GLOBAL EXISTENCE OF SOLUTION TO A NONLOCAL PARABOLIC PROBLEM MODELING LINEAR FRICTION WELDING

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ABSTRACT. We study a nonlocal parabolic problem airing in the modeling of linear friction welding. Using some a priori estimates, we derive the global in time existence of solution of this nonlocal problem.

1. INTRODUCTION

In this paper, we study the following nonlocal parabolic problem:

$$\begin{cases} u_t = u_{xx} - g(t)u^{-p}(x,t), & 0 < x < 1,\ t > 0, \\ u_x(0,t) = 0, \quad u(1,t) = 1, & t > 0, \\ u(x,0) = u_0(x), & 0 \le x \le 1. \end{cases} \tag{1.1}$$

where $\lambda > 0$, $p > 1$, $u_0(x)$ is a smooth function such that $0 < u_0(x) \le 1$ for all $x \in [0,1]$, $u_0'(x) > 0$ for all $x \in (0,1]$, $u_0'(0) = 0$, $u_0(1) = 1$, and

$$g(t) := \lambda \left(\int_0^1 u^{-p}(x,t)dx \right)^{-1-1/p}.$$

Under the above assumption it is clear that $u_x(x,t) > 0$ for $x \in (0,1]$. Also, it is clear that the solution exists and is unique as long as $u(0,t)$ remains positive. Assuming $[0,T)$ is the maximal existence interval, then either $\liminf_{t\to T-} u(0,t) = 0$, or $T = \infty$.

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The problem (1.1) arises in the study of linear friction welding for a hard material. The physical model is given by

$$u_t = u_{xx} - \left(\int_0^\infty u^{-p}(x,t)dx\right)^{-1-1/p} u^{-p}, \quad 0 < x < \infty,\ t > 0, \tag{1.2}$$

$$u_x(0,t) = 0, \quad u_x(\infty,t) = 1, \quad t > 0, \tag{1.3}$$

$$u(x,0) = u_0(x), \quad x \geq 0. \tag{1.4}$$

In the physical model, the parameter p is close to 4 (cf. [6] and references therein). For some related works on nonlocal parabolic problems, we also refer the reader to [2, 1, 3, 4, 5, 6].

In order to understand the model (1.2)-(1.4), it is proposed in [6] the following approximated problem:

$$u_t = u_{xx} - \left(\int_0^K u^{-p}(x,t)dx\right)^{-1-1/p} u^{-p}, \quad 0 < x < K,\ t > 0, \tag{1.5}$$

$$u_x(0,t) = 0, \quad u(K,t) = K, \quad t > 0, \tag{1.6}$$

$$u(x,0) = u_0(x), \quad 0 \leq x \leq K, \tag{1.7}$$

where K is any positive constant. Then, by a suitable re-scaling, (1.5)-(1.7) is reduced to the problem (1.1) with $\lambda := \lambda(K) := K^{1-1/p}$.

The steady states of (1.1) has been studied in [5]. The main purpose of this paper is to answer the question raised in [5], namely, whether the solution of (1.1) exists globally (in time). In [6], numerical simulations indicate that the solution of (1.1) exists globally. The main purpose of this paper is to prove this result rigorously as follows.

Theorem 1. *The solution of* (1.1) *exists for all time* $0 < t < \infty$*, and there exists a positive constant* c_2 *such that* $c_2 \leq u(x,t) \leq 1$ *for all* $0 \leq x \leq 1$, $0 < t < \infty$.

The details of proof of Theorem 1 is given in the next section.

2. Proof of Main Theorem

The proof of Theorem 1 is divided into the following lemmas. In this section, we shall let u be the solution of (1.1) with the maximal existence time interval $[0,T)$ for some $T \leq \infty$.

Lemma 2.1. *There exist positive constants η and C^*, independent of T, such that*

$$g(t) < C^* u^{p+\eta}(0,t) \quad \textit{for } \ 0 < t < T. \tag{2.1}$$

Proof. Since $p > 1$, we can choose $\alpha \in (0,1)$ such that

$$\frac{p+1}{(1+\alpha)p} < 1. \tag{2.2}$$

We take

$$\eta = 1 - \frac{p+1}{(1+\alpha)p}.$$

By parabolic estimates, for any $T_1 < T$,

$$\|u\|_{C^{1+\alpha,(1+\alpha)/2}([0,1]\times[0,T_1])} \le C_\alpha \sup_{0\le x\le 1,\ 0\le t\le T_1} g(t)u^{-p}(x,t), \tag{2.3}$$

where the constant C_α is independent of T_1 and T. In view of (2.2), we can choose C^* to be large enough so that

$$\lambda \cdot 2^{1+1/p}\Big[\frac{C_\alpha C^*}{1+\alpha}\Big]^{\frac{p+1}{(1+\alpha)p}} < C^*, \quad g(0) < C^* u_0^{p+\eta}(0).$$

With our choice of C^*, (2.1) is clearly valid for $t = 0$. If (2.1) is not valid, then there must be a $T_1 < T$ such that

$$g(t) < C^* u^{p+\eta}(0,t) \quad \text{for } \ 0 < t < T_1, \quad g(T_1) = C^* u^{p+\eta}(0,T_1). \tag{2.4}$$

Using this in (2.3) we find that

$$\|u\|_{C^{1+\alpha,(1+\alpha)/2}([0,1]\times[0,T_1])} \le C_\alpha C^*.$$

In particular,

$$0 \le u_x(x,t) = u_x(x,t) - u_x(x,0) \le C_\alpha C^* x^\alpha, \quad 0 \le x \le 1,\ 0 \le t \le T_1.$$

It follows that, for $0 \le x \le 1,\ 0 \le t \le T_1$,

$$u(x,t) \le u(0,t) + \frac{C_\alpha C^*}{1+\alpha} x^{1+\alpha} \le 2u(0,t) \quad \text{for } \ 0 \le x \le \bar{x} := \Big[\frac{(1+\alpha)u(0,t)}{C_\alpha C^*}\Big]^{1/(1+\alpha)}.$$

Thus, for $0 \le t \le T_1$,

$$\int_0^1 u^{-p}(x,t)dx \ge \int_0^{\bar{x}} 2^{-p}u^{-p}(0,t)dx = 2^{-p}\Big[\frac{(1+\alpha)}{C_\alpha C^*}\Big]^{1/(1+\alpha)}[u(0,t)]^{-p+1/(1+\alpha)},$$

which implies that, for $0 \le t \le T_1$,

$$g(t) \le \lambda 2^{1+1/p}\Big[\frac{C_\alpha C^*}{(1+\alpha)}\Big]^{(p+1)/[p(1+\alpha)]} u^{p+\eta}(0,t) < C^* u^{p+\eta}(0,t).$$

This is a contradiction to (2.4). Hence the lemma follows. ∎

Lemma 2.2. *There exists a positive constant c_0, independent of T, such that*

$$u(0,t) < 1 - c_0 \quad \textit{for } \ 0 \le t < T. \tag{2.5}$$

Proof. We take positive constants c_1 and c_2 such that

$$u_0(0) < c_1 < c_2 < 1.$$

In view of (2.3), if $u(0,t_1) = c_1$ and $u(0,t_2) \ge c_2$, then

$$|t_1 - t_2| \ge \gamma := \left[\frac{c_2 - c_1}{C_\alpha C^*}\right]^{2/(1+\alpha)}. \tag{2.6}$$

Let φ be the solution of

$$\begin{cases} \varphi_t = \varphi_{xx} - \lambda c_1^{p+1}, & 0 < x < 1, \ t > 0, \\ \varphi_x(0,t) = 0, \quad \varphi(1,t) = 1, & t > 0, \\ \varphi(x,0) \equiv 1, & 0 \le x \le 1. \end{cases}$$

We then take c_0 such that

$$0 < c_0 < \min\Big(1 - c_2, \inf_{\gamma < t < \infty} \{1 - \varphi(0,t)\}\Big).$$

It is clear that (2.5) is true for small t. If (2.5) is not always true, then there exists t_1 and t_2 such that

$$u(0,t_1) = c_1, \quad c_1 < u(0,t) < 1 - c_0 \quad \text{for } \ t_1 < t < t_2, \quad u(0,t_2) = 1 - c_0.$$

Note that we always have

$$g(t)u^{-p}(x,t) \ge g(t) \ge \lambda u^{p+1}(0,t) > \lambda c_1^{p+1} \quad \text{for } \ t_1 < t \le t_2,$$

so that, by comparison principle,

$$u(x,t) \le \varphi(x, t - t_1) \quad \text{for } \ t_1 < t \le t_2.$$

In particular, recalling (2.6) ($t_2 - t_1 \ge \gamma$) and the definition of c_0, we conclude

$$u(0,t_2) \le \varphi(0, t_2 - t_1) < 1 - c_0,$$

which is a contradiction. ∎

Lemma 2.3. *There exists a positive constant c_0^*, independent of T, such that*

$$u_x(x,t) \ge c_0^* x \quad \textit{for } \ 0 \le x \le 1, \ 0 \le t < T. \tag{2.7}$$

Proof. We take $c_0^* = c_0$ in (2.5) so that (2.5) holds. Take a smaller c_0^* if necessary so that $1 - c_0^* + c_0^* x \geq u_0(x)$. Then the comparison principle implies that

$$u(x,t) \leq 1 - c_0^* + c_0^* x \quad \text{for } \ 0 < t < T.$$

In particular, this implies that

$$u_x(1,t) \geq c_0^* \quad \text{for } \ 0 < t < T.$$

Take a smaller c_0^* if necessary so that $u_0'(x) \geq c_0^* x$. Differentiate the equation for u with respect to x and apply comparison principle, we derive (2.7). ∎

Lemma 2.4. *There exists a positive constant $\bar{c}_0$, independent of T, such that*

$$u(0,t) > \bar{c}_0 \quad \textit{for } \ 0 \leq t < T.$$

Proof. Let c_0^* be given by the above lemma. Take $\bar{c}_0$ and $\bar{c}_1$ such that

$$C^* \bar{c}_1^{\eta} < c_0^*, \quad \bar{c}_0 < \bar{c}_1 < u_0(0).$$

If the conclusion is not true, then there exist $t_2 > t_1 > 0$ such that

$$u(0,t_1) = \bar{c}_1, \quad \bar{c}_0 < u(0,t) < \bar{c}_1 \quad \text{for } \ t_1 < t < t_2, \quad u(0,t_2) = \bar{c}_0.$$

Using Lemma 2.1 we find that

$$g(t) u^{-p}(x,t) \leq C^* u^{\eta}(0,t) < c_0^* \quad \text{for } \ 0 < x < 1, \ t_1 < t < t_2.$$

Using Lemma 2.3 we find that

$$u(x,t_1) \geq \bar{c}_1 + \frac{c_0^*}{2} x^2.$$

Therefore by comparison principle

$$u(x,t) \geq \bar{c}_1 + \frac{c_0^*}{2} x^2 \quad \text{for } \ 0 < x < 1, \ t_1 < t < t_2,$$

which implies that $u(0,t_2) \geq \bar{c}_1 > \bar{c}_0$, which is a contradiction. ∎

Combining these lemmas, we conclude the proof of Theorem 1.

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