) + v(x_j, t_j) \to 0$ as $j \to \infty$.

**Theorem 3.** *Let $(u, v)$ be a solution of (1.2) and (1.5) with the initial data $(u_0, v_0)$ satisfying (1.7) and (1.8) with $M > 0$ and $N = 0$. Then the solution of (1.2) with (1.5) quenches at the finite time $t = T = M^2/2$. Moreover, the solution quenches only at space infinity.*

Note that the problem of quenching at space infinity for scalar equation was studied by Giga-Seki-Umeda [2, 3]. In [2], they characterized that, with suitable initial data, solutions of the following Cauchy problem $u_t = u_{xx}/(1 + u_x^2) - (n - 1)/u$ quenching only at space infinity. In [3], they estimated its profile at the quenching time from above and below.

The motivation of this study is from a work of Guo-Ninomiya-Shimojo-Yanagida [5]. In [5], they considered, instead of (1.1), the following complex-valued equation:

$$z_t = \Delta z + z^2 \tag{1.9}$$

where $z = z(x, t) = u(x, t) + iv(x, t)$ is a complex-valued function of $x \in \mathbb{R}^m$ ($m \in \mathbb{N}$) and $t \geq 0$. To obtain the asymptotical behavior of the solution, our method is close to that in [5] by using an invariant set argument. But, instead of considering the invariant subset in

$(u,v)$-plane, we transform our problem in $(u,w)$-plane where $w := 1/v$. Also, the solution blows up non-simultaneously at space infinity for the case (1.9) with asymptotically constant initial data. But, in our case (1.1), quenching can only occurs simultaneously.

This paper is organized as follows. In section 2, we provide a sufficient condition for the existence of global solutions and study the asymptotic behavior of solutions as $t \to \infty$. In section 3, we study the solution of (1.2) with asymptotically constant initial data.

## 2. Global existence and Convergence

In this section we give a proof of Theorem 1. Let us first recall some properties about invariant sets (cf. [17]).

**Lemma 2.1.** *Suppose that $\Omega(t) \subset \mathbb{R}^2$ is convex for each $t \geq 0$ and $\{\Omega(t)\}_{t\geq 0}$ is (positively) invariant under the flow* (1.3) *in the sense that $(U(t), V(t)) \in \Omega(t)$ for all $t > 0$, if $(U(0), V(0)) \in \Omega(0)$. Then $\{\Omega(t)\}_{t\geq 0}$ is also invariant under the flow* (1.2)*. That is, if $\mathcal{R}(t_0) \subset \Omega(t_0)$ for some $t_0 \geq 0$, then $\mathcal{R}(t) \subset \Omega(t)$ for all $t > t_0$.*

To construct invariant sets, the following lemma is very useful.

**Lemma 2.2.** *Let $\{F_i\}_{1\leq i\leq m}$ be a set of $C^1$ functions from $\mathbb{R}^3$ to $\mathbb{R}$. Suppose that $\Omega(t)$ is expressed as*

$$\Omega(t) = \bigcap_{i=1}^{m} \{(u,v) \in \mathbb{R}^2 \ ; \ F_i(u,v,t) < 0\}, \quad t \geq 0.$$

*Then $\{\Omega(t)\}_{t\geq 0}$ is invariant under the flow* (1.3) *if*

$$\frac{d}{dt} F_i(U(t), V(t), t) \leq 0 \quad \text{on} \quad \{(u,v) \in \partial\Omega \ ; F_i(u,v,t) = 0\}$$

*for all $i = 1, \ldots, m$.*

With these lemmas, we are ready to prove the global existence of the solution of (1.2) and (1.5).

**Proof of Theorem 1.** Set

$$D_1 := \left\{(u,v) \in \mathbb{R}^2; \ u > 0, \ v > 0 \ \text{ and } \ -uv + K \leq 0\right\}.$$

By assumption, we have $\mathcal{R}(0) \subset D_1$. For $(U,V) \in \partial D_1$, we compute

$$\frac{d}{dt}(-UV + K) = -(U_t V + U V_t) = 0.$$

Thus $D_1$ is invariant under the flow (1.3) by Lemma 2.2.

Since $u_0$ is bounded, there exists a constant $A > 0$ such that $u_0(x) \leq A$, $\forall x \in \mathbb{R}$. Set

$$D_2 := \left\{(u,v) \in \mathbb{R}^2; \ u > 0, \ v > 0 \text{ and } u \leq A\right\}.$$

Note that $D_1 \cap D_2$ is convex. For $(U, V) \in D_1 \cap \partial D_2$, we compute

$$\frac{d}{dt}(U - A) = -\frac{U}{U^2 + V^2} < 0.$$

Therefore, $D_1 \cap D_2$ is invariant under the flow (1.3) by Lemma 2.2.

It follows from Lemma 2.1 that $u(x,t) > 0$, $v(x,t) > 0$, $u(x,t)v(x,t) \geq K$ and $u(x,t) \leq A$ for all $x \in \mathbb{R}$ and $t \geq 0$, as long as $v$ stays finite. Using $u^2 + v^2 \geq 2uv \geq 2K$, we have

$$v_t \leq v_{xx} + v/(2K).$$

From this, it follows that the solution of (1.2) and (1.5) with (1.6) exists globally in time.

Next, we shall prove the asymptotic behavior of the solution $(u, v)$ as $t \to \infty$. We set $w := 1/v$. Then (1.2) is equivalent to

$$\begin{cases} u_t &= \quad u_{xx} - uw^2/(u^2w^2 + 1), \\ w_t &= w_{xx} - 2w_x^2/w - w^3/(u^2w^2 + 1). \end{cases}$$

Moreover, it follows from (1.6) that

$$\begin{cases} u_0(x) > 0, \ w_0(x) := 1/v_0(x) > 0, \ u_0(x), w_0(x) \text{ are bounded, } \forall x \in \mathbb{R}, \\ u_0(x) \geq K w_0(x), \ \forall x \in \mathbb{R}, \quad \text{for some constant } K > 0. \end{cases} \tag{2.1}$$

Therefore, it is enough to prove that $(u, w)$ converges to $(0, 0)$ as $t \to \infty$.

For this, we first consider the spatially homogeneous solution $(u, w) = (U(t), W(t))$. Then $(U, W)$ satisfies the following ODE system:

$$\begin{cases} U_t = -UW^2/(U^2W^2 + 1), \\ W_t = -W^3/(U^2W^2 + 1). \end{cases} \tag{2.2}$$

We set

$$D_3 := \left\{(u, w) \in \mathbb{R}^2; \ u > 0, \quad w > 0 \ \text{ and } Kw - u \leq 0\right\}.$$

Then, by (2.1), we obtain that $\mathcal{S}(0) \subset D_3$. Hereafter $\mathcal{S}(t) := \{(u(x,t), w(x,t)) \in \mathbb{R}^2; \, x \in \mathbb{R}\}$.

For $(U, W) \in \partial D_3$, we have

$$\begin{aligned} \frac{d}{dt}(KW - U) &= KW_t - U_t \\ &= -\frac{KW^3}{U^2W^2 + 1} + \frac{UW^2}{U^2W^2 + 1} \\ &= \frac{-W^2(KW - U)}{U^2W^2 + 1} = 0. \end{aligned}$$

Hence $D_3$ is invariant under the flow (2.2) by Lemma 2.2.

Next, we set

$$D_4 := \left\{(u, w) \in \mathbb{R}^2; \ u > 0, \quad w > 0 \ \text{ and } \ -w + au^2 \leq 0\right\}$$

for some positive constant $a$ such that $\mathcal{S}(0) \subset D_4$. This can be done due to (2.1). Note that $D_3 \cap D_4$ is convex and

$$\partial D_3 \cap \partial D_4 = \left\{(0,0),\ (1/(aK), 1/(aK^2))\right\}.$$

For $(U,W) \in D_3 \cap \partial D_4$, we have

$$\begin{aligned}\frac{d}{dt}(-W + aU^2) &= -W_t + 2aUU_t \\ &= \frac{W^3}{U^2W^2+1} + 2aU\left(\frac{-UW^2}{U^2W^2+1}\right) \\ &= \frac{W^2}{U^2W^2+1}[W - 2aU^2] \\ &= \frac{W^2}{U^2W^2+1}[aU^2 - 2aU^2] = \frac{-aU^2W^2}{U^2W^2+1} \le 0.\end{aligned}$$

Hence $D_3 \cap D_4$ is invariant under the flow (2.2).

Finally, we set

$$D_5(t) := \left\{(u,w) \in \mathbb{R}^2;\ u > 0, \quad w > 0 \text{ and } w - h(t) \le 0\right\},\ t \ge 0,$$

where $h(t)$ is a positive smooth decreasing function to be specified later. Note that $D_3 \cap D_4 \cap D_5(t)$ is convex. We choose $h(0) = 1/(aK^2)$ such that $\mathcal{S}(0) \subset D_3 \cap D_4 \cap D_5(0)$. For $(U,W) \in D_3 \cap D_4 \cap \partial D_5(t)$, we compute

$$\frac{d}{dt}(W - h) = W_t - h_t = \frac{-W^3}{U^2W^2+1} - h_t.$$

Hence $\{D_3 \cap D_4 \cap D_5(t)\}_{t\ge 0}$ is invariant under the flow (2.2), if

$$h_t = \sup_{(U,W)\in D_3\cap D_4\cap\partial D_5(t)} \frac{-W^3}{U^2W^2+1} = \frac{-h^3}{c^2h^2+1}, \text{ where } c := 1/(aK) > 0. \tag{2.3}$$

Therefore, let $h(t)$ be the solution of

$$c^2 \ln h(t) - \frac{1}{2h^2(t)} = c^2 \ln h(0) - \frac{1}{2h^2(0)} - t, \tag{2.4}$$

we have that $h(t)$ satisfies (2.3) and $\{D_3 \cap D_4 \cap D_5(t)\}_{t\ge 0}$ is invariant under the flow (2.2). Moreover, by (2.3) and (2.4) we obtain that $h(t)$ decreases to 0 as $t \to \infty$. Therefore, $(u,w)$ converges to $(0,0)$ as $t \to \infty$. Since $v = 1/w$, we have $(u,v)$ converges to $(0,\infty)$ as $t \to \infty$. This completes the proof of the theorem. $\square$

## 3. Asymptotically constant initial data

This section is devoted to the study the solution of (1.2) with asymptotically constant initial data. We first consider the following ODE system:

$$\left\{ \begin{aligned} U_t &= -U/(U^2+V^2), \\ V_t &= V/(U^2+V^2), \end{aligned} \right. \tag{3.1}$$

for $t \geq 0$ with the initial condition $(U(0), V(0)) = (M, 0)$ for some constant $M > 0$. Then it is easy to see that the solution is given by $(U(t), V(t)) := (\sqrt{M^2 - 2t}, 0)$. Note that the quenching time of this ODE system is $T = T(M) := M^2/2$.

Next, in order to estimate $u(x,t)$ from below, we consider the following Cauchy problem:

$$\left\{ \begin{array}{l} \overline{u}_t = \overline{u}_{xx} - 1/\overline{u}, \quad x \in \mathbb{R}, \quad t \in [0, \overline{T}), \\ \overline{u}(x,0) = \overline{u}_0(x), \quad x \in \mathbb{R}. \end{array} \right. \tag{3.2}$$

where $[0, \overline{T})$ is the maximal existence interval of $\overline{u}$. Also, we consider the following ODE problem corresponding to the problem (3.2):

$$\overline{U}_t = -1/\overline{U}, \quad t \in [0, T), \quad \overline{U}(0) = M. \tag{3.3}$$

Note that the solution of (3.3) is given by $\overline{U}(t) = \sqrt{M^2 - 2t}$ with $T = T(M) := M^2/2$.

Motivated by an idea from [5], we have the following lemma. We also refer the reader to [10] for the Fujita equation, [14] for a quasilinear parabolic equation, and [15] for a cooperative parabolic system.

**Lemma 3.1.** *Let $\overline{U}$ be the solution of (3.3) and let $\overline{u}$ be the solution of (3.2) defined on $\mathbb{R} \times [0, \overline{T})$. Suppose that there exist $t_0 \in [0, \widehat{T})$, $r_0 \in (0, \infty)$ and a constant $\theta > 1$ such that*

$$\overline{u}(x,t) \geq \theta \overline{U}(t), \quad for \quad |x| \leq r_0, \quad t_0 \leq t < \widehat{T}.$$

*where $\widehat{T} := \min\{T, \overline{T}\}$. Then $\overline{u}$ has a positive lower bound in $\{|x| \leq r_0/2\} \times [t_0, \widehat{T})$.*

**Proof.** We shall construct a suitable subsolution of (3.2) as follows

$$w(x,t) := \widehat{\theta} \sqrt{M^2 - 2t + h(x)},$$

where $\widehat{\theta} \in (1, \theta)$ and

$$h(x) := \varepsilon cos^2(\frac{\pi x}{2r_0})$$

with small $\varepsilon > 0$ to be specified later.

By a simple computation, we obtain that

$$\begin{aligned} w_t - w_{xx} + \frac{1}{w} &= \frac{\widehat{\theta}^2}{w} \Big\{ -1 - \frac{1}{2}h'' + \frac{h'^2}{4(M^2 - 2t + h)} + \Big(\frac{1}{\widehat{\theta}}\Big)^2 \Big\} \\ &\leq \frac{\widehat{\theta}^2}{w} \Big\{ -1 - \frac{1}{2}h'' + \frac{h'^2}{4h} + \Big(\frac{1}{\widehat{\theta}}\Big)^2 \Big\}. \end{aligned}$$

By the choice of $h$, we obtain that both $|h''|$ and $h'^2/h$ are of order $\varepsilon$ for $|x| \le r_0$. Hence, if we choose $\varepsilon > 0$ sufficiently small such that

$$\varepsilon \le (M^2 - 2t_0)\Big[\Big(\frac{\theta}{\widehat{\theta}}\Big)^2 - 1\Big], \quad -1 - \frac{1}{2}h'' + \frac{h'^2}{4h} + \Big(\frac{1}{\widehat{\theta}}\Big)^2 \le 0,$$

then we have

$$\begin{cases} w_t \le w_{xx} - 1/w, & |x| \le r_0, \quad t_0 \le t < \widehat{T}, \\ w(x,t_0) \le \overline{u}(x,t_0), & |x| \le r_0, \\ w(x,t) \le \overline{u}(x,t), & |x| = r_0, \quad t_0 \le t < \widehat{T}, \end{cases} \tag{3.4}$$

where we have used the fact $\widehat{\theta} \in (1,\theta)$.

Then it follows from (3.4) and the comparison principle that $w(x,t) \le \overline{u}(x,t)$ for $|x| \le r_0$ and $t_0 \le t < \widehat{T}$. Therefore, we have

$$\overline{u}(x,t) \ge \widehat{\theta}\sqrt{M^2 - 2t + h(r_0/2)} = \widehat{\theta}\sqrt{M^2 - 2t + \varepsilon/2} \ge \widehat{\theta}\sqrt{\varepsilon/2} > 0$$

for any $|x| \le r_0/2$ and $t_0 \le t < \widehat{T}$. The lemma follows. □

Hereafter, we assume

$$\overline{u}_0 \in C^1(\mathbb{R}), \quad \overline{u}_0 \ge M, \quad \overline{u}_0 \not\equiv M, \tag{3.5}$$

$$\lim_{|x|\to\infty} \overline{u}_0(x) = M. \tag{3.6}$$

Note that by (3.2), (3.3), and (3.5) we have $\overline{U} \le \overline{u}$. Therefore, we obtain $\overline{T} \ge T$ and so $\widehat{T} = T$.

The following lemma shows that quenching can occur only at space infinity.

**Lemma 3.2.** *Let $\overline{u}$ be a solution of* (3.2) *satisfying* (3.5) *and* (3.6) *for some constant $M > 0$. Then $\overline{u}$ has a positive lower bound in $\Omega \times [0,T)$ for any compact set $\Omega \subset \mathbb{R}$.*

**Proof.** In view of Lemma 3.1, since $\widehat{T} = T$, it suffices to show that, for any given $R > 0$ there exist $t_0 \in [0,T)$ and $\theta > 1$ such that

$$\overline{u}(x,t) \ge \theta\sqrt{M^2 - 2t}, \quad |x| \le 2R, \quad t_0 \le t < T. \tag{3.7}$$

For this purpose, we let $\gamma(x,t) := \overline{u}(x,t)/\overline{U}(t)$. Then the function $\gamma = \gamma(x,t)$ satisfies

$$\gamma_t = \gamma_{xx} + \frac{1}{\overline{U}^2}\Big(-\frac{1}{\gamma} + \gamma\Big) \ge \gamma_{xx},$$

since $\gamma \ge 1$. Moreover, by (3.5) and (3.6) we obtain

$$\gamma(\cdot,0) = \frac{\overline{u}_0}{M} \ge 1, \quad \gamma(\cdot,0) \not\equiv 1.$$

From the strong maximum principle, we have that $\gamma(x,t) > 1$ for all $x \in \mathbb{R}$ and $t > 0$. Therefore, for any given $R > 0$, there exist $\theta > 1$ and $t_0 \in (0,T)$ such that

$$\gamma(x,t) \ge \theta, \quad |x| \le 2R, \quad t_0 \le t < T.$$

This gives (3.7). Therefore we complete the proof. □

To investigate the behavior of the solution of (1.2) at space infinity, we recall the following useful property (cf. [5]). We also refer the reader to [15] for the blow-up problem for a cooperative parabolic system.

**Theorem 4.** *Let* $\mathbf{u}$ *and* $\widehat{\mathbf{u}}$ *be solutions of*

$$\begin{cases} \mathbf{u}_t = D\mathbf{u}_{xx} + \mathbf{f}(\mathbf{u}), & x \in \mathbb{R}, \quad t > 0, \\ \mathbf{u}(x,0) = \mathbf{u}_0(x), & x \in \mathbb{R}. \end{cases} \tag{3.8}$$

*where* $\mathbf{u}(x,t) = (u(x,t), v(x,t)) \in \mathbb{R}^2$, $\mathbf{f} = (f_1, f_2)$ *is a smooth mapping from* $\mathbb{R}^2$ *to* $\mathbb{R}^2$, $D = diag(1,1)$, *with initial data* $\mathbf{u}_0$, $\widehat{\mathbf{u}}_0 \in (L^\infty(\mathbb{R}) \cap C(\mathbb{R}))^2$, *respectively. Suppose that there exist sequences* $\{r_n\}_{n=1}^\infty \subset (0,\infty)$ *and* $\{a_n\}_{n=1}^\infty \subset \mathbb{R}$ *with* $r_n \to \infty$ *as* $n \to \infty$ *such that*

$$\lim_{n\to\infty} ||\mathbf{u}_0 - \widehat{\mathbf{u}}_0||_{L^\infty(B_{2r_n}(a_n))} = 0.$$

*Then*

$$\lim_{n\to\infty} ||\mathbf{u}(\cdot,t) - \widehat{\mathbf{u}}(\cdot,t)||_{L^\infty(B_{r_n}(a_n))} = 0.$$

*for any* $t \in (0, \widetilde{T})$, *where* $\widetilde{T} = \min\{T(\mathbf{u}_0), T(\widehat{\mathbf{u}}_0)\}$.

Notice that the following corollary is applicable to our system (1.2). Since its proof is exactly the same as the one given in [5, Corollary 4.2], we omit is here.

**Corollary 3.3.** *If some solutions of*

$$\mathbf{U}_t = \mathbf{f}(\mathbf{U}) \tag{3.9}$$

*quenches in a finite time, then there exists a spatially inhomogeneous solutions of* (3.8) *which quenches in a finite time.*

In the following, we shall focus on the Cauchy problem for (1.2) with initial data satisfying (1.7) and (1.8).

**Lemma 3.4.** *Let* $\overline{u}$ *be a solution of* (3.2) *satisfying* (3.5) *and* (3.6) *for some constant* $M > 0$. *Then* $\overline{u}$ *quenches at the finite time* $T = M^2/2$.

**Proof.** First, we set $\mathbf{u}(x,t) = \overline{u}(x,t)$, $\widehat{\mathbf{u}}(x,t) = \overline{U}(t)$, $|a_n| = 4n$, and $r_n = n$. By (3.6), we have

$$\lim_{n\to\infty} ||\mathbf{u}_0 - \widehat{\mathbf{u}}_0||_{L^\infty(B_{2r_n}(a_n))} = 0. \tag{3.10}$$

Notice that $\mathbf{u}$ and $\widehat{\mathbf{u}}$ are solutions of (3.2) and (3.3) with initial data $\mathbf{u}_0$ and $\widehat{\mathbf{u}}_0$, respectively. Let $\mathbf{f}(\mathbf{u}) = -1/\overline{u}$, $\mathbf{f}(\widehat{\mathbf{u}}) = -1/\overline{U}$. Applying Theorem 4 to (3.2) and (3.3), we obtain

$$\lim_{|x|\to\infty} \overline{u}(x,t) = \overline{U}(t), \quad \forall t \in [0,T).$$

On the other hand, by (3.2), (3.3), (3.5), and the comparison principle, we have $\overline{u}(x,t) \geq \overline{U}(t)$ for all $x \in \mathbb{R}$ and $t > 0$. Combining the above two facts, we have the quenching time $\overline{T} = T = M^2/2$. □

Now we prove the Theorem 2 by using Theorems 1 and 4.

**Proof of Theorem 2.** First, we have the local existence of $(u,v)$ for $t \in [0,\sigma]$ for some $\sigma > 0$. Let $\mathbf{u}(x,t) = (u(x,t), v(x,t))$, $\widehat{\mathbf{u}}(x,t) = (U(t), V(t))$ and

$$\mathbf{f}(\mathbf{u}) = \Big(\frac{-u}{u^2+v^2}, \frac{v}{u^2+v^2}\Big),$$

where $(u,v)$ and $(U,V)$ are solutions of (1.2) and (1.3), respectively. By applying Theorem 4 to (1.2) and (1.3) with $|a_n| = 4n$ and $r_n = n$, we have

$$\lim_{|x|\to\infty} u(x,t) = U(t), \quad \text{and} \quad \lim_{|x|\to\infty} v(x,t) = V(t), \quad \forall t \in [0,\sigma]. \tag{3.11}$$

Also, it follows from (1.4) and (1.8) with $N > 0$ that

$$\lim_{|x|\to\infty} u(x,t)v(x,t) = U(t)V(t) = U(0)V(0) = \lim_{|x|\to\infty} u_0(x)v_0(x) = MN > 0.$$

Hence the assumption (1.6) is satisfied for all $x$ with $|x| \geq R$ at $t = \sigma$ for some constants $R$ sufficient large and $K > 0$.

Moreover, by the strong maximum principle, we obtain $v > 0$ in $\mathbb{R} \times [0,\sigma]$. It implies that the assumption (1.6) holds for all $x$ with $|x| \leq R$ at $t = \sigma$ with the positive constant $K$ (taking a smaller one if necessary). Therefore, by applying Theorem 1 to the Cauchy problem (1.2) starting at $t = \sigma$, we obtain that the solution $(u,v)$ exists globally in time and $(u,v)$ converges to $(0,\infty)$ as $t \to \infty$. This completes the proof of Theorem 2. □

Finally, we give a proof of Theorem 3.

**Proof of Theorem 3.** We choose $\overline{u}_0 = u_0$. Then, by the comparison principle, we obtain

$$u(x,t) \geq \overline{u}(x,t), \quad x \in \mathbb{R}, \quad \text{for } t > 0 \text{ such that } u \text{ and } \overline{u} \text{ exist.} \tag{3.12}$$

Suppose that the solution $(u,v)$ quenches at time $T^*$. By (3.12), we have $T^* \geq T$. On the other hand, by Lemmas 3.2 and 3.4, the solution $\overline{u}$ quenches at finite time $T = M^2/2$ only at space infinity. Thus the inequality (3.12) implies that

$$u \geq \overline{u} > 0 \text{ in } \mathbb{R} \times [0,T). \tag{3.13}$$

Moreover, we set $\mathbf{u}(x,t) = (u(x,t), v(x,t))$, $\widehat{\mathbf{u}}(x,t) = (U(t), V(t)) = (U(t), 0)$ and

$$\mathbf{f}(\mathbf{u}) = \Big(\frac{-u}{u^2+v^2}, \frac{v}{u^2+v^2}\Big),$$

where $(u,v)$ and $(U,V)$ are solutions of (1.2) and (3.1), respectively. Applying Theorem 4 to (1.2) and (3.1) with $|a_n| = 4n$ and $r_n = n$ again, we have

$$\lim_{|x|\to\infty} u(x,t) = \overline{U}(t), \quad \lim_{|x|\to\infty} v(x,t) = V(t) = 0, \quad \forall t \in [0,T). \tag{3.14}$$

Hence we obtain $T^* = T$. From Lemma 3.2, $\overline{u}$ quenches only at space infinity. Combining this with (3.13), we conclude that the quenching of the solution $(u, v)$ occurs only at space infinity. This proves the theorem. $\square$

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