Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

Website: http://AIMsciences.org

pp. **X–XX**

HYPERBOLIC QUENCHING PROBLEM WITH DAMPING IN THE MICRO-ELECTRO MECHANICAL SYSTEM DEVICE

Jong-Shenq Guo

Department of Mathematics, Tamkang University
Tamsui, New Taipei City 25137, Taiwan

Bo-Chih Huang

Department of Mathematics, Tamkang University
Tamsui, New Taipei City 25137, Taiwan

(Communicated by the associate editor name)

Abstract. We study the initial boundary value problem for the damped hyperbolic equation arising in the micro-electro mechanical system device with local or nonlocal singular nonlinearity. For both cases, we provide some criteria for quenching and global existence of the solution. We also derive the existence of the quenching curve for the corresponding Cauchy problem with local source.

1. **Introduction.** In this paper, we consider the following initial boundary value problem arising in the study of the micro-electro mechanical system (MEMS) device:

$$\begin{cases} \varepsilon u_{tt} + u_t = \Delta u + F(x,t,u), & \text{in } \Omega \times (0,\infty) \\ u = 0, & \text{on } \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x), & \text{for } x \in \overline{\Omega} \\ u_t(x,0) = u_1(x), & \text{for } x \in \overline{\Omega} \end{cases} \tag{1.1}$$

where $\varepsilon > 0$, $\Omega \subset \mathbb{R}^N$, $u_0 < 1$ on $\overline{\Omega}$, $u_0, u_1 \in C(\overline{\Omega})$,

$$F(x,t,u) := \frac{\lambda f(x,t)}{(1-u)^2\big(1+\alpha\int_\Omega \frac{1}{1-u}dx\big)^2}$$

with $\lambda > 0$, $\alpha \geq 0$ and $f(x,t) > 0$ on $\overline{\Omega}$.

In (1.1), u stands for the deflection of the membrane, ε is the ratio of the inertial and damping terms in the model, while

$$\lambda = \frac{V^2 L^2 \varepsilon_0}{2\mathcal{T} l^2}$$

in which V stands for the applied voltage, $\mathcal{T}$ is the tension in the membrane, L the characteristic length (diameter) of the fixed ground plate Ω, l the characteristic width of the gap between the membrane and the ground plate, and ε_0 the

2000 *Mathematics Subject Classification.* Primary: 35L71, 35L81; Secondary: 35A01, 35L20, 74H35, 74K15.

Key words and phrases. Wave equations, Nonlocal hyperbolic problem, Quenching of solutions, Micro-electro mechanical system.

This work was partially supported by the National Science Council of the Republic of China under the grants NSC 99-2115-M-032-006-MY3 and NSC 101-2811-M-032-012.

permittivity of the free space. The function $f(x,t)$ represents the varying dielectric properties of the membrane. The appearance of the integral in F depends on whether the device is embedded in an electrical circuit with a capacitor of fixed capacitance or not. Here we have normalized the distance between the edge of the membrane and the ground plate to be 1.

We say that u is a weak solution of (1.1) on $D_T := \Omega \times (0,T)$ for some $T > 0$, if

(i) $u \in C^0(\overline{D}_T)$ such that the initial and boundary conditions in (1.1) are satisfied,
(ii) $|u| \le 1-\delta$ on $\overline{D}_T$ for some $\delta \in (0,1)$, and
(iii) the first order weak derivatives $u_t, \nabla u$ of u are in $L^2(D_T)$ such that

$$\begin{aligned}&\int_\Omega \psi(x,t)[\varepsilon u_t(x,t)+u(x,t)]dx - \int_\Omega \psi(x,0)[\varepsilon u_1(x)+u_0(x)]dx\\ &= \int_0^t\int_\Omega [\psi_\tau(x,\tau)(\varepsilon u_\tau(x,\tau)+u(x,\tau)) - \nabla\psi(x,\tau)\cdot\nabla u(x,\tau)]dxd\tau\\ &\quad + \int_0^t\int_\Omega \psi(x,\tau)F(x,\tau,u(x,\tau))dxd\tau\end{aligned}$$

for all $t \in (0,T)$ for any function $\psi(x,t) \in C^1(\overline{D}_T)$ with $\psi = 0$ on $\partial\Omega \times [0,T)$.

We call a (weak) solution quenches (in finite time) if there is a $T < \infty$ such that

$$\limsup_{t\uparrow T}\{\max_{x\in\overline{\Omega}} u(x,t)\} = 1.$$

In this case, the device breaks down, since a singularity occurs.

When $\varepsilon = 0$, (1.1) is reduced to the following parabolic problem

$$\begin{cases} u_t = \Delta u + F(x,t,u), & \text{in } \Omega\times(0,\infty),\\ u = 0, & \text{on } \partial\Omega\times(0,\infty),\\ u(x,0) = u_0(x), & \text{for } x \in \overline{\Omega}.\end{cases} \tag{1.2}$$

We refer the reader to the work [10] and the references cited therein for the study of (1.2) when F is independent of (x,t). In fact, the study of quenching for parabolic problems has a long history back to the work by Kawarada [14]. After this pioneer work, there has been extensive study on quenching of solution for parabolic problems. We refer the reader to [2, 3, 7, 8, 9, 10, 11, 14, 15, 16, 17] and the references therein. In particular, when F is independent of (x,t), some criteria for quenching and global existence of solution of (1.2) for 1-dimensional local source case were given in [2, 3, 16, 17]. The higher dimensional local source case was studied by [11]. The nonlocal source case was studied by [9] for 1-dimensional case and [10] for higher dimensional case. For the study of the phenomena beyond quenching and the quenching profile, we refer the reader to [16, 7]. Although little is known for (1.2) with F depending on (x,t).

On the other hand, most classical works on the singularities (e.g., quenching, blow-up, etc.) of hyperbolic problems are dealing with wave equation with a nonlinearity without damping term. We refer the reader to [4, 5, 6, 13, 15, 18, 19, 20, 21] and the references therein. We also refer the reader to [1, 22] for works with damping. The main purpose of this paper is to derive some criteria of quenching and global existence of solutions of (1.1) with a damping term. Without loss of generality, by rescaling the variables x and t, we may assume that $\varepsilon = 1$. In this work, we shall mainly consider the case that $N = 1$ and $f \equiv 1$. Some quenching criteria for the higher dimensional case are also given.

For $N = 1$, we let $\Omega = (0, 1)$. Then the first equation in (1.1) becomes

$$u_{tt} + u_t = u_{xx} + \frac{\lambda h^2(u)}{[1 + \alpha I(u)]^2}, \tag{1.3}$$

where for convenience we let

$$h(u) := \frac{1}{1-u}, \quad I(u)(t) := \int_0^1 h(u(x,t))dx.$$

Let $v = e^{t/2}u$. Then we can rewrite (1.3) as

$$v_{tt} = v_{xx} + \frac{1}{4}v + \frac{\lambda e^{t/2} h^2(e^{-t/2}v)}{[1 + \alpha I(e^{-t/2}v)]^2}.$$

Thus we end up with an equation with no damping term. Hence the local (in time) existence and uniqueness of weak solution can be easily deduced by the contraction mapping principle (cf. e.g., [6, 13] for details).

It is well-known that the global vs non-global existence of solutions of evolution problems is strongly related to the structure of the stationary solutions. For the stationary solutions, we refer it to [16] for $\alpha = 0$ and [9, 13] for $\alpha > 0$. In either case, there is a critical value λ_* such that a stationary solution exists if and only if $\lambda \le \lambda_*$. Therefore, it is natural to expect that we have the global (in time) existence when λ is small and quenching occurs when λ is large enough. However, due to the dependence of time variable for the nonlinear term and the lack of comparison principle, certain difficulties arise when we derive the quenching criteria and global existence.

This paper is organized as follows. In §2, we deal with the case without capacitor, i.e., the case of local source. By using a convexity argument (cf. [6]), we derive some quenching criteria for the local problem. Then, by employing an energy argument, we provide a criterion of the global existence for the local problem. Next, we study the case with a capacitor, i.e., the case when $\alpha > 0$ in §3. In this section, we first provide some criteria for which the solution exists globally in time by using an energy argument (cf. [6, 13]). Then we modify the method of [13] to obtain a quenching criterion for the problem with zero initial data. Here we prove the spatially independent lemma (see Lemma 3.2 below) without the symmetry condition. The quenching criterion with nonzero initial data is also derived by applying an energy argument (cf. [10]). Finally, in §4, we study the quenching curve for the corresponding Cauchy problem when $\alpha = 0$. We prove that there is a curve such that $u(x, t)$ reaches 1 as (x, t) tends to this curve.

2. **Local source case: without capacitor.** In this section, we assume that $\alpha = 0$. Then (1.1) with $N = 1$ becomes the following initial-boundary value problem:

$$\begin{cases} u_{tt} + u_t = u_{xx} + \lambda h^2(u), & 0 < x < 1,\ t > 0, \\ u(0,t) = u(1,t) = 0, & t > 0, \\ u(x,0) = u_0(x), & 0 \le x \le 1, \\ u_t(x,0) = u_1(x), & 0 \le x \le 1, \end{cases} \tag{2.1}$$

where $h(z) = 1/(1 - z)$.

We shall give some quenching criteria for large λ as follows. First, we consider the case $u_0 \equiv 0$ and $u_1 \equiv 0$. Let

$$H(z) = -\pi^2 z + \lambda h^2(z), \quad -\infty < z < 1. \tag{2.2}$$

Note that $\lim_{z \to 1^-} H(z) = +\infty$.

Employing the standard convexity argument (cf. [12, 6]), we give the following quenching criterion.

Theorem 2.1. *Suppose that $N=1$, $u_0\equiv 0$ and $u_1\equiv 0$. If $\lambda\geq \pi^2/2$, then the solution of (2.1) quenches in finite time.*

Proof. Suppose not, that is, $0\leq u(x,t)<1$ for $(x,t)\in[0,1]\times[0,\infty)$. Define

$$G(t):=\int_0^1 u(x,t)\rho(x)dx,\quad \rho(x):=\frac{\pi}{2}\sin(\pi x). \tag{2.3}$$

Then $0\leq G(t)<1$ for all $t\in[0,\infty)$.

Set $\psi(x,t):=t\rho(x)$. Multiplying the first equation in (2.1) by ψ and integrating over $[0,1]\times[0,t]$, the integration by parts gives

$$\begin{aligned} tG'(t) &= \frac{t\pi}{2}\int_0^1 \sin(\pi x)u_t dx=\int_0^1 \psi u_t dx\\ &= -\int_0^1 \psi u dx+\int_0^t\!\!\int_0^1[\psi_\tau(u_\tau+u)+\psi_{xx}u]dxd\tau+\lambda\int_0^t\!\!\int_0^1\psi h^2(u)dxd\tau\\ &= -tG(t)+\int_0^t[G'(\tau)+G(\tau)-\pi^2\tau G(\tau)]d\tau+\lambda\int_0^t\tau\int_0^1 h^2(u)\rho(x)dxd\tau. \end{aligned}$$

By differentiating the above equation with respect to t once, we deduce that

$$G''(t)=-G'(t)-\pi^2G(t)+\lambda\int_0^1 h^2(u)\rho(x)dx.$$

It follows from Jensen's inequality that

$$G''(t)+G'(t)\geq H(G(t)) \tag{2.4}$$

for all $t\in[0,\infty)$.

Suppose that $\lambda\geq\pi^2/2$. Then, by the definition of H, we have $H(0)=\lambda$, $H'(0)\geq 0$ and $H''(z)>0$ for all $z<1$. Hence $H(z)$ is increasing on $[0,1]$. Since $u(x,0)=u_t(x,0)=0$, we have $G(0)=G'(0)=0$. Also, $G''(0)\geq H(G(0))=H(0)=\lambda>0$, so there exists $t_0>0$ such that $G'(t)>0$ on $(0,t_0]$.

We claim that $G'(t)>0$ for all $t>0$. If not, let $t_1\in(t_0,\infty)$ be the smallest positive number such that $G'(t_1)=0$. Then $G(t)$ is increasing on $[0,t_1]$. So we have $G(t)>0$ for all $t\in(0,t_1]$. Since H is increasing on $[0,1]$, it follows from (2.4) that

$$G''(t)+G'(t)\geq H(G(t))\geq H(0)=\lambda,\ \forall\, t\in[0,t_1]$$

which is equivalent to

$$\left(e^tG'(t)\right)'\geq\lambda e^t,\ \forall\, t\in[0,t_1].$$

Thus

$$e^tG'(t)\geq\lambda(e^t-1)>0$$

for all $t\in(0,t_1]$, a contradiction. Therefore, $G'(t)>0$ for $t>0$.

Recall that $G(0)=0$, we then have $G(t)>0$ on $(0,\infty)$. Also, by (2.4),

$$G''(t)+G'(t)\geq H(0)=\lambda,\ \forall\, t\in[0,\infty)$$

and so

$$G'(t)+G(t)\geq\lambda t,\ \forall\, t\in[0,\infty).$$

Since $G(t)<1$ for $t\geq 0$, we have

$$G'(t)\geq\lambda t-G(t)>\lambda t-1,\ \forall\, t\in[0,\infty).$$

Hence we get

$$1 > G(t) > \frac{\lambda}{2}t^2 - t, \ \forall \, t \in [0, \infty),$$

which is impossible. Thus the theorem is proved. $\square$

For a general bounded domain $\Omega \subset \mathbb{R}^N$ with $N > 1$, let (λ^*, ρ) be the first eigen-pair of the problem

$$-\Delta\rho = \lambda^*\rho \quad \text{in } \Omega, \quad \rho = 0 \quad \text{on } \partial\Omega \tag{2.5}$$

such that $\int_\Omega \rho(x)dx = 1$. Then, by a similar argument as the proof of Theorem 2.1, we have the following quenching criterion.

Theorem 2.2. *Suppose that $N > 1$. Then the solution of (1.1) with $\varepsilon = 1$, $f(x,t) \equiv 1$ and $u_0 = u_1 = 0$ quenches in finite time, provided that $\lambda \geq \lambda^*/2$.*

Proof. As in (2.3), we set

$$G(t) := \int_\Omega u(x,t)\rho(x)dx,$$

where ρ is defined by (2.5). We redefine H in (2.2) by $H(z) = -\lambda^* z + \lambda h^2(z)$. Then a similar argument as in the proof of Theorem 2.1 leads to the inequality (2.4). Also, $H(z)$ is increasing on $[0,1]$ due to the assumption $\lambda \geq \lambda^*/2$. Hence the theorem follows by the same argument as that of Theorem 2.1. $\square$

For the general case $u_0 \not\equiv 0$ or $u_1 \not\equiv 0$, we have

$$G(0) = \frac{\pi}{2}\int_0^1 \sin(\pi x)u_0(x)dx, \quad G'(0) = \frac{\pi}{2}\int_0^1 \sin(\pi x)u_1(x)dx$$

when $N = 1$.

Theorem 2.3. *Suppose that $N = 1$, $G(0) \geq 0$ and $G'(0) \geq 0$ such that $G(0)^2 + G'(0)^2 \neq 0$. Then the solution of (2.1) must quench in finite time, provided that*

$$\begin{cases} \lambda \geq \pi^2[1 - G(0)]^3/2, & \text{if } \ 0 \leq G(0) < 1/3, \\ \lambda > \pi^2 G(0)[1 - G(0)]^2, & \text{if } \ 1/3 \leq G(0) < 1. \end{cases} \tag{2.6}$$

Proof. Observe that

$$\pi^2 G(0)[1 - G(0)]^2 - \frac{\pi^2[1 - G(0)]^3}{2} = \frac{\pi^2[1 - G(0)]^2}{2}[3G(0) - 1].$$

Suppose that (2.6) holds. In either case, we always have that

$$H(G(0)) > 0 \quad \text{and} \quad H'(G(0)) \geq 0.$$

Suppose that $|u| < 1$ for $(x,t) \in [0,1] \times [0,\infty)$. Note that (2.4) holds for all $t \geq 0$. By assumption, there exists $t_0 > 0$ such that $G'(t) > 0$ on $(0, t_0]$. Indeed, this is trivial if $G'(0) > 0$ due to the continuity. If $G'(0) = 0$, then $G''(0) \geq H(G(0)) > 0$ also implies the existence of such t_0.

We claim that $G'(t) > 0$ for all $t > 0$. Suppose not. Let $t_1 \in (t_0, \infty)$ be the smallest number such that $G'(t_1) = 0$. Then $G'(t) > 0$ on $(0, t_1)$ and so $G(t) \geq G(0)$ for all $t \in [0, t_1]$. Since $H'' > 0$ and $H'(G(0)) \geq 0$, we see that H is increasing on $[G(0), 1)$. Therefore, we obtain

$$G''(t) + G'(t) \geq H(G(t)) \geq H(G(0)) \tag{2.7}$$

so that

$$\left(e^t G'(t)\right)' \geq H(G(0))e^t \tag{2.8}$$

for all $t \in [0, t_1]$. Integrating (2.8) from 0 to t_1 and using $G'(t_1) = 0$, we find

$$-G'(0) \geq H(G(0)) \int_0^{t_1} e^s ds = H(G(0))(e^{t_1} - 1) > 0,$$

a contradiction. Hence $G'(t) > 0$ for all $t > 0$ and so (2.7) holds for all $t > 0$.

Now, integrating (2.7) twice and using the fact $G(t) < 1$, we have

$$1 > G(t) \geq \frac{1}{2} H(G(0))t^2 + [G(0) + G'(0) - 1]t + G(0), \ \forall\, t > 0,$$

which is impossible. Hence the solution of (2.1) must quench in finite time provided (2.6) holds. □

In the rest of this section, we shall derive some criteria for the global existence when $N = 1$ by the energy method (cf. [6]). To find the energy, we multiply the first equation of (2.1) by u_t and integrate it over $[0, 1]$. Then we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^1 (u_t^2 + u_x^2)dx + \int_0^1 u_t^2 dx = \lambda \frac{d}{dt}\int_0^1 \Phi(u)dx$$

where $\Phi(z) = -1 + 1/(1 - z)$. Define

$$E(t) = \int_0^1 \left[\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2 - \lambda\Phi(u)\right]dx$$

then

$$\frac{dE}{dt}(t) = -\int_0^1 u_t^2 dx \leq 0$$

and hence $E(t) \leq E(0)$ for $t > 0$.

We now state and prove the following criterion for the global existence.

Theorem 2.4. *Suppose that $u = u(x, t; \lambda)$ is the solution of (2.1) such that*

$$0 < \lambda < \max_{0 \leq \delta \leq 1} \frac{\pi^2 \delta(1 - \delta)}{\pi\delta - 2\delta + 2}.$$

If $E(0) \leq 0$, then there exists $\delta \in (0, 1)$ such that $|u(x, t; \lambda)| < 1 - \delta$ for all $x \in [0, 1]$ and $t \geq 0$.

Proof. By assumption, there exists a $\delta > 0$ such that

$$\lambda < \frac{\pi^2 \delta(1 - \delta)}{\pi\delta - 2\delta + 2} < \frac{\pi^2 \delta}{2}. \tag{2.9}$$

We claim that $|u(x, t; \lambda)| < 1 - \delta$ for all $x \in [0, 1]$ and $t \geq 0$.

For contradiction, we assume that T is the finite number such that

$$\max_{(x,t) \in [0,1] \times [0,T]} u(x, t; \lambda) = 1 - \delta.$$

Note that $E(t) \leq E(0) \leq 0$ for $t > 0$ and we write

$$\Phi(z) = \frac{z}{1 - z} = z + \frac{z^2}{1 - z}.$$

Then, by Poincaré and Schwartz inequalities, we get

$$\begin{aligned}\pi^2\int_0^1 u^2dx &\le \int_0^1 u_x^2dx \le 2\lambda\int_0^1 \Phi(u)dx\\ &\le 2\lambda\int_0^1 udx + 2\lambda\int_0^1 \frac{u^2}{1-u}dx\\ &\le 2\lambda\Big(\int_0^1 u^2dx\Big)^{1/2} + \frac{2\lambda}{\delta}\int_0^1 u^2dx.\end{aligned}$$

Hence we obtain

$$\pi^2\int_0^1 u^2dx \le 2\lambda\Big(\int_0^1 u^2dx\Big)^{1/2}\Big[1+\frac{1}{\delta}\Big(\int_0^1 u^2dx\Big)^{1/2}\Big]. \tag{2.10}$$

for $t\in[0,T]$. This implies that

$$\Big(\int_0^1 u^2dx\Big)^{1/2} \le \frac{2\lambda}{\pi^2-2\lambda/\delta}$$

If this bound is used on the right hand side of (2.10) and if the inequality

$$4u^2 \le \int_0^1 u_x^2dx \tag{2.11}$$

is also employed, we obtain that

$$\begin{aligned}4u^2 &\le \int_0^1 u_x^2dx \le 2\lambda\Big(\int_0^1 u^2dx\Big)^{1/2}\Big[1+\frac{1}{\delta}\Big(\int_0^1 u^2dx\Big)^{1/2}\Big]\\ &\le \frac{4\lambda^2}{\pi^2-2\lambda/\delta}\Big(1+\frac{1}{\delta}\cdot\frac{2\lambda}{\pi^2-2\lambda/\delta}\Big) = \frac{4\pi^2\delta^2\lambda^2}{(\pi^2\delta-2\lambda)^2}.\end{aligned}$$

It follows from (2.9) that $u^2(x,t;\lambda) < (1-\delta)^2$ for all $(x,t)\in[0,1]\times[0,T]$, which contradicts the choice of T. Hence the theorem is proved. □

3. **Nonlocal source case: with capacitor.** In this section, we consider the case $\alpha>0$ so that the problem has a nonlocal source. Then the problem (1.1) with $N=1$ becomes the following initial-boundary value problem:

$$\begin{cases} u_{tt}+u_t = u_{xx}+\frac{\lambda h^2(u)}{[1+\alpha I(u)]^2}, & 0<x<1,\ t>0,\\ u(0,t)=u(1,t)=0, & t>0,\\ u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), & 0\le x\le 1.\end{cases} \tag{3.1}$$

We first derive the criterion for the global existence. For convenient, we denote

$$\Psi(u)(t) = \frac{1}{1+\alpha I(u)(t)}.$$

Then

$$\frac{d}{dt}\Psi(u) = -\alpha\int_0^1 \frac{h^2(u)u_t}{[1+\alpha I(u)]^2}dx$$

Using the energy method (cf. [6, 13]), we multiply the first equation of (3.1) by u_t and integrate it over $[0,1]$. Then we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^1 (u_t^2+u_x^2)dx + \int_0^1 u_t^2dx = -\frac{\lambda}{\alpha}\frac{d}{dt}\Psi(u)(t).$$

Define

$$E(t) = \int_0^1 \Big[\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2\Big] dx + \frac{\lambda}{\alpha}\Psi(u)(t). \tag{3.2}$$

Then

$$\frac{dE}{dt}(t) = -\int_0^1 u_t^2 dx \le 0$$

and hence $E(t) \le E(0)$ for $t > 0$.

Theorem 3.1. *If $u_0 = u_1 = 0$ and*

$$0 < \lambda < 2(1+\alpha) \max_{0\le\delta\le1} (1-\delta)(\alpha+\delta),$$

then the solution $u = u(x,t;\lambda)$ of (3.1) exists globally in time.

Proof. Consider the function

$$\Lambda(\delta) = 2(1+\alpha)(1-\delta)(\alpha+\delta), \quad \delta \in [0,1].$$

We have $\Lambda(0) = 2\alpha(1+\alpha)$, $\Lambda(1) = 0$, and $\Lambda'(\delta) = -2(1+\alpha)(2\delta+\alpha-1)$. Then $\Lambda(\delta)$ is strictly decreasing on $[0,1]$ if $\alpha \ge 1$, while $\Lambda(\delta)$ has a unique maximum point on $(0,1)$ if $0 < \alpha < 1$.

By assumption, there exists $\delta \in (0,1)$ such that $\lambda < \Lambda(\delta)$. We claim that $|u(x,t;\lambda)| < 1-\delta$ for all $x \in [0,1]$ and $t \ge 0$. Assume on the contrary that there is $T > 0$ such that

$$\max_{(x,t)\in[0,1]\times[0,T]} u(x,t;\lambda) = 1-\delta.$$

Since $u \le 1-\delta$, we have

$$\Psi(u) = \frac{1}{1+\alpha I(u)} \ge \frac{\delta}{\alpha+\delta}$$

for $t \in [0,T]$. By (2.11) and using $u_0 = u_1 \equiv 0$, we obtain

$$4u^2 \le \int_0^1 u_x^2 dx \le 2E(0) - \frac{2\lambda}{\alpha}\Psi(u) \le \frac{2\lambda}{\alpha(1+\alpha)} - \frac{2\lambda\delta}{\alpha(\alpha+\delta)} = \frac{2\lambda(1-\delta)}{(1+\alpha)(\alpha+\delta)}.$$

Since $\lambda < \Lambda(\delta)$, so $u^2(x,t;\lambda) < (1-\delta)^2$ for all $t \in [0,T]$, which contradicts the choice of T. Hence the theorem follows. $\square$

For the case of non-zero small initial data, we have the global existence if

$$0 < \lambda < [2 - (\|u_0'\|^2 + \|u_1\|^2)/2]\alpha. \tag{3.3}$$

Hereafter $\|\cdot\|$ denotes the L^2 norm. Indeed, suppose that $u \le M < 1$ in $[0,1]\times[0,T]$. Then we have

$$\Psi(u)(t) \ge \frac{1}{1+\alpha/(1-M)} \quad \text{for all } t \in [0,T].$$

Using $E(t) \le E(0)$, (2.11) and $\Psi(u) \le 1$, we obtain

$$4M^2 + \frac{2\lambda}{\alpha}\frac{1-M}{1+\alpha-M} \le \|u_0'\|^2 + \|u_1\|^2 + \frac{2\lambda}{\alpha} \tag{3.4}$$

for all $t \in [0,T]$. If u quenches in finite time, by letting $M \to 1$ in (3.4), then it contradicts (3.3). Hence u exists globally if the condition (3.3) is assumed. Note that the condition (3.3) is meaningful if the initial data are small enough in the sense that $\|u_0'\|^2 + \|u_1\|^2 < 4$.

Next, we study the quenching criteria for large λ. For the case of zero initial and boundary conditions, let $v = e^{t/2}u$, then v satisfies

$$\begin{cases} v_{tt} - v_{xx} = \frac{1}{4}v + \frac{\lambda e^{t/2}h^2(e^{-t/2}v)}{[1+\alpha I(e^{-t/2}v)]^2}, & 0 < x < 1,\ t > 0, \\ v(0,t) = v(1,t) = 0, & t > 0, \\ v(x,0) = v_t(x,0) = 0, & 0 \le x \le 1. \end{cases} \tag{3.5}$$

Then we have the following lemma. Although the proof is quite similar to the one given in [13], we provide the details here for the reader's convenience.

Lemma 3.2. *There exists $t_0 \in (0, 1/2]$ such that the solution v of (3.5) satisfies*

$$v(x,t) = V(t) = \max_{x\in[0,1/2]} v(x,t)$$

in the set $\{(x,t) \in \mathcal{R} \mid x \ge t,\ t < t_0\}$, where $\mathcal{R} := \{(x,t) \mid 0 < x < 1/2, 0 < t < 1/2\}$.

Proof. Following [13], we shall prove this lemma by applying a Picard iteration.

Initially, we define v_0 to be the solution of

$$\partial_{tt}v_0 - \partial_{xx}v_0 = g_0(t), \quad (x,t) \in [0,1] \times [0,1/2],$$

with the initial and boundary conditions defined as in (3.5), where

$$g_0(t) = \frac{\lambda e^{t/2}h^2(0)}{[1+\alpha I(0)]^2} = \frac{\lambda e^{t/2}}{(1+\alpha)^2}.$$

Then, by Duhamel's Principle and the domain of dependence for 1-d wave equation, $v_0(x,t)$ can be solved explicitly as

$$\begin{aligned} v_0(x,t) &= \begin{cases} \dfrac{1}{2}\displaystyle\int_0^t \int_{x-t+\tau}^{x+t-\tau} g_0(\tau)dyd\tau, & \text{if } x \ge t, \\ \dfrac{1}{2}\displaystyle\int_{t-x}^t \int_{x-t+\tau}^{x+t-\tau} g_0(\tau)dyd\tau + \dfrac{1}{2}\int_0^{t-x}\int_{t-x-\tau}^{x+t-\tau} g_0(\tau)dyd\tau, & \text{if } x \le t, \end{cases} \\ &= \begin{cases} \displaystyle\int_0^t (t-\tau)g_0(\tau)d\tau := V_0(t), & \text{if } x \ge t, \\ V_0(t) - V_0(t-x), & \text{if } x \le t \end{cases} \end{aligned}$$

for $(x,t) \in \mathcal{R}$. Note that

$$\partial_x v_0(x,t) = \begin{cases} 0, & \text{if } x \ge t, \\ \displaystyle\int_0^{t-x} g_0(\tau)d\tau > 0, & \text{if } x \le t. \end{cases}$$

Then $\partial_x v_0 \ge 0$ in $\mathcal{R}$ and we have

$$0 \le v_0(x,t) \le V_0(t) = \max_{x\in[0,1/2]} v_0(x,t), \quad (x,t) \in \mathcal{R},$$

and $v_0(x,t) = V_0(t)$ in $\{(x,t) \in \mathcal{R} \mid x \ge t\}$.

Next, we define $v_1(x,t)$ to be the solution of

$$\partial_{tt}v_1 - \partial_{xx}v_1 = g_1(x,t), \quad (x,t) \in [0,1] \times [0,1/2],$$

with the same initial and boundary conditions as in (3.5), where

$$\begin{aligned} g_1(x,t) &:= \frac{1}{4}v_0(x,t) + \lambda e^{t/2}k_0(t)h^2(e^{-t/2}v_0)(x,t), \\ k_0(t) &:= 1/\big[1 + \alpha I(e^{-t/2}v_0)(t)\big]^2. \end{aligned}$$

Since $v_0 \geq 0$, it is easy to see that $g_1(x,t) > 0$. Moreover, $v_1(x,t)$ can be written as

$$v_1(x,t) = \begin{cases} \frac{1}{2}\int_0^t \int_{x-t+\tau}^{x+t-\tau} g_1(y,\tau)dyd\tau, & \text{if } x \geq t, \\ \frac{1}{2}\int_{t-x}^t \int_{x-t+\tau}^{x+t-\tau} g_1(y,\tau)dyd\tau + \frac{1}{2}\int_0^{t-x}\int_{t-x-\tau}^{x+t-\tau} g_1(y,\tau)dyd\tau, & \text{if } x \leq t \end{cases}$$

for $(x,t) \in \mathcal{R}$. We compute that

$$\partial_x v_1(x,t) = \begin{cases} \frac{1}{2}\int_0^t [g_1(x+t-\tau,\tau) - g_1(x-t+\tau,\tau)]d\tau, & \text{if } x \geq t, \\ \frac{1}{2}\int_{t-x}^t [g_1(x+t-\tau,\tau) - g_1(x-t+\tau,\tau)]d\tau & \\ \quad +\frac{1}{2}\int_0^{t-x} [g_1(t+x-\tau,\tau) + g_1(t-x-\tau,\tau)]d\tau, & \text{if } x \leq t. \end{cases}$$

Note that $h(e^{-t/2}v_0)$ is well-defined if $v_0 < e^{t/2}$. Since $k_0(t) > 0$ and $h' > 0$, so

$$\partial_x g_1 = [1/4 + 2\lambda k_0(t) h(e^{-t/2}v_0)h'(e^{-t/2}v_0)]\partial_x v_0 \geq 0$$

and we obtain that $\partial_x v_1 \geq 0$ in $\mathcal{R}$. Hence

$$0 \leq v_1(x,t) \leq V_1(t) = \max_{x\in[0,1/2]} v_1(x,t), \quad (x,t) \in \mathcal{R},$$

where

$$\begin{aligned} V_1(t) &= \int_0^t (t-\tau)\tilde{g}_1(\tau)d\tau, \\ \tilde{g}_1(t) &:= \frac{1}{4}V_0(t) + \lambda e^{t/2}\tilde{k}_0(t)h^2(e^{-t/2}V_0(t)), \\ \tilde{k}_0(t) &:= 1/\big[1+\alpha I(e^{-t/2}V_0(t))\big]^2. \end{aligned}$$

Note also that $v_1(x,t) = V_1(t)$ in $\{(x,t) \in \mathcal{R} \mid x \geq t\}$, due to g_1 is independent of x for $x \geq t$. Moreover, since $\tilde{g}_1 \geq g_0$, we have $V_1 \geq V_0$, if $V_0 < e^{t/2}$.

In general, for $n \geq 2$, having (v_k, V_k) for $k = 0, \cdots, n-1$, with

$$\begin{aligned} &V_{n-1} \geq V_{n-2},\ \partial_x v_{n-1} \geq 0,\ 0 \leq v_{n-1} \leq V_{n-1} \text{ in } \mathcal{R}, \text{ and} \\ &v_{n-1} = V_{n-1} \text{ in } \{(x,t) \in \mathcal{R} \mid x \geq t\}, \end{aligned}$$

we inductively define $v_n(x,t)$ to be the solution of

$$\partial_{tt}v_n - \partial_{xx}v_n = g_n(x,t), \quad (x,t) \in [0,1]\times[0,1/2],$$

with the same initial and boundary conditions as in (3.5), where

$$\begin{aligned} g_n(x,t) &:= \frac{1}{4}v_{n-1}(x,t) + \lambda e^{t/2}k_{n-1}(t)h^2(e^{-t/2}v_{n-1})(x,t), \\ k_{n-1}(t) &:= 1/\big[1+\alpha I(e^{-t/2}v_{n-1})(t)\big]^2. \end{aligned}$$

By the induction hypothesis, $\partial_x g_n \geq 0$ and hence $\partial_x v_n \geq 0$ in $\mathcal{R}$, if $V_{n-1} < e^{t/2}$. Moreover,

$$0 \leq v_n(x,t) \leq V_n(t) = \max_{x\in[0,1/2]} v_n(x,t), \quad (x,t) \in \mathcal{R},$$

and $v_n(x,t) = V_n(t)$ for $x \geq t$, where

$$\begin{aligned} V_n(t) &= \int_0^t (t-\tau)\tilde{g}_n(\tau)d\tau, \\ \tilde{g}_n(t) &:= \frac{1}{4}V_{n-1}(t) + \lambda e^{t/2}\tilde{k}_{n-1}(t)h^2(e^{-t/2}V_{n-1}(t)), \\ \tilde{k}_{n-1}(t) &:= 1/\big[1+\alpha I(e^{-t/2}V_{n-1}(t))\big]^2. \end{aligned}$$

Moreover, since h is increasing, the induction hypothesis implies that $\tilde{g}_n \geq \tilde{g}_{n-1}$ and so $V_n \geq V_{n-1}$, if $V_{n-1} < e^{t/2}$.

Therefore, we obtain an increasing sequence $\{V_n(t)\}_{n=1}^{\infty}$ such that it converges to a function $V(t)$ as $n \to \infty$. It is easy to check that V satisfies

$$V(t) = \int_0^t (t-\tau)\left\{\frac{1}{4}V(\tau) + \lambda e^{\tau/2}h^2(e^{-\tau/2}V(\tau))/[1+\alpha I(e^{-\tau/2}V(\tau))]^2\right\}d\tau.$$

Note that $V(0) = V'(0) = 0$ and $V''(t) > 0$ for $t \geq 0$. Then there exists $t_0 \in (0,1/2]$ such that $V(t) < e^{t/2}$ for all $t \in [0,t_0)$. Moreover, we have

$$0 \leq v(x,t) = V(t) = \max_{x\in[0,1/2]} v(x,t), \quad \text{for } x \geq t,$$

for the solution v of (3.5). Hence the lemma is proved. $\square$

With this lemma, we give the following quenching criterion for the problem with zero initial data.

Theorem 3.3. *Suppose that* $\lambda \geq \hat{\lambda}(\alpha)$, *where*

$$\hat{\lambda}(\alpha) := \frac{(1+\alpha)^2}{1/\sqrt{e} - 1/2}.$$

Then the solution u *to* (3.1) *with* $u_0 = u_1 = 0$ *quenches in a finite time* $T \leq 1/2$.

Proof. Since $u = e^{-t/2}v$, we let $U(t) = e^{-t/2}V(t)$, where V is defined in Lemma 3.2. Then the function $U(t)$ is the solution of the problem (3.1) with $u_0 = u_1 = 0$ for $x \geq t$. Hence there holds

$$\begin{aligned} &U'' + U' \geq \frac{\lambda h^2(U)}{[1+\alpha I(U)]^2} = \frac{\lambda}{(1+\alpha-U)^2}, \\ &U(0) = 0,\ U'(0) = 0. \end{aligned}$$

Suppose that $U(t) < 1$ for $t \in [0,1/2]$. It is clear that

$$\frac{\lambda}{(1+\alpha-U)^2} \geq \frac{\lambda}{(1+\alpha)^2}$$

and thus

$$\left(e^t U'\right)' \geq \frac{\lambda e^t}{(1+\alpha)^2} \quad \text{for all } t \in [0,1/2].$$

Using $U(0) = U'(0) = 0$, by integrating the above inequality from 0 to t, we get

$$U(t) \geq \frac{\lambda}{(1+\alpha)^2}(e^{-t} + t - 1) \quad \text{for all } t \in [0,1/2].$$

In particular, $U(1/2) \geq 1$, if $\lambda \geq \hat{\lambda}(\alpha)$, a contradiction. Therefore we deduce that $U(t)$ reaches 1 before the time $t = 1/2$, if $\lambda \geq \hat{\lambda}(\alpha)$. This proves the theorem. $\square$

Now, we derive a quenching criterion for the case $u_0 \not\equiv 0$ and/or $u_1 \not\equiv 0$ such that the following quantity

$$\lambda^+(\alpha) = \lambda^+(\alpha; u_0, u_1) := \begin{cases} \frac{\alpha[\alpha I(u_0)+1](\|u_0'\|^2+\|u_1\|^2)}{\alpha I(u_0)-1}, & 0 < \alpha \le 1/2, \\ \frac{2\alpha^2[\alpha I(u_0)+1](\|u_0'\|^2+\|u_1\|^2)}{1-4\alpha+\alpha I(u_0)}, & \alpha > 1/2 \end{cases} \tag{3.6}$$

is well-defined and is positive. In fact, this is true only when the initial function u_0 is sufficiently close to 1 so that the denominators in (3.6) are positive.

Theorem 3.4. *Assume that the initial function u_0 satisfies*

$$I(u_0) > 1/\alpha, \quad \textit{if} \ \ \alpha \in (0, 1/2]; \quad I(u_0) > 4 - 1/\alpha, \quad \textit{if} \ \ \alpha > 1/2.$$

Suppose that either $\lambda > \lambda^+(\alpha; u_0, u_1)$, or

$$\lambda = \lambda^+(\alpha; u_0, u_1) \quad \textit{and} \quad \int_0^1 u_0^2(x)dx + 2\int_0^1 u_0(x)u_1(x)dx > 1. \tag{3.7}$$

Then the solution of (3.1) quenches in finite time.

Proof. Suppose that $|u| < 1$ for $(x,t) \in [0,1] \times [0,\infty)$. Following [10], we set

$$A(t) = \int_0^1 u^2(x,t)dx.$$

Then we compute that

$$A'(t) = 2\int_0^1 uu_t dx, \quad A''(t) = 2\int_0^1 uu_{tt}dx + 2\int_0^1 u_t^2 dx.$$

It follows from (3.1) and an integration by parts that

$$A''(t) + A'(t) = -2\int_0^1 u_x^2 dx + 2\int_0^1 u_t^2 dx + 2\lambda \frac{\int_0^1 uh^2(u)dx}{[1+\alpha I(u)]^2}.$$

Using (3.2) and $h(u) = 1/(1-u)$, we can deduce that

$$\begin{aligned} A''(t) + A'(t) &= 4\int_0^1 u_t^2 dx - 4E(t) + \frac{4\lambda}{\alpha[1+\alpha I(u)]} + 2\lambda\frac{\int_0^1 uh^2(u)dx}{[1+\alpha I(u)]^2} \\ &= 4\int_0^1 u_t^2 dx - 4E(t) + 2\lambda\frac{2+\alpha\int_0^1(2-u)h^2(u)dx}{\alpha[1+\alpha I(u)]^2}. \end{aligned}$$

Moreover, using Young's inequality and Hölder's inequality, we have

$$[1+\alpha I(u)]^2 \le 2[1+\alpha^2 I(u)^2] \le 2 + 2\alpha^2\int_0^1 h^2(u)dx.$$

Since $E(t) \le E(0)$ for $t \ge 0$ and $|u| < 1$ for $(x,t) \in [0,1] \times [0,\infty)$, we end up with the following inequality

$$A''(t) + A'(t) \ge -4E(0) + \frac{\lambda}{\alpha}\theta\left(\int_0^1 h^2(u)dx\right), \tag{3.8}$$

where $\theta(z) := (2+\alpha z)/(1+\alpha^2 z)$ for $z \ge 0$.

Since

$$\theta'(z) = \alpha(1-2\alpha)/(1+\alpha^2 z)^2,$$

we see that $\theta(z) \ge \theta(0) = 2$ for all $z \ge 0$, if $\alpha \in (0, 1/2]$, while $\theta(z) \ge \theta(+\infty) = 1/\alpha$ for all $z \ge 0$, if $\alpha > 1/2$. It follows from (3.8) that

$$A''(t) + A'(t) \ge Q(\lambda, \alpha), \tag{3.9}$$

where

$$Q(\lambda,\alpha):=\begin{cases} -4E(0)+2\lambda/\alpha, & 0<\alpha\le 1/2,\\ -4E(0)+\lambda/\alpha^2, & \alpha>1/2.\end{cases}$$

Integrating (3.9) twice and using the fact that $A(t)<1$ for all $t\ge 0$, it follows that

$$A(t)\ge \frac{1}{2}Q(\lambda,\alpha)t^2+[A(0)+A'(0)-1]t+A(0),\ \forall\, t\ge 0. \tag{3.10}$$

Recall that

$$E(0)=(\|u_0'\|^2+\|u_1\|^2)/2+\lambda/\{\alpha[1+\alpha I(u_0)]\}.$$

Then

$$Q(\lambda,\alpha)=\begin{cases} \lambda\frac{2[\alpha I(u_0)-1]}{\alpha[\alpha I(u_0)+1]}-2(\|u_0'\|^2+\|u_1\|^2), & 0<\alpha\le\frac{1}{2},\\ \lambda\frac{1-4\alpha+\alpha I(u_0)}{\alpha^2[\alpha I(u_0)+1]}-2(\|u_0'\|^2+\|u_1\|^2), & \alpha>\frac{1}{2}.\end{cases}$$

Therefore, if $\lambda>\lambda^+(\alpha)$, then $Q(\lambda,\alpha)>0$. It follows from (3.10) that $A(t)\to\infty$ as $t\to\infty$, a contradiction. On the other hand, if $\lambda=\lambda^+(\alpha)$, then $Q(\lambda,\alpha)=0$. But, the condition (3.7) implies that $A(0)+A'(0)>1$. Hence we also obtain from (3.10) that $A(t)\to\infty$ as $t\to\infty$, a contradiction. This completes the proof of the theorem. □

For general bounded $\Omega\subset\mathbb{R}^N$ with $N>1$, let

$$\lambda^+(\alpha;u_0,u_1):=\begin{cases} \frac{\alpha[\alpha I(u_0)+1](\|\nabla u_0\|^2+\|u_1\|^2)}{\alpha I(u_0)-1}, & 0<\alpha\le 1/(2|\Omega|),\\ \frac{2\alpha^2|\Omega|[\alpha I(u_0)+1](\|\nabla u_0\|^2+\|u_1\|^2)}{1-4|\Omega|\alpha+\alpha I(u_0)}, & \alpha>1/(2|\Omega|).\end{cases}$$

Then, by a similar argument as the proof of Theorem 3.4, we have the following quenching criterion. We safely omit the proof.

Theorem 3.5. *Under the assumption that the initial function u_0 satisfies*

$$I(u_0)>1/\alpha,\quad \textit{if}\ \ 0<\alpha\le 1/(2|\Omega|);\qquad I(u_0)>4|\Omega|-1/\alpha,\quad \textit{if}\ \ \alpha>1/(2|\Omega|),$$

if either $\lambda>\lambda^+(\alpha;u_0,u_1)$, or

$$\lambda=\lambda^+(\alpha;u_0,u_1)\quad \textit{and}\quad \|u_0\|^2+2\int_\Omega u_0(x)u_1(x)dx>1,$$

then the solution of (1.1) with $\varepsilon=1$, $f(x,t)\equiv 1$ quenches in finite time.

4. **Quenching curve for the Cauchy problem.** In this section, we study the quenching curve for the following Cauchy problem:

$$\begin{cases} u_{tt}+u_t=u_{xx}+\lambda h^2(u) & \text{in } \mathbb{R}\times\{t>0\},\\ u(x,0)=u_0(x) & \text{on } \mathbb{R}\times\{t=0\},\\ u_t(x,0)=u_1(x) & \text{on } \mathbb{R}\times\{t=0\}.\end{cases} \tag{4.1}$$

As before, we set $v(x,t)=e^{t/2}u(x,t)$. Then v satisfies

$$\begin{cases} v_{tt}-v_{xx}=\frac{1}{4}v+\lambda e^{t/2}h^2(e^{-t/2}v) & \text{in } \ \mathbb{R}\times\{t>0\},\\ v(x,0)=u_0(x)=:v_0(x) & \text{on } \ \mathbb{R}\times\{t=0\},\\ v_t(x,0)=\frac{1}{2}u_0(x)+u_1(x)=:v_1(x) & \text{on } \ \mathbb{R}\times\{t=0\}.\end{cases}$$

Then v can be expressed as

$$v(x,t)=\bar{v}(x,t)+\frac{\lambda}{2}\int_0^t\int_{x-t+\tau}^{x+t-\tau}W(v(y,\tau),\tau)dyd\tau,$$

where

$$\bar{v}(x,t) := \frac{1}{2}[v_0(x+t) + v_0(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} v_1(y)dy,$$
$$W(z,s) := \frac{1}{4\lambda}z + e^{s/2}h^2(e^{-s/2}z),$$

as long as $v(y,\tau) < e^{\tau/2}$ in the domain of dependence at (x,t).

First, we prove the following lemma.

Lemma 4.1. *Let $U(t)$ be the solution of*

$$\begin{cases} U'' + U' = \lambda h^2(U), \\ U(0) = U'(0) = 0. \end{cases}$$

Then there exists $T_0 < \infty$ such that $U(t) \to 1$ as $t \to T_0$.

Proof. Suppose that $U(t) < 1$ for all $t > 0$. Since

$$(e^t U')' = \lambda e^t h^2(U) > 0$$

and $U'(0) = 0$, we have $U'(t) > 0$ for all $t > 0$. Hence $U(t)$ is increasing and so $U(t) > 0$ for all $t > 0$ due to $U(0) = 0$. Note that $h(U)$ is increasing for $U < 1$. Then we have

$$U''(t) + U'(t) > \frac{\lambda}{[1 - U(0)]^2} = \lambda.$$

for any $t > 0$. Integrating the above inequality from 0 to t and using the initial condition, we get

$$U'(t) + U(t) > \lambda t, \ \forall\, t > 0.$$

Since $U < 1$, we have

$$U'(t) > \lambda t - U(t) > \lambda t - 1.$$

Integrating this inequality from 0 to t, we obtain

$$1 > U(t) > \frac{\lambda t^2}{2} - t, \ \forall\, t > 0,$$

a contradiction. Hence there exists $T_0 < \infty$ such that $U(t) \to 1$ as $t \to T_0$. □

Now, let $V(t) = e^{t/2}U(t)$. Then $V(t)$ satisfies

$$V'' = \frac{1}{4}V + \lambda e^{t/2}h^2(e^{-t/2}V), \quad V(0) = V'(0) = 0,$$

and $V(t) \to e^{T_0/2}$ as $t \to T_0$. Note that we can represent $V(t)$ in the form

$$V(t) = \lambda \int_0^t (t-\tau)W(V(\tau),\tau)d\tau = \frac{\lambda}{2}\int_0^t \int_{x-t+\tau}^{x+t-\tau} W(V(\tau),\tau)dyd\tau$$

for any $x \in \mathbb{R}$.

Now, we state and prove the following theorem for the existence of quenching curve.

Theorem 4.2. *Suppose that $0 \le u_0(x) < 1$ and $u_1(x) > 0$ in $\mathbb{R}$. Let u be the solution of the Cauchy problem (4.1). Then there exists a function ϕ defined in $\mathbb{R}$ with $0 < \phi(x) \le T_0$ such that $u(x,t) < 1$ for $0 \le t < \phi(x), x \in \mathbb{R}$ and $u(x,t) \to 1$ as $t \uparrow \phi(x)$ for each $x \in \mathbb{R}$.*

Proof. First, given a positive number T_1 such that $T_1 \leq T_0$. Suppose that $u < 1$ in the set

$$K_{\xi,T_1} := \{(y,\tau) : |y-\xi| \leq T_1 - \tau, \tau > 0\}$$

for some $\xi \in \mathbb{R}$. Then $v(x,t) < e^{t/2}$ in K_{ξ,T_1}. Since

$$v(x,0) - V(0) = u_0(x) \geq 0, \quad v_t(x,0) - V'(0) = \frac{1}{2}u_0(x) + u_1(x) \geq \nu$$

for $x \in [-\xi - T_0, \xi + T_0]$ for some positive constant ν, we have $v(x,t) > V(t)$ for all $x \in [-\xi - T_0, \xi + T_0]$ and $t \in (0, \delta]$ for some sufficiently small $\delta > 0$.

We claim that $v(x,t) > V(t)$ for all $(x,t) \in K_{\xi,T_1}$. Otherwise, let t_0 be the smallest positive time such that $v(x_0,t_0) = V(t_0)$ for some $(x_0,t_0) \in K_{\xi,T_1}$. Note that $W(z,s)$ is monotone increasing in z. Then

$$0 = v(x_0,t_0) - V(t_0) = \bar{v}(x_0,t_0) + \frac{\lambda}{2}\int_0^{t_0}\int_{x_0-t_0+\tau}^{x_0+t_0+\tau} [W(v,\tau) - W(V,\tau)]dyd\tau > 0,$$

a contradiction. Hence $v(x,t) > V(t)$ for all $(x,t) \in K_{\xi,T_1}$. This implies that $U(t) < u(x,t) < 1$ for $(x,t) \in K_{\xi,T_1}$.

Now, for contradiction, we assume that there is a point $x_* \in \mathbb{R}$ such that $u(x_*,t) < 1$ for all $t \leq T_0$. Then, by the domain of dependence, $u(x,t) < 1$ for all $(x,t) \in K_{x_*,T_0}$. Furthermore, by Lemma 4.1, we have

$$1 = U(T_0) < u(x_*,T_0) < 1,$$

a contradiction. Hence the theorem is proved. $\square$

Acknowledgments. The authors thank the anonymous referees for their valuable comments and suggestions that help the improvement of the manuscript.

REFERENCES

[1] K. Agre, M.A. Rammaha, *Quenching and non-quenching for nonlinear wave equations with damping*, Canad. Appl. Math. Quart., **9** (2001), 203–223.

[2] A. Andrew, W. Wolfgang, *The quenching problem for nonlinear parabolic differential equations*, in “Ordinary and Partial Differential Equations”, Lecture Notes in Math., **564**, Springer, Berlin, 1976, 1–12.

[3] A. Andrew, W. Wolfgang, *On the global existence of solutions of parabolic differential equations with a singular nonlinear term*, Nonlinear Anal., **2** (1978), 499–504.

[4] L.A. Caffarelli, A. Friedman, *Differentiability of the blow-up curve for one-dimensional nonlinear wave equations*, Arch. Rational Mech. Anal., **91** (1985), 83–98.

[5] L.A. Caffarelli, A. Friedman, *The blow-up boundary for nonlinear wave equations*, Trans. Amer. Math. Soc., **297** (1986), 223–241.

[6] P.H. Chang, H.A. Levine, *The quenching of solutions of semiliear hyperbolic equations*, SIAM J. Math. Anal., **12** (1981), 893–903.

[7] S. Filippas, J.-S. Guo, *Quenching profiles for one-dimensional semilinear heat equations*, Quart. Appl. Math., **51** (1993), 713–729.

[8] J.-S. Guo, *On the quenching behavior of the solution of a semilinear parabolic equation*, J. Math. Anal. Appl., **151** (1990), 58–79.

[9] J.-S. Guo, B. Hu, and C.-J. Wang, *A nonlocal quenching problem arising in a micro-electro mechanical system*, Quarterly Appl. Math., **67** (2009), 725–734.

[10] J.-S. Guo, N.I. Kavallaris, *On a nonlocal parabolic problem arising in electrostatic MEMS control*, Discrete Contin. Dyn. Syst., **32** (2012), 1723–1746.

[11] N. Ghoussoub, Y. Guo, *On the partial differential equations of electrostatic MEMS devices II: dynamic case*, Nonlinear Diff. Eqns. Appl., **15** (2008), 115–145.

[12] S. Kaplan, *On the growth of solutions of quasi-linear parabolic equations*, Comm. Pure Appl. Math., **16** (1963), 305–330.

[13] N.I. Kavallaris, A.A. Lacey, C.V. Nikolopoulos, and D.E. Tzanetis, *A hyperbolic non-local problem modelling MEMS technology*, Rocky Mountain J. Math., **41** (2011), 505–534.

[14] H. Kawarada, *On solutions of initial boundary value problem for $u_t = u_{xx} + 1/(1-u)$*, RIMS. Kyoto Univ., **10** (1975), 729–736.

[15] H.A. Levine, *The phenomenon of quenching: a survey*, in "Trends in the theory and practice of nonlinear analysis", North-Holland Math. Stud., **110**, North-Holland, Amsterdam, 1985, 275–286.

[16] H.A. Levine, *Quenching, nonquenching, and beyond quenching for solution of some parabolic equations*, Ann. Mat. Pura Appl., **155** (1989), 243–260.

[17] H.A. Levine, J.T. Montgomery *The quenching of solutions of some nonlinear parabolic equations*, SIAM J. Math. Anal., **11** (1980), 842–847.

[18] F. Merle, H. Zaag, *Determination of the blow-up rate for the semilinear wave equation*, Amer. J. Math., **125** (2003), 1147–1164.

[19] F. Merle, H. Zaag, *Determination of the blow-up rate for a critical semilinear wave equation*, Math. Ann., **331** (2005), 395–416.

[20] F. Merle, H. Zaag, *Blow-up behavior outside the origin for a semilinear wave equation in the radial case*, Bull. Sci. Math., **135** (2011), 353–373.

[21] R.A. Smith, *On a hyperbolic quenching problem in several dimensions*, SIAM J. Math. Anal., **20** (1989), 1081–1094.

[22] J. Zhu, *Quenching of solutions of nonlinear hyperbolic equations with damping*, in "Differential and difference equations and applications", Hindawi Publ. Corp., New York, 2006, 1187–1194.

Received xxxx 20xx; revised xxxx 20xx.

E-mail address: jsguo@mail.tku.edu.tw
E-mail address: huangbz@math.ncu.edu.tw