THE MINIMAL SPEED OF TRAVELING WAVE SOLUTIONS FOR A DIFFUSIVE THREE SPECIES COMPETITION SYSTEM

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ABSTRACT. In this paper, we study the minimal speed of traveling wave solutions for a diffusive three species competition system. Our main concern is the linear determinacy for the minimal speed. We provide some conditions on the parameters of the competition system such that the linear determinacy is assured. The main idea is by studying the linear determinacy of the corresponding approximated lattice dynamical systems and using the discrete Fourier transform.

1. INTRODUCTION

To understand the interaction between multiple competing species on population dynamics is one of the important issue in mathematical biology. One of typical mathematical models describing such phenomenon is the following so-called Lotka-Volterra type competition system:

$$u^i_t = D_i u^i_{xx} + r_i u^i (1 - \sum_{j=1}^{N} b_{ij} u^j), \quad x, t \in \mathbb{R},\ i = 1, ..., N, \tag{1.1}$$

which describes how N species compete to each other, where $D_i, r_i, b_{ij} > 0$ for $i, j = 1, ..., N$. To investigate the invasion phenomenon for (1.1), it is very nature to look for traveling wave solutions. Indeed, there have been tremendous works devoted to the existence of traveling wave solutions for (1.1), see, for example, [1, 6, 8, 13, 17, 18, 19, 21, 22] and the references cited therein. However, most of them were devoted to two-species case ($N = 2$).

In this paper, we shall consider (1.1) with three species case ($N = 3$). We envision that there are three species u, v and w living together such that each species has the preference of food resource so that the competition occurs only between species u and v and between species v and w, respectively. In other words, species u and w have different preferences of food resource. But, species v has both preferences so that it needs to compete with both species u and w.

Date: May 4, 2015. Corresponding Author: C.-C. Wu.

This work was partially supported by the National Science Council of the Republic of China under the grants NSC 102-2115-M-032-003-MY3, NSC 102-2115-M-024-001 and NSC 101-2115-M-005-001. The work of the second author was partially supported by NSF of China with grant 91130016, 11371338.

Key words and phrases: three species competition system, lattice dynamical system, traveling wave, minimal speed, linear determinacy.

More precisely, we study the following diffusive three species competition system:

$$u_t = D_1 u_{xx} + r_1 u(1 - u - b_{12} v), \quad x, t \in \mathbb{R}, \tag{1.2}$$

$$v_t = D_2 v_{xx} + r_2 v(1 - b_{21} u - v - b_{23} w), \quad x, t \in \mathbb{R}, \tag{1.3}$$

$$w_t = D_3 w_{xx} + r_3 w(1 - b_{32} v - w), \quad x, t \in \mathbb{R}, \tag{1.4}$$

where $D_i > 0, r_i > 0, b_{ij} > 0$. Here u, v, w are the population densities of species $1, 2, 3$, respectively, b_{ij} is the competition coefficient of species j to species i, r_i is the growth rate of species i and D_i is the diffusion coefficient of species i. Also, we have taken the scales of species so that the carrying capacity of each species is normalized to be 1 and the states $(u, v, w) = (0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 0, 1)$ are equilibria of the system (1.2)-(1.4).

Throughout this paper, we shall always assume that

(A) $b_{12}, b_{32} > 1$, $b_{21} + b_{23} < 1$,

which means that the species u, w are weak competitors to the species v. Therefore, it is expected that the species v shall win the competition eventually. We thus are interested in the traveling wave solution of the system (1.2)-(1.4), connecting the equilibria $(1, 0, 1)$ and $(0, 1, 0)$, in the form

$$(u, v, w)(x, t) := (\phi, \psi, \theta)(y), \quad y := x + st,$$

where s is the wave speed and (ϕ, ψ, θ) is the wave profile. It is easy to see that (ϕ, ψ, θ) satisfies the following problem:

$$\begin{cases} s\phi' = D_1 \phi'' + r_1 \phi(1 - \phi - b_{12}\psi), \quad y \in \mathbb{R}, \\ s\psi' = D_2 \psi'' + r_2 \psi(1 - b_{21}\phi - \psi - b_{23}\theta), \quad y \in \mathbb{R}, \\ s\theta' = D_3 \theta'' + r_3 \theta(1 - b_{32}\psi - \theta), \quad y \in \mathbb{R}, \\ (\phi, \psi, \theta)(-\infty) = (1, 0, 1), \quad (\phi, \psi, \theta)(+\infty) = (0, 1, 0), \\ 0 \le \phi, \psi, \theta \le 1. \end{cases} \tag{1.5}$$

Notice that, by a linearization of the corresponding kinetic systems to (1.2)-(1.4), we can easily check that near the equilibrium $(1, 0, 1)$ the stable manifold is of dimension 2 and the unstable manifold is of dimension 1; and the equilibrium $(0, 1, 0)$ is stable such that the stable manifold is of dimension 3, under the assumption **(A)**.

If we consider the linearization of the second equation of (1.5) around the state $(1, 0, 1)$, the corresponding characteristic equation is given by

$$D_2 \mu^2 - s\mu + r_2(1 - b_{21} - b_{23}) = 0. \tag{1.6}$$

We easily obtain that (1.6) has a positive solution if and only if $s \ge s_*$, where

$$s_* := 2\sqrt{D_2 r_2 (1 - b_{21} - b_{23})}.$$

Thus, the minimal speed $s_{\min}$ (if it exists) for the diffusive model (1.2)-(1.4) with $\phi' < 0, \psi' > 0, \theta' < 0$, must satisfy that $s_{\min} \ge s_*$. Indeed, since the limiting linear equation of the second equation in (1.5) as $y \to -\infty$ is given by

$$D_2\psi'' - s\psi' + r_2(1 - b_{21} - b_{23})\psi = 0$$

which has a monotone solution near $y = -\infty$ only if $s \geq s_*$. Hence we should have $s_{\min} \geq s_*$.

Our main purpose is to investigate the linear determinacy for the problem (1.5). By linear determinacy, it means that $s_{\min} = s_*$. In fact, the definition of linear determinacy is first defined in [19], which means that the minimal speed is determined by the linearization of the problem at some unstable equilibrium. For the works related to linear determinacy, we refer to [10, 14, 15, 16, 19] for partial differential equations and [10, 12] for lattice dynamical systems.

We now state our main theorem of this paper, the linear determinacy theorem for (1.5), as follows.

Theorem 1. *Assume that* **(A)** *holds. Also, let $D_2, r_2, b_{21}, b_{23} > 0$ be given. Then $s_{\min} = s_*$ as long as*

$$(D_j, r_j, b_{j2}) \in B_j^1 \cup B_j^2, \quad j = 1, 3, \tag{1.7}$$

where

$$B_j^1 := \{D_j \in (0, 2D_2],\ b_{j2}(b_{21} + b_{23}) \leq 1,\ r_j > 0\}, \tag{1.8}$$

$$B_j^2 := \left\{D_j \in (0, 2D_2),\ b_{j2}(b_{21} + b_{23}) > 1,\ 0 < r_j < \left(2 - \frac{D_j}{D_2}\right)\frac{r_2(1 - b_{21} - b_{23})}{b_{j2}(b_{21} + b_{23}) - 1}\right\}, \tag{1.9}$$

for $j = 1, 3$.

To prove this main theorem, following a method developed in [10], we first consider the corresponding discrete diffusive system of (1.2)-(1.4) in the following form

$$u_j'(t) = d_1\mathcal{D}[u_j](t) + r_1u_j(t)[1 - u_j(t) - b_{12}v_j(t)],\ j \in \mathbb{Z},\ t \in \mathbb{R}, \tag{1.10}$$

$$v_j'(t) = d_2\mathcal{D}[v_j](t) + r_2v_j(t)[1 - b_{21}u_j - v_j(t) - b_{23}w_j(t)],\ j \in \mathbb{Z},\ t \in \mathbb{R}, \tag{1.11}$$

$$w_j'(t) = d_3\mathcal{D}[w_j](t) + r_3w_j(t)[1 - b_{32}v_j(t) - w_j],\ j \in \mathbb{Z},\ t \in \mathbb{R}, \tag{1.12}$$

where d_j is the discrete diffusion rate and $\mathcal{D}[u_j] := (u_{j+1} - u_j) + (u_{j-1} - u_j)$ and so on. The system (1.10)-(1.12) is a so-called lattice dynamical system. For the study of lattice dynamical systems, we refer to the book of Fife [7] and survey papers by Chow [5] and Mallet-Paret [20].

A traveling wave of (1.10)-(1.12) is a solution in the form

$$(u_j(t), v_j(t), w_j(t)) = (\hat{U}(\xi), \hat{V}(\xi), \hat{W}(\xi)), \quad \xi = j + ct,$$

where c is the wave speed and $\{\hat{U}, \hat{V}, \hat{W}\}$ are the wave profiles. Therefore, the problem of finding traveling wave of (1.10)-(1.12) is equivalent to find $(c, \hat{U}, \hat{V}, \hat{W}) \in \mathbb{R} \times [C^1(\mathbb{R})]^3$ such that

$$
\begin{cases}
c\hat{U}' = d_1\mathcal{D}[\hat{U}] + r_1\hat{U}(1-\hat{U}-b_{12}\hat{V}), \quad \xi \in \mathbb{R},\\
c\hat{V}' = d_2\mathcal{D}[\hat{V}] + r_2\hat{V}(1-b_{21}\hat{U}-\hat{V}-b_{23}\hat{W}), \quad \xi \in \mathbb{R},\\
c\hat{W}' = d_3\mathcal{D}[\hat{W}] + r_3\hat{W}(1-b_{32}\hat{V}-\hat{W}), \quad \xi \in \mathbb{R},\\
(\hat{U},\hat{V},\hat{W})(-\infty) = (1,0,1), \quad (\hat{U},\hat{V},\hat{W})(+\infty) = (0,1,0),\\
0 \le \hat{U},\hat{V},\hat{W} \le 1,
\end{cases} \tag{1.13}
$$

where $\mathcal{D}[u](\xi) := u(\xi+1)+u(\xi-1)-2u(\xi)$ and so on.

Following [3, 12], we first have the following theorem on the existence of traveling waves and the minimal wave speed for (1.13).

Theorem 2. *Assume* **(A)**. *Then there exists a positive constant $c_{\min}$ such that the problem* (1.13) *admits a solution $(c,\hat{U},\hat{V},\hat{W})$ satisfying $\hat{U}'(\cdot)<0$, $\hat{V}'(\cdot)>0$ and $\hat{W}'(\cdot)<0$ on $\mathbb{R}$ if and only if $c \ge c_{\min}$.*

The main idea of proving Theorem 2 is to transform the problem into a monotone system. Based on the monotone property, a typical method to show the existence of traveling wave solution is to apply the monotone iteration scheme with the help of super-sub-solutions (cf. [23, 2]). Our approach here adopts an idea of [3] by truncating the original problem with the help of a super-solution. Then we are able to obtain the existence of traveling wave solution. For the 2-component system, we refer to [12]. In fact, the method of [3] (and [12]) works well to multiple component systems, as long as we can derive that the solutions of truncated problems can produce a desired solution with correct boundary conditions at $\pm\infty$. However, we were unable to accomplish this by using the definition of super-solution defined in [12] (or [3]). To overcome this difficulty, we introduce a suitable notion of super-solution (see Remark 2.1 and Proposition 1 below).

The related works about the minimal speed for lattice dynamical systems can be found in, for example, [2, 3, 9, 11, 12].

Next, to estimate the minimal speed for (1.13), we define

$$
c_* := \inf_{\lambda>0}\left\{\frac{d_2(e^{\lambda}+e^{-\lambda}-2)+r_2(1-b_{21}-b_{23})}{\lambda}\right\}.
$$

It is clear that

$$
c\lambda = d_2(e^{\lambda}+e^{-\lambda}-2)+r_2(1-b_{21}-b_{23}) \tag{1.14}
$$

has a positive solution if and only if $c \ge c_*$. Moreover, there exists $\lambda_* > 0$ such that λ_* is the unique solution of (1.14) when $c = c_*$. For $c > c_*$, (1.14) has exactly two solutions $\lambda_i(c)$, $i = 1,2$, with $0 < \lambda_1(c) < \lambda_2(c)$.

Then, based on a fundamental theory of [4] (see also [3, 12]), we have

Theorem 3. *Assume* **(A)**. *Then* $c_{\min} \geq c_*$.

By applying an idea used in [10, 12], the linear determinacy for (1.13) is given as follows.

Theorem 4. *Assume* **(A)**. *Let* $r_2 > 0$, $b_{21} > 0$ *and* $b_{23} > 0$ *be given. Then there exists a constant* $d_* = d_*(d_2) > 2d_2$ *such that* $c_{\min} = c_*$ *as long as*

$$(d_j, r_j, b_{j2}) \in A_j^1 \cup A_j^2, \quad j = 1, 3, \tag{1.15}$$

where

$$A_j^1 := \{d_j \in (0, d_*],\ b_{j2}(b_{21} + b_{23}) \leq 1,\ r_j > 0\}, \tag{1.16}$$

$$A_j^2 := \left\{d_j \in (0, d_*],\ b_{j2}(b_{21} + b_{23}) > 1,\ 0 < r_j \leq \frac{d_* - d_j}{d_* - d_2} \cdot \frac{r_2(1 - b_{21} - b_{23})}{b_{j2}(b_{21} + b_{23}) - 1}\right\} \tag{1.17}$$

for $j = 1, 3$.

With this discrete linear determinacy theorem, we can apply the method of discretization with the help of discrete Fourier transform used in [10] to finish the proof of our main theorem, Theorem 1. However, to prove Theorem 1, we need a detailed analysis of the quantity $d_*(d_2)$ defined in Theorem 4. See the two key lemmas (Lemmas 4.1 and 4.2) in §4.

The rest of this paper is organized as follows. In §2, we study the existence of minimal speed for the discrete model (1.13). In §3, we characterize the linear determinacy for the discrete model (1.13). Finally, in §4, we study the continuous PDE system (1.2)-(1.4) and prove the linear determinacy theorem for the continuous system (1.5) by using an idea from [10].

2. Discrete problem: existence of minimal speed

This section is devoted to the proofs of Theorem 2 and Theorem 3.

First, if $(c, \hat{U}, \hat{V}, \hat{W})$ is a solution of (1.13), then it is easy to see that (cf. [3, 12])

$$0 < \hat{U}(\cdot), \hat{V}(\cdot), \hat{W}(\cdot) < 1 \quad \text{in } \mathbb{R}, \quad c > 0. \tag{2.1}$$

For convenience, we introduce the new variable $U := 1 - \hat{U}, V := \hat{V}, W := 1 - \hat{W}$ so that (1.13) is transformed into a cooperative system as follows:

$$\begin{cases} cU' = d_1\mathcal{D}[U] + r_1(1 - U)(-U + b_{12}V), \quad \xi \in \mathbb{R}, \\ cV' = d_2\mathcal{D}[V] + r_2V(1 - b_{21} - b_{23} - V + b_{21}U + b_{23}W), \quad \xi \in \mathbb{R}, \\ cW' = d_3\mathcal{D}[W] + r_3(1 - W)(-W + b_{32}V), \quad \xi \in \mathbb{R}, \\ (U, V, W)(-\infty) = (0, 0, 0), \quad (U, V, W)(+\infty) = (1, 1, 1), \\ 0 \leq U, V, W \leq 1. \end{cases} \tag{2.2}$$

Note that the new problem (2.2) enjoys the monotone property. In fact, for given $c, \mu > 0$, we let

$$H_1(U,V,W)(\xi) := \left\{ \mu U + \frac{d_1}{c}\mathcal{D}[U] + \frac{1}{c} r_1(1-U)(-U+b_{12}V) \right\}(\xi),$$
$$H_2(U,V,W)(\xi) := \left\{ \mu V + \frac{d_2}{c}\mathcal{D}[V] + \frac{1}{c} r_2 V(1-b_{21}-b_{23}-V+b_{21}U+b_{23}W) \right\}(\xi),$$
$$H_3(U,V,W)(\xi) := \left\{ \mu W + \frac{d_3}{c}\mathcal{D}[W] + \frac{1}{c} r_3(1-W)(-W+b_{32}V) \right\}(\xi).$$

Then, by choosing a sufficient large constant μ, the operator $T := (T_1, T_2, T_3)$ defined by

$$T_1(U,V,W)(\xi) := e^{-\mu\xi} \int_{-\infty}^{\xi} e^{\mu s} H_1(U,V,W)(s)ds,$$
$$T_2(U,V,W)(\xi) := e^{-\mu\xi} \int_{-\infty}^{\xi} e^{\mu s} H_2(U,V,W)(s)ds,$$
$$T_3(U,V,W)(\xi) := e^{-\mu\xi} \int_{-\infty}^{\xi} e^{\mu s} H_3(U,V,W)(s)ds,$$

is a monotone operator on the space $C^0(\mathbb{R};[0,1]) \times C^0(\mathbb{R};[0,1]) \times C^0(\mathbb{R};[0,1])$. It is clear that (c,U,V,W) is a solution of (2.2) if and only if $(U,V,W) = T(U,V,W)$ and

$$(U,V,W)(-\infty) = (0,0,0), \quad (U,V,W)(+\infty) = (1,1,1).$$

We now introduce the notion of super-solution as follows.

Definition 2.1. *Given a constant $c > 0$. A continuous function (U_+, V_+, W_+) from $\mathbb{R}$ to $(0,1] \times (0,1] \times (0,1]$ is called a* super-solution *of* (2.2)*, if the followings hold:*

(i) *There exists some $\xi_0 \in \mathbb{R}$ such that $U_+(\xi_0) < 1$ and $W_+(\xi_0) < 1$;*
(ii) *$U_+(+\infty) = V_+(+\infty) = W_+(+\infty) = 1$;*
(iii) *U_+, V_+ and W_+ are differentiable a.e. in $\mathbb{R}$ such that*

$$c(U_+)' \geq d_1\mathcal{D}[U_+] + r_1(1-U_+)(-U_+ + b_{12}V_+), \tag{2.3}$$
$$c(V_+)' \geq d_2\mathcal{D}[V_+] + r_2 V_+(1-b_{21}-b_{23}-V_+ + b_{21}U_+ + b_{23}W_+), \tag{2.4}$$
$$c(W_+)' \geq d_3\mathcal{D}[W_+] + r_3(1-W_+)(-W_+ + b_{32}V_+) \tag{2.5}$$

hold a.e. in $\mathbb{R}$.

Remark 2.1. *We note that the definition of super-solution here is a little different from the one used in* [12]*. Instead of requiring that U_+ and W_+ are non-constant in* [12]*, here we take a little stronger condition as* (i)*.*

In order to prove the existence of traveling waves, the following proposition plays an important role.

Proposition 1. *If there exists a super-solution* (U_+, V_+, W_+) *satisfying* $U_+(\cdot) = V_+(\cdot) = W_+(\cdot) = 1$ *on* $[0, +\infty)$*, then* (2.2) *admits a solution* (U, V, W) *with* $U'(\cdot) > 0$*,* $V'(\cdot) > 0$ *and* $W'(\cdot) > 0$ *in* $\mathbb{R}$*.*

To prove Proposition 1, following [3] we introduce the following truncated problem for the integral system

$$(U,V,W)(\xi) = (T_1^n(U,V,W), T_2^n(U,V,W), T_3^n(U,V,W))(\xi) \quad \text{for all } \xi \in [-n, 0] \tag{2.6}$$

with the boundary conditions:

$$U(\xi) = 1, V(\xi) = 1, W(\xi) = 1, \ \forall \ \xi \in (0, +\infty), \tag{2.7}$$

$$U(\xi) = V(\xi) = W(\xi) = \varepsilon, \ \forall \ \xi \in (-\infty, -n], \tag{2.8}$$

where $\varepsilon \in [0, 1)$, $n \in \mathbb{N}$ and

$$\begin{aligned}
T_1^n(U,V,W)(\xi) &:= e^{-\mu\xi} \int_{-\infty}^{-n} \varepsilon\mu e^{\mu s} ds + e^{-\mu\xi} \int_{-n}^{\xi} e^{\mu s} H_1(U,V,W)(s) ds,\\
T_2^n(U,V,W)(\xi) &:= e^{-\mu\xi} \int_{-\infty}^{-n} \varepsilon\mu e^{\mu s} ds + e^{-\mu\xi} \int_{-n}^{\xi} e^{\mu s} H_2(U,V,W)(s) ds,\\
T_3^n(U,V,W)(\xi) &:= e^{-\mu\xi} \int_{-\infty}^{-n} \varepsilon\mu e^{\mu s} ds + e^{-\mu\xi} \int_{-n}^{\xi} e^{\mu s} H_3(U,V,W)(s) ds.
\end{aligned}$$

Note that (U, V, W) satisfies the differential equations in (2.2) on $(-n, 0)$ if it satisfies (2.6).

Since T_i^n also enjoy the monotone property, we can derive the following lemma by a similar argument as that for [12, Lemma 2.2]. We shall not repeat the proof here.

Lemma 2.1. *For each* $n \in \mathbb{N}$ *and* $\varepsilon \in [0, 1)$*, there exists a unique function* $(U^{n,\varepsilon}, V^{n,\varepsilon}, W^{n,\varepsilon})$ *from* $\mathbb{R}$ *to* $[\varepsilon, 1] \times [\varepsilon, 1] \times [\varepsilon, 1]$ *that satisfies* (2.6)-(2.8) *and has the following properties:*

(1) $U^{n,\varepsilon}, V^{n,\varepsilon}, W^{n,\varepsilon} \in C^1((-n, 0)) \cap C((-\infty, 0])$.

(2) $(U^{n,\varepsilon})', (V^{n,\varepsilon})', (W^{n,\varepsilon})' > 0$ *on* $(-n, 0)$ *for any* $\varepsilon \in [0, 1)$*.*

(3) $\dfrac{d}{d\varepsilon} U^{n,\varepsilon}(\xi), \dfrac{d}{d\varepsilon} V^{n,\varepsilon}(\xi), \dfrac{d}{d\varepsilon} W^{n,\varepsilon}(\xi) \geq e^{-\mu(\xi+n)}$ *for* $\xi \in [-n, 0]$*.*

To proceed further, we also recall the following Helly's Lemma.

Proposition 2 (Helly's Lemma). *Let* $\{U_n\}_{n\in\mathbb{N}}$ *be a sequence of uniformly bounded and non-decreasing functions defined in* $\mathbb{R}$*. Then there exist a subsequence* $\{U_{n_i}\}$ *of* $\{U_n\}$ *and a non-decreasing function* U *such that* $U_{n_i} \to U$ *as* $i \to +\infty$ *point-wise in* $\mathbb{R}$*.*

Based on Lemma 2.1 and Helly's Lemma, we can modify the proof in [12, Lemma 2.4] to show Proposition 1 by using the new definition (comparing with that in [12]) of super-solution. For reader's convenience, we provide the details of proof as follows.

Proof of Proposition 1. First, we choose $n_0 > 0$ such that $U_+(-n_0) = \varepsilon_1$ and $W_+(-n_0) = \varepsilon_2$ for some $\varepsilon_i \in (0,1)$ $(i = 1,2)$. Note that ε_1 and ε_2 exist because of the definition of super-solution. Then we shall prove that there exists a subsequence $\{n_k\}$ of $\{n\}$ such that one of the followings must hold:

(i) There exists $\varepsilon = \varepsilon(n_k) \in (0,1)$ such that

$$U^{n_k,\varepsilon(n_k)}(-n_k/2) = \varepsilon_1,\ W^{n_k,\varepsilon(n_k)}(-n_k/2) \le \varepsilon_2 \text{ for all } n_k > 2n_0;$$

(ii) There exists $\varepsilon = \varepsilon(n_k) \in (0,1)$ such that

$$U^{n_k,\varepsilon(n_k)}(-n_k/2) \le \varepsilon_1,\ W^{n_k,\varepsilon(n_k)}(-n_k/2) = \varepsilon_2 \text{ for all } n_k > 2n_0.$$

In order to do this, we consider

$$\begin{aligned}\eta^* \ &:= \ \inf\{\eta > 0 |\, U_+(\xi) \ge U^{n,0}(\xi - \eta),\, V_+(\xi) \ge V^{n,0}(\xi - \eta), \\ &\qquad W_+(\xi) \ge W^{n,0}(\xi - \eta) \text{ for all } \xi \in (-\infty, 0]\,\}.\end{aligned}$$

We can see that η^* is well-defined and $\eta^* \in [0,n]$ since $U_+(\cdot) = V_+(\cdot) = W_+(\cdot) = 1$ on $[0,+\infty)$ and $U^{n,0}(\cdot) = V^{n,0}(\cdot) = W^{n,0}(\cdot) = 0$ on $(-\infty,-n]$. By continuity, we have

$$U_+(\xi) \ge U^{n,0}(\xi - \eta^*),\ V_+(\xi) \ge V^{n,0}(\xi - \eta^*),\ W_+(\xi) \ge W^{n,0}(\xi - \eta^*)$$

for all $\xi \in (-\infty, 0]$. This implies that

$$H_i(U_+, V_+, W_+)(\xi) \ge H_i(U^{n,0}, V^{n,0}, W^{n,0})(\xi - \eta^*) \text{ for } \xi \in (-\infty, 0],\ i = 1,2,3.$$

Using this monotone property of H_i and the same process of [11, Lemma 2.4], we have $\eta^* = 0$. In particular, $U_+(\cdot) \ge U^{n,0}(\cdot)$ and $W_+(\cdot) \ge W^{n,0}(\cdot)$ on $(-\infty, 0]$. Thus, we have

$$\begin{aligned}&U^{n,0}(-\frac{n}{2}) < U^{n,0}(-n_0) \le U_+(-n_0) = \varepsilon_1,\\ &W^{n,0}(-\frac{n}{2}) < W^{n,0}(-n_0) \le W_+(-n_0) = \varepsilon_2\end{aligned}$$

for any $n > 2n_0$.

Consequently, for each $n > 2n_0$, by using Lemma 2.1(3) and the continuity of $U^{\varepsilon,n}$ and $W^{\varepsilon,n}$ in ε, there exists a unique $\varepsilon = \varepsilon(n) \in (0,1)$ such that

$$U^{n,\varepsilon(n)}(-n/2) = \varepsilon_1,\ W^{n,\varepsilon(n)}(-n/2) \le \varepsilon_2 \quad \text{or} \quad U^{n,\varepsilon(n)}(-n/2) \le \varepsilon_1,\ W^{n,\varepsilon(n)}(-n/2) = \varepsilon_2$$

must hold. By choosing a suitable subsequence $\{n_k\}$ of $\{n\}$, one of (i) and (ii) must hold.

We now consider the sequence of functions

$$\{U^{n_k,\varepsilon(n_k)}(-n_k/2 + \cdot),\ V^{n_k,\varepsilon(n_k)}(-n_k/2 + \cdot),\ W^{n_k,\varepsilon(n_k)}(-n_k/2 + \cdot)\}_{n_k > 2n_0},$$

in $\mathbb{R}$. Then Helly's Lemma gives

$$(U^{n_k,\varepsilon(n_k)}, V^{n_k,\varepsilon(n_k)},\ W^{n_k,\varepsilon(n_k)})(-n_k/2 + \cdot) \to (U,V,W)(\cdot) \text{ in } \mathbb{R}$$

as $k \to \infty$ (up to take a subsequence), where (U, V, W) is a non-decreasing function from $\mathbb{R}$ to $[0,1] \times [0,1] \times [0,1]$ and satisfies

$$U(\xi) = T_1(U,V,W)(\xi),\ V(\xi) = T_1(U,V,W)(\xi), W(\xi) = T_2(U,V,W)(\xi) \text{ for all } \xi \in \mathbb{R}.$$

Furthermore, by (i) and (ii), one of the following must hold:

$$U(0) = \varepsilon_1 \quad \text{and } W(0) \le \varepsilon_2, \tag{2.9}$$

$$U(0) \le \varepsilon_1 \quad \text{and } W(0) = \varepsilon_2. \tag{2.10}$$

To prove that (U, V, W) is a solution of (2.2)-(2.3), it suffices to show that (U, V, W) satisfies the boundary conditions. Since U, V and W are non-decreasing in $\mathbb{R}$ and $0 \le U,\ W \le 1$ in $\mathbb{R}$, we see that $U(\pm\infty)$, $V(\pm\infty)$ and $W(\pm\infty)$ exist. By using $U = T_1(U, V, W)$, $V = T_1(U, V, W)$, $W = T_2(U, V, W)$ and L'Hospital's rule, we have

$$(1 - U)(b_{12}V - U)(\pm\infty) = 0, \tag{2.11}$$

$$V\{\,[1 - V] + b_{21}[U - 1] + b_{23}[W - 1]\,\}(\pm\infty) = 0, \tag{2.12}$$

$$(1 - W)(b_{32}V - W)(\pm\infty) = 0. \tag{2.13}$$

Hence $U(\pm\infty), V(\pm\infty), W(\pm\infty) \in \{0, 1\}$.

Recall that we have (2.9) or (2.10). Without loss of generality, we may assume that (2.9) occurs. The same argument can apply to the other case. When (2.9) occurs, we have $U(-\infty) = 0$ and $U(+\infty) = 1$ since U is non-decreasing in $\mathbb{R}$. Note that $U(-\infty) = 0$ implies $V(-\infty) = 0$ because of (2.11). Then we can show that $V(+\infty) = 1$. Otherwise, $V(\pm\infty) = 0$ implies that $V \equiv 0$. Integrating the first equation of (2.2) over $(-\infty, +\infty)$ gives

$$0 < c = -r_1 \int_{-\infty}^{+\infty} U(s)(1 - U(s))ds < 0,$$

a contradiction. Thus, we must have $(V(-\infty), V(+\infty)) = (0, 1)$.

Finally, we show that $(W(-\infty), W(+\infty)) = (0, 1)$. Indeed, by (2.12) and using $U(+\infty) = V(+\infty) = 1$, we have $W(+\infty) = 1$. Recall that $W(0) \le \varepsilon_2 \in (0, 1)$ because of (2.9), it follows that $W(-\infty) = 0$ since W is non-decreasing. Thus, we have $(W(-\infty), W(+\infty)) = (0, 1)$. Thus, (U, V, W) is a solution of (2.2).

Finally, it suffices to show that $U' > 0$ $V' > 0$ and $W' > 0$ in $\mathbb{R}$. For this, note that U, V and W are non-decreasing in $\mathbb{R}$, and $\mu \gg 1$, it follows that

$$H_i(U, V, W)(s) \ge H_i(U, V, W)(\xi) \quad \text{for all } s \ge \xi \text{ and } i = 1, 2, 3.$$

Differentiating $U(\xi) = T_1(U, W)(\xi)$ gives

$$U'(\xi) = -\mu e^{-\mu\xi} \int_{-\infty}^{\xi} e^{\mu s} \{H_1(U, V, W)(s) - H_1(U, V, W)(\xi)\}\, ds \ge 0 \ \text{ in } \mathbb{R}.$$

If there exists $U'(\xi_0) = 0$ for some $\xi_0 \in \mathbb{R}$, we obtain $H_1(U,V,W)(s) \equiv H_1(U,V,W)(\xi_0)$ for all $s \in (-\infty, \xi_0]$. By taking $s \to -\infty$,

$$\mu U(\xi_0) + U'(\xi_0) = H_1(U,V,W)(\xi_0) = H_1(U,V,W)(-\infty) = 0.$$

Note that $U'(\xi_0) = 0$ we obtain $U(\xi_0) = 0$, a contradiction with (2.1). Thus $U'(\xi) > 0$ for all $\xi \in \mathbb{R}$. Similarly, we also have $V'(\cdot) > 0$ and $W'(\cdot) > 0$ in $\mathbb{R}$. Then we complete the proof of Proposition 1. □

As an application of Proposition 1, we have

Corollary 2.2. *The problem* (2.2) *admits a solution* (c,U,V,W) *with* $U'(\cdot) > 0$*,* $V'(\cdot) > 0$ *and* $W'(\cdot) > 0$ *in* $\mathbb{R}$ *as long as* $c \geq \hat{c}$*, where*

$$\begin{aligned} \hat{c} \ := \ & \max\{d_1(e + e^{-1} - 2) + r_1(b_{12} - 1), d_2(e + e^{-1} - 2) + r_2, \\ & \qquad d_3(e + e^{-1} - 2) + r_3(b_{32} - 1)\}. \end{aligned}$$

Proof. Set

$$U_+(\xi) = V_+(\xi) = W_+(\xi) = \min\{e^{\xi}, 1\}.$$

Then by some simple computations, it is not hard to check that (U_+, V_+, W_+) forms a super-solution of (2.2) as long as $c \geq \hat{c}$. Then Corollary 2.2 follows from Proposition 1. □

The proof of the following lemma is similar to that of [12, Lemma 2.5], we omit it here.

Lemma 2.3. *If there exists a super-solution* (U_+, V_+, W_+) *of* (2.2) *with* $(U_+)', (V_+)', (W_+)' > 0$ *for a given* $c > 0$*, then* (2.2) *admits a solution* (c,U,V,W) *with* $U' > 0, V' > 0, W' > 0$ *in* $\mathbb{R}$*.*

We are ready to prove Theorem 2.

Proof of Theorem 2. Due to Corollary 2.2, the constant

$$\begin{aligned} c_{\min} \ := \ & \inf\{c > 0 \mid \ (2.2) \text{ has a solution } (c,U,V,W) \\ & \qquad \text{with } U' > 0, V' > 0 \text{ and } W' > 0 \text{ in } \mathbb{R}\} \end{aligned}$$

is well-defined and $c_{\min} \geq 0$.

Note that the wave profile of a monotone front with wave speed c_0 is a super-solution of (2.2) for any $c > c_0$. Thus, Lemma 2.3 implies that (2.2) admits a solution (c,U,V,W) with $U' > 0$, $V' > 0$ and $W' > 0$ in $\mathbb{R}$ for any $c > c_{\min}$. To complete the proof of Theorem 2, it suffices to show that (2.2) also has a strictly monotone solution (c,U,V,W) for $c = c_{\min}$.

To do so, we choose $\{c_i, U_i, V_i, W_i\}$ be a sequence of strictly monotone solutions of (2.2) such that $c_i \downarrow c_{\min}$ and one of the following cases occurs:

(i) $U_i(0) = 1/2$ and $W_i(0) \leq 1/2$ for all $i \in \mathbb{N}$;
(ii) $U_i(0) \leq 1/2$ and $W_i(0) = 1/2$ for all $i \in \mathbb{N}$.

Note that we can choose U_i and W_i such that either (i) or (ii) holds since the wave profiles are monotone and if necessary, we take a subsequence.

By Helly's Lemma, there exists a subsequence $\{c_{i_j}, U_{i_j}, V_{i_j}, W_{i_j}\}$ and a monotone non-decreasing function $(U_{\min}, V_{\min}, W_{\min})$ such that

$$(c_{i_j}, U_{i_j}, V_{i_j}, W_{i_j}) \to (c_{\min}, U_{\min}, V_{\min}, W_{\min})$$

pointwise in $\mathbb{R}$ as $j \to \infty$. Since one of (i) and (ii) holds, we have either $U_{\min}(0) = 1/2$ and $W_{\min}(0) \leq 1/2$; or $U_{\min}(0) \leq 1/2$ and $W_{\min}(0) = 1/2$. Thus, we can apply the same argument in the proof of Proposition 1 to derive that

$$(U_{\min}, V_{\min}, W_{\min})(-\infty) = (0,0,0), \quad (U_{\min}, V_{\min}, W_{\min})(+\infty) = (1,1,1)$$

and $U'_{\min} > 0$, $V'_{\min} > 0$ and $W'_{\min} > 0$ in $\mathbb{R}$. Consequently, (2.2) also has a strictly monotone solution for $c = c_{\min}$. Hence $c_{\min} > 0$ and so we have completed the proof of Theorem 2. $\square$

In order to prove Theorem 3, we study the asymptotic behavior of wave profile as $\xi \to -\infty$ based on the following fundamental theory developed in [3, 4].

Proposition 3. *Let $a > 0$ be a constant and $B(\cdot)$ be a continuous function having finite $B(\pm\infty) := \lim_{\xi \to \pm\infty} B(\xi)$. Let $W(\cdot)$ be a measurable function satisfying*

$$az(\xi) = e^{\int_\xi^{\xi+1} z(s)ds} + e^{\int_\xi^{\xi-1} z(s)ds} + B(\xi), \ \forall \xi \in \mathbb{R}. \tag{2.14}$$

Then z is uniformly continuous and bounded. In addition, $\omega^{\pm} = \lim_{\xi \to \pm\infty} z(\xi)$ exist and are real roots of the characteristic equation

$$a\omega = e^{\omega} + e^{-\omega} + B(\pm\infty).$$

Proof of Theorem 3. The theorem can be easily proved by applying Proposition 3 to $z = V'/V$. Indeed, it follows from (2.2) that the function $z = V'/V$ satisfies the equation (2.14) with

$$B(\xi) = -2d_2 + r_2[1 - b_{21} - b_{23} - V(\xi) + b_{21}U(\xi) + b_{23}W(\xi)].$$

Since $B(-\infty) = -2d_2 + r_2(1 - b_{21} - b_{23})$, we have

$$c\lambda = d_2(e^{\lambda} + e^{-\lambda} - 2) + r_2(1 - b_{21} - b_{23})$$

for some $\lambda > 0$. It follows that $c \geq c_*$. Hence Theorem 3 follows. $\square$

3. Characterization of linear determinacy for discrete model

This section is devoted to the proof of Theorem 4.

To begin with, recall that $\lambda_1 = \lambda_1(c)$ be the smallest root of (1.14) for any given $c \geq c_*$. Then we consider the function

$$g(c, d) := c\lambda_1(c) - d(e^{\lambda_1(c)} + e^{-\lambda_1(c)} - 2). \tag{3.1}$$

Note that $g(c, d)$ is strictly decreasing in d for any fixed c. Also, we have

$$g(c, d_2) = r_2(1 - b_{21} - b_{23}) > 0.$$

This implies that, for $c = c_*$, there exists a unique constant $d_* = d_*(d_2) > d_2$ such that $g(c_*, d_*) = 0$. Furthermore, we can have $d_* > 2d_2$. Indeed, following the same process in [10, Lemma 2.2], we have

Lemma 3.1. *Let r_2, b_{21} and b_{23} be fixed and let the function g be given in* (3.1). *Then there exists a unique $d_* = d_*(d_2) > 2d_2$ such that*

$$g(c_*, d_*) = 0 < g(c_*, d) \quad \textit{for all } d \in (0, d_*).$$

Furthermore, for each $c > c_*$, there exists a unique $d_c > d_*$ such that

$$g(c, d_c) = 0 < g(c, d) \quad \textit{for all } d \in (0, d_c).$$

We are ready to prove Theorem 4.

Proof of Theorem 4. Thanks to Theorem 3, it suffices to show that $c_{\min} \leq c_*$ as long as (1.15) holds. Furthermore, by Proposition 1, it suffices to show that a super-solution exists for each $c \geq c_*$. For this, given $c \geq c_*$. We introduce the functions

$$U_+(\xi) = W_+(\xi) := \min\left\{1, \frac{e^{\lambda_1 \xi}}{b_{21} + b_{23}}\right\}, \quad V_+(\xi) := \min\{1, e^{\lambda_1 \xi}\}.$$

Then it is easy to check that (2.4) holds a.e. in $\mathbb{R}$, using (3.1). Moreover, (2.3) and (2.5) hold for all $\xi > [\ln(b_{21} + b_{23})]/\lambda_1 := \xi_1$, since $U_+(\xi) = W_+(\xi) = 1$ for all $\xi > \xi_1$.

It remains to check (2.3) and (2.5) for $\xi < \xi_1$. Indeed, for $\xi < \xi_1$, we have

$$U_+(\xi) = W_+(\xi) = \frac{e^{\lambda_1 \xi}}{b_{21} + b_{23}}, \quad V_+(\xi) = e^{\lambda_1 \xi}.$$

Let us focus on the U-equation first. Direct calculations yield

$$\begin{aligned} &c(U_+)'(\xi) - d_1\mathcal{D}[U_+](\xi) - r_1[(1 - U_+)(-U_+ + b_{12}V_+)](\xi) \\ \geq\ &\frac{e^{\lambda_1 \xi}}{b_{21} + b_{23}}\left\{c\lambda_1 - d_1(e^{\lambda_1} + e^{-\lambda_1} - 2) + r_1\left(1 - \frac{e^{\lambda_1 \xi}}{b_{21} + b_{23}}\right)[1 - b_{12}(b_{21} + b_{23})]\right\} \\ =:\ &\frac{e^{\lambda_1 \xi}}{b_{21} + b_{23}}\mathcal{Q}(\xi) \quad \text{for } \xi < \xi_1. \end{aligned} \tag{3.2}$$

If $b_{12}(b_{21}+b_{23}) \le 1$, using $e^{\lambda_1\xi}/(b_{21}+b_{23})<1$ and Lemma 3.1 give us

$$\mathcal{Q}(\xi) \ge c\lambda_1 - d_1(e^{\lambda_1}+e^{-\lambda_1}-2) \ge 0 \quad \text{in } (-\infty,\xi_1)$$

for all $d_1 \in (0,d_*]$ and for all $r_1>0$. Hence (2.3) is verified whenever $(d_1,r_1,b_{12}) \in A_1^1$, where A_1^1 is given by (1.16).

Now, we shall consider the case that $b_{12}(b_{21}+b_{23})>1$. For $\xi<\xi_1$, we see from (3.2) that

$$\begin{aligned}\mathcal{Q}(\xi) &\ge c\lambda_1 - d_1(e^{\lambda_1}+e^{-\lambda_1}-2) - r_1[b_{12}(b_{21}+b_{23})-1]\\ &:= g(c,d_1) - r_1[b_{12}(b_{21}+b_{23})-1].\end{aligned}$$

We shall use an idea from the proof of [10, Theorem 1]. By Lemma 3.1, there is a unique d_c such that

$$\frac{c\lambda_1(c)}{d_c} = e^{\lambda_1(c)}+e^{-\lambda_1(c)}-2. \tag{3.3}$$

Together with (1.14), we obtain

$$c\lambda_1(c) = r_2(1-b_{21}-b_{23})\frac{d_c}{d_c-d_2}. \tag{3.4}$$

Combining (3.3) and (3.4) give

$$g(c,d_1) = r_2(1-b_{21}-b_{23})\frac{d_c-d_1}{d_c-d_2}.$$

Thus, $\mathcal{Q}(\xi)\ge 0$ for $\xi<\xi_1$ as long as

$$b_{12}(b_{21}+b_{23})>1, \quad 0<r_1<\frac{r_2(1-b_{21}-b_{23})}{b_{12}(b_{21}+b_{23})-1}\cdot\frac{d_c-d_1}{d_c-d_2}.$$

Recall that we have $d_c>d_*>2d_2$ (Lemma 3.1). Hence we have

$$\frac{d_*-d_1}{d_*-d_2} \le \frac{d_c-d_1}{d_c-d_2}.$$

Then $\mathcal{Q}(\xi)\ge 0$ for $\xi<\xi_1$ as long as $(d_1,r_1,b_{12})\in A_1^2$, where A_1^2 is given by (1.17).

Next, we turn to the W-equation. Direct computations give

$$\begin{aligned}&c(W_+)'(\xi) - d_3\mathcal{D}[W_+](\xi) - r_3[(1-W_+)(-W_+ + b_{32}V_+)](\xi)\\ \ge\ &\frac{e^{\lambda_1\xi}}{b_{21}+b_{23}}\left\{c\lambda_1 - d_3(e^{\lambda_1}+e^{-\lambda_1}-2) + r_3\left(1-\frac{e^{\lambda_1\xi}}{b_{21}+b_{23}}\right)[1-b_{32}(b_{21}+b_{23})]\right\}.\end{aligned} \tag{3.5}$$

The same process as above, we can derive (3.5) is non-negative for $\xi<\xi_1$ as long as $(d_3,r_3,b_{32})\in A_3^1\cup A_3^2$.

Consequently, for each $c\ge c_*$, we see that (U_+,V_+,W_+) is a super-solution of (1.13) and satisfies $U_+(\cdot)=V_+(\cdot)=W_+(\cdot)=1$ on $[0,+\infty)$ if (1.15) holds. By Proposition 1, the problem (1.13) has a solution for $c\ge c_*$ if (1.15) holds. Thus, we have $c_{\min}=c_*$. This completes the proof of Theorem 4. $\square$

4. Linear determinacy for continuous model (1.2)-(1.4)

In this section, we study the monotone traveling waves and linear determinacy for the continuous system (1.2)-(1.4). Our approach is based on the method used in [10] by approximating the continuous system with the following discrete system:

$$(u_j^\tau)'(t) = D_1\mathcal{D}[u_j^\tau](t)/\tau^2 + r_1 u_j^\tau(t)[1 - u_j^\tau(t) - b_{12}v_j^\tau(t)],\ j \in \mathbb{Z},\ t \in \mathbb{R}, \tag{4.1}$$

$$(v_j^\tau)'(t) = D_2\mathcal{D}[v_j^\tau](t)/\tau^2 + r_2 v_j^\tau(t)[1 - b_{21}u_j^\tau - v_j^\tau(t) - b_{23}w_j^\tau(t)],\ j \in \mathbb{Z},\ t \in \mathbb{R}, \tag{4.2}$$

$$(w_j^\tau)'(t) = D_3\mathcal{D}[w_j^\tau](t)/\tau^2 + r_3 w_j^\tau(t)[1 - b_{32}v_j^\tau(t) - w_j^\tau],\ j \in \mathbb{Z},\ t \in \mathbb{R}, \tag{4.3}$$

for any $\tau > 0$ small.

Let

$$c_*(\tau^{-2}D_2) := \min_{\lambda>0}\left\{\frac{\tau^{-2}D_2(e^\lambda + e^{-\lambda} - 2) + r_2(1 - b_{21} - b_{23})}{\lambda}\right\}. \tag{4.4}$$

Then, using the same argument as in [12, Section 5], we can easily show that

$$\tau c_*(\tau^{-2}D_2) \to 2\sqrt{D_2 r_2(1 - b_{21} - b_{23})} := s_* \quad \text{as } \tau \to 0^+. \tag{4.5}$$

To prove Theorem 1, we first prepare two key lemmas as follows.

Lemma 4.1. *$d_*(d_2)/d_2$ is decreasing in d_2, where $d_*(d_2)$ is given by Lemma 3.1.*

Proof. From Lemma 3.1, we have $g(c_*, d_*) = 0$, which implies

$$c_*(d_2)\lambda_*(d_2) = d_*(d_2)(e^{\lambda_*(d_2)} + e^{-\lambda_*(d_2)} - 2).$$

Also, recall from (1.14) that

$$c_*(d_2)\lambda_*(d_2) = d_2(e^{\lambda_*(d_2)} + e^{-\lambda_*(d_2)} - 2) + r_2(1 - b_{21} - b_{23}). \tag{4.6}$$

It follows that

$$\frac{d_*(d_2)}{d_2} = 1 + \frac{r_2(1 - b_{21} - b_{23})}{d_2(e^{\lambda_*(d_2)} + e^{-\lambda_*(d_2)} - 2)}. \tag{4.7}$$

Thus, to prove Lemma 4.1, it suffices to show

$$d_2(e^{\lambda_*(d_2)} + e^{-\lambda_*(d_2)} - 2) \quad \text{is increasing in } d_2. \tag{4.8}$$

For this, recall that

$$\begin{aligned}
&c_*(d_2) = \min_{\lambda>0}\Phi(\lambda, d_2), \quad \Phi(\lambda, d_2) := \frac{d_2(e^\lambda + e^{-\lambda} - 2) + r_2(1 - b_{21} - b_{23})}{\lambda},\\
&c_*(d_2)\lambda_*(d_2) = \Psi(d_2), \quad \Psi(d_2) := d_2(e^{\lambda_*(d_2)} + e^{-\lambda_*(d_2)} - 2) + r_2(1 - b_{21} - b_{23}).
\end{aligned}$$

For a given $d_2 > 0$, since Φ is strictly convex and $\Phi(0^+, d_2) = \Phi(\infty, d_2) = \infty$, there exists a unique $\lambda_*(d_2)$ satisfying

$$\frac{\partial}{\partial\lambda}\Phi(\lambda, d_2)\big|_{\lambda=\lambda_*(d_2)} = 0.$$

It follows that

$$d_2(e^{\lambda_*(d_2)} - e^{-\lambda_*(d_2)})\lambda_*(d_2) = \Psi(d_2). \tag{4.9}$$

By differentiating (4.9) with respect to d_2, we arrive at

$$\lambda_*'(d_2) = \frac{e^{\lambda_*(d_2)} + e^{-\lambda_*(d_2)} - 2 - \lambda_*(d_2)(e^{\lambda_*(d_2)} - e^{-\lambda_*(d_2)})}{d_2\lambda_*(d_2)(e^{\lambda_*(d_2)} + e^{-\lambda_*(d_2)})}. \tag{4.10}$$

By differentiating $\Psi(d_2)$ with respect to d_2, we arrive at

$$\Psi'(d_2) = e^{\lambda_*(d_2)} + e^{-\lambda_*(d_2)} - 2 + \lambda_*'(d_2)d_2(e^{\lambda_*(d_2)} - e^{-\lambda_*(d_2)}). \tag{4.11}$$

Putting (4.10) into (4.11), we have

$$\begin{aligned}\Psi'(d_2) &= (e^{\lambda_*(d_2)} + e^{-\lambda_*(d_2)} - 2) \\ &\quad + \frac{(e^{\lambda_*(d_2)} - e^{-\lambda_*(d_2)})}{\lambda_*(d_2)(e^{\lambda_*(d_2)} + e^{-\lambda_*(d_2)})}[e^{\lambda_*(d_2)} + e^{-\lambda_*(d_2)} - 2 - \lambda_*(d_2)(e^{\lambda_*(d_2)} - e^{-\lambda_*(d_2)})].\end{aligned}$$

In order to determine the sign of $\Psi'(d_2)$, we consider

$$G(x) := (e^x + e^{-x} - 2) + \frac{(e^x - e^{-x})}{x(e^x + e^{-x})}[e^x + e^{-x} - 2 - x(e^x - e^{-x})] = \frac{Q(x)}{x(e^x + e^{-x})},$$

where

$$Q(x) := e^{2x} - e^{-2x} - 2(e^x - e^{-x}) - 2x(e^x + e^{-x}) + 4x.$$

Note that if $G(x) > 0$ for all $x > 0$, then $\Psi'(d_2)$ for all $d_2 > 0$ since $\lambda_*(d_2) > 0$ for given any $d_2 > 0$.

To show $G(x) > 0$ for all $x > 0$, it suffices to show that

$$Q(x) > 0 \quad \text{for all } x > 0. \tag{4.12}$$

However, to show (4.12), we shall show that

(i) there exists small $\delta > 0$ such that $Q(x) > 0$ for all $x \in (0, \delta)$, and
(ii) any positive critical point of Q must be a minimal point.

Combining (i) and (ii), we obtain (4.12) since $Q(0) = 0$.

We now set $y(x) := e^x + e^{-x}$. Then it is easy to see that $y(0) = 2$, $y(x) > 2$ and $y'(x) > 0$ for all $x > 0$. Also, note that

$$e^x - e^{-x} = \sqrt{y^2 - 4} = y', \quad y'' = y. \tag{4.13}$$

By (4.13), we can easily derive

$$\begin{aligned} Q(y) = yy' - 2y' - 2xy + 4x, \quad Q'(x) &= 2[y^2 - 2y - xy'], \\ Q''(x) = 2[2yy' - 3y' - xy], \quad Q^{(3)}(x) &= 2[2(y')^2 + 2y^2 - 4y - xy'], \\ Q^{(4)}(x) = 2[8yy' - 5y' - xy], \quad Q^{(5)}(x) &= 2[8(y')^2 + 8y^2 - 6y - xy']. \end{aligned}$$

Then we have $Q^{(5)}(0) = 40 > 0$ using $y(0) = 2$ and $y'(0) = 0$. Together with the fact that $Q^{(n)}(0) = 0$ for $n = 0, \ldots, 4$, we obtain that $Q(\cdot) > 0$ in $(0, \delta)$ for some $\delta > 0$. So we have proved (i).

For (ii), let $x_0 > 0$ such that $Q'(x_0) = 0$. The above equality of Q' gives us

$$x_0 = \frac{y^2(x_0) - 2y(x_0)}{y'(x_0)}.$$

Putting this into the above equality of Q'' and using (4.13), we obtain

$$Q''(x_0) = \frac{2}{y'(x_0)}[y^3(x_0) - y^2(x_0) - 8y(x_0) + 12] := \frac{2}{y'(x_0)} R(y(x_0)).$$

It is easy to see that $R(y) > 0$ for all $y > 2$, since

$$R'(y) = (y - 2)(3y + 4).$$

Recall that $x_0 > 0$ implies $y(x_0) > 2$ and $y'(x_0) > 0$, we have $R(y(x_0)) > 0$, which gives $Q''(x_0) > 0$. Thus, (ii) holds. Combining (i) and (ii), we have proved (4.12) and then $\Psi'(d_2) > 0$ for all $d_2 > 0$. Hence (4.8) holds. This completes the proof of Lemma 4.1. □

Moreover, we have

Lemma 4.2. *There holds*

$$\inf_{d_2 > 0} \frac{d_*(d_2)}{d_2} = \lim_{d_2 \to \infty} \frac{d_*(d_2)}{d_2} = 2.$$

Proof. By Lemma 3.1 and Lemma 4.1, we see

$$\inf_{d_2 > 0} \frac{d_*(d_2)}{d_2} = \lim_{d_2 \to \infty} \frac{d_*(d_2)}{d_2} \geq 2. \tag{4.14}$$

Putting $\tau = \sqrt{D_2}/\sqrt{d_2}$ into (4.5), we have

$$\frac{c_*(d_2)}{\sqrt{d_2}} \to 2\sqrt{r_2(1 - b_{21} - b_{23})} \quad \text{as } d_2 \to \infty. \tag{4.15}$$

Recall from the proof of Lemma 4.1 that $c_*(d_2)\lambda_*(d_2)$ is increasing in d_2. On the other hand, it follows from (4.7) and (4.14) that $d_2(e^{\lambda_*(d_2)} + e^{-\lambda_*(d_2)} - 2)$ must have a positive upper bound for all $d_2 \in (0, \infty)$. Thus, (4.6) implies that $\lim_{d_2 \to \infty} c_*(d_2)\lambda_*(d_2)$ exists and is finite. Together with (4.15), there exists a constant $\gamma \in (0, \infty)$ such that

$$\lim_{d_2 \to \infty} \sqrt{d_2}\lambda_*(d_2) = \gamma. \tag{4.16}$$

Now, the fact that $\lim_{\lambda \to 0}[e^\lambda + e^{-\lambda} - 2]/\lambda^2 = 1$ and $\lambda_*(d_2) \to 0$ as $d_2 \to \infty$ gives

$$\lim_{d_2 \to \infty} \frac{d_2(e^{\lambda_*(d_2)} + e^{-\lambda_*(d_2)} - 2)}{d_2[\lambda_*(d_2)]^2} = 1.$$

It then follows from (4.16) that

$$\lim_{d_2 \to \infty} d_2(e^{\lambda_*(d_2)} + e^{-\lambda_*(d_2)} - 2) = \gamma^2 \tag{4.17}$$

Using (4.15), (4.16) and (4.17) and taking $d_2 \to \infty$ in

$$c_*(d_2)\lambda_*(d_2) = d_2(e^{\lambda_*(d_2)} + e^{-\lambda_*(d_2)} - 2) + r_2(1 - b_{21} - b_{23})$$

yield that

$$[2\sqrt{r_2(1 - b_{21} - b_{23})}\,]\gamma = \gamma^2 + r_2(1 - b_{21} - b_{23}).$$

Hence we have $\gamma = \sqrt{r_2(1 - b_{21} - b_{23})}$. By (4.7) and (4.17), we have

$$\lim_{d_2\to\infty} \frac{d_*(d_2)}{d_2} = 2.$$

This completes the proof of Lemma 4.2. □

Consequently, we obtain the following result.

Corollary 4.3. *Suppose that* $(D_j, r_j, b_{j2}) \in B_j^1 \cup B_j^2$ *for* $j = 1, 3$. *Let* $d_j(\tau) := D_j/\tau^2$ *and* $d_*(\tau) := d_*(d_2(\tau))$ *for* $\tau > 0$. *Then* $(d_j(\tau), r_j, b_{j2}) \in A_j^1 \cup A_j^2$ *for* $j = 1, 3$, *for all small* $\tau > 0$.

Proof. It is easy to see that $(D_j, r_j, b_{j2}) \in B_j^1$ implies that $(d_j(\tau), r_j, b_{j2}) \in A_j^1$ for all $\tau > 0$ and $j = 1, 3$, since $d_* > 2d_2$.

For $(D_j, r_j, b_{j2}) \in B_j^2$, it follows from Lemma 4.2 that

$$\frac{d_*(\tau) - d_j(\tau)}{d_*(\tau) - d_2(\tau)} = \frac{[d_*(\tau)/d_2(\tau)] - D_j/D_2}{[d_*(\tau)/d_2(\tau)] - 1} \to 2 - \frac{D_j}{D_2} \quad \text{as } \tau \to 0^+.$$

Thus, $(d_j(\tau), r_j, b_{j2}) \in A_j^2$ for all small $\tau > 0$, if $(D_j, r_j, b_{j2}) \in B_j^2$, for $j = 1, 3$. Then Corollary 4.3 follows. □

Since the system (1.5) has no solution when $s < s_*$, to prove Theorem 1 it suffices to prove the existence of monotone traveling front with speed s for any $s > s_*$ under the conditions (1.8)-(1.9), where

$$s_* := 2\sqrt{D_2 r_2(1 - b_{21} - b_{23})}.$$

Hereafter, for convenience, we write $c_*(\tau^{-2}D_2)$ in (4.4) as $c_*(\tau)$ without any confusion. For any given $s > s_*$, we can choose small $\eta > 0$ such that $s_* + \eta < s$. Then from (4.5) there exists $\tau_0 > 0$ sufficiently small such that

$$\tau c_*(\tau) < s_* + \eta < s \quad \forall\, \tau \in (0, \tau_0].$$

For any $\tau \in (0, \tau_0]$, since $s/\tau > c_*(\tau)$, it follows from Corollary 4.3 (if necessary, we choose τ_0 smaller) and Theorem 4 that there exists a traveling front of the system (4.1)-(4.3) connecting $(1, 0, 1)$ to $(0, 1, 0)$ with speed s/τ and profile $(\hat{U}^{\tau,s}, \hat{V}^{\tau,s}, \hat{W}^{\tau,s})$ under the assumption (1.7). Namely, we have

$$\begin{cases} \frac{s}{\tau}(\hat{U}^{\tau,s})' = D_1\mathcal{D}[\hat{U}^{\tau,s}]/\tau^2 + r_1\hat{U}^{\tau,s}(1-\hat{U}^{\tau,s}-b_{12}\hat{V}^{\tau,s}), \quad y\in\mathbb{R}, \\ \frac{s}{\tau}(\hat{V}^{\tau,s})' = D_2\mathcal{D}[\hat{V}^{\tau,s}]/\tau^2 + r_2\hat{V}^{\tau,s}(1-b_{21}\hat{U}^{\tau,s}-\hat{V}^{\tau,s}-b_{23}\hat{W}^{\tau,s}), \quad y\in\mathbb{R}, \\ \frac{s}{\tau}(\hat{W}^{\tau,s})' = D_3\mathcal{D}[\hat{W}^{\tau,s}]/\tau^2 + r_3\hat{W}^{\tau,s}(1-b_{32}\hat{V}^{\tau,s}-\hat{W}^{\tau,s}), \quad y\in\mathbb{R}, \\ (\hat{U}^{\tau,s},\hat{V}^{\tau,s},\hat{W}^{\tau,s})(-\infty)=(1,0,1), \quad (\hat{U}^{\tau,s},\hat{V}^{\tau,s},\hat{W}^{\tau,s})(+\infty)=(0,1,0), \\ 0\le \hat{U}^{\tau,s},\hat{V}^{\tau,s},\hat{W}^{\tau,s}\le 1 \quad \text{in } \mathbb{R}. \end{cases}$$

Next, we define

$$\begin{aligned} (\phi^{\tau,s},\psi^{\tau,s},\theta^{\tau,s})(y) &:= (\hat{U}^{\tau,s},\hat{V}^{\tau,s},\hat{W}^{\tau,s})(y/\tau), \\ (u^{\tau,s},v^{\tau,s},w^{\tau,s})(x,t) &:= (\phi^{\tau,s},\psi^{\tau,s},\theta^{\tau,s})(x+st). \end{aligned}$$

Then it is easy to check that

$$\begin{gathered} u^{\tau,s}(x,t+1/s)=u^{\tau,s}(x+1,t), \quad v^{\tau,s}(x,t+1/s)=v^{\tau,s}(x+1,t), \\ w^{\tau,s}(x,t+1/s)=w^{\tau,s}(x+1,t), \quad x,t\in\mathbb{R}, \\ u_x^{\tau,s}=u_t^{\tau,s}/s, \quad v_x^{\tau,s}=v_t^{\tau,s}/s, \quad w_x^{\tau,s}=w_t^{\tau,s}/s. \end{gathered}$$

It follows that

$$\begin{aligned} u_t^{\tau,s} &= \frac{D_1[u^{\tau,s}(\cdot+\tau,t)+u^{\tau,s}(\cdot-\tau,t)-2u^{\tau,s}]}{\tau^2} + r_1u^{\tau,s}(1-u^{\tau,s}-b_{12}v^{\tau,s}), \\ v_t^{\tau,s} &= \frac{D_2[v^{\tau,s}(\cdot+\tau,t)+v^{\tau,s}(\cdot-\tau,t)-2v^{\tau,s}]}{\tau^2} + r_2v^{\tau,s}(1-v^{\tau,s}-b_{21}u^{\tau,s}-b_{23}w^{\tau,s}), \\ w_t^{\tau,s} &= \frac{D_3[w^{\tau,s}(\cdot+\tau,t)+w^{\tau,s}(\cdot-\tau,t)-2w^{\tau,s}]}{\tau^2} + r_3w^{\tau,s}(1-w^{\tau,s}-b_{32}v^{\tau,s}). \end{aligned}$$

Also, we have

$$s(\phi^{\tau,s})' = \frac{D_1[\phi^{\tau,s}(\cdot+\tau)+\phi^{\tau,s}(\cdot-\tau)-2\phi^{\tau,s}]}{\tau^2} + r_1\phi^{\tau,s}(1-\phi^{\tau,s}-b_{12}\psi^{\tau,s}), \tag{4.18}$$

$$s(\psi^{\tau,s})' = \frac{D_2[\psi^{\tau,s}(\cdot+\tau)+\psi^{\tau,s}(\cdot-\tau)-2\psi^{\tau,s}]}{\tau^2} + r_2\psi^{\tau,s}(1-\psi^{\tau,s}-b_{21}\phi^{\tau,s}-b_{23}\theta^{\tau,s}), \tag{4.19}$$

$$s(\theta^{\tau,s})' = \frac{D_3[\theta^{\tau,s}(\cdot+\tau)+\theta^{\tau,s}(\cdot-\tau)-2\theta^{\tau,s}]}{\tau^2} + r_3\theta^{\tau,s}(1-\theta^{\tau,s}-b_{32}\psi^{\tau,s}). \tag{4.20}$$

To proceed further, we consider the initial value problem for the system (4.1)-(4.3) for $t\ge 0$ such that $0\le u_j^\tau(0),v_j^\tau(0),w_j^\tau(0)\le 1$ for all j. For simplicity, we use the notation

$$(u^n,v^n,w^n) := \{(u_j^n,v_j^n,w_j^n)\}_{j\in\mathbb{Z}}, \quad (u_j^n,v_j^n,w_j^n) := (u_j^{1/n},v_j^{1/n},w_j^{1/n})$$

to represent the solution of the initial value problem for (4.1)-(4.3) for $t\ge 0$ with $\tau=1/n$, $n\in\mathbb{N}$.

Applying the discrete Fourier transform, the solution of the following linear lattice equation

$$(z_j^n)'(t) = n^2d[z_{j+1}^n(t)+z_{j-1}^n(t)-2z_j^n(t)], \quad j\in\mathbb{Z},$$

with the initial data $\{z_j^n(0)\}_{j\in\mathbb{Z}}$ is given by

$$z_j^n(t) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\left(\int_{-\pi}^{\pi} e^{\mathrm{i}(j-k)\omega+2n^2dt(\cos\omega-1)}d\omega\right) z_k^n(0), \quad \mathrm{i} := \sqrt{-1}.$$

Then we have the following estimate which can be found in [10, Lemma 3.1].

Lemma 4.4. *For any $\alpha, M > 0$, there is a constant $L = L(\alpha, M)$ such that if $|z_k^n(0)| \le M$ for all $k \in \mathbb{Z}$, then $|z_j^n(t) - z_{j+2}^n(t)| \le L/n$ for any $j \in \mathbb{Z}$ and $t \ge \alpha/d$.*

Based on Lemma 4.4, the following estimate can be derived similarly as [10, Lemma 3.2].

Lemma 4.5. *Suppose that $0 \le u_j^n(0), v_j^n(0), w_j^n(0) \le 1$ for all $j \in \mathbb{Z}$. Then, for any $\epsilon > 0$, there is a constant $\delta > 0$ such that for any $n \in \mathbb{N}$*

$$|u_{j_1}^n(1) - u_{j_2}^n(1)|,\ |v_{j_1}^n(1) - v_{j_2}^n(1)|,\ |w_{j_1}^n(1) - w_{j_2}^n(1)| < \epsilon,$$

if $|j_1 - j_2| < n\delta$ and $(j_1 - j_2)$ is even.

Now, we consider the sequence

$$(U_n, V_n, W_n) := (\hat{U}^{\tau_n,s_n}, \hat{V}^{\tau_n,s_n}, \hat{W}^{\tau_n,s_n}),\ (\phi_n, \psi_n, \theta_n) := (\phi^{\tau_n,s_n}, \psi^{\tau_n,s_n}, \theta^{\tau_n,s_n})$$

with $\tau_n = 1/n$ and $s_n \downarrow s$ as $n \to \infty$. Then from (4.18)-(4.20) we see that $(\phi_n, \psi_n, \theta_n)$ satisfies

$$s_n(\phi_n)'(\xi) = n^2D_1[\phi_n(\xi+\frac{1}{n}) + \phi_n(\xi-\frac{1}{n}) - 2\phi_n(\xi)] + r_1\{\phi_n[1-\phi_n-b_{12}\psi_n]\}(\xi),$$
$$s_n(\psi_n)'(\xi) = n^2D_2[\psi_n(\xi+\frac{1}{n}) + \psi_n(\xi-\frac{1}{n}) - 2\psi_n(\xi)] + r_2\{\psi_n[1-\psi_n-b_{21}\phi_n-b_{23}\theta_n]\}(\xi),$$
$$s_n(\theta_n)'(\xi) = n^2D_3[\theta_n(\xi+\frac{1}{n}) + \theta_n(\xi-\frac{1}{n}) - 2\theta_n(\xi)] + r_3\{\theta_n[1-\theta_n-b_{32}\psi_n]\}(\xi)$$

for $\xi \in \mathbb{R}$. Also, the following lemma about the equicontinuity of $(\phi_n, \psi_n, \theta_n)$ follows from Lemma 4.5.

Lemma 4.6. *The families $\{\phi_n\}_{n\in\mathbb{N}}$, $\{\psi_n\}_{n\in\mathbb{N}}$ and $\{\theta_n\}_{n\in\mathbb{N}}$ are equicontinuous functions.*

Proof. We only consider $\{\phi_n\}_{n\in\mathbb{N}}$, because the cases for $\{\psi_n\}_{n\in\mathbb{N}}$ and $\{\theta_n\}_{n\in\mathbb{N}}$ are similar. Given $\epsilon > 0$. Since $(U_n, V_n, W_n)(j + ns_nt)$ is a traveling wave solution of (4.1)-(4.3), by Lemma 4.5, there is $\delta > 0$ such that $|U_n(j_1 + ns_n) - U_n(j_2 + ns_n)| < \epsilon$ for any $|j_1 - j_2| < n\delta$ and even j_1, j_2. This implies that

$$|\phi_n(j_1/n + s_n) - \phi_n(j_2/n + s_n)| < \epsilon \tag{4.21}$$

for any $|j_1 - j_2| < n\delta$ and even j_1, j_2.

Now given any $x, y \in \mathbb{R}$ with $|x - y| < \delta/2$. We can choose n large enough and two even integers j_1, j_2 with $|j_1 - j_2| < n\delta$ such that $j_1/n + s_n \le x, y \le j_2/n + s_n$. Since ϕ_n is a monotone continuous function, it follows from (4.21) that $|\phi_n(x) - \phi_n(y)| < \epsilon$. This proves the lemma. □

We are ready to give a proof of Theorem 1.

Proof of Theorem 1. First, Lemma 4.6 implies that (up to taking a subsequence) $\phi_n \to \phi, \psi_n \to \psi, \theta_n \to \theta$ as $n \to \infty$ uniformly in any bounded interval in $\mathbb{R}$ for some continuous functions ϕ, ψ, θ. Furthermore, it is easy to see that (ϕ, ψ, θ) satisfies (1.5) in the distribution sense except the boundary conditions. Indeed, the boundedness and continuity of (ϕ, ψ, θ) implies that (ϕ, ψ, θ) solves (1.5) in the classical sense except the boundary conditions.

It remains to show that (ϕ, ψ, θ) connects $(1, 0, 1)$ and $(0, 1, 0)$. In fact, the proof is similar to the one for Proposition 1. Without loss of generality, if necessary, we may take a subsequence such that one of the followings must occur:

(i) $\phi_n(0) = 1/2$ and $\theta_n(0) \geq 1/2$ for all $n \in \mathbb{N}$;
(ii) $\phi_n(0) \geq 1/2$ and $\theta_n(0) = 1/2$ for all $n \in \mathbb{N}$.

Note that we can obtain such a dichotomy due to the monotonicity of the profiles of $\phi_n(\cdot)$ and $\theta_n(\cdot)$ for all n.

If (i) holds, then $\phi(0) = 1/2$ and $\theta(0) \geq 1/2$. Since $(\phi, \psi, \theta)(\pm\infty)$ are constant steady states for (1.2)-(1.3), a direct calculation of

$$[\phi(1 - \phi - b_{12}\psi)](\pm\infty) = 0, \tag{4.22}$$

$$[\psi(1 - b_{21}\phi - \psi - b_{23}\theta)](\pm\infty) = 0, \tag{4.23}$$

$$[\theta(1 - b_{32}\psi - \theta)](\pm\infty) = 0, \tag{4.24}$$

yields that $(\phi, \psi, \theta)(\pm\infty)$ can only take values in $\{(1, 0, 1), (1, 0, 0), (0, 0, 0), (0, 1, 0), (0, 0, 1)\}$.

Since $\phi(+\infty) \leq \phi(0) \leq \phi(-\infty)$, we must have $\phi(-\infty) = 1$ and $\phi(+\infty) = 0$. By (4.22), $\psi(-\infty) = 0$. Then it follows that $\psi(+\infty) = 1$. Otherwise, $\psi \equiv 0$. Then, by integrating the ϕ-equation in (1.5) over $(-\infty, +\infty)$ gives $s < 0$, a contradiction. Thus, we have $(\psi(-\infty), \psi(+\infty)) = (0, 1)$. Next, from (4.23) we see that $\theta(+\infty) = 0$. Due to that $\theta(0) \geq 1/2$, we see that $\theta(-\infty) = 1$. Hence $(\phi, \psi, \theta)(x + st)$ is a monotone traveling front connecting $(1, 0, 1)$ and $(0, 1, 0)$ with speed s. The proof for the case (ii) is similar. This completes the proof of Theorem 1. □

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