ENTIRE SOLUTIONS FOR A DISCRETE DIFFUSIVE EQUATION WITH BISTABLE CONVOLUTION TYPE NONLINEARITY

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Abstract. We study entire solutions for a discrete diffusive equation with bistable convolution type nonlinearity. We construct three different types of entire solutions. Each of these entire solutions behaves as two traveling wavefronts connecting two of those three equilibria as time approaches minus infinity. Moreover, the first and second ones are solutions which behave as two traveling wavefronts approaching each other from both sides of x-axis. The behavior of the second one is like the first one except it connects two different wavefronts. The third one is a solution which behaves as two different traveling wavefronts and one chases another from the same side of x-axis.

1. Introduction

In this paper, we study the following discrete diffusive equation with convolution type nonlinearity.

$$u_t(x,t) = \mathcal{D}_2[u](x,t) - du(x,t) + \sum_{i\in\mathbb{Z}} J(i)b(u(x-i,t)), \qquad x\in\mathbb{R},\ t\in\mathbb{R}, \tag{1.1}$$

where $d>0$, $J(i)=J(-i)\geq 0$, $\sum_{i\in\mathbb{Z}} J(i)=1$, and

$$\mathcal{D}_2[u](x,t) := D[u(x+1,t)+u(x-1,t)-2u(x,t)]$$

for some positive constant D. Throughout this paper, we shall always assume that the function $b(\cdot)$ is an increasing smooth function on $[0,1]$ such that

- (P1) $b(0) = b(a) - ad = b(1) - d = 0$, where $0<a<1$,
- (P2) $b(t) < dt$ for $0<t<a$, $b(t) > dt$ for $a<t<1$,
- (P3) $\max\{b'(0), b'(1)\} < d < b'(a)$ (*bistable nonlinearity*),
- (P4) $\int_0^1 [b(u) - du]du > 0$ (*unbalanced case*).

When $J(0)=1$ and $J(i)=0$ for all $i\neq 0$, (1.1) is reduced to the classical equation

$$u_t(x,t) = \mathcal{D}_2[u](x,t) + f(u(x,t)), \quad f(u) := b(u) - du,$$

which has been studied recently in [5, 6].

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We also note that (1.1) is the continuum version of the following lattice dynamical system:

$$u_n'(t) = D[u_{n+1}(t) + u_{n-1}(t) - 2u_n(t)] - du_n(t) + \sum_{i\in\mathbb{Z}} J(i)b(u_{n-i}(t)), n \in \mathbb{Z}, t \in \mathbb{R}. \tag{1.2}$$

For (1.2), in ecology, u_n represents the population density at site n, D is the migration coefficient, d is the death rate and the nonlinear function b is the birth function of population density which is interacting with neighbors by the nonnegative weighted function J, if the habitat is divided into discrete regions and the population density is measured at the representative point in each region. In this model, we assume that the migration only happens to the nearest neighbors and the interaction happens with finite or infinite range.

We say that $\{u_n(t)\}$ is a traveling wavefront solution of (1.2) connecting two different equilibria $\{u_\pm\} \subset \{0, a, 1\}$ with speed c, if $u_n(t) = U(n + ct)$ for $n \in \mathbb{Z}$ and $t \in \mathbb{R}$ for some function U (called *wave profile*) such that $U(\pm\infty) = u_\pm$. Then (c, U) satisfies the following equation

$$cU'(y) = \mathcal{D}_2[U](y) - dU(y) + \sum_{i\in\mathbb{Z}} J(i)b(U(y-i)), \quad y \in \mathbb{R}, \tag{1.3}$$

where (as before)

$$\mathcal{D}_2[U](x) := D[U(x+1) + U(x-1) - 2U(x)].$$

Similarly, we can define the notion of traveling wavefront solution of (1.1) by setting $u(x,t) = U(x+ct)$, then U also satisfies the equation (1.3).

Recently, a more general version of (1.2) including time delay was studied in [11, 10]. In [11], they studied (1.2) with time delay for the bistable case. They proved that the problem admits a unique (up to a translation) strictly monotone increasing traveling wavefront solution connecting from 0 to 1 with a positive wave speed when $D \geq D_0$ for a certain positive constant D_0, under the following extra assumption

$$\sum_{i\in\mathbb{Z}} J(i) < \max\left\{ \frac{2\int_0^1 [b(u) - du]du}{\int_0^1 b(u)du}, \frac{2\int_0^1 [b(u) - du]du}{\int_0^1 b(u)du - d} \right\}. \tag{1.4}$$

More precisely, from [11, Theorem 1.1], under the above assumptions, there exist a unique speed $\hat{c} > 0$ and a unique (up to translations) wave profile $U(x)$ such that

$$\begin{cases} \hat{c}U'(x) = \mathcal{D}_2[U](x) - dU(x) + \sum_{i\in\mathbb{Z}} J(i)b(U(x-i)),\ x \in \mathbb{R}, \\ U(-\infty) = 0,\ U(+\infty) = 1, 0 < U < 1,\ U' > 0 \text{ in } \mathbb{R}, \end{cases} \tag{1.5}$$

if $D \geq D_0$. Note that a propagation failure occurs when D is small enough.

The monostable case for (1.2) with time delay was considered in [10]. In the present setting, it corresponding to the case for connecting two equilibria $\{a, 1\}$ or $\{0, a\}$. They obtained the existence of the asymptotic speed of propagation, the existence and (partial) uniqueness of the traveling wavefront and the minimal speed of the traveling wavefront for the delayed lattice dynamical system under the following extra condition at the unstable equilibrium a, namely,

$$b'(a)(u-a) - M|u-a|^{1+\alpha} \le b(u) - da \le b'(a)(u-a) + M|u-a|^{1+\alpha} \text{ for } u \in [0,1] \tag{1.6}$$

for some constants $M > 0$ and $\alpha \in (0,1]$. In fact, by [10, Theorem 1.2], there exist two constants c_*, c^* with $c^* > 0 > c_*$ such that for any c_1, c_2 (with $c_1 \ge c^*$, $c_2 \le c_*$) there exist $V_1(x)$ and $W_2(x)$ satisfying the following equations:

$$\begin{cases} c_1 V_1'(x) = \mathcal{D}_2[V_1](x) - dV_1(x) + \sum_{i\in\mathbb{Z}} J(i) b(V_1(x-i)), \ x \in \mathbb{R}, \\ V_1(-\infty) = a, \ V_1(+\infty) = 1, \ a < V_1 < 1, \ V_1' > 0 \text{ in } \mathbb{R}. \end{cases} \tag{1.7}$$

and

$$\begin{cases} c_2 W_2'(x) = \mathcal{D}_2[W_2](x) - dW_2(x) + \sum_{i\in\mathbb{Z}} J(i) b(W_2(x-i)), \ x \in \mathbb{R}, \\ W_2(-\infty) = 0, \ W_2(+\infty) = a, \ 0 < W_2 < a, \ W_2' > 0 \text{ in } \mathbb{R}, \end{cases} \tag{1.8}$$

where c^* (c_*, resp.) is the minimal (maximal, resp.) speed of (1.7) ((1.8), respectively).

The study of traveling wavefront solutions are important in many applications. It can describe certain dynamical behavior of the studied problem such as (1.2). But, the dynamics of reaction-diffusion equations or its discrete analogue is so rich that there might be other interesting patterns. Recently it is found that *two-front* entire solutions exist in many problems. Here an entire solution is meant by a solution defined for all $(x,t) \in \mathbb{R}^2$. In particular, traveling wavefront solutions are also entire solutions. For the study of entire solutions, we refer the reader to, for instance, [3, 5, 6, 7, 8, 9, 12, 13] and reference therein.

In a very interesting work by Morita and Ninomiya [12], they gave three different types of entire solutions for a bistable reaction-diffusion equation (see also [6] for the discrete diffusive case). The purpose of this work is to construct these three types of entire solutions for (1.1). Although the main idea and the methods of proofs in this paper are from [6, 12], there are certain difficulties in dealing with (1.1) (or (1.2)) due to the convolution type nonlinearity. For example, in the construction of super/sub solutions, we need to derive some estimations. In these estimations, the compactness (finite range interaction) assumption is needed in this study. So, from now on, besides the assumptions (1.4) and (1.6), we shall assume that

$$J(i) = 0 \text{ for } |i| > m \text{ for some } m \in \mathbb{N}. \tag{1.9}$$

We left the problem with infinite range interaction for the future study.

In fact, to construct these two-front entire solutions it is crucial to have a precise information on the asymptotic behavior of wave tails. More precisely, we need the following estimates for solutions U, V_1, W_2 of (1.5), (1.7), (1.8) respectively.

First, there exists a positive constant η such that

$$\inf_{y\le 0} \frac{U'(y)}{U(y)} \ge \eta, \quad \inf_{y\ge 0} \frac{U'(y)}{1-U(y)} \ge \eta. \tag{1.10}$$

Furthermore, there are positive constants K, k, γ, δ such that

$$ke^{\lambda y} \le U(y) \le Ke^{\lambda y},\ \forall\ y \le m; \quad \gamma e^{-\mu y} \le 1 - U(y) \le \delta e^{-\mu y},\ \forall\ y \ge -m, \tag{1.11}$$

where λ is the unique positive root of the characteristic equation

$$\hat{c}\lambda = D(e^{\lambda} + e^{-\lambda} - 2) - d + b'(0) \sum_{i=-m}^{m} J(i)e^{i\lambda}, \tag{1.12}$$

and μ is the unique positive root of the equation

$$-\hat{c}\mu = D(e^{\mu} + e^{-\mu} - 2) - d + b'(1) \sum_{i=-m}^{m} J(i)e^{i\mu}. \tag{1.13}$$

Next, for any $c_1 \ge c^*$ and $c_2 \le c_*$, let $(c_1, V_1(x))$ and $(c_2, W_2(x))$ be solutions of (1.7) and (1.8), respectively. Then there exist positive constants $\lambda_i, \mu_i, \kappa_i, \gamma_i$, $i = 1, 2$, such that the following inequalities hold:

$$V_1(y) - a \ge \kappa_1 e^{\lambda_1 y} \text{ on } (-\infty, 0]; \quad 1 - V_1(y) \ge \gamma_1 e^{-\mu_1 y} \text{ on } [0, \infty). \tag{1.14}$$

$$W_2(y) \ge \kappa_2 e^{\lambda_2 y} \text{ on } (-\infty, 0]; \quad a - W_2(y) \ge \gamma_2 e^{-\mu_2 y} \text{ on } [0, \infty). \tag{1.15}$$

Furthermore, there exist positive constants N, ρ such that

$$\rho[V_1(y) - a] \le V_1'(y) \le Ne^{\lambda_1 y} \text{ on } (-\infty, 0], \tag{1.16}$$

$$\rho[1 - V_1(y)] \le V_1'(y) \le Ne^{-\mu_1 y} \text{ on } [0, \infty), \tag{1.17}$$

$$\rho W_2(y) \le W_2'(y) \le Ne^{\lambda_2 y} \text{ on } (-\infty, 0], \tag{1.18}$$

$$\rho[a - W_2(y)] \le W_2'(y) \le Ne^{-\mu_2 y} \text{ on } [0, \infty). \tag{1.19}$$

The above asymptotic behavior of wave tail at the unstable equilibrium can be found in [4]. But, due to the technical difficulty arising from the convolution type nonlinearity, we need to assume that $m = 2$. As for the wave tail at the stable equilibrium, the method developed in [2] is well applicable here for any finite m.

Based on these asymptotic behaviors, we prove the following theorems on two-front entire solutions.

Theorem 1. *Let* (1.9) *be in force with $m = 2$ and let $(\hat{c}, U(x))$ be a solution of* (1.5). *Then for any real number θ there exists an entire solution $u(x,t)$ of* (1.1) *such that*

$$\lim_{t\to-\infty} \{\sup_{x\ge0} |u(x,t) - U(x + \hat{c}t + \theta)| + \sup_{x\le0} |u(x,t) - U(-x + \hat{c}t + \theta)|\} = 0. \tag{1.20}$$

Moreover, $u(x,t) \to 1$ as $t \to \infty$ for any x.

Theorem 2. *Let (1.9) be in force with $m = 2$. For any $c_1 > c^*$ and $c_2 < c_*$, let $(c_1, V_1(x))$ and $(c_2, W_2(x))$ be solutions of (1.7) and (1.8) respectively. Then there exist a constant ω and an entire solution $u(x,t)$ of (1.1) such that*

$$\lim_{t\to-\infty} \Big\{ \sup_{x\geq -(c_1+c_2)t/2} |u(x,t) - V_1(x + c_1 t + \omega)| + \sup_{x\leq -(c_1+c_2)t/2} |u(x,t) - W_2(x + c_2 t - \omega)| \Big\} = 0. \tag{1.21}$$

Moreover, there exists $\theta \in \mathbb{R}$ such that

$$\lim_{t\to\infty} \{\sup_{x\in\mathbb{R}} |u(x,t) - U(x + \hat{c}t + \theta)|\} = 0. \tag{1.22}$$

Theorem 3. *Let (1.9) be in force with $m = 2$. For any $c_2 < c_*$ with $-c_2 < \hat{c}$, let $(\hat{c}, U(x))$ and $(c_2, W_2(x))$ be solutions of (1.5) and (1.8) respectively. Then there exist a constant ω and an entire solution $u(x,t)$ of (1.1) such that*

$$\lim_{t\to-\infty} \Big\{ \sup_{x\geq (c_2-\hat{c})t/2} |u(x,t) - U(x + \hat{c}t + \omega)| + \sup_{x\leq (c_2-\hat{c})t/2} |u(x,t) - W_2(-x + c_2 t - \omega)| \Big\} = 0.$$

Moreover, we have

$$\lim_{t\to\infty} \{\inf_{x\in\mathbb{R}} u(x,t)\} = a, \qquad \lim_{t\to\infty} \{\sup_{x\geq -C} |u(x,t) - 1|\} = 0, \ \forall C > 0.$$

The above constructed entire solutions have some common characters. When $-t \gg 1$, they behave as two traveling wavefronts on the opposite sides or on the same side of x-axis. Note that, different from the previous works, we choose the distinguishing line of the initial conditions in the above theorems to be the mid-points of two front-positions of traveling wavefronts. For example, in Theorem 2, $x = -c_1 t$ and $x = -c_2 t$ are front-positions for two traveling wavefronts $V_1(x + c_1 t)$ and $W_2(x + c_2 t)$, respectively. It is nature to choose the distinguishing line to be $x = -(c_1 + c_2)t/2$ in (1.21).

We organize this paper as follows. In section 2, we give some proofs of the asymptotic behaviors of the traveling wavefronts stated above and some useful functions. Next, in section 3, we offer the proofs of Theorem 1, Theorem 2 and Theorem 3 by constructing suitable super/sub solutions.

2. Preliminaries

In this section, we first study the asymptotic behaviors of a solution $U(y)$ of (1.5) as $y \to \pm\infty$. Since the behavior near $y = \infty$ is similar to the one near $y = -\infty$, we shall only give the details for $y=-\infty$. For this, we use the following notation

$$\begin{aligned}\mathcal{N}[u_j](t) &:= u_j'(t) - D[u_{j+1}(t)+u_{j-1}(t)-2u_j(t)] \\ &\qquad +du_j(t) - \sum_{i=-m}^{m} J(i)b(u_{j-i}(t)),\ j\in\mathbb{Z}, t\in\mathbb{R}.\end{aligned}$$

First, we have the following strong comparison principle.

Lemma 2.1. *Let $c\in\mathbb{R}$, $j_0\in\mathbb{Z}$ and $t_0\in\mathbb{R}$. Assume that $u_j(t)$ and $v_j(t)$ are bounded and continuous in the set $\{(j,t)\in\mathbb{Z}\times\mathbb{R}|\ j\le j_0-ct, t\in[t_0,\infty)\}$ and satisfy*

$$\mathcal{N}[u_j](t)\ge\mathcal{N}[v_j](t) \quad \text{when} \quad j\le j_0-ct,\ t>t_0,$$

$$u_j(t_0)\ge v_j(t_0) \quad \text{when} \quad j\le j_0-ct_0.$$

Then $u_j(t)\ge v_j(t)$ for all $j\le j_0-ct$, $t>t_0$. In addition, if $u_{j_1}(t_0)>v_{j_1}(t_0)$ for some $j_1\le j_0-ct_0$, then $u_j(t)>v_j(t)$ for all $j\le j_0-ct$, $t>t_0$.

Since the proof is exactly the same as the one for [2, Lemma 1], we omit it here.

Using this comparison principle (Lemma 2.1), we can follow the proof of [2, Theorem 2] to prove the following theorem on the asymptotic behavior.

Theorem 4. *Assume that $(c,\{u_j(t)\})$ is a traveling wave solution of (1.2) connecting from 0 to 1 with positive speed c. Then there exists two positive constants C_1, C_2 such that*

$$C_1\le\frac{u_j(t)}{e^{\Lambda(j+ct)}}\le C_2,\ \forall j+ct\le -m, t\ge 0 \tag{2.1}$$

where Λ is the positive root of the following characteristic equation

$$P(c,\lambda):=c\lambda-D(e^{\lambda}+e^{-\lambda}-2)+d-b'(0)\sum_{i=-m}^{m}J(i)e^{-i\lambda}=0.$$

By the definition of Λ, the function $\psi(x):=e^{\Lambda x}$ is a solution of the following equation

$$c\psi'(x)-D[\psi(x+1)+\psi(x-1)-2\psi(x)]+d\psi(x)-b'(0)\sum_{i=-m}^{m}J(i)\psi(x-i)=0.$$

In the construction of sub/supersolutions, $\psi(x)$ play an important role. Indeed, we define

$$u_j^+(t;\epsilon_1,\theta,\epsilon_3):=\epsilon_1\psi(0)+\theta\psi(\Lambda)e^{\Lambda(j+ct)}-\epsilon_3\psi(2\Lambda)e^{2\Lambda(j+ct)},\ j\in\mathbb{Z},\ t\in\mathbb{R}.$$

where $\epsilon_1\ge 0,\ \epsilon_3\ge 0,\ \theta\in\mathbb{R}$. Hereafter the function b is suitably defined so that it is smooth with b,b',b'' bounded in $\mathbb{R}$. Since $P(c,0)>P(c,\Lambda)=0>P(c,2\Lambda)$ (due to the fact that $b'(0)<d$), we have

$$\mathcal{N}[u_j^+](t)\ge 0 \quad \text{when} \quad j+ct\le -m,\ t\in\mathbb{R},$$

if

$$0\le\epsilon_1\le E_1,\ \epsilon_3=E_3\theta^2,\ |\theta|\le E_2,$$

where

$$E_1 := \frac{P(c,0)}{2L\psi(0)},\ E_3 := \frac{8L\psi(\Lambda)^2 e^{2\Lambda m}}{-P(c,2\Lambda)\psi(2\Lambda)},\ E_2 := \frac{\psi(\Lambda)}{E_3\psi(2\Lambda)},\ L := \max_{u\in\mathbb{R}} |b''(u)|.$$

Similarly, by defining

$$u_j^-(t;\epsilon_1,\theta,\epsilon_3) := -\epsilon_1\psi(0) + \theta\psi(\Lambda)e^{\Lambda(j+ct)} + \epsilon_3\psi(2\Lambda)e^{2\Lambda(j+ct)},$$

we also get

$$\mathcal{N}[u_j^-](t) \le 0 \quad \text{when} \quad j+ct \le -m,\ t\in\mathbb{R}.$$

Then Theorem 4 can be proved by using the comparison principle as given in the proof of [2, Theorem 2]. We omit the details here.

Now, for a solution U of (1.5), using $u_n(t) = U(n+\hat{c}t)$ we obtain from (2.1) that

$$C_1 e^{\lambda y} \le U(y) \le C_2 e^{\lambda y},\ \forall\ y \le -m,$$

where λ is the unique positive root of the equation (1.12). Hence the first part of (1.11) follows. We remark that this process can be carried out as long as the equilibrium is stable. Therefore, all of the exponential tail behaviors of U, V_1, W_2 near the stable equilibria $\{0,1\}$ in (1.11), (1.14) and (1.15) can be derived similarly.

As for the exponential tail behavior near the unstable equilibrium a, we refer to [4, Theorem 5]. There it is assumed that $m = 2$. Therefore, we have the exponential tail behaviors of V_1, W_2 near the equilibrium a in (1.14) and (1.15) for $c_1 > c^*$ and $c_2 < c_*$ when $m = 2$.

For the estimates related the first derivatives of U, V_1, W_2, we recall from [4, Theorem 2] that the limits

$$\lim_{y\to-\infty} \frac{V_1'(y)}{V_1(y)-a},\ \lim_{y\to\infty} \frac{W_2'(y)}{a-W_2(y)}$$

exist and are positive. Here we need to assume that $m = 2$. This result is based on [4, Theorem 1] and is applicable to the case of stable equilibrium. Therefore, we also have the limits

$$\lim_{y\to-\infty} \frac{U'(y)}{U(y)},\ \lim_{y\to\infty} \frac{U'(y)}{1-U(y)},\ \lim_{y\to\infty} \frac{V_1'(y)}{1-V_1(y)},\ \lim_{y\to-\infty} \frac{W_2'(y)}{W_2(y)}$$

exist and are positive. Then the estimates (1.10) and (1.16)-(1.19) can be derived.

Next, we give some useful functions which were constructed in [5]. Given positive constants α, c, M and consider $p(t)$ and $q(t)$ solutions of

$$p'(t) = c + Me^{\alpha p(t)}, \quad q'(t) = c - Me^{\alpha q(t)}, \quad t \le 0, \tag{2.2}$$

$$p(0) \le 0, \quad q(0) < \min\{0, \ln(c/M)/\alpha\}, \quad e^{-\alpha q(0)} - e^{-\alpha p(0)} < 2M/c. \tag{2.3}$$

Indeed, $p(t)$ and $q(t)$ can be solved explicitly by

$$\begin{aligned} p(t) &= p(0) + ct - \ln[1 + Me^{\alpha p(0)}(1 - e^{c\alpha t})/c]/\alpha, \\ q(t) &= q(0) + ct - \ln[1 - Me^{\alpha q(0)}(1 - e^{c\alpha t})/c]/\alpha. \end{aligned}$$

Furthermore, there exists a positive constant κ such that

$$-\kappa e^{c\alpha t}/2 \leq q(t) - ct - \omega < 0 < p(t) - ct - \omega \leq \kappa e^{c\alpha t}/2, \quad \text{if} \quad t \leq 0, \tag{2.4}$$

where

$$\omega := p(0) - \ln(1 + Me^{\alpha p(0)}/c)/\alpha = q(0) - \ln(1 - Me^{\alpha q(0)}/c)/\alpha. \tag{2.5}$$

Hence,

$$0 < p(t) - q(t) \leq \kappa e^{c\alpha t} (\leq \kappa), \quad \text{if} \quad t \leq 0. \tag{2.6}$$

Finally, we give the following definitions about a supersolution and a subsolution.

Definition 2.1. *A function $\overline{u}(x,t)$ is called a supersolution (subsolution, resp.) of* (1.1) *on $(x,t) \in \mathbb{R} \times (-\infty, -T]$ for some $T \in \mathbb{R}$, if $\mathbb{L}[\overline{u}](x,t) \geq 0$ ($\mathbb{L}[\underline{u}](x,t) \leq 0$, resp.) for all $(x,t) \in \mathbb{R} \times (-\infty, -T]$, where*

$$\mathbb{L}[v](x,t) := v_t(x,t) - \mathcal{D}_2[v](x,t) + dv(x,t) - \sum_{i=-m}^{m} J(i) b(v(x-i,t)).$$

The following useful lemma can be found in [1, 3, 5].

Lemma 2.2. *Suppose that $\underline{u}(x,t)$ and $\overline{u}(x,t)$ are a subsolution and a supersolution of* (1.1) *on $(x,t) \in \mathbb{R} \times (-\infty, -T]$ for some $T \in \mathbb{R}$, respectively and satisfy that $\underline{u}(x,t) \leq \overline{u}(x,t)$ on $(x,t) \in \mathbb{R} \times (-\infty, -T]$. Then there exists an entire solution $u(x,t)$ of* (1.1) *such that*

$$\underline{u}(x,t) \leq u(x,t) \leq \overline{u}(x,t) \quad \textit{for all } (x,t) \in \mathbb{R} \times (-\infty, -T].$$

With this lemma, the construction of entire solutions is reduced to finding a suitable pair of super/sub solutions.

3. Entire solutions

This section is devoted to the proofs of main theorems stated in the introduction. Since the proofs work as long as the asymptotic behaviors (1.10), (1.11), (1.14)-(1.19) hold, we shall present the proof for general $m \in \mathbb{N}$ here.

3.1. **Proof of Theorem 1.** Let $p(t)$ and $q(t)$ be the solutions of (2.2)-(2.3) with $c = \hat{c}$, $\alpha = \lambda$ and a constant M to be determined later. We divide our discussion into two cases: $b'(0) \le b'(1)$ and $b'(0) > b'(1)$.

First, we consider the case that $b'(0) \le b'(1)$. In this case, we have $\lambda > \mu$, where λ and μ are positive roots of (1.12) and (1.13), respectively. Define

$$\begin{cases} \overline{u}(x,t) := U(x+p(t)) + U(-x+p(t)),\ x \in \mathbb{R},\ t \le 0 \\ \underline{u}(x,t) := U(x+q(t)) + U(-x+q(t)),\ x \in \mathbb{R},\ t \le 0. \end{cases} \tag{3.1}$$

Then

$$\begin{aligned} & \mathbb{L}[\overline{u}](x,t) \\ = \ & p'(t)[U'(x+p(t)) + U'(-x+p(t))] + (2D+d)[U(x+p(t)) + U(-x+p(t))] \\ & -D[U(x+1+p(t)) + U(-x-1+p(t)) + U(x-1+p(t)) + U(-x+1+p(t))] \\ & -\sum_{i=-m}^{m} J(i) b(U(x-i+p(t)) + U(-x+i+p(t))). \end{aligned}$$

By using (1.5), we obtain

$$\begin{aligned} \mathbb{L}[\overline{u}](x,t) &= (p'(t)-\hat{c})[U'(x+p(t)) + U'(-x+p(t))] - \sum_{i=-m}^{m} J(i) G(x,t,i) \\ &= [U'(x+p(t)) + U'(-x+p(t))] \left\{ Me^{\lambda p(t)} - \sum_{i=-m}^{m} J(i) P(x,t,i) \right\}, \end{aligned}$$

where

$$\begin{aligned} G(x,t,i) &:= b(U(x-i+p(t)) + U(-x+i+p(t))) \\ & \qquad -b(U(x-i+p(t)) - b(U(-x+i+p(t)), \\ P(x,t,i) &:= G(x,t,i)/[U'(x+p(t)) + U'(-x+p(t))]. \end{aligned}$$

From

$$|b(u+v) - b(u) - b(v)| \le Luv \quad \text{if } u, v \in (0,1),$$

it follows that

$$\mathbb{L}[\overline{u}](x,t) \ge [U'(x+p(t)) + U'(-x+p(t))] \left\{ Me^{\lambda p(t)} - L \sum_{i=-m}^{m} J(i) P_1(x,t,i) \right\},$$

where

$$P_1(x,t,i) := \frac{U(x-i+p(t))U(-x+i+p(t))}{U'(x+p(t)) + U'(-x+p(t))}.$$

For any $i \in \{-m, \cdots, -1, 0, 1, \cdots, m\}$, by using (1.10) and (1.11), we obtain

$$P_1(x,t,i) \le \frac{U(x-i+p(t))}{U'(-x+p(t))} \le \frac{K}{\eta\gamma} e^{-\lambda i} e^{(\lambda-\mu)x} e^{(\lambda+\mu)p(t)}, \text{ if } x < p(t), \tag{3.2}$$

$$P_1(x,t,i) \le \frac{K^2 e^{\lambda(x-i+p(t))} e^{\lambda(-x+i+p(t))}}{\eta k[e^{\lambda(x+p(t))} + e^{\lambda(-x+p(t))}]} \le \frac{K^2}{2\eta k} e^{\lambda p(t)}, \text{ if } p(t) \le x \le -p(t), \tag{3.3}$$

$$P_1(x,t,i) \le \frac{U(-x+i+p(t))}{U'(x+p(t))} \le \frac{K}{\eta\gamma} e^{\lambda i} e^{(\mu-\lambda)x} e^{(\lambda+\mu)p(t)}, \text{ if } x > -p(t). \tag{3.4}$$

By using the facts $\lambda > \mu$ and $p(t) < 0$, it follows from (3.2) and (3.4) that

$$P_1(x,t,i) \le \frac{K e^{\lambda m}}{\eta\gamma} e^{\lambda p(t)}, \text{ if } x < p(t) \text{ or } \ x > -p(t).$$

Therefore, if we choose

$$M \ge \max\{\frac{LK e^{\lambda m}}{\eta\gamma}, \frac{LK^2}{2\eta k}\},$$

then $\mathbb{L}[\overline{u}] \ge 0$ on $\mathbb{R} \times (-\infty, 0]$. By a similar estimation, we get $\mathbb{L}[\underline{u}] \le 0$ on $\mathbb{R} \times (-\infty, 0]$.

It follows from Lemma 2.2 that there exists an entire solution $u(x,t)$ of (1.1) such that

$$\underline{u}(x,t) \le u(x,t) \le \overline{u}(x,t), \quad \forall (x,t) \in \mathbb{R} \times (-\infty, 0].$$

Now, we derive the initial condition (1.20). By translation, we may only check $\theta = \omega$. For $x \ge 0$, by the mean-value theorem, (2.4) and (2.6), we get

$$\begin{aligned}
& |u(x,t) - U(x + \hat{c}t + \omega)| \\
\le \ & [u(x,t) - \underline{u}(x,t)] + U(-x+q(t)) + |U(x+q(t)) - U(x+\hat{c}t+\omega)| \\
\le \ & [\overline{u}(x,t) - \underline{u}(x,t)] + K e^{\lambda(-x+q(t))} + \frac{\kappa}{2} \sup\{U'(\cdot)\} e^{\hat{c}\lambda t} \\
\le \ & K e^{\lambda q(t)} + K_1 e^{\hat{c}\lambda t}
\end{aligned}$$

for some constant K_1. The case for $x \le 0$ is similar. Hence (1.20) holds.

Secondly, we consider the case that $b'(1) < b'(0)$. In this case, we define

$$\begin{cases} \overline{u}(x,t) := U(x+p(t)) + U(-x+p(t)), \ x \in \mathbb{R}, \ t \le 0 \\ \underline{u}(x,t) := \max\{U(x+\hat{c}t+\omega), U(-x+\hat{c}t+\omega)\}, \ x \in \mathbb{R}, \ t \le 0. \end{cases} \tag{3.5}$$

Note that the definition of $\overline{u}(x,t)$ is the same as the former case in (3.1). Also, (3.3) holds, since we do not need the fact that $\lambda > \mu$ when $x \in [p(t), -p(t)]$. Therefore, we focus on the other two ranges. Since $b'(1) < b'(0)$, by extending the definition of $b(\cdot)$ and taking a suitable translation of $U(\cdot)$, we may find $\delta_1 > 0$ such that

$$b'(u) < b'(0) \quad \text{if } u > 1 - \delta_1; \qquad U(z) \ge 1 - \delta_1 \quad \text{if } z \ge -m. \tag{3.6}$$

First, we consider the case $x \le p(t)$. From the equality

$$b(u+v) - b(u) - b(v) = v \int_0^1 [b'(u+sv) - b'(sv)] ds$$

and (3.6), it follows that

$$G(x,t,i) \le U(x-i+p(t)) \int_0^1 [b'(0) - b'(sU(x-i+p(t)))]ds \le L[U(x-i+p(t))]^2.$$

Therefore,

$$\mathbb{L}[\overline{u}](x,t) \ge [U'(x+p(t)) + U'(-x+p(t))] \left\{ Me^{\lambda p(t)} - L \sum_{i=-m}^{m} J(i) P_2(x,t,i) \right\},$$

where

$$P_2(x,t,i) := [U(x-i+p(t))]^2 / [U'(x+p(t)) + U'(-x+p(t))].$$

For any $i \in \{-m, \cdots, -1, 0, 1, \cdots, m\}$, by (1.10) and (1.11), we have

$$\begin{aligned} P_2(x,t,i) &\le \frac{U(x+p(t))}{U'(x+p(t))} \cdot \frac{U(x-i+p(t))}{U(x+p(t))} \cdot U(x-i+p(t)) \\ &\le \frac{K^2 e^{2\lambda m}}{\eta k} e^{\lambda(x+p(t))} \le \frac{K^2 e^{2\lambda m}}{\eta k} e^{\lambda p(t)} \end{aligned}$$

for $x \le p(t)$. So if we choose $M \ge LK^2 e^{2\lambda m}/(\eta k)$, then we have $\mathbb{L}[\overline{u}] \ge 0$ for $x \le p(t)$. The case when $x \ge -p(t)$ can be treated similarly. By the definition of $\underline{u}(x,t)$ in (3.5), we can easily check that it is a subsolution. Hence, by Lemma 2.2, there exists an entire solution $u(x,t)$ such that

$$\underline{u}(x,t) \le u(x,t) \le \overline{u}(x,t), \quad \forall (x,t) \in \mathbb{R} \times (-\infty, 0].$$

Finally, we study the asymptotic behavior of u near $t = -\infty$. For $x \ge 0$, by the definition of $\underline{u}(x,t)$, we obtain $\underline{u}(x,t) = U(x + \hat{c}t + \omega)$. So, by the estimation of Lemma 2.2 and (2.4),

$$\begin{aligned} 0 &\le u(x,t) - U(x+\hat{c}t+\omega) \le \overline{u}(x,t) - U(x+\hat{c}t+\omega) \\ &\le U(x+p(t)) - U(x+\hat{c}t+\omega) + U(-x+p(t)) \\ &\le \sup\{U'(\cdot)\}(p(t) - \hat{c}t - \omega) + Ke^{\lambda(-x+p(t))} \\ &\le \kappa \sup\{U'(\cdot)\} e^{\hat{c}\lambda t}/2 + Ke^{\lambda p(t)}. \end{aligned}$$

This implies that

$$\lim_{t\to-\infty} \sup_{x\ge 0} |u(x,t) - U(x+\hat{c}t+\omega)| = 0.$$

The case for $x \le 0$ is similar. Hence (1.20) holds and the proof of Theorem 1 is completed.

3.2. **Proof of Theorem 2.** Let $(c_1, V_1(x))$ and $(c_2, W_2(x))$ be solutions of (1.7) and (1.8) respectively. Set

$$\bar{c} := \frac{c_1 + c_2}{2},\ c_0 := \frac{c_1 - c_2}{2}. \tag{3.7}$$

Note that $c_0 > 0$. We define

$$f(u(x,t)) := \sum_{i=-m}^{m} J(i)[b(u(x-i,t)) - du(x,t)],$$
$$f(u(x)) := \sum_{i=-m}^{m} J(i)[b(u(x-i)) - du(x)],$$
$$\bar{f}(u(x)) := \sum_{i=-m}^{m} J(i)[b(u(x-i)) - du(x-i)].$$

By a simple computation, it is easy to see that $u(x,t) = R(x+\bar{c}t, t)$ is a solution of (1.1) if and only if $R(x,t)$ is a solution of

$$F[R](x,t) := R_t(x,t) + \bar{c}R_x(x,t) - \mathcal{D}_2[R](x,t) - f(R(x,t)) = 0. \tag{3.8}$$

Also, $V_1(x + c_0 t)$ and $W_2(x - c_0 t)$ are solutions of (3.8).

Let $p(t), q(t)$ be solutions of (2.2)-(2.3) with

$$\alpha = \min\{\lambda_1, \mu_2\},\ c = c_0,\ M > 0 \text{ (to be determined later)}.$$

Consider

$$\begin{cases} \overline{R}(x,t) := H(V_1(x+p(t)), W_2(x-q(t))),\ x \in \mathbb{R},\ t \le 0, \\ \underline{R}(x,t) := H(V_1(x+q(t)), W_2(x-p(t))),\ x \in \mathbb{R},\ t \le 0, \end{cases}$$

where

$$H(g,h) := \frac{(1-a)gh}{h(g-a) + a(1-g)}.$$

We shall claim that $(\overline{R}, \underline{R})(x,t)$ is a pair of supersolution and subsolution of (3.8).

For this, we denote

$$H_g := \frac{\partial H}{\partial g},\ H_h := \frac{\partial H}{\partial h},\ H_{gg} := \frac{\partial^2 H}{\partial g^2},\ H_{hh} := \frac{\partial^2 H}{\partial h^2},\ H_{gh} := \frac{\partial^2 H}{\partial h \partial g},$$
$$\widehat{H}(b,c) := H(V_1(y+b), W_2(z+c)),\ \widehat{H_g}(b,c) := H_g(V_1(y+b), W_2(z+c)),$$
$$\widehat{H_h}(b,c) := H_h(V_1(y+b), W_2(z+c)),\ \widehat{H_{gg}}(b,c) := H_{gg}(V_1(y+b), W_2(z+c)),$$
$$\widehat{H_{gh}}(b,c) := H_{gh}(V_1(y+b), W_2(z+c)),\ \widehat{H_{hh}}(b,c) := H_{hh}(V_1(y+b), W_2(z+c))$$

for $b, c \in \mathbb{R}$. Hereafter we denote $y := x + p(t)$ and $z := x - q(t)$.

By a simple computation, we have

$$H_g(g,h) = \frac{a(1-a)h(1-h)}{[h(g-a) + a(1-g)]^2},\ H_h(g,h) = \frac{a(1-a)g(1-g)}{[h(g-a) + a(1-g)]^2}.$$

Also, we have

$$\begin{aligned} H_{gg}(g,h) &= \frac{(-2)a(1-a)h(1-h)(h-a)}{[h(g-a)+a(1-g)]^3} := h(h-a)H_1(g,h), \\ H_{hh}(g,h) &= \frac{(-2)a(1-a)g(1-g)(g-a)}{[h(g-a)+a(1-g)]^3} := (g-1)(g-a)H_2(g,h), \\ H_{gh}(g,h) &= \frac{a(1-a)[(2a-1)gh+a(1-g-h)]}{[h(g-a)+a(1-g)]^3}. \end{aligned}$$

Because $0 < W(\cdot) < a < V(\cdot) < 1$, we have

$$\widehat{H_g}(b,c) > 0, \ \widehat{H_h}(b,c) > 0, \quad \forall \ b,c \in \mathbb{R}. \tag{3.9}$$

Now we are in a position to compute $F[\overline{R}]$. First, we have

$$\begin{aligned} F[\overline{R}](x,t) &= \overline{R}_t(x,t) + \bar{c}\overline{R}_x(x,t) - \mathcal{D}_2[\overline{R}](x,t) - f(\overline{R}(x,t)) \\ &= \widehat{H_g}(0,0)V_1'(y)[p'(t)+\bar{c}] + \widehat{H_h}(0,0)W_2'(z)[\bar{c}-q'(t)] \\ &\quad - \sum_{i=-m}^{m} \widetilde{J}(i)[H(V_1(y-i),W_2(z-i)) - H(V_1(y),W_2(z))] \\ &\quad - \sum_{i=-m}^{m} J(i)[b(H(V_1(y-i),W_2(z-i))) - dH(V_1(y-i),W_2(z-i))], \end{aligned}$$

where $\widetilde{J}(i) := dJ(i)$, if $|i| \neq 1$, $\widetilde{J}(i) := dJ(i) + D$, if $|i| = 1$.

Recall (1.7), (1.8), (3.7) and that $p(t), q(t)$ are solutions of (2.2)-(2.3) with $c = c_0$. Then we have

$$\begin{aligned}
& F[\overline{R}](x,t) \\
= \;& \widehat{H_g}(0,0)V_1'(y)[p'(t)-c_0] + \widehat{H_h}(0,0)W_2'(z)[c_0-q'(t)] \\
& +D\widehat{H_g}(0,0)[V_1(y+1)+V_1(y-1)-2V_1(y)] + \widehat{H_g}(0,0)f(V_1(y)) \\
& +D\widehat{H_h}(0,0)[W_2(z+1)+W_2(z-1)-2W_2(z)] + \widehat{H_h}(0,0)f(W_2(z)) \\
& -\sum_{i=-m}^{m} \widetilde{J}(i)[H(V_1(y-i),W_2(z-i)) - H(V_1(y),W_2(z))] \\
& -\sum_{i=-m}^{m} J(i)[b(H(V_1(y-i),W_2(z-i))) - dH(V_1(y-i),W_2(z-i))] \\
= \;& M\widehat{H_g}(0,0)V_1'(y)e^{\alpha p(t)} + M\widehat{H_h}(0,0)W_2'(z)e^{\alpha q(t)} \\
& +\widehat{H_g}(0,0)\sum_{i=-m}^{m} \widetilde{J}(i)[V_1(y-i)-V_1(y)] + \widehat{H_h}(0,0)\sum_{i=-m}^{m} \widetilde{J}(i)[W_2(z-i)-W_2(z)] \\
& -\sum_{i=-m}^{m} \widetilde{J}(i)[H(V_1(y-i),W_2(z-i)) - H(V_1(y),W_2(z))] \\
& +\widehat{H_g}(0,0)\bar{f}(V_1(y)) + \widehat{H_h}(0,0)\bar{f}(W_2(z)) \\
& -\sum_{i=-m}^{m} J(i)[b(H(V_1(y-i),W_2(z-i))) - dH(V_1(y-i),W_2(z-i))].
\end{aligned}$$

Recall that $V_1(\cdot)$ and $W_2(\cdot)$ are strictly increasing. It then follows from (2.6), (3.9) and the mean-value theorem that

$$\begin{aligned}
& F[\overline{R}](x,t) \\
\geq \;& Me^{\alpha q(t)}[\widehat{H_g}(0,0)V_1'(y) + \widehat{H_h}(0,0)W_2'(z)] \\
& +\widehat{H_g}(0,0)\sum_{i=-m}^{m} \widetilde{J}(i)[V_1(y-i)-V_1(y)] + \widehat{H_h}(0,0)\sum_{i=-m}^{m} \widetilde{J}(i)[W_2(z-i)-W_2(z)] \\
& -\sum_{i=-m}^{m} \widetilde{J}(i)\{H_g(V_1(y+\theta_{1i}),W_2(z+\theta_{2i}))[V_1(y-i)-V_1(y)] \\
& +H_h(V_1(y+\theta_{1i}),W_2(z+\theta_{2i}))[W_2(z-i)-W_2(z)]\} \\
& +\widehat{H_g}(0,0)\bar{f}(V_1(y)) + \widehat{H_h}(0,0)\bar{f}(W_2(z)) \\
& -\sum_{i=-m}^{m} J(i)[b(H(V_1(y-i),W_2(z-i))) - dH(V_1(y-i),W_2(z-i))] \\
\geq \;& Me^{\alpha q(t)}A(x,t) - B(x,t) - G(x,t),
\end{aligned}$$

where

$$
\begin{aligned}
A(x,t) &:= \widehat{H_g}(0,0)V_1'(y)+\widehat{H_h}(0,0)W_2'(z),\\
B(x,t) &:= \sum_{i=-m}^{m}\widetilde{J}(i)\{[\widehat{H_g}(\theta_{1i},\theta_{2i})-\widehat{H_g}(0,0)][V_1(y-i)-V_1(y)]\\
&\quad +[\widehat{H_h}(\theta_{1i},\theta_{2i})-\widehat{H_h}(0,0)][W_2(z-i)-W_2(z)]\}\\
&\quad +\sum_{i=-m}^{m}J(i)\{[\widehat{H_g}(-i,-i)-\widehat{H_g}(0,0)]l(V_1(y-i))\\
&\quad +[\widehat{H_h}(-i,-i)-\widehat{H_h}(0,0)]l(W_2(z-i))\}\\
&:= \sum_{i=-m}^{m}\widetilde{J}(i)B_1(x,t,i)+\sum_{i=-m}^{m}J(i)B_2(x,t,i),\\
G(x,t) &:= \sum_{i=-m}^{m}J(i)[l(H(V_1(y-i),W_2(z-i)))-\widehat{H_g}(-i,-i)l(V_1(y-i))\\
&\quad -\widehat{H_h}(-i,-i)l(W_2(z-i))]\\
&:= \sum_{i=-m}^{m}J(i)G(x,t,i),
\end{aligned}
$$

with θ_{1i},θ_{2i} are between 0 and $-i$, and $l(s):=b(s)-ds$.

By (3.9), we have $A(x,t)>0$ for all $(x,t)\in\mathbb{R}\times(-\infty,0]$. So in order to obtain $F[\overline{R}](x,t)\geq 0$, we must estimate

$$B_1(x,t,i)/A(x,t),\quad B_2(x,t,i)/A(x,t)\quad\text{and}\quad G(x,t,i)/A(x,t).$$

First, we consider $A(x,t)$. Since $p(-\infty)=q(-\infty)=-\infty$ and $W_2(\infty)=V_1(-\infty)=a$, we can choose $T>0$ such that

$$
\begin{cases}
V_1(x+p(t)+l_1)\leq(1+a)/2, \text{ if } x\leq 0,\ t\leq -T,\ |l_1|\leq m.\\
W_2(x-q(t)+l_2)\geq(a/2), \text{ if } x\geq 0,\ t\leq -T,\ |l_2|\leq m.
\end{cases}\tag{3.10}
$$

By the form of H_g, H_h and the estimation above, we have

$$
\begin{cases}
\widehat{H_g}(0,0)=H_g(V_1(x+p(t)),W_2(x-q(t)))\geq 1/8, \text{ if } x\geq 0, t\leq -T.\\
\widehat{H_h}(0,0)=H_h(V_1(x+p(t)),W_2(x-q(t)))\geq 1/8, \text{ if } x\leq 0, t\leq -T.
\end{cases}\tag{3.11}
$$

Secondly, we consider $B_1(x,t,i)$, $B_2(x,t,i)$ and $G(x,t,i)$. For this, we define

$$S(x,t,l_1,l_2):=W_2(x-q(t)+l_2)[V_1(x+p(t)+l_1)-a]+a[1-V_1(x+p(t)+l_1)].$$

By (3.10), if $x\geq 0, t\leq -T, |l_1|\leq m, |l_2|\leq m$, then

$$S(x,t,l_1,l_2)\geq\frac{a}{2}[V_1(x+p(t)+l_1)-a]+a[1-V_1(x+p(t)+l_1)]\geq\frac{a(1-a)}{2}.$$

On the other hand, if $x\leq 0, t\leq -T, |l_1|\leq m, |l_2|\leq m$, then

$$S(x,t,l_1,l_2)\geq\frac{1+a}{2}[W_2(x-q(t)+l_2)-a]+a[1-W_2(x-q(t)+l_2)]\geq\frac{a(1-a)}{2}.$$

Therefore,

$$S(x,t,l_1,l_2) \geq \frac{a(1-a)}{2}, \quad \text{if} \quad x \in \mathbb{R}, t \leq -T, |l_1| \leq m, |l_2| \leq m.$$

This implies that there exists a constant K_1 such that

$$| \,(H_1, H_2, H_{gh})(V_1(x+p(t)+l_1), W_2(x-q(t)+l_2))\, | \leq K_1, \tag{3.12}$$

for all $x \in \mathbb{R}, t \leq -T, |l_1| \leq m, |l_2| \leq m$.

Now, we are ready to estimate $B_1(x,t,i)/A(x,t)$. Consider the first term of $B_1(x,t,i,)$, by the mean-value theorem, we have

$$\begin{aligned}
& [\widehat{H_g}(\theta_{1i},\theta_{2i}) - \widehat{H_g}(0,0)][V_1(y-i) - V_1(y)] \\
= \; & [H_g(V_1(y+\theta_{1i}), W_2(z+\theta_{2i})) - H_g(V_1(y), W_2(z))]V_1'(y+\theta_{3i}) \\
= \; & [\theta_{1i} H_{gg}(V_1(y+\theta_{4i}), W_2(z+\theta_{5i}))V_1'(y+\theta_{6i}) \\
& + \theta_{2i} H_{gh}(V_1(y+\theta_{4i}), W_2(z+\theta_{5i}))W_2'(z+\theta_{7i})]V_1'(y+\theta_{3i}) \\
= \; & [\theta_{1i}\widehat{H_{gg}}(\theta_{4i},\theta_{5i})V_1'(y+\theta_{6i}) + \theta_{2i}\widehat{H_{gh}}(\theta_{4i},\theta_{5i})W_2'(z+\theta_{7i})]V_1'(y+\theta_{3i}),
\end{aligned}$$

where $\theta_{3i}, \theta_{4i}, \theta_{5i}, \theta_{6i}, \theta_{7i}$ are between 0 and $-i$. Therefore,

$$\begin{aligned}
B_1(x,t,i) \; = \; & [\theta_{1i}\widehat{H_{gg}}(\theta_{4i},\theta_{5i})V_1'(y+\theta_{6i}) + \theta_{2i}\widehat{H_{gh}}(\theta_{4i},\theta_{5i})W_2'(z+\theta_{7i})]V_1'(y+\theta_{3i}) \\
& + [\theta_{1i}\widehat{H_{hg}}(\tau_{4i},\tau_{5i})V_1'(y+\theta_{6i}) + \theta_{2i}\widehat{H_{hh}}(\tau_{4i},\tau_{5i})W_2'(z+\theta_{7i})]W_2'(z+\tau_{3i}),
\end{aligned}$$

where $\tau_{3i}, \tau_{4i}, \tau_{5i}$ are between 0 and $-i$. For $x \leq 0$, $t \leq -T$, by (3.11), we have

$$A(x,t) \geq \frac{1}{8} W_2'(x-q(t)).$$

Moreover, from (1.14)-(1.19) and (3.12), there exists a constant K_2 such that

$$\begin{aligned}
\left| \frac{\widehat{H_{gg}}(s_1,s_2)V_1'(y+s_3)V_1'(y+s_4)}{W_2'(z)} \right| \; & \leq \; K_2 e^{\lambda_1 p(t)} \\
\left| \frac{\widehat{H_{gh}}(s_1,s_2)W_2'(z+s_3)V_1'(y+s_4)}{W_2'(z)} \right| \; & \leq \; K_2 e^{\lambda_1 p(t)} \\
\left| \frac{\widehat{H_{hh}}(s_1,s_2)W_2'(z+s_3)W_2'(z+s_4)}{W_2'(z)} \right| \; & \leq \; K_2 e^{\lambda_1 p(t)},
\end{aligned}$$

for all $|s_1|, |s_2|, |s_3|, |s_4| \leq m$. Hence, there exists a constant K_3 such that

$$\left| \frac{B_1(x,t,i)}{A(x,t)} \right| \leq K_3 e^{\lambda_1 p(t)}, \quad \text{if} \quad x \leq 0, t \leq -T, |i| \leq m.$$

Using the same method to consider the case that $x \geq 0, t \leq -T$, we have

$$\left| \frac{B_1(x,t,i)}{A(x,t)} \right| \leq K_4 e^{\mu_2 q(t)}, \quad \text{if} \quad x \geq 0, t \leq -T, |i| \leq m.$$

for some constant K_4. Using the fact that $l'(\cdot)$ is bounded over $[0,1]$, we can choose a constant K_6 such that

$$|l(V_1(\cdot))| \le K_6 V_1'(\cdot), \quad |l(W_2(\cdot))| \le K_6 W_2'(\cdot).$$

By the same estimation of $B_1(x,t,i)$ for $B_2(x,t,i)$, we have

$$\left|\frac{B(x,t,i)}{A(x,t)}\right| \le K e^{\lambda_1 p(t)}, \quad \text{if} \quad x \le 0, t \le -T, |i| \le m,$$
$$\left|\frac{B(x,t,i)}{A(x,t)}\right| \le K e^{\mu_2 q(t)}, \quad \text{if} \quad x \ge 0, t \le -T, |i| \le m,$$

for some constant K.

Next, we estimate $G(x,t,i)/A(x,t)$. We define

$$\hat{G}(g,h) := l(H(g,h)) - H_g(g,h)l(g) - H_h(g,h)l(h).$$

By a simple computation, we get

$$\hat{G}(g,0) = \hat{G}(g,a) = \hat{G}(a,h) = \hat{G}(1,h) = 0, \quad \forall\ g \in (a,1), h \in (0,a). \tag{3.13}$$

For $x \le q(t), t \le 0$, by (3.13), we may write

$$\begin{aligned} G(x,t,i) &= \hat{G}(V_1(x+p(t)-i), W_2(x-q(t)-i)) \\ &= W_2(x-q(t)-i)[V_1(x+p(t)-i)-a]G_1(x,t,i). \end{aligned}$$

for some bounded function $G_1(x,t,i)$. When $x \le q(t), t \le -T, |i| \le m$, by (3.11), we have

$$|G(x,t,i)/A(x,t)| \le 8|G_1(x,t,i)|\frac{W_2(x-q(t)-i)[V_1(x+p(t)-i)-a]}{W_2'(x-q(t))} \le M_1 e^{\lambda_1 p(t)},$$

for some constant M_1 (independent of x, t and i). Similarly, for $q(t) \le x \le -p(t)$, we may write

$$G(x,t,i) = [a - W_2(x-q(t)-i)][V_1(x+p(t)-i)-a]G_2(x,t,i),$$

for some bounded function $G_2(x,t,i)$. When $q(t) \le x \le 0, t \le -T, |i| \le m$, we have

$$|G(x,t,i)/A(x,t)| \le 8|G_2(x,t,i)|\frac{[a-W_2(x-q(t)-i)][V_1(x+p(t)-i)-a]}{W_2'(x-q(t))} \le M_2 e^{\lambda_1 p(t)},$$

for some constant M_2 (independent of x, t and i). When $0 \le x \le -p(t), t \le -T, |i| \le m$, we have

$$|G(x,t,i)/A(x,t)| \le 8|G_2(x,t,i)|\frac{[a-W_2(x-q(t)-i)][V_1(x+p(t)-i)-a]}{V_1'(x+p(t))} \le M_3 e^{\mu_2 q(t)},$$

for some constant M_3 (independent of x, t and i). For $x \ge -p(t), t \le 0$, we may write

$$G(x,t,i) = [a - W_2(x-q(t)-i)][1 - V_1(x+p(t)-i)]G_3(x,t,i),$$

for some bounded function $G_3(x,t,i)$. When $x \ge -p(t), t \le -T, |i| \le m$, we have

$$|G(x,t,i)/A(x,t)| \le 8|G_3(x,t,i)|\frac{[a-W_2(x-q(t)-i)][1-V_1(x+p(t)-i)]}{V_1'(x+p(t))} \le M_4 e^{\mu_2 q(t)},$$

for some constant M_4 (independent of x, t and i).

Because $\alpha = \min\{\lambda_1, \mu_2\}$ and (2.6), we may choose

$$M \geq \max\{Ke^{\alpha\kappa}, M_1 e^{\alpha\kappa}, M_2 e^{\alpha\kappa}, M_3, M_4\}.$$

And then we obtain $F[\bar{R}] \geq 0$. Similarly, we can obtain $F[\underline{R}] \leq 0$.

Hence, by (2.6) and the identity

$$\begin{aligned}
& \overline{R}(x,t) - \underline{R}(x,t) \\
= \quad & r(t) \int_0^1 [H_g(V_1(x+q(t)+sr(t)), W_2(x-p(t)+sr(t)))V_1'(x+q(t)+sr(t)) \\
+ \quad & H_h(V_1(x+q(t)+sr(t)), W_2(x-p(t)+sr(t)))W_2'(x-p(t)+sr(t))]ds,
\end{aligned}$$

where $r(t) := p(t) - q(t)$, we obtain that

$$0 < \overline{R}(x,t) - \underline{R}(x,t) \leq M_5 e^{c_0\alpha t}, \quad \forall\ (x,t) \in \mathbb{R} \times (-\infty, 0],$$

for some constant M_5.

Define

$$\begin{cases} \overline{u}(x,t) := \overline{R}(x+\bar{c}t, t), & x \in \mathbb{R}, t \leq 0, \\ \underline{u}(x,t) := \underline{R}(x+\bar{c}t, t), & x \in \mathbb{R}, t \leq 0. \end{cases}$$

Because $F[\overline{R}] \geq 0$ and $F[\underline{R}] \leq 0$, $\overline{u}(x,t)$ and $\underline{u}(x,t)$ are a supersolution and a subsolution of (1.1) for $(x,t) \in \mathbb{R} \times (-\infty, -T]$ respectively and

$$0 < \overline{u}(x,t) - \underline{u}(x,t) \leq M_5 e^{c_0\alpha t}, \quad \forall\ (x,t) \in \mathbb{R} \times (-\infty, 0], \tag{3.14}$$

By Lemma 2.2, there exists an entire solution $u(x,t)$ of (1.1) such that

$$\underline{u}(x,t) \leq u(x,t) \leq \overline{u}(x,t), \quad \forall (x,t) \in \mathbb{R} \times (-\infty, -T].$$

Next, we consider (1.21) and recall w is defined on (2.5). If $x \geq -\bar{c}t$ and $t \leq -T$, then

$$\begin{aligned}
& |u(x,t) - V_1(x + c_1 t + \omega)| \\
\leq \quad & |u(x,t) - \underline{u}(x,t)| + |\underline{u}(x,t) - V_1(x + c_1 t + \omega)| \\
\leq \quad & [\overline{u}(x,t) - \underline{u}(x,t)] + |g(x,t) - V_1(x + c_1 t + \omega)| \\
& + |(a - h(x,t))(g(x,t) - 1)g(x,t)/[h(x,t)(g(x,t) - a) + a(1 - g(x,t))]|,
\end{aligned}$$

where $g(x,t) := V_1(x + \bar{c}t + q(t)),\ h(x,t) := W_2(x + \bar{c}t - p(t))$.

By the mean-value theorem and (2.4),

$$|g(x,t) - V_1(x + c_1 t + \omega)| \leq \sup V_1'(\cdot)|q(t) - c_0 t - \omega| \to 0 \quad \text{as} \quad t \to -\infty.$$

This implies

$$\lim_{t\to-\infty} [\sup_{x \geq -\bar{c}t} |g(x,t) - V_1(x + c_1 t + \omega)|] = 0. \tag{3.15}$$

On the other hand, because $x \geq -\bar{c}t$, we have

$$a > h(x,t) = W_2(x + \bar{c}t - p(t)) \geq W_2(-p(t)) \to a \quad \text{as} \quad t \to -\infty.$$

This implies

$$\lim_{t\to-\infty} [\sup_{x\geq -\bar{c}t} |a - h(x,t)|] = 0.$$

and

$$h(x,t)(g(x,t) - a) + a(1 - g(x,t))$$

has a positive low-bound if $x \geq -\bar{c}t$, $-t >> 1$. So

$$\lim_{t\to-\infty} \left\{ \sup_{x\geq -\bar{c}t} \left| \frac{(a - h(x,t))(g(x,t) - 1)g(x,t)}{h(x,t)(g(x,t) - a) + a(1 - g(x,t))} \right| \right\} = 0. \tag{3.16}$$

By (3.14), (3.15) and (3.16), we have

$$\lim_{t\to-\infty} \sup_{x\geq -\bar{c}t} |u(x,t) - V_1(x + c_1 t + \omega)| = 0.$$

Similarly, we get

$$\lim_{t\to-\infty} \sup_{x\leq -\bar{c}t} |u(x,t) - W_2(x + c_2 t - \omega)| = 0.$$

So (1.21) holds. Finally, from Zinner [14], the asymptotic behavior (1.22) follows. We have thus completed the proof of Theorem 2.

3.3. **Proof of Theorem 3.** Following the methods of [6, 12]), we consider the functions

$$\begin{cases} \overline{u}(x,t) := H(U(x + \bar{c}t + p(t)), W_2(-x - \bar{c}t - q(t))), \\ \underline{u}(x,t) := H(U(x + \bar{c}t + q(t)), W_2(-x - \bar{c}t - p(t))), \end{cases}$$

where $\bar{c} := (\hat{c} - c_2)/2$, $p(t)$ and $q(t)$ are the solutions of (2.2) and (2.3) with $c = c_0 := (\hat{c} + c_2)/2$ and suitable α, M, and

$$H(g,h) := \frac{a(g+h) - (1+a)gh}{a - gh}.$$

Then, by using a similar process as that of the proof of Theorem 2, we obtain the conclusion of Theorem 3. We safely omit the details here (see also [12, 6]).

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