# ON A FREE BOUNDARY PROBLEM FOR THE CURVATURE FLOW WITH DRIVING FORCE

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Abstract. We study a free boundary problem associated with the curvature dependent motion of planar curves in the upper half plane whose two endpoints slide along the horizontal axis with prescribed fixed contact angles. Our first main result concerns the classification of solutions; every solution falls into one of the three categories, namely, area expanding, area bounded and area shrinking types. We then study in detail the asymptotic behavior of solutions in each category. Among other things we show that solutions are asymptotically self-similar both in the area expanding and the area shrinking cases, while solutions converge to either a stationary solution or a traveling wave in the area bounded case. We also prove results on the concavity properties of solutions. One of the main tools of this paper is the intersection number principle; however, in order to deal with solutions with free boundaries, we introduce what we call "the extended intersection number principle", which turns out to be exceedingly useful in handling curves with moving endpoints.

## 1. Introduction

This paper deals with the following free boundary problem, which we shall call (P):

$$u_t = \frac{u_{xx}}{1+u_x^2} + c\sqrt{1+u_x^2}, \quad x \in (l_-(t), l_+(t)),\ t>0, \tag{1.1}$$

$$u(l_\pm(t),t) = 0, \quad t>0, \tag{1.2}$$

$$u_x(l_\pm(t),t) = \mp \tan\psi_\pm, \quad t>0, \tag{1.3}$$

$$u(x,0) = u^0(x),\ x\in[l_-^0, l_+^0], \quad l_\pm(0) = l_\pm^0, \tag{1.4}$$

where we assume that $\psi_\pm \in (0,\pi/2)$, $c>0$ and $-\infty < l_-^0 < l_+^0 < \infty$.

The problem (P) arises in the study of a curvature flow with driving force described as follows. Let $\Gamma^0$ be a smooth oriented curve in the upper half-plane whose endpoints lie on the $x$-axis with given contact angles $\psi_-$ on the left and $-\psi_+$ on the right:

$$\Gamma^0 := \{(x(\tau), y(\tau)) : 0\le\tau\le 1\}, \quad (x'(\tau), y'(\tau)) \ne (0,0) \text{ for } 0\le\tau\le 1,$$
$$y(0)=y(1)=0, \quad x'(0) = y'(0)\cot\psi_-, \quad x'(1) = -y'(1)\cot\psi_+.$$

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Here the "contact angle" refers to the angle of the tangent vector *measured from the positive $x$-axis with range* $(-\pi/2, \pi/2)$. Thus the interior angles between $\Gamma^0$ and the horizontal line are $\psi_-$ and $\psi_+$, respectively. Given such a curve $\Gamma^0$, we consider a problem of finding a family of oriented curves $\{\Gamma(t)\}_{t\geq 0}$, with $\Gamma(0) = \Gamma^0$, that evolve by the equation

$$V = \kappa + c, \tag{1.5}$$

while keeping the endpoints on the $x$-axis with the same fixed contact angles as above. Here $V$ denotes the normal velocity, $\kappa$ the (signed) curvature, and $c$ is a positive constant that represents a driving force. The signs of $V$, $\kappa$ are chosen in accordance with the orientation of the curve, in which the reference normal vector points toward the left-hand side of the tangent vector. When the curve $\Gamma(t)$ is a graph of a function $y = u(x,t)$, $x \in [l_-(t), l_+(t)]$, where $l_\pm(t)$ denote the position of the endpoints of the curve $\Gamma(t)$, this problem reduces to solving the problem (P). In this paper, we shall focus on this case. We also assume that

$$u^0(x) > 0 \ \text{ for } x \in (l_-^0, l_+^0)\ ,\ u^0(l_\pm^0) = 0,\ u_x^0(l_\pm^0) = \mp \tan\psi_\pm,\ u^0 \in C^{1+\alpha}([l_-^0, l_+^0]), \tag{1.6}$$

where $\alpha \in (0,1)$. For some technical reasons, we shall assume throughout this paper that $1/2 < \alpha < 1$. For the reader's convenience, we shall provide the proof of the local existence and uniqueness of the solution to problem (P) under (1.6) with $1/2 < \alpha < 1$ in the Appendix. As we shall see, the solution is $C^\infty$ up to the boundary.

The equation (1.5) comes from various fields. For example, it describes the motion of a superconducting vortex [14]. Also, it appears in the study of the traveling curved fronts (V-shaped waves) [35], the Belousov-Zhabotinsky reaction [6] and the Allen-Cahn model in chemistry [17]. As for the contact angle condition (1.3), it can be seen in the study of capillary drop dynamics and wetting phenomenon (cf. [18, 12, 5, 13]). We remark that our model has some similarities to the wetting phenomenon. For example, when we place a (partial wetting) liquid drop on a solid plane, we may regard $\Gamma(t)$ as the vertical slice of the free surface of the drop at time $t$. The drop can spread on a horizontal surface or slide down an inclined plane driven by surface tension. Also, shrinking drops can be observed from a wetting evaporating liquid on a smooth solid substrate. However, from the contact line dynamics, the contact angle usually changes in time (in equilibrium, the contact angle is determined by Young's law) depending on the velocity of the drop (cf. [12, 38, 29]). On the other hand, if one assumes that the dynamics is more of a global nature, then one ends up with constant contact angle condition on the free boundary, but with a thin-film equation in the wetted region (see, e.g., [36, 21]).

When $c = 0$, equation (1.5) is called the classical curve shortening flow, and there is extensive literature on this subject both for simple closed curves and non-simple ones (or hypersurfaces); we refer the reader to [4, 22, 23] and the references cited therein. As for the free boundary problem (P) with $c = 0$, it appears in the study of evolution of grain domains in polycrystals, see, e.g., [25, 26, 27, 34]. The intersection of two grain domains forms a grain boundary which is usually modeled by equation (1.5) with $c = 0$ (cf. [1, 2, 30]). For mathematically rigorous studies of problem (P) with $c = 0$, we refer to the work [9] and the references cited therein. See also [10, 24]. In particular, it is shown in [9] that the problem (P) with $c = 0$ has a unique self-similar shrinking solution and every solution $\Gamma(t)$ shrinks to a point in finite time in an asymptotically self-similar manner.

As for the problem (P) with $c > 0$, which is the subject of the present paper, one observes far richer phenomena than the case $c = 0$. The asymptotic behavior of the solution depends on the balance between the curvature and the driving force. If the curvature dominates the driving force, the curve $\Gamma(t)$ shrinks to a point in finite time, as in the case $c = 0$. On the other hand, if the driving force dominates the curvature eventually, the curve keeps expanding for all large time. It can also happen that the curvature remains in delicate balance with the driving force. In that case, $\Gamma(t)$ remains bounded and converges either to a stationary solution or to a traveling wave solution of (1.1)-(1.3). One of the main purposes of the present paper is to classify the behaviors of solutions into these three types and offer some criteria for these behaviors to occur. We also investigate detailed asymptotics of $\Gamma(t)$ for each type of behaviors.

Although our model is different from the classical wetting dynamics, this work may be useful for the study of wetting phenomena. In this paper, we only consider the case with a simply connected support. The case with multi-supports is still open (see [7] for the case of linear heat equation). It would be also very interesting to extend our work to the more general setting of anisotropic curvature flow and the case of surfaces of revolution (cf. [40, 16, 31]). We left these problems as future studies.

Now we summarize our main results. Here and in what follows, $[0, T)$ will denote the maximal time interval for the existence of a classical solution $(u, l_\pm)$ to the problem (P), where $T \in (0, \infty]$. We let $A(t)$ denote the area of the domain enclosed by $\Gamma(t)$ and the $x$-axis, and $L(t)$ the length of $\Gamma(t)$, namely,

$$A(t) := \int_{l_-(t)}^{l_+(t)} u(x,t)dx, \quad L(t) := \int_{l_-(t)}^{l_+(t)} \sqrt{1 + u_x^2(x,t)}dx. \tag{1.7}$$

Our first main result gives complete classification of the behavior of solutions:

**Theorem 1.1** (Classification)**.** *Any solution of* (P) *belongs to one of the following types:*

(A) [Expanding] $T = \infty$, *and both* $L(t)$ *and* $A(t)$ *tend to* $\infty$ *as* $t \to \infty$.

(B) [Bounded] $T = \infty$, *and both* $L(t)$ *and* $A(t)$ *remain bounded from above and below by two positive constants as* $t \to \infty$.

(C) [Shrinking] $T < \infty$, *and both* $L(t)$ *and* $A(t)$ *tend to* $0$ *as* $t \to T$.

In Section 5, we shall give some criteria for the above classification. Among other things it will be shown that if the initial data satisfies

$$A(0) > \frac{1}{\pi}\left(\frac{\psi_+ + \psi_-}{c}\right)^2,$$

then the solution is of type (A), while if

$$L(0) < \frac{2(1-\cos\psi_{\min})}{c}, \qquad \psi_{\min} := \min\{\psi_-, \psi_+\},$$

the solution is of type (C). See Propositions 5.1 and 5.2 for details.

The next results are concerned with the concavity of the solution.

**Theorem 1.2** (Preservation of concavity)**.** *Suppose that $u(x,t_0)$ is concave for some $t_0 \in [0,T)$, then it remains strictly concave for all $t \in (t_0,T)$. In particular, $u_{xx}(x,t) < 0$ for $x \in (l_-(t), l_+(t))$ for all $t \in (0,T)$, if $(u^0)_{xx} \le 0$ on $(l_-^0, l_+^0)$.*

**Theorem 1.3** (Eventual concavity in the bounded case)**.** *Let $(u, l_\pm)$ be a solution of type* (B)*. Then there exists $t^* \ge 0$ such that $u(\cdot,t)$ is strictly concave for all $t \in (t^*,\infty)$.*

**Theorem 1.4** (Eventual concavity in the shrinking case)**.** *Let $(u, l_\pm)$ be a solution of type* (C)*. Then there exists $t^* \in [0,T)$ such that $u(\cdot,t)$ is strictly concave for all $t \in (t^*,T)$.*

Our final results give more precise description of the asymptotic behavior of solutions for types (A)–(C). We first deal with type (A), the expanding case. In this case, as time passes, the effect of the curvature becomes smaller and smaller compared with the constant forcing term, so one may expect that the asymptotic behavior of the solution is well approximated by the solution of $V = c$. As we see below, this is indeed the case, and the profile of the solution approaches that of a self-similar solution of $V = c$.

To explain this result, we first note that the solution of $V = c$ is expressed by the graph of a function $y = g(x,t)$ satisfying

$$g_t = c\sqrt{1+g_x^2}. \tag{1.8}$$

Under the boundary conditions corresponding to (1.2) and (1.3), the above equation has a unique self-similar solution of the form $g(x,t) = tG(x/t)$, where $G(\zeta)$ is a function that is defined on some interval $\hat{p} \le \zeta \le \hat{q}$ and satisfies

$$G(\zeta) > 0, \qquad G(\zeta) - \zeta G'(\zeta) = c\sqrt{1+(G')^2(\zeta)}, \quad \hat{p} < \zeta < \hat{q}, \tag{1.9}$$

along with the boundary conditions

$$G(\hat{p}) = G(\hat{q}) = 0, \quad G'(\hat{p}) = \tan\psi_-, \quad G'(\hat{q}) = -\tan\psi_+. \tag{1.10}$$

The constants $\hat{p}, \hat{q} \in \mathbb{R}$ with $\hat{p} < \hat{q}$ are determined uniquely by the condition

$$-\hat{p}\sin\psi_- = \hat{q}\sin\psi_+ = c. \tag{1.11}$$

For any given $\psi_\pm \in (0,\pi/2)$ and $c>0$, the problem (1.9)-(1.10) is solvable if and only if (1.11) holds, and the solution is given by

$$G(\zeta)=G(\zeta;\psi_\pm):=\begin{cases}(\tan\psi_-)(\zeta-\hat{p}), & \hat{p}\le\zeta\le -c\,\sin\psi_-,\\ \sqrt{c^2-\zeta^2}, & -c\,\sin\psi_-\le\zeta\le c\,\sin\psi_+,\\ -(\tan\psi_+)(\zeta-\hat{q}), & c\,\sin\psi_+\le\zeta\le\hat{q}\end{cases}\tag{1.12}$$

with $\hat{p},\hat{q}$ as in (1.11). From (1.11) one sees that $\hat{p}<-c<0<c<\hat{q}$. Geometrically, the graph of $G$ consists of a part of the circle and two line segments in the upper half plane. In the expression (1.13) below, we understand that $G=0$ outside the interval $[\hat{p},\hat{q}]$.

**Theorem 1.5** (Asymptotics for the expanding case)**.** *Let* $(u,l_\pm)$ *be a type* (A) *solution of* (1.1)-(1.4) *and let* $G=G(\zeta;\psi_\pm)$ *be defined by* (1.12). *Then there exist a function* $\rho(t)$ *satisfying* $\lim_{t\to\infty}[\rho(t)/t]=1$ *and a constant* $t_0>0$ *such that*

$$\rho(t)G\Big(\frac{x}{\rho(t)}\Big)\le u(x,t)\le (t+t_0)G\Big(\frac{x}{t+t_0}\Big)\tag{1.13}$$

*for all* $x\in[l_-(t),l_+(t)]$ *and* $t>0$. *Consequently*

$$\lim_{t\to\infty}\frac{\pm l_\pm(t)}{t}=\frac{c}{\sin\psi_\pm},\tag{1.14}$$

$$\lim_{t\to\infty}\frac{u(x,t)}{tG(x/t)}=1 \quad \textit{uniformly on compact subsets of } \mathbb{R}.\tag{1.15}$$

Next, in the case of type (B), one may expect that the solution converges to a stationary solution if a stationary solution exists. Indeed, when $\psi_+=\psi_-$, the problem (P) admits a stationary solution whose shape is a portion of a circle with radius $1/c$. However, in the case $\psi_+\ne\psi_-$, there is no positive stationary solution. As we see from the theorem below, the solution converges to a traveling wave in this case.

A traveling wave of the problem (P) is a solution that has the form $u(x,t)=\Phi(x-\nu t-a)$, where $\nu$ denotes the wave speed, $\Phi(\xi)$ is a function that defines the profile of the wave and $a$ is an arbitrary constant that adjusts the phase. Substituting this form into (1.1)-(1.3) yields the following, where $\beta>0$ is some constant and $\Phi$ is normalized in such a way that the center of its support goes to the origin:

$$\begin{cases}\dfrac{\Phi_{\xi\xi}}{1+\Phi_\xi^2}+\nu\Phi_\xi+c\sqrt{1+\Phi_\xi^2}=0 \quad \text{in } (-\beta,\beta),\\ \Phi(\pm\beta)=0, \quad \Phi_\xi(\pm\beta)=\mp\tan\psi_\pm.\end{cases}\tag{1.16}$$

Multiplying (1.16) by $\Phi_\xi/\sqrt{1+\Phi_\xi^2}$ and integrating it over $[-\beta,\beta]$, we easily see that $\nu>0$ (resp. $=0$, $<0$) if and only if $\psi_- - \psi_+>0$ (resp. $=0$, $<0$); see Proposition 2.9 for a different proof. Thus, if $\psi_-=\psi_+=:\psi$, we have $\nu=0$, in which case $\Phi$ is a stationary solution. We shall distinguish this case by using the notation $\phi$ instead of $\Phi$:

$$\begin{cases} \dfrac{\phi_{xx}}{1+\phi_x^2} + c\sqrt{1+\phi_x^2} = 0 \quad \text{in } (-\beta,\beta), \\ \phi(\pm\beta) = 0, \quad \phi_x(\pm\beta) = \mp\tan\psi. \end{cases}$$

The solution $\phi$ represents a portion of a circle with radius $1/c$. In the following theorem, we understand that $\Phi = 0$ outside the interval $[-\beta,\beta]$.

**Theorem 1.6** (Asymptotics for the bounded case)**.** (i) *There exist unique constants $\beta > 0$ and $\nu \in \mathbb{R}$ and a unique function $\Phi(\xi)$ that satisfy* (1.16)*. Furthermore, the sign of $\nu$ coincides with the sign of $\psi_- - \psi_+$.*

(ii) *Let $(u, l_\pm)$ be a solution of type* (B)*. Then there exists a constant $a \in \mathbb{R}$ such that $u(x,t) \to \Phi(x - \nu t - a)$ uniformly for $x \in [l_-(t), l_+(t)]$, and that $l_\pm(t) - \nu t \to a \pm \beta$ as $t \to +\infty$, where $\Phi, \nu, \beta$ are as in* (1.16)*. In the special case where $\psi_- = \psi_+ =: \psi$, this means that $u$ converges to a stationary solution $\phi(x-a)$ as $t \to +\infty$.*

From the physical point of view, the condition $\psi_+ \neq \psi_-$ means that the surface tension on the floor that pulls the curve is different between the left and right endpoints, since the contact angle is determined by the relation between the surface tension on the floor and that on the curve. This explains intuitively why a traveling wave appears when $\psi_+ \neq \psi_-$.

Lastly, as for type (C), we shall show that the curve $\Gamma(t)$ shrinks to a point as $t \to T$ in a self-similar manner. In this case, as $t$ approaches $T$, the solution behaves like a solution of $V = \kappa$ under the same boundary conditions. In what follows, by spatial translation, we may assume without loss of generality that $x = 0$ is the limit point of the shrinking curve. As in [9], we introduce the following similarity transformation:

$$z = \frac{x}{\sqrt{2(T-t)}}, \quad s = -\frac{1}{2}\ln(T-t)\,,$$
$$u(x,t) = \sqrt{2(T-t)}\,w(z,s), \quad l_-(t) = \sqrt{2(T-t)}\,p(s), \quad l_+(t) = \sqrt{2(T-t)}\,q(s).$$

Then $u$ satisfies (1.1)-(1.4) if and only if $w$ satisfies

$$w_s = \frac{w_{zz}}{1+w_z^2} - zw_z + w + \sqrt{2}ce^{-s}\sqrt{1+w_z^2}, \quad z \in (p(s), q(s)),\ s > s_0 \tag{1.17}$$

$$w(p(s),s) = w(q(s),s) = 0, \quad s > s_0, \tag{1.18}$$

$$w_z(p(s),s) = \tan\psi_-, \quad w_z(q(s),s) = -\tan\psi_+, \quad s > s_0, \tag{1.19}$$

$$w(z,s_0) = w_0(z) := (2T)^{-1/2}u^0(z\sqrt{2T}\,), \quad z \in [l_-^0/\sqrt{2T}, l_+^0/\sqrt{2T}\,], \tag{1.20}$$

where $s_0 := -\frac{1}{2}\ln T$. If $c = 0$, the equation (1.17) is autonomous, and the stationary problem for (1.17)-(1.19) is given in the form

$$\frac{\varphi_{zz}}{1+(\varphi_z)^2} - z\varphi_z + \varphi = 0, \quad z \in (\bar{p}, \bar{q}), \tag{1.21}$$

$$\varphi(\bar{p}) = \varphi(\bar{q}) = 0, \tag{1.22}$$

$$\varphi_z(\bar{p}) = \tan\psi_-, \quad \varphi_z(\bar{q}) = -\tan\psi_+ \tag{1.23}$$

for some $\bar{p} < \bar{q}$. It is shown in [9] that, for any $\psi_\pm \in (0, \pi/2)$, the problem (1.21)– (1.23) has a unique solution $\varphi(z)$; the constants $\bar{p}, \bar{q}$ are also uniquely determined by $\psi_\pm$. Note that, when $c \neq 0$, the equation (1.17) is no longer autonomous but is asymptotically autonomous as $s \to +\infty$.

The following theorem gives asymptotics for type (C) in terms of the rescaled solution $w$. Here we understand $\varphi = 0$ outside the interval $(\bar{p}, \bar{q})$.

**Theorem 1.7** (Asymptotics for the shrinking case)**.** *Let* $(u, l_\pm)$ *be a solution of type* (C) *and* $(w, p, q)$ *be the corresponding solution of* (1.17)-(1.20)*. Then* $(w(z,s), p(s), q(s))$ *converges to the unique solution* $(\varphi(z), \bar{p}, \bar{q})$ *of* (1.21)-(1.23) *as* $s \to +\infty$ *uniformly for* $z \in [p(s), q(s)]$.

The solution of (1.21)-(1.23) corresponds to a self-similar shrinking solution of the form

$$u(x,t) = \sqrt{2(T-t)}\,\varphi\Big(\frac{x}{\sqrt{2(T-t)}}\Big)$$

to the problem ($\mathrm{P}_0$), that is, the problem (P) with $c = 0$. As mentioned above, the existence and uniqueness of such $\varphi$ has been established in [9]. Theorem 1.7 asserts that any shrinking solution of (P) behaves like the unique self-similar solution of ($\mathrm{P}_0$) as $t$ approaches $T$. Intuitively this sounds reasonable as the curvature tends to infinity in the case (C) and therefore the driving force $c$ should become negligible.

One of the main tools in this paper is the intersection number argument. However, as we are dealing with a free boundary problem in which the endpoints of the curve can slide freely along the $x$-axis, the standard intersection number principle (cf. [3, 33]) dose not work. Indeed, the number of intersections between two solutions may increase in time. To overcome this difficulty, we introduce the notion of *extended intersection number* by extending the solutions linearly below the $x$-axis outside their domain of definition, and counting the number of intersections between the extended solutions. It turns out that this extended intersection number does not increase in time; moreover, it drops strictly each time a multiple zero occurs (see §2). We call this property the *extended intersection number principle*, which turns out to be exceedingly useful in analyzing the problem (P).

The rest of this paper is organized as follows. In Section 2, we provide some preliminaries, which include some a priori estimates, proof of the extended intersection number principle, and formulation of the curvature equation for a strictly concave solution. Section 3 is devoted to the classification of solutions, while Section 4 deals with concavity properties of solutions. In Section 5, we provide some criteria for determining the types of solutions (among (A), (B), (C)) from their initial data. We shall also prove a result concerning the sharp transition between types (A) and (C) (Theorem 5.1).

In Section 6, we study more detailed asymptotics for each of the types (A)–(C) and prove Theorems 1.5, 1.6 and 1.7. The methods for proving these three theorems are all different. For the expanding case, we use the method of super-sub-solutions. For the area bounded case, we apply the extended intersection number principle to show the convergence of the solution to a traveling wave or a steady-state. It may sound somewhat surprising that such a convergence result follows simply by counting the intersection numbers, without constructing a Lyapunov functional. As for the shrinking case, the proof of Theorem 1.7 goes in two steps: We first show that the aspect ratio of the curve remains bounded (Proposition 6.2) by using an idea similar to Grayson [23]. The boundedness of the aspect ratio implies that the rescaled solution $w$ possesses certain compactness properties. We then use a Lyapunov functional borrowed from [28]. Here, the Lyapunov functional is not necessarily decreasing, partly because of the presence of the free boundary, partly because (1.17) is non-autonomous since $c \neq 0$. However, since the perturbation term decays exponentially as $s \to +\infty$, it creates no problem in proving the convergence.

Finally, Section 7 is an appendix, where we prove the local existence and uniqueness of the solution to the problem (P) and also prove the continuous dependence of solutions on the initial data.

## 2. Preliminaries

In this section, we provide some preliminaries for later purposes. Before going into specific topics, we remark that the existence of the local-in-time solution of (P), along with its regularity, is discussed in Theorem 7.1 in Appendix. In particular, the solution $(u(x,t), l_\pm(t))$ is $C^\infty$ in $x, t$ for $t > 0$.

### 2.1. **Some a priori estimates.**

Let $[0, T)$ be the maximal time interval for the existence of a solution $u$ to (P). In this subsection, we shall use the idea of [24] (or [9]) to derive some a priori bounds for $u_x$, $u_t$ and $u_{xx}$, respectively. We first introduce some notation:

$$
\begin{aligned}
&Q_\tau := \{(x,t) \mid x \in (l_-(t), l_+(t)),\ t \in (0,\tau)\}, \quad \text{for} \ \ \tau \in (0,T],\\
&\psi_{\max} := \max\{\psi_+, \psi_-\}, \quad \psi_{\min} := \min\{\psi_+, \psi_-\},\\
&M_1 := \|u_x^0\|_{L^\infty[l_-^0, l_+^0]},\\
&M_2 := \max\left\{ \frac{c}{\cos\psi_{\max}},\ \left\| \frac{u_{xx}^0}{1+(u_x^0)^2} + c\sqrt{1+(u_x^0)^2} \right\|_{L^\infty[l_-^0, l_+^0]} \right\},\\
&M_3 := M_2(1+M_1^2) - c,\\
&M_4 := -\min_{x \in [l_-^0, l_+^0]} u_{xx}^0 > 0.
\end{aligned}
$$

The following lemma is an easy consequence of the maximum principle.

**Lemma 2.1.** $|u_x(x,t)| \le M_1,\ \ u_t(x,t) \le M_2,\ \ u_{xx}(x,t) \le M_3$ *for all* $(x,t) \in Q_T$.

*Proof.* First, let $v := u_x$. Then, by differentiating (1.1) in $x$, we obtain

$$
v_t = \frac{1}{1+u_x^2} v_{xx} + \left[ -\frac{2u_x u_{xx}}{(1+u_x^2)^2} + \frac{c u_x}{\sqrt{1+u_x^2}} \right] v_x. \tag{2.1}
$$

Since $-M_1 \le v \le M_1$ on the parabolic boundary of $Q_T$, it follows from the maximum principle that $|v| \le M_1$ in $Q_T$.

Next, set $v := u_t$. Differentiating (1.1) in $t$ yields the same equation (2.1) for this $v$. One can also deduce the following boundary conditions from (1.1), (1.2) and (1.3):

$$
v_x(l_\pm(t), t) = \mp \frac{2}{\sin 2\psi_\pm} \left[ v(l_\pm(t), t) - \frac{c}{\cos\psi_\pm} \right] v(l_\pm(t), t).
$$

Hence any constant larger than $c/\cos\psi_{\max}$ is a super-solution of (2.1), which gives the bound $u_t \le \max\{c/\cos\psi_{\max},\ \|u_t(\cdot,0)\|_{L^\infty}\}$. Finally, the upper bound of $u_{xx}$ follows from (1.1) and the upper bound of $u_t$. This completes the proof of the lemma. □

Next we provide lower bounds for $u_t$ and $u_{xx}$. We first introduce the following notation:

$$
\begin{aligned}
\eta_-(t) &:= \min\{x \in (l_-(t), l_+(t)) \mid u_x(x,t) = 0\},\\
\eta_+(t) &:= \max\{x \in (l_-(t), l_+(t)) \mid u_x(x,t) = 0\}.
\end{aligned}
$$

Note that the functions $\eta_\pm(t)$ are not necessarily continuous. Indeed, $\eta_-(t)$ (resp. $\eta_+(t)$) has a positive (resp. negative) jump each time the left-most (resp. right-most) local maximum of $u(x,t)$ disappears. However, by the result of [3], such a situation can occur at most at discrete time moments, since $v = u_x$ satisfies the equation (2.1).

**Lemma 2.2.**

$$u_t(x,t) \geq C_1 \min\Big\{ -\frac{\psi_{\max}}{d(t)}, -K\Big\} + c, \quad 0 \leq t < T,\ l_-(t) \leq x \leq l_+(t), \tag{2.2}$$

$$u_{xx}(x,t) \geq C_2 \min\Big\{ -\frac{\psi_{\max}}{d(t)}, -K\Big\}, \qquad 0 \leq t < T,\ l_-(t) \leq x \leq l_+(t), \tag{2.3}$$

*where* $C_1 = \tan\psi_{\max}$, $K = 4M_4/\psi_{\min}$, $C_2 = (1+M_1^2)C_1$ *and*

$$d(t) := \min_{\tau\in[0,t]} \min\big\{u(\eta_-(\tau),\tau), u(\eta_+(\tau),\tau)\big\}.$$

*Proof.* Let $(u, l_\pm)$ be a solution of (1.1)-(1.4). Define

$$J(x,t) := K_-(t)u(x,t) + \arctan u_x(x,t) - \psi_-,$$

where $K_-(t)$ is a positive non-decreasing function that will be specified later. We define a differential operator $\mathcal{A}$ by

$$\mathcal{A}U := \frac{1}{1+u_x^2}U_{xx} + \frac{cu_x}{\sqrt{1+u_x^2}}U_x.$$

Then we have

$$\Big(\frac{\partial}{\partial t} - \mathcal{A}\Big)\arctan u_x = \frac{u_{xt}}{1+u_x^2} - \frac{1}{1+u_x^2}\Big(\frac{u_{xx}}{1+u_x^2}\Big)_x - \frac{cu_x}{\sqrt{1+u_x^2}}\frac{u_{xx}}{1+u_x^2} = 0,$$
$$\Big(\frac{\partial}{\partial t} - \mathcal{A}\Big)u = u_t - \frac{u_{xx}}{1+u_x^2} - \frac{cu_x^2}{\sqrt{1+u_x^2}} = \frac{c}{\sqrt{1+u_x^2}} > 0.$$

It follows that

$$J_t - \mathcal{A}J = K_-'(t)u + \frac{cK_-(t)}{\sqrt{1+u_x^2}} > 0, \tag{2.4}$$

where the derivative $K_-'(t)$ is to be understood in a generalized sense. Now we set

$$K_-(t) := \max\Big\{\frac{\psi_-}{d(t)}, \frac{4M_4}{\psi_-}\Big\}, \tag{2.5}$$

where $M_4$ is the constant defined above. Then, since $d(t)$ is non-increasing, $K_-(t)$ is non-decreasing; hence (2.4) holds on $Q_T$. Furthermore, since $u(\eta_-(t),t) \geq d(t)$, we have

$$J(\eta_-(t),t) \geq \frac{\psi_-}{d(t)}u(\eta_-(t),t) - \psi_- \geq 0, \quad 0 \leq t < T. \tag{2.6}$$

Note also that

$$J(l_-(t),t) = 0, \qquad 0 \leq t < T, \tag{2.7}$$

$$J(x,0) \geq \frac{4M_4}{\psi_-}u^0(x) + \arctan u_x^0(x) - \psi_- \geq 0, \quad x \in [l_-^0, \eta_-(0)], \tag{2.8}$$

the latter being a consequence of Lemma 2.3 below. If $\eta_-(t)$ is continuous, then (2.6), (2.7) and (2.8) imply that $J \geq 0$ on the parabolic boundary of the domain $Q_T^- := \{x \in (l_-(t), \eta_-(t)),\ t \in (0,T)\}$. This, together with (2.4) and the maximum principle, yields

$$J(x,t) \geq 0 \quad \text{for } 0 \leq t < T \text{ and } x \in [l_-(t), \eta_-(t)] \tag{2.9}$$

provided that $\eta_-(t)$ is continuous. If $\eta_-(t)$ is not continuous, then, as mentioned above, discontinuity may occur only at discrete time moments. Suppose $\eta_-(t)$ is discontinuous at $t = t_0$. Then the horizontal line segment $[\eta_-(t_0), \eta_-(t_0+0)] \times \{t_0\}$ forms part of the parabolic boundary of $Q_T^-$. Since $u_x \geq 0$ on this line segment, we have $J(x,t_0) \geq J(\eta_-(t_0), t_0) \geq 0$ for $x \in [\eta_-(t_0), \eta_-(t_0+0)]$. Thus, even if $\eta_-(t)$ is not continuous, $J \geq 0$ on the entire parabolic boundary of $Q_T^-$, which establishes (2.9).

Now, by (2.7) and (2.9), we have $J_x(l_-(t), t) \geq 0$ for $0 \leq t < T$, which implies

$$\frac{u_{xx}}{1+u_x^2} \geq -K_-(t) \tan \psi_-, \quad 0 \leq t < T, \tag{2.10}$$

where $K_-(t)$ is given in (2.5). Similarly, replacing $J$ by the function

$$K_+(t)u - \arctan u_x - \psi_+, \qquad \text{where} \ \ K_+(t) := \max\left\{\frac{\psi_+}{d(t)}, \frac{4M_4}{\psi_+}\right\},$$

and arguing as above in the region $\eta_+(t) < x < l_+(t)$, we obtain

$$\frac{u_{xx}}{1+u_x^2} \geq -K_+(t) \tan \psi_+, \quad 0 \leq t < T. \tag{2.11}$$

Now we define $w := u_{xx}/(1+u_x^2)$. Then a direct calculation shows

$$\begin{aligned} w_t &= \frac{1}{1+u_x^2} w_{xx} + \Big[ -\frac{2u_x u_{xx}}{(1+u_x^2)^2} + \frac{c u_x}{\sqrt{1+u_x^2}} \Big] w_x + \frac{c}{\sqrt{1+u_x^2}} w^2 \\ &\geq \frac{1}{1+u_x^2} w_{xx} + \Big[ -\frac{2u_x u_{xx}}{(1+u_x^2)^2} + \frac{c u_x}{\sqrt{1+u_x^2}} \Big] w_x. \end{aligned}$$

Note also that $w(x,0) \geq \min_x u_{xx}(x,0) = -M_4$. Combining these with (2.10) and (2.11), we see, by the maximum principle, that

$$w(x,t) \geq \min\left\{ -\frac{\psi_{\max}}{d(t)}, -\frac{4M_4}{\psi_{\min}} \right\} \tan \psi_{\max}.$$

The estimates (2.2) and (2.3) then follow immediately from (1.1) and the estimate $|u_x| \leq M_1$. The proof of Lemma 2.2 is complete. □

**Lemma 2.3.** *Let $w(x)$ be a $C^2$ function defined on an interval $[0,r]$ satisfying*

$$w(0) = 0, \quad w'(0) = \tan\theta, \quad w'(x) > 0 \ (x \in [0,r)\,), \quad w'(r) = 0$$

*for some $\theta \in (0, \pi/2)$. Then*

$$\sup_{0<x\leq r} \frac{\theta - \arctan w'(x)}{w(x)} \leq -\frac{4}{\theta} \min_{0\leq x\leq r} w''(x).$$

*Proof.* Let $r_1 := \min\{x \in (0, r) \mid \arctan w'(x) = \theta/2\}$. Then, for $x \in [r_1, r]$,

$$\frac{\theta - \arctan w'(x)}{w(x)} \leq \frac{\theta}{w(r_1)} = 2\,\frac{\theta - \arctan w'(r_1)}{w(r_1)}.$$

On the other hand, for $x \in (0, r_1]$, we have

$$\frac{\theta - \arctan w'(x)}{w(x)} \leq \frac{-\int_0^x \frac{w''}{1+(w')^2}dx}{(\tan\frac{\theta}{2})x} \leq -\frac{2}{\theta}\min_{0\leq z\leq r} w''(z).$$

This proves the lemma. □

2.2. **Intersection number.** One of the principal tools in our analysis is the intersection number argument. The idea is to obtain information about the behavior of solutions by counting the number of intersections between two solutions. For this method to work, the number of intersections should not increase as time passes. However, as the endpoints of the curve can slide freely in problem (P), the number of intersections may increase. In order to overcome this difficulty, we introduce what we call *the extended intersection number principle*, which we shall explain below.

Given a continuous function $f : I \to \mathbb{R}$ defined on some interval $I \subset \mathbb{R}$, by the *zero number* of $f$ on $I$ we mean the number of sign changes of $f$ on $I$; that is, the supremum of all integers $N_m \geq 1$ such that there exists an increasing sequence $x_0 < x_1 < \cdots < x_{N_m}$ in $I$ satisfying

$$f(x_k) \cdot f(x_{k+1}) < 0, \quad \text{for } k = 0, \ldots, N_m - 1.$$

We denote this number by $\mathcal{Z}_I[f]$. If there is no such sequence $\{x_k\}_{k=0}^{N_m}$, we set $\mathcal{Z}_I[f] = 0$. This happens, for instance, if $f$ has a constant sign or is identically equal to 0.

In order to define the extended intersection number, we assume that $f^1$ and $f^2$ are both $C^1$ functions defined on closed intervals $I_1, I_2$, respectively. We then extend each $f^i$ linearly outside $I_i$ to form a $C^1$ function on $\mathbb{R}$ and denote this unique extension by $f^i_*$, $i = 1, 2$. Then we define *the extended intersection number* between $f^1$ and $f^2$ by

$$\mathcal{Z}_*[f^1, f^2] := \mathcal{Z}_{\mathbb{R}}[f^1_* - f^2_*]. \tag{2.12}$$

The goal of this subsection is to prove that the extended intersection number of two solutions of the problem (P), or its generalized version, does not increase in time.

To be more precise, let us consider the following generalized version of problem (P), which we shall call (Q):

$$w_t = \frac{w_{xx}}{1 + w_x^2} + c\sqrt{1 + w_x^2}, \quad x \in (\sigma_-(t), \sigma_+(t)), \quad t > 0, \tag{2.13}$$

$$w(\sigma_\pm(t), t) = 0, \quad t > 0, \tag{2.14}$$

$$w_x(\sigma_\pm(t), t) = \mp \tan\theta_\pm(t), \quad t > 0, \tag{2.15}$$

$$w(x, 0) = w^0(x), \quad x \in [\sigma^0_-, \sigma^0_+], \qquad \sigma_\pm(0) = \sigma^0_\pm, \tag{2.16}$$

where $\theta_\pm(t)$ are given positive smooth functions with values in $(0,\pi/2)$. The problem (P) is a special case of the above problem where $\theta_\pm(t) \equiv \psi_\pm$. Then the above-mentioned linear extension of $w$, which we denote by $w_*$, is given by

$$w_*(x,t) = \begin{cases} (\tan\theta_-(t))(x-\sigma_-(t)), & x \in (-\infty, \sigma_-(t)), \\ w(x,t), & x \in [\sigma_-(t), \sigma_+(t)], \\ -(\tan\theta_+(t))(x-\sigma_+(t)), & x \in (\sigma_+(t), \infty). \end{cases} \tag{2.17}$$

We shall also consider the problem in which one of the two free boundaries in the problem (Q) is not present. If the left free boundary is missing, the solution is defined on the interval $-\infty < x \le \sigma_+(t)$. We denote this problem by ($\mathrm{Q}^+$). The other case, where the solution is defined on the interval $\sigma_-(t) \le x < \infty$ will be denoted by ($\mathrm{Q}^-$). A particularly important class of solutions of ($\mathrm{Q}^-$) is an upper portion of the following straight line:

$$y = w^-(x,t) := (\tan\theta_-)(x-\sigma_-(t)), \quad \sigma_-(t) := \sigma_0 - ct/\sin\theta_-, \tag{2.18}$$

where $\theta_- \in (0,\pi/2)$ and $\sigma_0 \in \mathbb{R}$ are constants. Similarly, an upper portion of the following straight line forms an important class of solutions of ($\mathrm{Q}^+$):

$$y = w^+(x,t) := -(\tan\theta_+)(x-\sigma_+(t)), \quad \sigma_+(t) := \sigma_0 + ct/\sin\theta_+, \tag{2.19}$$

where $\theta_+ \in (0,\pi/2)$ and $\sigma_0$ are constants. In both cases, their extended solutions $w_*^\pm$ represent entire lines that move upward with normal speed $c$.

We are now ready to state the extended intersection number principle.

**Proposition 2.4** (Extended intersection-number principle)**.** *Let $(w^i, \sigma^i_\pm)$, $i = 1, 2$, be solutions of* (Q) *for $\theta_\pm(t) = \theta^i_\pm(t)$ defined on some time interval $0 \le t < T_1$. Assume that:*

$$\text{(a)}\ \ \theta^1_-(t) \ne \theta^2_-(t)\ \ \forall\, t \in [0,T_1) \quad \textit{or} \quad \text{(b)}\ \ \theta^1_-(t) \equiv \theta^2_-(t) \ \ \textit{on}\ \ [0,T_1),$$

*and also that*

$$\text{(a}'\text{)}\ \ \theta^1_+(t) \ne \theta^2_+(t)\ \ \forall\, t \in [0,T_1) \quad \textit{or} \quad \text{(b}'\text{)}\ \ \theta^1_+(t) \equiv \theta^2_+(t) \ \ \textit{on}\ \ [0,T_1).$$

*Then the following holds:*

(i) *$\mathcal{Z}_*[w^1(\cdot,t), w^2(\cdot,t)]$ is non-increasing in $t \in [0,T_1)$ and is finite for each $t \in (0,T_1)$.*

(ii) *If, for some $t_0 \in (0,T_1)$, the curves $y = w^1(x,t_0)$ and $y = w^2(x,t_0)$ become tangential at an interior point $x_0 \in (\sigma^1_-(t_0), \sigma^1_+(t_0)) \cap (\sigma^2_-(t_0), \sigma^2_+(t_0))$ and if $w^1(x,t_0) \not\equiv w^2(x,t_0)$, then $\mathcal{Z}_*[w^1(\cdot,t), w^2(\cdot,t)]$ drops at $t = t_0$ at least by 2.*

(iii) *Let* (b) *or* (b′) *above hold. If, for some $t_0 \in (0,T_1)$, the curves $y = w^1(x,t_0)$ and $y = w^2(x,t_0)$ become tangential at one of their endpoints and if $w^1(x,t_0) \not\equiv w^2(x,t_0)$, then $\mathcal{Z}_*[w^1(\cdot,t), w^2(\cdot,t)]$ drops at $t = t_0$ at least by 1.*

*The same conclusion holds if $w^1$ or $w^2$ is a solution of* ($\mathrm{Q}^-$) *or* ($\mathrm{Q}^+$) *instead of* (Q).

*Proof.* We shall only consider the case where both $w^1$ and $w^2$ are solutions of (Q). The case where one of them is a solution of ($Q^-$) or ($Q^+$) is basically the same and is easier.

In order to prove the assertion (i), it suffices to show that, for every $t_1 \in (0, T_1)$ there exists an $\varepsilon > 0$ such that $\mathcal{Z}_*[w^1(\cdot,t), w^2(\cdot,t)]$ is finite and non-increasing on the interval $t_1 - \varepsilon < t < t_1 + \varepsilon$. Let $\gamma^1(t), \gamma^2(t)$ denote the curves $y = w^i(x,t)$ $(i = 1, 2)$, respectively, and $\gamma^1_*(t), \gamma^2_*(t)$ the extended curves. We consider the following three cases separately:

- (Case 1) The four endpoints $\sigma^i_\pm(t)$ $(i = 1, 2)$ are all different at $t = t_1$.
- (Case 2) Some of the endpoints coincide at $t = t_1$, but $\gamma^1(t_1)$ and $\gamma^2(t_1)$ are not tangential at any of the common endpoints.
- (Case 3) $\gamma^1(t_1)$ and $\gamma^2(t_1)$ are tangential at some common endpoint.

We first consider (Case 1). Since $\sigma^i_\pm(t)$ $(i = 1, 2)$ are continuous in $t$, there exists $\varepsilon > 0$ such that these four endpoints are all different for every $t \in [t_1 - \varepsilon, t_1 + \varepsilon]$. It follows that

$$\mathcal{Z}_*[w^1(\cdot,t), w^2(\cdot,t)] = \mathcal{Z}_{I(t)}[w^1(\cdot,t) - w^2(\cdot,t)] + h_*(t) \quad \text{for } \forall t \in [t_1 - \varepsilon, t_1 + \varepsilon], \tag{2.20}$$

where $I(t) = [\sigma^1_-(t), \sigma^1_+(t)] \cap [\sigma^2_-(t), \sigma^2_+(t)]$ and $h_*(t)$ denotes the number of intersections between the extended portion of the curves $\gamma^1_*(t), \gamma^2_*(t)$, which consists of two pairs of half lines below the $x$-axis with slopes $\mp \tan \theta^i_\pm(t)$ $(i = 1, 2)$. By the assumption, $\theta^1_-(t) - \theta^2_-(t)$ is either never zero or identically equal to zero, and the same holds for $\theta^1_+(t) - \theta^2_+(t)$. In view of this and the fact that the four endpoints remain different from one another, one easily sees that $h_*(t)$ is independent of $t \in [t_1 - \varepsilon, t_1 + \varepsilon]$. On the other hand, note that $w^1(x,t) - w^2(x,t)$ never vanishes on the boundary of $I(t)$ for any $t \in [t_1 - \varepsilon, t_1 + \varepsilon]$ and $w^1(x,t) - w^2(x,t)$ satisfies a linear parabolic equation (by the help of the mean value theorem). Then the result in [3] shows that $\mathcal{Z}_{I(t)}[w^1(\cdot,t) - w^2(\cdot,t)]$ is non-increasing in $t \in [t_1 - \varepsilon, t_1 + \varepsilon]$ and is finite for $t \in (t_1 - \varepsilon, t_1 + \varepsilon]$. Consequently $\mathcal{Z}_*[w^1(\cdot,t), w^2(\cdot,t)]$ is finite and non-increasing in $t \in (t_1 - \varepsilon, t_1 + \varepsilon]$.

In (Case 2), we only consider the case where $x_1 := \sigma^1_-(t_1) = \sigma^2_-(t_1)$ and $\sigma^1_+(t_1) \neq \sigma^2_+(t_1)$, as other cases can be treated similarly. Set $W(x,t) := w^1_*(x,t) - w^2_*(x,t)$. By the assumption, the curves $\gamma^1(t_1)$ and $\gamma^2(t_1)$ are not tangential at the common endpoint $x_1$. Therefore we have $W(x_1, t_1) = 0$ and $W_x(x_1, t_1) = \tan \theta^1_-(t_1) - \tan \theta^2_-(t_1) \neq 0$. Without loss of generality we may assume $W_x(x_1, t_1) > 0$. Then, by the assumption, we have $\theta^1_-(t) - \theta^2_-(t) > 0$ for all $t \in [0, T_1)$ and there exist positive constants $\delta, \varepsilon$ such that

$$\begin{aligned}
&w^1_*(x_1 - \delta, t) < w^2_*(x_1 - \delta, t) < 0 \quad \text{for } t \in [t_1 - \varepsilon, t_1 + \varepsilon],\\
&w^1_*(x_1 + \delta, t) > w^2_*(x_1 + \delta, t) > 0 \quad \text{for } t \in [t_1 - \varepsilon, t_1 + \varepsilon],\\
&W_x(x,t) > 0 \quad \text{for } x \in [x_1 - \delta, x_1 + \delta], \quad \text{for } t \in [t_1 - \varepsilon, t_1 + \varepsilon],\\
&\qquad \sigma^1_+(t) \neq \sigma^2_+(t) \quad \text{for } t \in [t_1 - \varepsilon, t_1 + \varepsilon].
\end{aligned}$$

Since $w_*^1(x,t)$, $w_*^2(x,t)$ are straight lines in the region $x \le x_1 - \delta$ with slopes $\tan\theta_-^1(t)$ and $\tan\theta_-^2(t)$, respectively, we see that $W(x,t)$ never vanishes for $x \in (-\infty, x_1-\delta]$, while it vanishes precisely once in $[x_1-\delta, x_1+\delta]$ for each $t \in [t_1-\varepsilon, t_1+\varepsilon]$. It follows that

$$\mathcal{Z}_*[w^1(\cdot,t), w^2(\cdot,t)] = 1 + \mathcal{Z}_{I_1(t)}[w^1(\cdot,t) - w^2(\cdot,t)] + h_*^+(t) \quad \text{for } \forall t \in [t_1-\varepsilon, t_1+\varepsilon], \tag{2.21}$$

where $I_1(t) = [x_1+\delta, \min\{\sigma_+^1(t), \sigma_+^2(t)\}]$ and $h_*^+(t)$ denotes the number of intersections between the extended portion of the curves $\gamma_*^1(t), \gamma_*^2(t)$ that lie on the right-hand side of $\sigma_+^i(t)$ $(i=1,2)$. As in (Case 1), $h_*^+(t)$ remains constant and $\mathcal{Z}_{I_1(t)}[w^1(\cdot,t) - w^2(\cdot,t)]$ is non-increasing and finite for $t \in (t_1-\varepsilon, t_1+\varepsilon]$. Consequently, $\mathcal{Z}_*[w^1(\cdot,t), w^2(\cdot,t)]$ is non-increasing and finite for $t \in (t_1-\varepsilon, t_1+\varepsilon]$.

Next we consider (Case 3). In this case we have (b) or (b′) (or both). Without loss of generality, we may assume (b) and that $\sigma_-^1(t_1) = \sigma_-^2(t_1)$. For simplicity, we also assume $\sigma_+^1(t_1) \neq \sigma_+^2(t_1)$, as the case $\sigma_+^1(t_1) = \sigma_+^2(t_1)$ can be treated with minor modification of the argument. Then, by Lemma 2.5 below, there exist positive constants $\delta, \varepsilon$ such that

$$\begin{aligned} \mathcal{Z}_*[w^1(\cdot,t), w^2(\cdot,t)] &= \mathcal{Z}_{[x_1-\delta, x_1+\delta]}[w_*^1(\cdot,t) - w_*^2(\cdot,t)] \\ &\quad + \mathcal{Z}_{I_1(t)}[w^1(\cdot,t) - w^2(\cdot,t)] + h_*^+(t) \quad \text{for } \forall t \in [t_1-\varepsilon, t_1+\varepsilon], \end{aligned} \tag{2.22}$$

where $I_1(t), h_*^+(t)$ are as in (2.21), and $\mathcal{Z}_{[x_1-\delta,x_1+\delta]}[w_*^1(\cdot,t) - w_*^2(\cdot,t)] = j \ge 1$ for $t \in [t_1-\varepsilon, t_1)$, while $\mathcal{Z}_{[x_1-\delta,x_1+\delta]}[w_*^1(\cdot,t) - w_*^2(\cdot,t)] = 0$ for $t \in (t_1, t_1+\varepsilon]$. As in (Case 2), $h_*^+(t)$ remains constant and $\mathcal{Z}_{I_1(t)}[w^1(\cdot,t) - w^2(\cdot,t)]$ is non-increasing and finite for $t \in (t_1-\varepsilon, t_1+\varepsilon]$. Consequently, $\mathcal{Z}_*[w^1(\cdot,t), w^2(\cdot,t)]$ is non-increasing and finite for $t \in (t_1-\varepsilon, t_1+\varepsilon]$.

To prove the assertion (ii), suppose first that $w^1(x,t_0) - w^2(x,t_0)$ changes sign at $x = x_0$. Then the result in [3] states that the number of zeros of $w^1(x,t_0) - w^2(x,t_0)$ in a small neighborhood of $x_0$ changes from $2k+1$ to $1$ at $t = t_0$ for some positive integer $k$, therefore, by (2.20), (2.21) and (2.22), $\mathcal{Z}_*[w^1(\cdot,t), w^2(\cdot,t)]$ drops at least by $2k$ at $t = t_0$. Similarly, if $w^1(x,t_0) - w^2(x,t_0)$ does not change sign at $x = x_0$, then the number of zeros of $w^1(x,t_0) - w^2(x,t_0)$ in a small neighborhood of $x_0$ changes from $2k$ to $0$ at $t = t_0$ for some positive integer $k$, so again $\mathcal{Z}_*[w^1(\cdot,t), w^2(\cdot,t)]$ drops at least by $2k$ at $t = t_0$.

Finally the proof of the assertion (iii) is already included in the proof of (i) for (Case 3). This completes the proof of the Proposition 2.4. □

**Lemma 2.5.** *Let $(w^i, \sigma_\pm^i)$, $i = 1,2$, be as in Proposition 2.4. Assume that* (b) *holds, and that $x_1 := \sigma_-^1(t_1) = \sigma_-^2(t_1)$ for some $t_1 \in (0, T_1)$, while $w^1(x,t_1) \not\equiv w^2(x,t_1)$. Then there exist constants $\delta, \varepsilon > 0$ and a positive integer $j$ such that*

$$\mathcal{Z}_{[x_1-\delta, x_1+\delta]}[w_*^1(\cdot,t) - w_*^2(\cdot,t)] = \begin{cases} j & \text{for } t \in [t_1-\varepsilon, t_1), \\ 0 & \text{for } t \in (t_1, t_1+\varepsilon]. \end{cases}$$

*Proof.* Let $\theta(t) := \theta_-^1(t) \equiv \theta_-^2(t)$. Since $w_x^i(\sigma_-^i(t), t) = \tan\theta(t) > 0$, we see from the inverse function theorem that the functions $y = w^i(x,t)$ $(i = 1,2)$ can be expressed as

$x = v_i(y,t)$ $(i = 1,2)$ locally around $x = \sigma_-^i(t)$. These functions $v_1, v_2$ satisfy

$$v_t = \frac{v_{yy}}{1+v_y^2} - c\sqrt{1+v_y^2}, \quad y \in [0,\delta_1],\ t \in [t_0-\varepsilon, t_0+\varepsilon],$$
$$v_y(0,t) = \cot\theta, \quad t \in [t_0-\varepsilon, t_0+\varepsilon]$$

for some sufficiently small $\delta_1, \varepsilon > 0$. Set $W(y,t) := v_1(y,t) - v_2(y,t)$. Then $W$ satisfies

$$\begin{cases} W_t = a(y,t)W_{yy} + b(y,t)W_y, & y \in [0,\delta],\ t \in [t_0-\varepsilon, t_0+\varepsilon], \\ W_y(0,t) = 0, & t \in [t_0-\varepsilon, t_0+\varepsilon], \end{cases} \tag{2.23}$$

where

$$a(y,t) = \frac{1}{1+(\partial_y v_1)^2},$$
$$b(y,t) = -\frac{\partial_{yy} v_2(\partial_y v_1 + \partial_y v_2)}{[1+(\partial_y v_1)^2][1+(\partial_y v_2)^2]} - \frac{c(\partial_y v_1 + \partial_y v_2)}{\sqrt{1+(\partial_y v_1)^2} + \sqrt{1+(\partial_y v_2)^2}}.$$

Consequently the even extension

$$\overline{W}(y,t) := \begin{cases} W(y,t), & y \in [0,\delta_1],\ t \in [t_0-\varepsilon, t_0+\varepsilon], \\ W(-y,t), & y \in [-\delta_1, 0],\ t \in [t_0-\varepsilon, t_0+\varepsilon], \end{cases}$$

satisfies (2.23) for $y \in [-\delta_1, \delta_1]$ with both coefficients $a$ and $b$ replaced by their even extensions. Since $\overline{W}(0,t_0) = \overline{W}_y(0,t_0) = 0$, and since $\overline{W}(y,t)$ is an even function, we see from the result in [3] that the number of zeros of $\overline{W}(\cdot,t)$ on the interval $[-\delta_1, \delta_1]$ is equal to some even integer $2j$ for $t \in [t_0-\varepsilon, t_0)$, while it is equal to 0 for $t \in (t_0, t_0+\varepsilon]$, provided that $\delta_1, \varepsilon$ are chosen sufficiently small. This is equivalent to the statement of the lemma with an appropriate choice of $\delta > 0$, and the proof is complete. □

One of the typical situations in which the above proposition can be applied is when one of $w^1, w^2$ is a solution of the problem (P) and the other is a straight line of the form (2.18) or (2.19). These two types of lines do not include horizontal lines, but similar results hold also for the horizontal ones. We summarize these results as follows.

**Corollary 2.6.** *Let $u$ be a solution of problem* (P) *defined for $t \in [0,T)$ and let $u_*$ denote its extension defined in* (2.17). *For arbitrary constants $\theta \in (-\pi/2, \pi/2)$ and $b \in \mathbb{R}$, define*

$$w(x,t) := (\tan\theta)x + \frac{ct}{\cos\theta} + b \quad \textit{for } \ x \in \mathbb{R},\ t \geq 0.$$

*Then the following holds:*

(i) $\mathcal{Z}_{\mathbb{R}}[u_*(\cdot,t) - w(\cdot,t)]$ *is non-increasing in $t \in [0,T)$ and is finite for each $t \in (0,T)$.*

(ii) *If, for some $t_0 \in (0,T)$, the curves $y = u(x,t_0)$ and $y = w(x,t_0)$ become tangential at some $x_0 \in (l_-(t_0), l_+(t_0))$, then $\mathcal{Z}_{\mathbb{R}}[u_*(\cdot,t) - w(\cdot,t)]$ drops at $t = t_0$ at least by 2.*

(iii) *Let* $\theta = \psi_-$ *or* $\theta = -\psi_+$. *If, for some* $t_0 \in (0,T)$, *the curves* $y = u(x,t_0)$ *and* $y = w(x,t_0)$ *become tangential at one of the endpoints* $l_\pm(t_0)$, *then* $\mathcal{Z}_{\mathbb{R}}[u_*(\cdot,t) - w(\cdot,t)]$ *drops at* $t = t_0$ *at least by* 1.

*Proof.* If $\theta \neq 0$, then $w$ can be rewritten as either (2.18) or (2.19) with $\sigma_0 = b\cot\theta$. Therefore the conclusion follows immediately from Proposition 2.4.

Next we consider the case $\theta = 0$, where $w$ represent a horizontal line. If $b > 0$, then the intersection between $u_*$ and $w$ can occur only in the upper half plane where $u_*$ coincides with $u$. Therefore the conclusion follows from the standard zero-number principle of [3]. If $b \leq 0$, then the line $y = w(x,t)$ initially lies below the $x$-axis. Thus $\mathcal{Z}_{\mathbb{R}}[u_*(\cdot,t) - w(\cdot,t)] = 2$ until this line comes slightly above the $x$-axis. After that, the situation reduces to the case $b > 0$, which is already discussed above. Hence the conclusion of the corollary holds for all $t \in [0,T)$. The proof of the corollary is complete. □

The following corollary will play an important role in proving the convergence results in Section 5.

**Corollary 2.7.** *Let* $(w^i, \sigma^i_\pm)$, $i = 1,2$, *be solutions of* (Q) *for* $\theta_\pm(t) = \theta^i_\pm(t)$ *on some time interval* $[0,T_1)$ *and assume that* $\theta^1_-(t) \equiv \theta^2_-(t)$, $\theta^1_+(t) \equiv \theta^2_+(t)$. *Then, for any* $t^* \in [0,T_1)$, $\sigma^1_-(t) - \sigma^2_-(t)$ *and* $\sigma^1_+(t) - \sigma^2_+(t)$ *change sign at most finitely many times on* $[t^*, T_1)$.

*Proof.* By Proposition 2.4 (iii), $\mathcal{Z}_*[w^1(\cdot,t_0), w^2(\cdot,t_0)]$ drops at least by one each time $\sigma^1_-(t) - \sigma^2_-(t)$ or $\sigma^1_+(t) - \sigma^2_+(t)$ vanishes. Furthermore, $\mathcal{Z}_*[w^1(\cdot,t), w^2(\cdot,t)]$ is non-increasing in $t$ and finite for each $t \in (0,T_1)$. Hence $\sigma^1_\pm(t) - \sigma^2_\pm(t)$ can change sign at most finitely many times on the interval $[t^*, T_1)$. □

**Corollary 2.8.** *Let* $(w^i, \sigma^i_\pm)$, $i = 1,2$, *be solutions of* (Q) *for* $\theta_\pm(t) = \theta^i_\pm(t)$. *If* $\theta^1_-(t) = \theta^2_-(t)$ *and* $\sigma^1_-(t) = \sigma^2_-(t)$ *or* $\theta^1_+(t) = \theta^2_+(t)$ *and* $\sigma^1_+(t) = \sigma^2_+(t)$ *on some interval* $[t_1,t_2]$ *with* $t_1 < t_2$, *then* $w^1(x,t) \equiv w^2(x,t)$ *for* $t \in [t_1,t_2]$.

*Proof.* Suppose that $w^1(x,t) \not\equiv w^2(x,t)$ for some $t \in [t_1,t_2]$. By continuity we have $w^1(x,t) \not\equiv w^2(x,t)$ for every $t \in [t_3,t_4]$ with $t_1 \leq t_3 < t_4 \leq t_2$ Then we obtain a contradiction from Lemma 2.5. □

2.3. **Equation for the curvature.** In this subsection, we assume that the solution $u$ is strictly concave. Then one can convert (1.1) into an equation for the curvature function

$$\kappa := u_{xx}/(1+u_x^2)^{3/2}$$

with independent variables $\theta, t$, where $\theta$ is defined by

$$\theta = \theta(x,t) := \arctan u_x(x,t).$$

Note that there is a one-to-one correspondence between $x \in [l_-(t), l_+(t)]$ and $\theta \in [-\psi_+, \psi_-]$ for each fixed $t$, since $u$ is strictly concave. It is well known that, for a general motion of concave curves, $\kappa(\theta, t)$ satisfies the following equation:

$$\kappa_t = \kappa^2(V_{\theta\theta} + V),$$

where $V$ denotes the normal velocity (see [22]). It follows from (1.5) that

$$\kappa_t = \kappa^2(\kappa_{\theta\theta} + \kappa + c), \quad -\psi_+ < \theta < \psi_-,\ t \in [0, T). \tag{2.24}$$

In order for the solution of (2.24) to represent a curve whose endpoints have the same $y$ coordinate (which is the case for our problem (P) since $u(l_\pm(t), t) = 0$), one must have

$$\int_{-\psi_+}^{\psi_-} \frac{\sin\theta}{\kappa(\theta, t)}\, d\theta = 0. \tag{2.25}$$

Thus our attention is restricted to the class of solutions of (2.24) that satisfy (2.25).

Next, we derive the boundary conditions. We first note

$$\kappa(\theta(x,t), t) = \frac{u_{xx}}{(1+u_x^2)^{3/2}} = \frac{u_t}{\sqrt{1+u_x^2}} - c.$$

Differentiating this equation in $x$ gives

$$\kappa_\theta \theta_x = \frac{u_{xt}}{\sqrt{1+u_x^2}} - \frac{u_t u_x u_{xx}}{(1+u_x^2)^{3/2}} = \frac{u_{xt}}{\sqrt{1+u_x^2}} - \kappa u_x u_t.$$

Since $\theta_x = \kappa/\cos\theta$ and $\theta_t = u_{xt}/(1+u_x^2)$, we obtain

$$\frac{\kappa\kappa_\theta}{\cos\theta} = \frac{\theta_t}{\cos\theta} - \frac{\tan\theta}{\cos\theta}\kappa(\kappa + c).$$

Here we have used the fact that $u_t = (\kappa + c)/\cos\theta$. This leads to the equation

$$\kappa\kappa_\theta = \theta_t - \tan\theta\, \kappa(\kappa + c). \tag{2.26}$$

Determining the value of $\theta_t$ at the endpoints of the curve is slightly tricky. It has to reflect the fact that the contact angles are fixed at $\psi_-, -\psi_+$ and that the endpoints remains on the $x$-axis as $t$ varies. For this purpose, we introduce a new variable $\Theta$, which is defined near the two endpoints of the curve through the following identity:

$$x = X(u, t) \qquad \text{(local inverse of} \quad u = u(x,t)\,), \tag{2.27}$$

$$\Theta(u, t) = \theta(X(u,t), t).,\ t \in [0, T), \tag{2.28}$$

Differentiating (2.27) yields

$$1 = X_u u_x, \qquad 0 = X_u u_t + X_t;$$

hence

$$\Theta_t = \theta_t - \theta_x X_u u_t = \theta_t - \frac{\theta_x u_t}{u_x}.$$

Consequently

$$\theta_t = \Theta_t + \frac{\kappa(\kappa + c)}{\sin\theta\cos\theta}.$$

By the boundary conditions $u(l_\pm(t), t) = 0$, $u_x(l_\pm(t), t) = \mp\tan\psi_\pm$, we have $\Theta_t(0, t) = 0$ on the both ends of the curve. Thus we have

$$\theta_t = \frac{\kappa(\kappa + c)}{\sin\theta\cos\theta}$$

at the two end points of the curve. Putting this into (2.26), we obtain

$$\kappa\kappa_\theta = \frac{\kappa(\kappa + c)}{\sin\theta\cos\theta} - \tan\theta\,\kappa(\kappa + c) = \frac{\cos\theta}{\sin\theta}\kappa(\kappa + c).$$

Thus we have established the following boundary conditions

$$\kappa_\theta = (\kappa + c)\cot\theta, \qquad \text{for } \ \theta = \mp\psi_\pm, \quad t > 0. \tag{2.29}$$

**Remark 2.1.** The boundary condition (2.29) reflects the fact that the two endpoints of the curve slide along the horizontal axis while keeping the same contact angles. If we consider another situation in which the two endpoints slide vertically, then we have $\theta_t = 0$ at the endpoints; therefore the boundary condition for this case would become

$$\kappa_\theta = -(\kappa + c)\tan\theta, \qquad \text{for } \ \theta = \mp\psi_\pm, \quad t > 0.$$

**Remark 2.2.** Under the boundary condition (2.29), we have

$$\begin{aligned} \frac{d}{dt}\int_{-\psi_+}^{\psi_-} \frac{\sin\theta}{\kappa}\,d\theta &= -\int_{-\psi_+}^{\psi_-} \frac{\sin\theta\,\kappa_t}{\kappa^2}\,d\theta = -\int_{-\psi_+}^{\psi_-} \sin\theta(\kappa_{\theta\theta} + \kappa + c)\,d\theta \\ &= -[\sin\theta\,\kappa_\theta - \cos\theta(\kappa + c)]_{-\psi_+}^{\psi_-} = 0. \end{aligned}$$

Consequently, the condition (2.25) is satisfied for all $t \in [0, T)$ if it is satisfied at $t = 0$.

In summary, our problem (P) reduces to the following problem for the curvature:

$$\begin{cases} \kappa_t = \kappa^2(\kappa_{\theta\theta} + \kappa + c), & -\psi_+ < \theta < \psi_+, \quad t > 0, \\ \kappa_\theta = \cot\theta\,(\kappa + c), & \theta = \mp\psi_\pm, \quad t > 0, \\ \kappa(\theta, 0) = \kappa_0(\theta), & -\psi_+ < \theta < \psi_-, \end{cases} \tag{2.30}$$

where the initial data $\kappa_0$ satisfies

$$\int_{-\psi_+}^{\psi_-} \frac{\sin\theta}{\kappa_0(\theta)}\,d\theta = 0.$$

Note that the problem (2.30) does not carry information about the location of the curve, but it gives complete information about the shape of the curve $\Gamma(t)$.

It is easy to see that the set of stationary solutions of (2.30) is given by

$$\kappa = -\nu\sin\theta - c \qquad (\nu \in \mathbb{R}), \tag{2.31}$$

under the constraint

$$\int_{-\psi_+}^{\psi_-} \frac{\sin\theta}{\nu\sin\theta + c}\, d\theta = 0. \tag{2.32}$$

Let us determine the value of $\nu$. First, since $\kappa < 0$, (2.31) implies $\nu\sin\theta + c > 0$ ($\theta \in [-\psi_+, \psi_-]$). Consequently

$$-\frac{c}{\sin\psi_-} < \nu < \frac{c}{\sin\psi_+}. \tag{2.33}$$

Next, let $F(\nu)$ denote the left-hand side of (2.32). Then

$$F'(\nu) = \int_{-\psi_+}^{\psi_-} \frac{-\sin^2\theta}{(c+\nu\sin\theta)^2}\, d\theta < 0,$$

$$F(\nu) \to +\infty \quad \text{as} \quad \nu \searrow -\frac{c}{\sin\psi_-}, \qquad F(\nu) \to -\infty \quad \text{as} \quad \nu \nearrow \frac{c}{\sin\psi_+}.$$

Consequently, for any given $\psi_\pm \in (0, \pi/2)$, there exists a unique constant $\nu$ for which the solution (2.31) satisfying the constraint (2.32) exists. Furthermore, $F(0) = (\cos\psi_+ - \cos\psi_-)/c$. Thus this unique constant $\nu$ satisfies $\nu > 0$ (resp. $= 0$, $< 0$) if $F(0) > 0$ (resp. $= 0$, $< 0$), or, equivalently, if $\psi_- - \psi_+ > 0$ (resp. $= 0$, $< 0$).

Now, since $\kappa = u_{xx}/(1+u_x^2)^{3/2}$ and $\sin\theta = u_x/(1+u_x^2)^{1/2}$, the solution $u$ corresponding to the above stationary solution (2.31) satisfies

$$\frac{u_{xx}}{1+u_x^2} + \nu u_x + c\sqrt{1+u_x^2} = 0.$$

Consequently, $u_t = -\nu u_x$, which means that $u$ is a traveling wave solution of the form $u(x,t) = \Phi(x - \nu t + a)$, where $\Phi$ is a solution of (1.16) and $a$ is some constant. Conversely, if $u$ is a traveling wave solution of the form $u(x,t) = \Phi(x - \nu t + a)$ with a strictly concave function $\Phi$, then it is clear that the corresponding curvature function $\kappa$ satisfies (2.31) along with the constraint (2.32). Thus we have established the following proposition:

**Proposition 2.9.** *For any given $\psi_\pm \in (0, \pi/2)$, the problem* (1.16) *has a unique strictly concave solution. Furthermore $\nu$ satisfies* (2.33) *and*

$$\nu \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0 \quad \textit{if and only if} \quad \psi_- \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} \psi_+. \tag{2.34}$$

**Remark 2.3.** As we shall see in Section 4, every solution of type (B) (the bounded case) becomes strictly concave within finite time. This means that any traveling wave solution of (P) is strictly concave. Therefore the above proposition covers all the solutions of (1.16).

2.4. **Area and length.** Before closing this section, we present some basic identities and inequalities concerning the area and length. First, we recall the notation (1.7):

$$A(t) := \int_{l_-(t)}^{l_+(t)} u(x,t)dx, \qquad L(t) := \int_{l_-(t)}^{l_+(t)} \sqrt{1+u_x^2(x,t)}dx.$$

We shall also use the notation

$$h(t) := \max_{x\in[l_-(t),l_+(t)]} u(x,t), \qquad l(t) := l_+(t) - l_-(t), \tag{2.35}$$

which represent the height and width of the curve $\Gamma(t)$, respectively.

A direct calculation yields

$$\begin{cases} A'(t) = -(\psi_+ + \psi_-) + cL(t) \\ L'(t) = l'_+(t)\cos\psi_+ - l'_-(t)\cos\psi_- + c(\psi_+ + \psi_-) - \displaystyle\int_0^{L(t)} \kappa^2\, ds, \end{cases} \tag{2.36}$$

where $ds = \sqrt{1+u_x^2}\, dx$. In deriving the formula for $L'(t)$, we have used the identity

$$l'_\pm(t) = \pm\cot\psi_\pm\, u_t(l_\pm(t),t), \tag{2.37}$$

which follows from differentiating $u(l_\pm(t),t) = 0$ in $t$ and using the boundary condition $u_x(l_\pm(t),t) = \mp\tan\psi_\pm$. Combining the above identity and Lemma 2.1, we obtain

$$l'_-(t) \geq -M_2\cot\psi_-, \quad l'_+(t) \leq M_2\cot\psi_+, \quad l'(t) \leq 2M_2\cot\psi_{\min} \quad \text{for } \ t\in[0,T). \tag{2.38}$$

The above estimates imply that the support of a solution $u(x,t)$ cannot expand too fast. In the special case where $u(x,t)$ is concave, we have $u_{xx}\leq 0$ ; hence

$$u_t(l_\pm(t),t) \leq c\sqrt{1+u_x^2}\,\big|_{x=l_\pm(t)} = \frac{c}{\cos\psi_\pm},$$

which gives the following sharper estimates:

$$l'_-(t) \geq -\frac{c}{\sin\psi_-}, \quad l'_+(t) \leq \frac{c}{\sin\psi_+}, \quad l'(t) \leq c\left(\frac{1}{\sin\psi_-} + \frac{1}{\sin\psi_+}\right). \tag{2.39}$$

We also note that the estimate $|u_x|\leq M_1$ in Lemma 2.1 implies

$$h(t) \leq \frac{M_1}{2}\, l(t), \qquad \frac{1}{M_1}\, h^2(t) \leq A(t) \leq h(t)l(t) \leq \frac{M_1}{2}\, l^2(t). \tag{2.40}$$

## 3. Classification of solutions

This section is devoted to the proof of Theorem 1.1, which classify solutions of (P) into three different types, namely, expanding, bounded and shrinking ones. In what follows, given a solution $u$ of problem (P), $[0,T)$ will denote, as before, the maximal time interval for the existence of the solution $u$. We shall use the notation (2.35) and set

$$I(t) := [l_-(t),\, l_+(t)].$$

As before, $\Gamma(t)$ denotes the curve $y = u(x,t)$ and $\kappa(x,t)$ denotes the curvature.

3.1. **Main lemmas.** We first present three main lemmas for the classification:

**Lemma 3.1.** *If $T = \infty$, the following four conditions are equivalent:*

(a) *There exists a sequence $t_n \to \infty$ such that $\lim_{n\to\infty} l(t_n) = \infty$.*
(b) *There exists a sequence $t_n \to \infty$ such that $\lim_{n\to\infty} h(t_n) = \infty$.*
(c) $\lim_{t\to\infty} l(t) = \infty$.
(d) $\lim_{t\to\infty} h(t) = \infty$.

*In particular, if $T = \infty$ and $\limsup_{t\to\infty} l(t) = \infty$, the solution is of type* (A).

**Lemma 3.2.** *The following four conditions are equivalent:*

(a) *There exists a sequence $\{t_n\}$ with $t_n \uparrow T$ as $n \to \infty$ such that $\lim_{n\to\infty} l(t_n) = 0$.*
(b) *There exists a sequence $\{t_n\}$ with $t_n \uparrow T$ as $n \to \infty$ such that $\lim_{n\to\infty} h(t_n) = 0$.*
(c) $\lim_{t\to T} l(t) = 0$.
(d) $\lim_{t\to T} h(t) = 0$.

*Furthermore, $T < \infty$ under any one of the above four conditions. In particular, if $T = \infty$, then $\liminf_{t\to\infty} h(t) > 0$ and $\liminf_{t\to\infty} l(t) > 0$.*

**Lemma 3.3.** *Assume that $T < \infty$ and let $\kappa(x,t)$ denote the curvature. Then:*

(i) $\lim_{t\to T} \|\kappa(\cdot,t)\|_{L^\infty([l_-(t),l_+(t)])} = \infty$.
(ii) *The solution curve $\Gamma(t)$ shrinks to a point as $t \to T$. That is, both $A(t)$ and $L(t)$ tend to zero as $t \to T$.*

Once the above three lemmas are established, Theorem 1.1 will follow easily.

*Proof of Theorem 1.1.* If $T < \infty$, then, by Lemma 3.3, the solution is of type (C). If $T = \infty$ and $\limsup_{t\to\infty} l(t) = \infty$, then, by Lemma 3.1, the solution is of type (A). Thus, all we need to show is that the solution is of type (B) if $T = \infty$ and $\limsup_{t\to\infty} l(t) < \infty$.

By the estimate $|u_x| \le M_1$ in Lemma 2.1, $A(t) \le M_1 l^2(t)/2$ and $L(t) \le \sqrt{1+M_1^2}\, l(t)$. Therefore, both $A(t)$ and $L(t)$ remain bounded from above as $t \to \infty$. On the other hand, since $T = \infty$, we see, by Lemma 3.2, that

$$\liminf_{t\to\infty} h(t) > 0, \qquad \liminf_{t\to\infty} l(t) > 0.$$

Since $A(t) \ge h^2(t)/M_1$ by (2.40) and since $L(t) \ge l(t)$, both $A(t)$ and $L(t)$ remain bounded from below by positive constants; hence the solution is of type (B). This completes the proof of the theorem. $\square$

The rest of this section is devoted to the proof of the above three lemmas.

3.2. **Comparison principle.** Here we introduce the notion of super- and sub-solutions for our free boundary problem. Let $\sigma_\pm(t)$ be $C^1$ functions on some interval $[T_0, T_1)$ satisfying $\sigma_-(t) < \sigma_+(t)$ and let $v(x,t)$ be a function that belongs to $C^{2,1}(\Omega_{T_0,T_1}) \cap C(\bar{\Omega}_{T_0,T_1})$, where

$$\Omega_{T_0,T_1} := \{(x,t) : \sigma_-(t) \le x \le \sigma_+(t),\ T_0 < t < T_1\}.$$

We say that $(v, \sigma_\pm)$ is a *sub-solution* of the problem (1.1)-(1.3) for $t \in [T_0, T_1)$ if

$$v_t \le \frac{v_{xx}}{1+v_x^2} + c\sqrt{1+v_x^2} \quad \text{in } \Omega_{T_0,T_1}, \tag{3.1}$$

$$v(\sigma_\pm(t), t) = 0, \quad t \in [T_0, T_1), \tag{3.2}$$

$$v_x(\sigma_-(t), t) \ge \tan\psi_-, \quad v_x(\sigma_+(t), t) \le -\tan\psi_+, \quad t \in [T_0, T_1). \tag{3.3}$$

Also, $(v, \sigma_\pm)$ is called a *super-solution* for $t \in [T_0, T_1)$ if the reversed inequalities hold in (3.1) and (3.3).

**Proposition 3.4** (Comparison principle). *Let $(v^1, \sigma^1_\pm)$ and $(v^2, \sigma^2_\pm)$ be a sub-solution and a super-solution of (1.1)-(1.3) for $t \in [T_0, T_1)$, respectively, and assume that*

$$[\sigma^1_-(T_0), \sigma^1_+(T_0)] \subset [\sigma^2_-(T_0), \sigma^2_+(T_0)], \quad v^1(x, T_0) \le v^2(x, T_0) \ \ \textit{for}\ \ x \in [\sigma^1_-(T_0), \sigma^1_+(T_0)].$$

*Then*

$$[\sigma^1_-(t), \sigma^1_+(t)] \subset [\sigma^2_-(t), \sigma^2_+(t)],\ v^1(x,t) \le v^2(x,t) \ \ \textit{for}\ \ x \in [\sigma^1_-(t), \sigma^1_+(t)],\ t \in [T_0, T_1).$$

The proof of the above proposition is rather standard, so we omit it. The only slightly delicate part is to prove that the endpoints $\sigma^1_\pm(t)$ and $\sigma^2_\pm(t)$ do not cross each other, but this can be done by using the same coordinate change as in the proof of Lemma 2.5 and apply the Hopf boundary lemma at $y = 0$.

The next result is a stronger version of Proposition 3.4 and can be shown by the strong maximum principle. The proof is again standard, so we omit it. We shall need this lemma only in the proof of Theorem 5.1.

**Proposition 3.5** (Strong comparison principle). *Let the assumptions of Proposition 3.4 hold, and assume further that $v^1(x, T_0) \not\equiv v^2(x, T_0)$. Then*

$$\sigma^2_-(t) < \sigma^1_-(t) < \sigma^1_+(t) < \sigma^2_+(t),\ v^1(x,t) < v^2(x,t) \ \ \textit{for}\ \ x \in [\sigma^1_-(t), \sigma^1_+(t)],\ t \in (T_0, T_1).$$

Needless to say, Propositions 3.4 and 3.5 hold if $v^1$ or $v^2$ is a solution of problem (P), since any exact solution is a sub- and super-solution at the same time.

**Remark 3.1.** We say that $(v, \sigma_\pm)$ is a *weak sub-solution* of (1.1)-(1.3) if $v$ belong to the class $W^{2,1}_\infty(\Omega_{T_0,T_1})$ and satisfies (3.1) in the weak sense along with the boundary conditions (3.2) and (3.3), where

$$W^{2,1}_\infty(\Omega_{T_0,T_1}) := \Big\{ u \in L^\infty(\Omega_{T_0,T_1}) \Big| \sum_{0 \le \alpha + 2\beta \le 2} \|\partial_x^\alpha \partial_t^\beta u\|_{L^\infty(\Omega_{T_0,T_1})} < \infty \Big\}.$$

A *weak super-solution* is defined similarly. Proposition 3.4 remains valid for weak sub- and super-solutions; see, e.g., [15].

A simple example of sub-solution (resp. super-solution) can be constructed by using a portion of a growing (resp. shrinking) circle. More precisely, for each $R_0 > 0$, let $R(t)$ denote a solution of the following equation:

$$R'(t) = c - \frac{1}{R(t)}, \quad t \geq 0, \qquad R(0) = R_0. \tag{3.4}$$

As is easily seen, $R'(t) > 0$ and $R(t) \to \infty$ as $t \to \infty$ if $R_0 > 1/c$, while $R'(t) < 0$ and $R(t) \to 0$ in finite time if $R_0 < 1/c$. Note that a circle of radius $R(t)$ satisfies the equation (1.5), since its normal speed is equal to $R'(t)$ and its curvature is equal to $-1/R(t)$.

Given any constant $\theta_0 \in (0, \pi/2)$, we now introduce the following function:

$$\begin{aligned} W(x,t) &:= \sqrt{R(t)^2 - x^2} - R_0 \cos\theta_0, \quad \sigma_-(t) \leq x \leq \sigma_+(t),\ t > 0, \\ \text{where} \quad & \sigma_\pm(t) := \pm\sqrt{R(t)^2 - (R_0\cos\theta_0)^2}. \end{aligned} \tag{3.5}$$

This function is defined so long as $|\sigma_\pm(t)| > 0$. The graph $y = W(x,t)$ is an upper portion of a circle with radius $R(t)$ centered at $(x, y) = (0, -R_0 \cos\theta_0)$, and the endpoints of this curve meet the $x$-axis at $x = \sigma_\pm(t)$ with the following contact angles:

$$\theta_\pm(t) := -\arctan\frac{\sigma_\pm(t)}{R_0\cos\theta_0} = \mp\arctan\frac{\sqrt{R(t)^2 - (R_0\cos\theta_0)^2}}{R_0\cos\theta_0}.$$

Note that $\theta_\pm(0) = \mp\theta_0$, and that $\theta'_-(t) > 0$ if $R_0 > 1/c$, while $\theta'_-(t) < 0$ if $R_0 < 1/c$.

**Lemma 3.6.** *For $R_0 > 0, \theta_0 \in (0, \pi/2)$, let $W(x,t)$ be the function defined in* (3.5) *which depends on $R_0$ and $\theta_0$.*

(i) *If $R_0 > 1/c$ and $\theta_0 \geq \psi_\pm$, then $W$ is a sub-solution of problem* (P) *for $t \in [0, \infty)$. Furthermore, $W \to \infty$ as $t \to \infty$ uniformly on every compact set of $\mathbb{R}$.*

(ii) *If $R_0 < 1/c$ and $0 < \theta_0 \leq \psi_\pm$, then $W$ is a super-solution of problem* (P) *for $t \in [0, T_1)$, where $T_1$ is determined by $R(T_1) = R_0 \cos\theta_0$. Furthermore, the curve $y = W(x,t)$ shrinks to a point as $t \nearrow T_1$.*

*Proof.* Let us first prove (i). Since $R(t)$ satisfies (3.4), $W$ satisfies the same equation as (1.1). Furthermore, since $R_0 > 1/c$, $R(t)$ is monotonically increasing, therefore, so do $|\sigma_\pm(t)|$ and $|\theta_\pm(t)|$. Consequently

$$\theta_-(t) \geq \theta_0 \geq \psi_- > 0 > -\psi_+ \geq -\theta_0 \geq -\theta_+(t) \quad \text{for} \quad t \geq 0.$$

Hence $W$ is a subsolution. Since $R(t) \to \infty$ as $t \to \infty$, we have $W \to \infty$ as $t \to \infty$ uniformly on compact sets. This proves (i). The statement (ii) can be proved similarly by using the fact that $R'(t) < 0$ and $R(t) \to 0$ as $t \to T_1$. The lemma is proved. □

**Remark 3.2.** Obviously Lemma 3.6 (i) remains true for $\theta_0 = \pi/2$. In this case, the graph of $W(x,t)$ is a growing half circle with $\theta(t) \equiv \pi/2$.

3.3. **Proof of Lemmas 3.1 and 3.2.** We first present the following lemma, which will play a useful role in this subsection. The proof of this lemma is based on the extended intersection number principle.

**Lemma 3.7.** *For any $\tau \in (0,T)$ and $M \in (0, c\tau/2)$, there exists $l_{M,\tau} > 0$ such that, whenever $l(t_0) > l_{M,\tau}$ for some $t_0 \in [\tau, T)$, it holds that $h(t_0) > M$.*

*Proof.* Given $\tau \in (0,T)$ and $M \in (0, c\tau/2)$, we choose $R_0$ such that

$$R_0 \geq \max\left\{\frac{2}{c}, \frac{c\tau}{1/\cos\psi_{\min} - 1}\right\}, \tag{3.6}$$

where $\psi_{\min} := \min\{\psi_+, \psi_-\}$. Let $R(t)$ be the solution of (3.4) with $R(0) = R_0$. Since $R_0 > 1/c$, $R(t)$ is increasing in $t$; hence $R'(t) \geq c - 1/R_0 \geq c/2$, which implies

$$R(\tau) \geq R_0 + c\tau/2 \geq R_0 + M.$$

On the other hand, since $R'(t) < c$, we have

$$\frac{R(t)}{R_0} < \frac{R_0 + ct}{R_0} = 1 + \frac{ct}{R_0} \leq \frac{1}{\cos\psi_{\min}} \quad \text{for } \ t \in [0,\tau].$$

Consequently, there exists $\tau_1 \in (0,\tau]$ such that

$$R(\tau_1) = R_0 + M, \qquad R(t)/R_0 < \frac{1}{\cos\psi_{\min}} \quad \text{for } \ t \in [0,\tau_1]. \tag{3.7}$$

Now let $W(x,t)$ be the function in (3.5) for the above choice of $R_0$ and with $\theta_0 = 0$:

$$W(x,t) := \sqrt{R(t)^2 - x^2} - R_0, \quad x \in [\sigma_-(t), \sigma_+(t)], \ t \in (0,\tau_1],$$
$$\text{where} \quad \sigma_\pm(t) := \pm\sqrt{R(t)^2 - R_0^2}.$$

As before, the contact angle of the curve $y = W(x,t)$ at $x = \sigma_\pm(t)$ is given by

$$\theta_\pm(t) = -\arctan\frac{\sigma_\pm(t)}{R_0} = \mp\arctan\frac{\sqrt{R(t)^2 - R_0^2}}{R_0}.$$

Hence, by the second inequality of (3.7), we have

$$\psi_- \geq \psi_{\min} > \theta_-(t) > 0 > \theta_+(t) > -\psi_{\min} \geq -\psi_+ \quad \text{for } \ t \in (0,\tau_1]. \tag{3.8}$$

Note also that

$$\sigma_+(\tau_1) - \sigma_-(\tau_1) = 2\sqrt{R(\tau_1)^2 - R_0^2} = 2\sqrt{M^2 + 2R_0M} := l_{M,\tau}. \tag{3.9}$$

By (3.8) and Proposition 2.4, we have, for any $t_0 \in [\tau, T)$ and any $a \in \mathbb{R}$,

$$\mathcal{Z}_*[u(\cdot,t_0), W(\cdot - a, \tau_1)] \leq \mathcal{Z}_*[u(\cdot, t_0 - s), W(\cdot - a, \tau_1 - s)] \quad \text{for } \ s \in [0,\tau_1).$$

Since the extended curve $y = W_*(x-a, \tau_1 - s)$ converges to the $x$-axis as $s \to \tau_1$, the right-hand side of the above inequality equals 2 for $s$ sufficiently close to $\tau_1$. Consequently

$$\mathcal{Z}_*[u(\cdot, t_0), W(\cdot - a, \tau_1)] \leq 2 \quad \text{for any} \;\; a \in \mathbb{R}. \tag{3.10}$$

Now suppose that $l(t_0) > l_{M,\tau}$ for some $t_0 \in [\tau, T)$, where $l_{M,\tau}$ is the constant defined in (3.9). Then, by (3.9), the endpoints of the curve $y = W(x-a, \tau_1)$, namely $\sigma_\pm(\tau_1)+a$, satisfy the following if we set $a = (l_-(t_0) + l_+(t_0))/2$:

$$l_-(t_0) < \sigma_-(\tau_1) + a, \quad \sigma_+(\tau_1) + a < l_+(t_0).$$

This and (3.8) imply that the two extended graphs $y = u_*(x, t_0)$ and $y = W_*(x-a, \tau_1)$ intersect at two points below the $x$-axis. Hence, by (3.10), they do not intersect above the $x$-axis. This means that $u(x, t_0) > W(x-a, \tau_1)$ on the interval $[l_-(t_0), l_+(t_0)]$, therefore $h(t_0) > M$. The lemma is proved. □

**Corollary 3.8.** *Let $T = \infty$ and suppose that there exists a sequence $t_n \to \infty$ such that $l(t_n) \to \infty$ as $n \to \infty$. Then $h(t_n) \to \infty$ as $n \to \infty$.*

*Proof.* Let $M > 0$ be arbitrary and set $\tau > 2M/c$. Define $R_0$ as in (3.6) and $l_{M,\tau}$ as in (3.9). Then, since $l(t_n) > l_{M,\tau}$ for all large $n$, we see from Lemma 3.7 that $h(t_n) > M$ for all large $n$. Since $M > 0$ is arbitrary, the proof of the corollary is complete. □

**Corollary 3.9.** *Suppose that there exists a sequence $t_n \nearrow T$ such that $h(t_n) \to 0$ as $n \to \infty$. Then $l(t_n) \to 0$ as $n \to \infty$.*

*Proof.* Fix any $\tau \in (0, T)$. Since $h(t_n) \to 0$ and $t_n \to T$ as $n \to \infty$, we have $h(t_n) < c\tau/2$ and $t_n \in [\tau, T)$ for all sufficiently large $n$. Thus, by Lemma 3.7 with $M = h(t_n)$,

$$l(t_n) \leq l_{h(t_n),\tau} = 2\sqrt{h(t_n)^2 + 2R_0 h(t_n)},$$

where $R_0 = \max\{2/c, c\tau/(1/\cos\psi_{\min} - 1)\}$. This proves the corollary. □

Now we are ready to prove Lemmas 3.1 and 3.2.

*Proof of Lemma 3.1.* By Corollary 3.8, we have (a)⇒(b). Next we prove (b)⇒(d). By the estimate $|u_x| \leq M_1$ in Lemma 2.1, one easily sees that $\Gamma(t_n)$ lies above a half-circle of radius $R_0 > 1/c$ for all large $n$. Thus, by Remark 3.2 and the comparison principle, we have $h(t) \to \infty$ as $t \to \infty$. The assertion (d)⇒(c) follows from the estimate $|u_x| \leq M_1$, and the assertion (c)⇒(a) is obvious. The lemma is proved. □

*Proof of Lemma 3.2.* By Corollary 3.9, we have (b)⇒(a). The assertion (a)⇒(b) follows from the estimate $|u_x| \leq M_1$. Thus we have (a)⇔(b). Now, for an arbitrary choice of $R_0 \in (0, 1/c)$, let $W(x,t)$ be the super-solution given in Lemma 3.6 (ii) with $\theta_0 = \psi_{\min}$. Then, under the assumptions (a), (b), the curve $\Gamma(t_n)$ lies below $y = W(x - a_n, 0)$ for all large $n$, where $a_n = (l_-(t_n) + l_+(t_n))/2$. Choose one such $n$ and fix it. By the comparison principle, $u(x,t) \leq W(x - a_n, t - t_n)$ for all $t \in [t_n, T)$. Since this super-solution $W$ shrinks to a point in finite time, we have $T < \infty$. Furthermore, the above inequality and (3.5) imply $l(t) \leq 2R_0 \sin\psi_{\min}$ for $t \in [t_n, T)$. Since $R_0$ can be chosen arbitrarily small, we see that (c) holds. This establishes (a)⇒(c). The assertion (c)⇒(d) follows from the estimate $|u_x| \leq M_1$, and the assertion (d)⇒(b) is obvious. The lemma is proved. □

3.4. **Proof of Lemma 3.3.** We begin with the following lemma:

**Lemma 3.10.** *Assume that there exist* $t_0 \in (0,T)$, $x_0 \in (l_-(t_0), l_+(t_0))$ *such that* $u_x(x_0,t_0) = 0$ *and that* $u(x_0,t_0) \leq ct_0$. *Then* $u(x_0,t_0) = \max_{x\in I(t_0)} u(x,t_0)$.

*Proof.* Define $w(x,t) = ct + u(x_0,t_0) - ct_0$. Then by Corollary 2.6 (i),

$$\mathcal{Z}_{\mathbb{R}}[u_*(\cdot,t) - w(\cdot,t)] \leq \mathcal{Z}_{\mathbb{R}}[u_*(\cdot,0) - w(\cdot,0)] \quad \text{for all } \ t \in [0,T). \tag{3.11}$$

Since $u(x_0,t_0) - ct_0 \leq 0$, the right-hand side of (3.11) equals 2. By the assumption, the graphs of $u(x,t_0)$ and $w(x,t_0)$ are tangential at $x = x_0$. Consequently, by Corollary 2.6 (ii),

$$\mathcal{Z}_{\mathbb{R}}[u_*(\cdot,t) - w(\cdot,t)] = 0 \quad \text{for all } \ t \in (t_0,T),$$

which is possible only if $u(x,t_0)$ attains its maximum at $x = x_0$. □

**Corollary 3.11.** *Suppose that there exist some sequences* $t_n \to T$ *and* $x_n \in I(t_n)$ *such that* $u_x(x_n,t_n) = 0$ $(n = 1,2,3,\ldots)$ *and* $u(x_n,t_n) \to 0$ *as* $n \to \infty$. *Then* $h(t_n) \to 0$ *as* $n \to \infty$.

*Proof.* For all sufficiently large $n$, we have $u(x_n,t_n) \leq ct_n$; hence, by Lemma 3.10, $h(t_n) = u(x_n,t_n)$. Consequently $h(t_n) \to 0$ as $n \to \infty$. The corollary is proved. □

Now we are ready to prove the main lemma:

*Proof of Lemma 3.3.* We first deal with (i). Let $I(t) = [l_-(t), l_+(t)]$ and $l(t) := l_+(t) - l_-(t)$. We argue by contradiction. Suppose there exists a sequence $t_n \nearrow T$ such that $\|\kappa(\cdot,t_n)\|_{L^\infty}$ remains bounded as $n \to \infty$. Then, since $\kappa = u_{xx}/(1 + u_x^2)^{3/2}$ and since $|u_x| \leq M_1$ by Lemma 2.1, there exists a constant $M > 0$ such that

$$\|u(\cdot,t_n)\|_{C^2(I(t_n))} \leq M \quad (n = 1,2,3,\ldots).$$

By (2.38), there exists a constant $M' > 0$ such that

$$l(t_n) \leq M' \quad (n = 1,2,3,\ldots).$$

Consequently, by the local existence result of Theorem 7.1, there exists a constant $\tau > 0$ independent of $n$ such that the solution can be continued over the interval $[t_n, t_n + \tau]$ for all $n$. Since $t_n \nearrow T$, we have $t_n + \tau > T$ for large $n$, which contradicts the fact that $[0,T)$ is the maximal interval for the existence of the solution $u$. This proves (i).

Next, we prove (ii). By (i), we have $\|u(\cdot,t_n)\|_{C^2(I(t_n))} \to \infty$, but since $|u_x| \le M_1$ and $u_{xx}$ is bounded from above by Lemma 2.1, we have

$$\min_{x\in I(t)} u_{xx}(x,t) \to -\infty \quad \text{as} \;\; t \to T.$$

Thus, by (2.3), we have $\lim_{t\to T} d(t) = 0$, which implies, by virtue of Lemma 3.10, that $\liminf_{t\to T} h(t) = 0$. Hence, by Lemma 3.2, $\lim_{t\to T} l(t) = 0$, from which it follows that $A(t) \to 0$, $L(t) \to 0$ as $t \to T$. The lemma is proved. □

3.5. **Some useful corollaries.** Before closing this section, we summarize some results that follow immediately from our earlier arguments in this section. The first result gives a simple criterion for type (A) and type (C) behaviors. It follows immediately from Lemmas 3.3, 3.6 and the comparison principle, so we omit the proof.

**Corollary 3.12.** *If, for some $t_0 \in [0,T)$, the solution curve $y = u(x,t_0)$ lies above (resp. below) a circular arc of radius $R_0 > 1/c$ (resp. $R_0 < 1/c$) with contact angle $\theta_0 \in [\psi_{\max}, \pi/2]$ (resp. $\theta_0 \in (0, \psi_{\min}]$), then $u$ is of type* (A) *(resp.* (C)*).*

The next result concerns the shape of the curve $\Gamma(t)$. It is a weaker version of Theorems 1.3 and 1.4, but is worth noting here because of the simplicity of its proof.

**Corollary 3.13.** *Let $(u, l_\pm)$ be a solution of* (P) *that is of type* (B) *or type* (C)*. Then there exists $t^* \in [0,T)$ and $\xi \in C^1[t^*,T)$ such that for each $t \in [t^*,T)$,*

$$\begin{cases} u_x(x,t) > 0 \quad \textit{for} \;\; x \in [l_-(t), \xi(t)), \\ u_x(x,t) < 0 \quad \textit{for} \;\; x \in (\xi(t), l_+(t)], \\ u(\xi(t),t) = \max_{x\in I(t)} u(x,t). \end{cases} \tag{3.12}$$

*Proof.* First, assume that $(u, l_\pm)$ is of type (B). Set $M := \sup_{t\ge 0} \|u(\cdot,t)\|_{L^\infty}$ and $t^* = M/c$. Then, since $u(x,t) \le ct$ for all $t \in [t^*,\infty)$ and $x \in I(t)$, we see by Lemma 3.10 that $u_x(x,t)$ can vanish only where $u(x,t)$ attains its maximum on $I(t)$. Furthermore, since $v := u_x$ satisfies the linear parabolic equation (2.1), we see from the result in [3] that the zeros of $u_x$ are isolated. This implies that the maximum is attained at a single point, which proves (3.12). By the same result in [3], $u_x$ has a simple zero at $x = \xi(t)$ for $t \in (t^*,T)$. Thus, by the implicit function theorem, $\xi(t)$ is $C^1$ in $t$.

Next, suppose that $(u, l_\pm)$ is of type (C). Then, since $h(t) \to 0$ as $t \to T$, there exists $t^* \in [0,T)$ such that $u(x,t) \le ct$ for all $t \in [t^*,T)$ and $x \in I(t)$. The rest of the proof is then the same as above. This corollary is proved. □

## 4. Concavity properties

In this section we prove results on the concavity properties of solutions. Most of the arguments here rely exclusively on the extended intersection number principle.

4.1. **Preservation of concavity.**

*Proof of Theorem 1.2.* Suppose that $u$ becomes non-concave at some $t = t_1 \in (t_0, T)$. Then the extended graph $y = u_*(x, t_1)$ (see (2.17)) has at least three intersections with a line $y = (\tan\theta)x + b$ for some $\theta \in (-\pi/2, \pi/2)$ and $b \in \mathbb{R}$. Define

$$w(x,t) = (\tan\theta)x + \frac{c(t-t_1)}{\cos\theta} + b.$$

By Corollary 2.6, $\mathcal{Z}_{\mathbb{R}}[u_*(\cdot,t) - w(\cdot,t)]$ is non-increasing in $t$. Hence

$$\mathcal{Z}_{\mathbb{R}}[u_*(\cdot,t_0) - w(\cdot,t_0)] \geq \mathcal{Z}_{\mathbb{R}}[u_*(\cdot,t_1) - w(\cdot,t_1)] \geq 3.$$

On the other hand, since $u(x,t_0)$ is concave, the same holds for $u_*(x,t_0)$. Therefore the number of intersections between the curve $y = u_*(x,t_0)$ and the line $y = w(x,t_0)$ is at most two, which contradicts the above inequality. This contradiction proves that $u(\cdot,t)$ is concave for all $t \in [t_0, T)$. The inequality $u_{xx}(\cdot,t) < 0$ then follows from the strong maximum principle and the fact that $u_{xx}$ satisfies a linear parabolic equation, which can be obtained by differentiating (1.1) in $x$ twice. This complete the proof of Theorem 1.2. □

4.2. **Eventual concavity for type (B).** Next we prove Theorem 1.3 on the eventual concavity of type (B) solutions. We begin with the following lemma:

**Lemma 4.1.** *Let $u$ be a type* (B) *solution of problem* (P)*. Then there exist constants $a < b$ such that*

$$\nu t + a \leq l_-(t) < l_+(t) \leq \nu t + b \quad \text{for} \;\; 0 \leq t < +\infty, \tag{4.1}$$

*where $\nu$ is the traveling wave speed defined in* (1.16)*.*

*Proof.* Let $\Phi$ denote the unique concave solution of (1.16) and define

$$w^1(x,t) = \Phi(x - \nu t - \bar{a}), \qquad w^2(x,t) = \Phi(x - \nu t - \bar{b}),$$

where the constants $\bar{a}, \bar{b}$ are chosen to satisfy

$$\bar{a} + \beta < l_-^0 < l_+^0 < \bar{b} - \beta$$

so that the support of $w^1(\cdot,0)$, $u^0$, $w^2(\cdot,0)$, namely $[\bar{a}-\beta,\, \bar{a}+\beta]$, $[l_-^0,\, l_+^0]$, $[\bar{b}-\beta,\, \bar{b}+\beta]$, respectively, are mutually disjoint. Then it is clear that

$$\mathcal{Z}_*[w^1(\cdot,0), u^0] = \mathcal{Z}_*[w^2(\cdot,0), u^0] = 1.$$

Hence, by Proposition 2.4 (i), we have

$$\mathcal{Z}_*[w^1(\cdot,t), u(\cdot,t)] \leq 1, \quad \mathcal{Z}_*[w^2(\cdot,t), u(\cdot,t)] \leq 1 \qquad \text{for} \;\; t \geq 0.$$

Now, if $\nu t + \bar{a} - \beta < l_-(t)$ for all $t \geq 0$, it implies the first part of (4.1) with $a = \bar{a} - \beta$. On the other hand, if $\nu t_0 + \bar{a} - \beta = l_-(t_0)$ for some $t_0 > 0$, then the curves $y = w_1(x,t_0)$

and $y = u(x, t_0)$ become tangential at their left endpoint. Hence, by Proposition 2.4 (iii), $\mathcal{Z}_*[w^1(\cdot, t), u(\cdot, t)] = 0$ for all $t > t_0$. This means that the graph of $u(x, t)$ lies entirely below the graph of $w^1(x, t)$ for every $t > t_0$, or the other way around for every $t > t_0$. In the former case, we obtain the same lower bound for $l_-(t)$ as before. In the latter case, we have

$$l_-(t) \geq \nu t + \bar{a} + \beta - \sup_{\tau \geq t_0} l(\tau).$$

Here $\sup_{\tau \geq t_0} l(\tau) < \infty$ since $u$ is of type (B). This proves the first inequality of (4.1). The last inequality of (4.1) follows by a similar argument. The lemma is proved. □

**Remark 4.1.** As we shall see in Section 5, any solution that lies below a traveling wave is of type (C), while any solution that lies above a traveling wave is of type (A). Therefore $u(x, t)$ and $w^i(x, t)$ $(i = 1, 2)$ in the above proof actually never become tangential.

Now, for each constant $\theta \in (0, \psi_-]$, we define a function

$$w_\theta(x, t) := \tan\theta \left( x + \frac{ct}{\sin\theta} - \sigma \right),$$

while, for $\theta \in [-\psi_+, 0)$, we define

$$w_\theta(x, t) := \tan\theta \left( x + \frac{ct}{\sin\theta} + \sigma \right),$$

where $\sigma$ is a constant satisfying

$$-\sigma < l_-^0 < l_+^0 < \sigma. \tag{4.2}$$

We also set $w_\theta(x, t) := ct$ if $\theta = 0$. Thus $w_\theta$ can be expressed in the following unified form:

$$w_\theta(x, t) = (\tan\theta) x + \frac{ct}{\cos\theta} - \sigma |\tan\theta|. \tag{4.3}$$

The function $w_\theta$ represents a line of slope $\tan\theta$ that moves upward with normal speed $c$. This line intersects with the $x$-axis at $x = \sigma - ct/\sin\theta$ if $\theta > 0$ and at $x = -\sigma - ct/\sin\theta$ if $\theta < 0$. By (4.2), the intersection point is initially (i.e. at $t = 0$) located on the right-hand (resp. left-hand) side of the interval $[l_-^0, l_+^0]$ and moves to the left (resp. right) if $\theta > 0$ (resp. $\theta < 0$). The same is true of the intersection between the line $y = w_\theta(x, t) - m$ and the $x$-axis for any $m \geq 0$. In the case where $\theta = 0$, the line $y = w_\theta(x, t) - m = ct - m$ is horizontal and is initially located below the $x$-axis.

The above observation about the initial position of the line $y = w_\theta(x, 0) - m$ implies:

$$\mathcal{Z}_{\mathbb{R}}[u_*^0 - (w_\theta(\cdot, 0) - m)] = \begin{cases} 2 & \text{if } \theta \in (-\psi_+, \psi_-), \\ 1 & \text{if } \theta = \mp\psi_\pm \end{cases}$$

for any constant $m \geq 0$. Consequently, by Corollary 2.6 (i),

$$\mathcal{Z}_{\mathbb{R}}[u_*(\cdot, t) - (w_\theta(\cdot, t) - m)] \leq \left\{ \begin{array}{ll} 2 & \text{if } \theta \in (-\psi_+, \psi_-), \\ 1 & \text{if } \theta = \mp\psi_\pm \end{array} \right\} \quad \text{for all } t \geq 0, \tag{4.4}$$

for any choice of the constant $m \geq 0$.

**Lemma 4.2.** *Let $w_\theta(x,t)$ be as above. Then there exists $t^* > 0$ such that*

$$(4.5)\qquad \begin{aligned} w_\theta(x,0) &\le u^0(x) \quad \textit{for all}\ \ x \in [l_-^0,\, l_+^0],\ \ \theta \in [-\psi_+,\, \psi_-], \\ w_\theta(x,t) &\ge u(x,t) \quad \textit{for all}\ \ t \in [t^*,\infty),\ \ x \in I(t),\ \ \theta \in [-\psi_+,\, \psi_-]. \end{aligned}$$

*Proof.* The first line in (4.5) is clear from the previous argument, so we shall prove only the second line. In the moving frame $\xi := x - \nu t$, the support of the solution $u(\cdot,t)$ is always contained in the interval $[a,b]$, by virtue of (4.1). On the other hand, by (4.3),

$$w_\theta = (\tan\theta)\xi + \frac{c + \nu\sin\theta}{\cos\theta}\, t - \sigma|\tan\theta|.$$

By (2.33), there exists a constant $\delta > 0$ such that $c + \nu\sin\theta \ge \delta$ for $\theta \in [-\psi_+,\, \psi_-]$. Hence $w_\theta \to \infty$ as $t \to \infty$ uniformly in $\xi \in [a,b]$ and $\theta \in [-\psi_+,\, \psi_-]$. This implies (4.5) if $t^*$ is chosen sufficiently large, since $u$ remains bounded. The lemma is proved. □

Now we are ready to prove Theorem 1.3.

*Proof of Theorem 1.3.* Let $t^*$ be as in Lemma 4.2. We first prove that

$$(4.6)\qquad -\tan\psi_+ < u_x(x,t) < \tan\psi_- \quad \text{for all}\ \ t \ge t^*,\ \ x \in (l_-(t),\, l_+(t)).$$

Suppose that (4.6) is violated at some $t_0 \ge t^*$. Then, by the intermediate value theorem, there exist some $x_0 \in (l_-(t_0),\, l_+(t_0))$ such that

$$u_x(x_0,t_0) = \tan\psi_- \quad \text{or} \quad u_x(x_0,t_0) = -\tan\psi_+.$$

Without loss of generality, we may assume that the former holds. Then the tangent line of the curve $y = u(x,t_0)$ at $x = x_0$ is given in the form $y = w_{\psi_-}(x,t_0) - m$ for some $m \in \mathbb{R}$. By Lemma 4.2, we have $m \ge 0$. Hence (4.4) holds. On the other hand, by Corollary 2.6 (ii), $\mathcal{Z}_{\mathbb{R}}[u_*(\cdot,t) - \big(w_{\psi_-}(\cdot,t) - m\big)]$ must drop at least by two at $t = t_0$, but this is impossible because of (4.4). This contradiction proves (4.6).

Next choose $t_1 \ge t^*$ and $x_1 \in (l_-(t_1),\, l_+(t_1))$ arbitrarily, and put $\theta_1 := \arctan u_x(x_1,t_1)$. Then by (4.6) we have $-\psi_+ < \theta_1 < \psi_-$. Arguing as above, we see that the tangent line of the curve $y = u(x,t_1)$ at $x = x_1$ is given in the form $y = w_{\theta_1}(x,t_1) - m$ for some $m \ge 0$. Thus, by Corollary 2.6 (ii) and (4.4), we have

$$\mathcal{Z}_{\mathbb{R}}[u_*(\cdot,t) - \big(w_{\theta_1}(\cdot,t) - m\big)] = 0 \quad \text{for}\ \ t > t_1.$$

This implies that $u(x,t_1) - \big(w_{\theta_1}(x,t_1) - m\big)$ does not change sign at $x = x_1$. Consequently, the curve $y = u(x,t_1)$ lies below the tangent line at $x = x_1$, for every $x_1 \in (l_-(t_1),\, l_+(t_1))$. Hence $u_{xx}(x,t_1) \le 0$ on $[l_-(t_1),\, l_+(t_1)]$. The inequality $u_{xx} < 0$ follows from the strong maximum principle. This completes the proof of the theorem. □

4.3. **Semi-concavity result for type (C).** In the case of type (C) solutions, we are not able to prove the eventual concavity result (Theorem 1.4) by simply using the intersection number argument as we have done for type (B) solutions. Instead we shall prove a slightly weaker result by a similar argument. This result will be used to establish the uniform boundedness of the aspect ratio (Proposition 6.2), which leads to the proof of Theorem 1.7 on the convergence to self-similar shrinking solutions. The proof of Theorem 1.4 will then be completed in Section 6 by using Theorem 1.7.

**Lemma 4.3.** *Let* $(u, l_\pm)$ *be a type* (C) *solution of* (P)*. Then there exist* $t^* \in [0, T)$ *and a constant* $\delta_0 > 0$ *such that, for every* $t \in (t^*, T)$*, there exist* $\xi_-(t), \xi_+(t)$ *with* $l_-(t) < \xi_-(t) < \xi_+(t) < l_+(t)$ *such that*

$$\begin{cases} u_x(x,t) \geq \delta_0 & \textit{for} \;\; x \in [l_-(t),\, \xi_-(t)], \\ |u_x(x,t)| \leq \delta_0, \;\; u_{xx}(x,t) < 0 & \textit{for} \;\; x \in [\xi_-(t),\, \xi_+(t)], \\ u_x(x,t) \leq -\delta_0 & \textit{for} \;\; x \in [\xi_+(t),\, l_+(t)]. \end{cases} \tag{4.7}$$

*Proof.* By the estimate (2.38), there exists a constant $M > 0$ such that $I(t) \subset [-M, M]$ for all $t \in [0, T)$. Since $u^0 > 0$ in the interior of $I^0 = I(0)$ and since $\psi_\pm \neq 0$, we can find constants $\delta_0, h_0 > 0$ such that any straight line of the form

$$y = (\tan\theta)x + h_0 - m \qquad (\,|\tan\theta| \leq \delta_0,\; m \geq 0\,) \tag{4.8}$$

intersects the extended graph $y = u^0_*(x)$ precisely at two points. Replacing $\delta_0, h_0 > 0$ by smaller constants if necessary, we may assume that $h_0 = 2M\delta_0$. Then, as is easily seen, any straight line that passes through or under the rectangle $D := \{|x| \leq M,\, 0 \leq y \leq h_0/2\}$ with slope between $-\delta_0$ and $\delta_0$ belongs to the family of lines given in (4.8), therefore it intersects $u^0_*$ precisely twice. Now let $t^* \in (0, T)$ be such that $\|u(\cdot, t)\|_{L^\infty} \leq h_0/2$ for all $t \in [t^*, T)$. Then the graph of $u(x, t)$ is contained in the rectangle $D$ for $t \in [t^*, T)$. Arguing as in the latter part of the proof of Theorem 1.3, we see that, for any $t \in [t^*, T)$, the graph $y = u(x, t)$ lies below any tangent line whose slope $\tan\theta$ satisfies $|\tan\theta| \leq \delta_0$. This proves (4.7). □

## 5. Some criteria for shrinking, bounded and expanding

In this section, we provide some criteria for shrinking, bounded and expanding solutions. We have already given a simple criterion in Corollary 3.12 using shrinking and expanding circles. Here we shall offer two other criteria. Another topic to be discussed in this section concerns the sharp transition between shrinking and expanding solutions when a family of initial data are given.

We begin with a sufficient condition for the expanding case.

**Proposition 5.1.** *Let $(u, l_\pm)$ be a solution of problem* (P). *Suppose that*

$$A(0) > \frac{1}{2\pi}\left(\frac{\psi_+ + \psi_-}{c}\right)^2. \tag{5.1}$$

*Then $(u, l_\pm)$ is of type* (A). *Furthermore, $A(t)$ is strictly increasing for all $t > 0$.*

*Proof.* Let $\tilde{\Gamma}(t)$ denote the closed curve consisting of $\Gamma(t)$ and its mirror image below the $x$-axis. The length of $\tilde{\Gamma}(t)$ equals $2L(t)$ and the area enclosed by $\tilde{\Gamma}(t)$ equals $2A(t)$. By the isoperimetric inequality (see, for example, [11, p.33]), we have

$$2L(t) \geq \sqrt{4\pi \times 2A(t)} = 2\sqrt{2\pi A(t)}.$$

Hence, by (2.36),

$$A'(t) \geq -(\psi_+ + \psi_-) + c\sqrt{2\pi A(t)}.$$

Thus, if (5.1) holds, then $\delta := A'(0) > 0$, which implies $A'(t) \geq \delta$ for all $t \in [0, T)$. The desired result then follows from Theorem 1.1. $\square$

Next we give criteria for the shrinking case. The first one follows from Corollary 3.12:

**Proposition 5.2.** *Let $(u, l_\pm)$ be a solution of problem* (P). *Suppose that*

$$\left(\frac{l(0)}{2}\right)^2 + \big(h(0)\big)^2 + \frac{2\cos\psi_{\min}}{c}\, h(0) < \frac{\sin^2\psi_{\min}}{c^2} \tag{5.2}$$

*or that*

$$L(0) < \frac{2(1-\cos\psi_{\min})}{c}, \tag{5.3}$$

*where $\psi_{\min} = \min\{\psi_-, \psi_+\}$. Then $(u, l_\pm)$ is of type* (C).

*Proof.* The condition (5.2) is equivalent to $(l/2)^2 + (h + R_0\cos\psi_{\min})^2 < R_0^2$, where $R_0 = 1/c$. Therefore it guarantees that a rectangle of width $l(0)$ and height $h(0)$ (in which $\Gamma(0)$ can be confined) lies strictly below a circular arc of radius $R_0 = 1/c$ with contact angle $\theta_0 = \psi_{min}$. Thus the conclusion follows from Corollary 3.12. Next, since $L \geq \sqrt{l^2 + 4h^2}$, we have

$$\left(\frac{l(0)}{2}\right)^2 + \big(h(0)\big)^2 + \frac{2\cos\psi_{\min}}{c}\, h(0) \leq \left(\frac{L(0)}{2}\right)^2 + \frac{\cos\psi_{\min}}{c}\, L(0).$$

Hence the condition (5.3) implies (5.2). The proposition is proved. $\square$

The following two criteria for type (C) are restricted to concave curves. The first one is a modification of (5.3). The second one is derived by a totally different method.

**Proposition 5.3.** *Let $(u, l_\pm)$ be a solution of problem* (P) *such that $(u^0)_{xx} \leq 0$ on $[l_0^-, l_0^+]$. If either of the following holds, then $(u, l_\pm)$ is of type* (C).

$$L(0) < \frac{2\sin\psi_{\min}}{c}\left(\sqrt{1 + \left(\frac{2\cos\psi_{\min}}{1+\sin\psi_{\min}}\right)^2} - \frac{2\cos\psi_{\min}}{1+\sin\psi_{\min}}\right), \tag{5.4}$$

$$L(0) < \frac{(\psi_+ + \psi_-)^2}{c(\cot\psi_+ + \cot\psi_- + \psi_+ + \psi_-)}. \tag{5.5}$$

*Proof.* The concavity of $\Gamma(0)$ implies $h \leq \dfrac{\sin\psi_{\min}}{1+\sin\psi_{\min}} L$. In view of this and $L \geq \sqrt{l^2 + 4h^2}$, one can easily deduce (5.2) from (5.4), which proves the first part of the proposition.

Next, by Theorem 1.2, $u(\cdot, t)$ is concave for all $t \in [0, T)$. Thus, by (2.36) and (2.39),

$$L'(t) \leq c(\cot\psi_+ + \cot\psi_- + \psi_+ + \psi_-) - \int_0^{L(t)} \kappa^2\, ds.$$

On the other hand, by the Cauchy inequality,

$$(\psi_+ + \psi_-)^2 = \Big|\int_0^{L(t)} \frac{d\theta}{ds}\, ds\Big|^2 = \Big|\int_0^{L(t)} \kappa\, ds\Big|^2 \leq L(t) \int_0^{L(t)} \kappa^2\, ds.$$

Consequently

$$L'(t) \leq c(\cot\psi_+ + \cot\psi_- + \psi_+ + \psi_-) - \frac{(\psi_+ + \psi_-)^2}{L(t)}.$$

Thus (5.5) implies $L(t) \to 0$ as $t \to T$. The proof of the proposition is complete. $\square$

**Remark 5.1.** The condition (5.4) is only marginally better than (5.3) when $\psi_{\min}$ is close to $\pi/2$, but it is much better when $\psi_{\min}$ is close to 0. On the other hand, the condition (5.5) is not a good criterion when both $\psi_+, \psi_-$ are close to 0, but it is better than (5.3) and (5.4) when $\psi_+, \psi_-$ are close to $\pi/2$.

Next we consider a one-parameter family of initial data with monotone dependence on the parameter. More precisely, let $\{v_\lambda\}_{\lambda>0}$ be a family of functions defined on $I_\lambda^0 := [l_{\lambda,-}^0, l_{\lambda,+}^0]$ that belong to $C^2(I_\lambda^0)$ for some $0 < \alpha < 1$ and satisfy

$$v_\lambda(x) > 0 \ (x \in (l_{\lambda,-}^0, l_{\lambda,+}^0)), \quad v_\lambda(l_{\lambda,-}^0) = v_\lambda(l_{\lambda,+}^0) = 0, \quad v_\lambda'(l_{\lambda,\pm}^0) = \mp\tan\psi_\pm \tag{5.6}$$

for all $\lambda > 0$. Assume that

$$\begin{cases} \text{(V1)} \ \ I_\lambda^0 \subset I_\mu^0, \quad v_\lambda(x) \leq v_\mu(x),\ x \in I_\lambda^0, \quad v_\lambda \not\equiv v_\mu \quad \text{for any} \ \ 0 < \lambda < \mu\,; \\ \text{(V2)} \ \ l_{\lambda,\pm}^0 \text{ is continuous in } \lambda \text{ and so is } v_\lambda(x) \text{ for each fixed } x; \\ \qquad\ \ \text{here we understand that } v_\lambda \equiv 0 \text{ outside } I_\lambda^0. \\ \text{(V3)} \ \ \|v_\lambda\|_{C^2(I_\lambda^0)} \text{ is bounded on any compact interval of } \lambda\,; \\ \text{(V4)} \ \ \lim\limits_{\lambda\to 0} \|v_\lambda\|_{L^\infty} = \lim\limits_{\lambda\to 0} (l_{\lambda,+}^0 - l_{\lambda,-}^0) = 0\,; \\ \text{(V5)} \ \ \lim\limits_{\lambda\to\infty} \displaystyle\int_{I_\lambda^0} v_\lambda(x)dx = \infty. \end{cases} \tag{5.7}$$

A typical example of such a family is given by

$$v_\lambda(x) = \lambda u^0(x/\lambda), \qquad \lambda > 0,$$

where $u^0(x)$ is a function satisfying (1.6), along with an extra condition that guarantees that $v_\lambda$ depends on $\lambda$ monotonically. This last condition is satisfied if the graph $y = u^0(x)$ is star-shaped with respect to the origin.

Our goal is to show that all the three types of behaviors (A), (B), (C) appear in this family and that the transition from type (A) to type (C) is sharp.

**Theorem 5.1.** *Let* $\{v_\lambda\}_{\lambda>0}$ *satisfy* (5.6), (5.7) *and let* $(u_\lambda, l_{\lambda,\pm})$ *be the solution of* (P) *for* $u^0 = v_\lambda$ *and* $l^0_\pm = l^0_{\lambda,\pm}$. *Then there exists a constant* $\lambda^*$ *such that*

(i) $(u_\lambda, l_{\lambda,\pm})$ *is of type* (A) *for all* $\lambda > \lambda^*$,
(ii) $(u_\lambda, l_{\lambda,\pm})$ *is of type* (B) *for* $\lambda = \lambda^*$,
(iii) $(u_\lambda, l_{\lambda,\pm})$ *is of type* (C) *for all* $0 < \lambda < \lambda^*$.

*Proof.* Let $A_\lambda(t)$, $L_\lambda(t)$ denote the quantities in (1.7) for $u = u_\lambda$ and define

$$\mathcal{A} := \{\lambda > 0 \mid (u_\lambda, l_{\lambda,\pm}) \text{ is of type (A)}\}, \quad \mathcal{C} := \{\lambda > 0 \mid (u_\lambda, l_{\lambda,\pm}) \text{ is of type (C)}\}.$$

Clearly $\mathcal{A} \neq \emptyset$ and $\mathcal{C} \neq \emptyset$. Indeed, by the condition (V5) in (5.7) and Proposition 5.1, $(u_\lambda, l_{\lambda,\pm})$ is of type (A) for all sufficiently large $\lambda$. On the other hand, if $\lambda$ is sufficiently small, then, by the condition (V4) and Proposition 5.2, $(u_\lambda, l_{\lambda,\pm})$ is of type (C). Now set

$$\lambda^* := \inf \mathcal{A}, \qquad \lambda_* := \sup \mathcal{C}.$$

Then, by the monotonicity of $\lambda \mapsto v_\lambda$ and the comparison principle, we have

$$(0, \lambda_*) \subset \mathcal{C}, \qquad (\lambda^*, \infty) \subset \mathcal{A}.$$

Next we show that both $\mathcal{A}$ and $\mathcal{C}$ are open subsets of $\mathbb{R}$. Choose $\lambda_0 \in \mathcal{A}$ arbitrarily. Then, since $A_{\lambda_0}(t) \to \infty$ as $t \to \infty$, we have

$$A_{\lambda_0}(t_0) > \frac{1}{2\pi}\left(\frac{\psi_+ + \psi_-}{c}\right)^2$$

for some $t_0 \geq 0$. By the conditions (V2), (V3) above and the Ascoli-Arzelà theorem, the zero extension of $v_\lambda$ over $\mathbb{R}$ depends on $\lambda$ continuously in $L^\infty(\mathbb{R})$. Therefore, by Theorem 7.2, $A_\lambda(t_0)$ is continuous at $\lambda = \lambda_0$. Consequently, the same inequality as above holds for all $\lambda$ sufficiently close to $\lambda_0$; hence, by Proposition 5.1, $\lambda_0$ is an interior point of $\mathcal{A}$. The openness of $\mathcal{C}$ can be shown similarly by using Proposition 5.2. Thus

$$\mathcal{C} = (0, \lambda_*), \qquad \mathcal{A} = (\lambda^*, \infty).$$

Hence, by Theorem 1.1, $(u_\lambda, l_{\lambda,\pm})$ is of type (B) for every $\lambda \in [\lambda_*, \lambda^*]$.

It remains to show that $\lambda_* = \lambda^*$. For this, we need Theorem 1.6, which is to be proved in Section 6. We argue by contradiction. Suppose that $\lambda_* < \lambda^*$. Then by (V1) and the strong comparison principle (Proposition 3.5),

$$l_{\lambda^*,-}(t) < l_{\lambda_*,-}(t) < l_{\lambda_*,+}(t) < l_{\lambda^*,+}(t), \qquad u_{\lambda_*}(x,t) < u_{\lambda^*}(x,t) \ \ (x \in I_{\lambda_*}(t))$$

for all $t > 0$. Take any $t_1 > 0$. Then, by the above inequality, $u_{\lambda_*}(x+\varepsilon, t_1) < u_{\lambda^*}(x, t_1)$ for any sufficiently small constant $\varepsilon > 0$. Hence, by the comparison principle,

$$u_{\lambda_*}(x+\varepsilon, t) \le u_{\lambda^*}(x,t) \quad \text{for all } x \in I_{\lambda_*}(t_1),\ t \ge t_1,\ 0 < \varepsilon \ll 1.$$

By Theorem 1.6, both $u_{\lambda_*}$ and $u_{\lambda^*}$ converge to a traveling wave (or a stationary solution) of the form $\Phi(x - \nu t - a)$ and $\Phi(x - \nu t - b)$, respectively. Letting $t \to \infty$ in the above inequality, we obtain $\Phi(\xi + \varepsilon - a) \le \Phi(\xi - b)$ for all $\xi \in \mathbb{R}$. Hence $a - \varepsilon = b$ for all small $\varepsilon > 0$, which is clearly impossible. This contradiction shows $\lambda_* = \lambda^*$ and the proof of the theorem is complete. □

## 6. Asymptotic behaviors

In this section we shall prove Theorems 1.5, 1.6 and 1.7 concerning the asymptotic behavior of solutions for each of the types (A), (B) and (C). As before, $[0, T)$, $0 < T \le +\infty$, will denote the maximal time interval for the existence of the solution $(u, l_\pm)$.

6.1. **The expanding case.** First we deal with type (A) solutions, for which $T = +\infty$ and $A(t), L(t) \to +\infty$ as $t \to +\infty$.

*Proof of Theorem 1.5.* Define

$$g(x,t) := tG(x/t) = \begin{cases} \tan\psi_-(x - \hat{p}t), & \hat{p}t \le x \le -ct\sin\psi_-, \\ \sqrt{(ct)^2 - x^2}, & -ct\sin\psi_- \le x \le ct\sin\psi_+, \\ -\tan\psi_+(x - \hat{q}t), & ct\,\sin\psi_+ \le x \le \hat{q}t, \end{cases} \tag{6.1}$$

where $G$ is as in (1.12) and $(\hat{p}, \hat{q}) = (-c/\sin\psi_-, c/\sin\psi_+)$. The function $g$ represents a curve consisting of a portion of a circle of radius $ct$ and two line segments. This curve moves with normal velocity $V = c$. Thus, as we have seen in Section 1, this is a self-similar solution of (1.8).

Since $-\hat{p}t, \hat{q}t \to \infty$ and $ct \to \infty$ as $t \to \infty$, we can choose $t_0 > 0$ such that

$$-\hat{p}t_0 \le l_-^0, \quad l_+^0 \le \hat{q}t_0, \quad u^0(x) \le g(x, t_0) \quad \text{for all } x \in [l_-^0, l_+^0].$$

For each $t \ge 0$, $g(x,t)$ is a concave function on $\{-\hat{p}t \le x \le \hat{q}t\}$ which is $C^1$ in $(x,t)$ and is also $C^2$ in $x$ except at $x = \pm ct\sin\psi_\pm$. This and (1.8) imply

$$g_t \ge \frac{g_{xx}}{1+g_x^2} + c\sqrt{1+g_x^2}$$

on $\{-\hat{p}t < x < \hat{q}t, t > 0\}$ in the sense of distributions. Moreover, we have

$$g(-\hat{p}t, t) = g(\hat{q}t, t) = 0, \quad g_x(-\hat{p}t, t) = \tan\psi_-, \quad g_x(\hat{q}t, t) = -\tan\psi_+.$$

Since $g(x,t)$ loses $C^2$ smoothness at $\{x = \pm ct\sin\psi_\pm\}$, we cannot use Proposition 3.4 directly. However, since $g \in W^{2,1}_\infty$, we can apply Remark 3.1, to obtain

$$-\hat{p}(t+t_0) \le l_-(t), \quad l_+(t) \le \hat{q}(t+t_0), \qquad \text{for } \ t \ge 0, \tag{6.2}$$

$$u(x,t) \le g(x,t+t_0) \qquad \text{for } \ x \in [l_-(t), l_+(t)], \ t \ge 0. \tag{6.3}$$

To find a sub-solution, we first consider the following function:

$$\sqrt{R(t)^2 - x^2}, \quad x \in [-R(t), R(t)], \ t > 0,$$

where $R(t)$ is the solution of (3.4) with $R_0 > 1/c$. This function represents a half circle of radius $R(t)$, and $R(t) \to \infty$ as $t \to \infty$. Moreover, l'Hôpital's rule gives

$$\lim_{t\to\infty} \frac{R(t)}{ct} = \lim_{t\to\infty} \left[1 - \frac{1}{cR(t)}\right] = 1. \tag{6.4}$$

Now we set $\underline{\rho}(t) = R(t)/c$ and define

$$\underline{g}(x,t) := \underline{\rho}(t) G(x/\underline{\rho}(t)). \tag{6.5}$$

Then we have $\underline{g}(x,t) > 0$ for $x \in (r_-(t), r_+(t)), \ t \ge 0$, and

$$\underline{g}(r_\pm(t), t) = 0, \quad \underline{g}_x(r_\pm(t), t) = \mp \tan\psi_\pm,$$

where $r_\pm(t) := \pm R(t)/\sin\psi_\pm$. The function $\underline{g}$ represents a curve consisting of a portion of a circle of radius $R(t)$ and two line segments. As is easily seen, this curve moves with normal velocity $V = \dot{R}(t) = c - 1/R(t)$ everywhere. Furthermore, the curvature of this curve satisfies

$$\kappa = \begin{cases} -\dfrac{1}{R(t)}, & -R(t)\sin\psi_- < x < R(t)\sin\psi_+, \\ 0, & r_-(t) < x < -R(t)\sin\psi_-, \quad R(t)\sin\psi_+ < x < r_+(t). \end{cases}$$

Thus $V \le \kappa + c$ except at $x = \pm R(t)\sin\psi_\pm$. In view of these and the fact that $\underline{g} \in W^{2,1}_\infty$, we easily find that

$$\underline{g}_t - \frac{\underline{g}_{xx}}{1 + \underline{g}_x^2} - c\sqrt{1 + \underline{g}_x^2} \le 0$$

in the sense of distributions. Now $(u, l_\pm)$ is of type (A), we can find $t_1 > 0$ such that $u(x, t_1) \ge \underline{g}(x, 0)$ for all $x \in [r_-(0), r_+(0)]$. Hence, by Remark 3.1,

$$l_-(t+t_1) \le r_-(t), \quad r_+(t) \le l_+(t+t_1), \qquad \text{for } \ t \ge 0, \tag{6.6}$$

$$u(x, t+t_1) \ge \underline{g}(x,t) \qquad \text{for } \ x \in [r_-(t), r_+(t)], \ t \ge 0. \tag{6.7}$$

Combining (6.2) and (6.6), we obtain

$$R(t-t_1)/\sin\psi_+ \le l_+(t) \le c(t+t_0)/\sin\psi_+ \qquad \text{for all } \ t \ge t_1,$$
$$-c(t+t_0)/\sin\psi_- \le l_-(t) \le -R(t-t_1)/\sin\psi_- \qquad \text{for all } \ t \ge t_1.$$

Hence (1.14) follows by using (6.4). The inequalities (1.13) follow from (6.3) and (6.7) by setting $\rho(t) = \underline{\rho}(t-t_1)$. Finally (1.15) follows from (1.13) and the fact that $\lim_{t\to\infty} \rho(t)/t = 1$. This completes the proof of the theorem. □

6.2. **The bounded case.** In this subsection, we shall prove Theorem 1.6 concerning type (B) solutions. As shown in Remark 2.3, Theorem 1.6 (i) follows from Proposition 2.9 and Theorem 1.3. Therefore, we only need to prove the assertion (ii) of Theorem 1.6. Let us first state the following lemma on the uniform regularity of type (B) solutions:

**Lemma 6.1.** *Let* $(u, l_\pm)$ *be a type* (B) *solution of* (P). *Then*

$$\limsup_{t_0\to\infty} \Big( \|u\|_{C^{3,3/2}(D_{t_0,t_0+1})} + \|l_\pm\|_{C^{3/2}([t_0,t_0+1])} \Big) < \infty,$$

*where* $D_{t_0,t_0+1} := \{(x,t) \mid l_-(t) \le x \le l_+(t),\ t_0 \le t \le t_0+1\}$.

*Proof.* If $(u, l_\pm)$ is a type (B) solution, then, by Corollary 3.13, there exists $t^* \ge 0$ such that, for each fixed $t \ge t^*$, the function $u_x(x,t)$ vanishes only at its maximal point, say $x = \xi(t)$. Since $h(t) := u(\xi(t), t)$ remains away from zero as $t \to \infty$ by Lemma 3.2, we see from Lemmas 2.1 and 2.2 that $u_{xx}(x,t)$ remains uniformly bounded as $t \to \infty$. Consequently, for any $\alpha \in (1/2, 1)$, we have

$$\limsup_{t\to\infty} \big( \|u(\cdot,t)\|_{C^{1+\alpha}([l_-(t),l_+(t)])} + l_+(t) - l_-(t) \big) < \infty.$$

The conclusion of the lemma then follows from (7.4) in Theorem 7.1. □

*Proof of Theorem 1.6 (ii).* We first show that $\lim_{t\to\infty}\{l_-(t) - \nu t\}$ exists. By (4.1), $l_-(t) - \nu t$ remains bounded as $t \to \infty$. Suppose $\lim_{t\to\infty}\{l_-(t) - \nu t\}$ does not exist, and let $x_0$ be a point satisfying

$$\liminf_{t\to\infty}\{l_-(t) - \nu t\} < x_0 < \limsup_{t\to\infty}\{l_-(t) - \nu t\}.$$

Then $l_-(t) - (x_0 + \nu t)$ changes sign infinitely many times as $t$ varies over $[0, \infty)$. This, however, contradicts Corollary 2.7, since $x = x_0 + \nu t$ is the left endpoint of $\Phi(x - \nu t - x_0 - \beta)$ and both $u$ and $\Phi(x - \nu t - x_0 - \beta)$ satisfy (P). Hence $\lim_{t\to\infty}\{l_-(t) - \nu t\}$ exists and is finite. Set

$$\mu := \lim_{t\to\infty}\{l_-(t) - \nu t\}. \tag{6.8}$$

In what follows we shall prove that $u(x + \nu t, t) \to \Phi(x - \beta - \mu)$ as $t \to \infty$ in an appropriate sense. For this, we take any sequence $\{t_n\}$ tending to $\infty$ and set $u_n(x,t) := u(x + \nu t_n, t + t_n)$ for each $n$. Then, by Lemma 6.1, there is a subsequence of $\{t_n\}$, again denoted by $\{t_n\}$ for notational simplicity, such that

$$u_n(x,t) \to\ w^1(x,t) \quad \text{as} \quad n \to \infty,$$

where the convergence is taken place in the set of the closure of the support of $u$ in the $C^{2,1}$ sense. Moreover, the limit function $w^1$ is a solution of (1.1) satisfying

$$w^1(\pm\mu+\nu t,t)=0,\quad w^1_x(\pm\mu+\nu t,t)=\mp\tan\psi_\pm,$$
$$w^1(x,t)>0\quad\text{for}\quad -\mu+\nu t<x<\mu+\nu t.$$

This last statement is a consequence of (6.8) and the fact that

$$u(x+\nu t_n,t+t_n)\big|_{x=l_\pm(t+t_n)-\nu t_n}=0,\quad u_x(x+\nu t_n,t+t_n)\big|_{x=l_\pm(t+t_n)-\nu t_n}=\mp\tan\psi_\pm$$

for all $n$. On the other hand, the function $w^2(x,t):=\Phi(x-\nu t-\beta-\mu)$ satisfies $w^2(\mu+\nu t,t)=0$ and $w^2_x(\mu+\nu t,t)=-\tan\psi_+$ for all $t$. Thus, by Corollary 2.8, we have $w^1(x,t)\equiv w^2(x,t)$. Since this limit is independent of the choice of $\{t_n\}$, part (ii) of Theorem 1.6 is proved. □

6.3. **The shrinking case.** To prove Theorem 1.7, we first derive the boundedness of the aspect ratio $r(t):=l(t)/h(t)$. A lower bound of $r(t)$ is given in (2.40). The following result provides an upper bound for the aspect ratio (see [9] for the case $c=0$).

**Proposition 6.2.** *Let $(u,l_\pm)$ be a type* (C) *solution of* (P). *Then $r(t)$ is uniformly bounded for $t\in[0,T)$. Consequently there exist constants $0<K_1<K_2$ such that*

$$K_1\le\frac{l(t)}{\sqrt{T-t}}\le K_2,\quad K_1\le\frac{h(t)}{\sqrt{T-t}}\le K_2\quad\textit{for}\quad t\in[0,T).\tag{6.9}$$

To prove Proposition 6.2, we adopt the argument in [9], in which the method of Grayson [23] is modified to deal with free boundaries with different contact angles. However, because of the presence of the driving force $c>0$, further modification will be needed. We begin with a simple lemma concerning concave shapes.

**Lemma 6.3.** *Let $v(x)$ be a concave function on an interval $[x_0,x_2]$ satisfying $v(x)>0$ for $x\in(x_0,x_2)$, and let $x_1$ be a point in $(x_0,x_2)$. Set $a_1:=x_1-x_0$, $a_2:=x_2-x_1$ and*

$$S_1:=\int_{x_0}^{x_1}v(x)dx,\qquad S_2:=\int_{x_1}^{x_2}v(x)dx.$$

*Then*

$$\frac{a_1^2}{(a_1+a_2)^2}\le\frac{S_1}{S_1+S_2},\qquad\frac{a_2^2}{(a_1+a_2)^2}\le\frac{S_2}{S_1+S_2}.$$

The above lemma can be proved easily by an elementary geometric argument, so the proof is omitted. Note that it also follows from the fact that $h^{-2}\int_{x_0}^{x_0+h}v(x)dx$ is a decreasing function of $h>0$. The next lemma is due to [9], with slight modification:

**Lemma 6.4.** *Let $v(x)$ be a concave function on an interval $[x_0,x_3]$ satisfying $v(x)>0$ for $x\in(x_0,x_3)$ and choose points $x_0<x_1<x_2<x_3$ such that $S_2=S_1+S_3$, where*

$$S_1:=\int_{x_0}^{x_1}v(x)dx,\quad S_2:=\int_{x_1}^{x_2}v(x)dx,\quad S_3:=\int_{x_2}^{x_3}v(x)dx.$$

*Set* $a_1 := x_1 - x_0,\ a_2 := x_2 - x_1,\ a_3 := x_3 - x_2$ *and* $\mu := S_1/(S_1 + S_3) \in (0, 1)$. *Then there exist constants* $\omega, \omega' \in (0, \frac{1}{\sqrt{2}})$ *depending only on* $\mu$ *such that*

$$a_2 \leq \omega(a_1 + a_2 + a_3), \qquad a_1 \leq \omega'(a_1 + a_2 + a_3), \quad a_3 \leq \omega'(a_1 + a_2 + a_3).$$

*Proof.* By Lemma 6.3, we have

$$\frac{a_2^2}{(a_1 + a_2)^2} \leq \frac{S_2}{S_1 + S_2} = \frac{1}{1 + \mu}, \qquad \frac{a_2^2}{(a_2 + a_3)^2} \leq \frac{S_2}{S_2 + S_3} = \frac{1}{2 - \mu}.$$

Consequently

$$\frac{a_1 + a_2 + a_3}{a_2} \geq \sqrt{1 + \mu} + \sqrt{2 - \mu} - 1 > \sqrt{2}.$$

This proves the first inequality. The remaining inequalities follow from

$$\frac{a_1^2}{(a_1 + a_2 + a_3)^2} \leq \frac{S_1}{S_1 + S_2 + S_3} = \frac{\mu}{2}, \qquad \frac{a_3^2}{(a_1 + a_2 + a_3)^2} \leq \frac{S_3}{S_1 + S_2 + S_3} = \frac{1 - \mu}{2}.$$

The lemma is proved. □

Next, from (2.36) and the fact that $A(t) \to 0, L(t) \to 0$ as $t \nearrow T$, we see that

$$A(t) = (\psi_+ + \psi_-)(T - t) - c\int_t^T L(\tau)d\tau = (\psi_+ + \psi_-)(T - t) + o(T - t). \tag{6.10}$$

This and (2.40) imply

$$h(t) = O\big(\sqrt{T - t}\big). \tag{6.11}$$

Again by (2.40),

$$\left(\frac{l^2(t)}{M_1 A(t)}\right)^{1/2} \leq \frac{l(t)}{h(t)} \leq \frac{l^2(t)}{A(t)}. \tag{6.12}$$

Consequently, the aspect ratio $r(t) := l(t)/h(t)$ remains bounded as $t \nearrow T$ if and only if

$$\limsup_{t \to T} \frac{l(t)}{\sqrt{T - t}} < \infty. \tag{6.13}$$

Once the boundedness of $r(t)$ is established, then the estimates (6.9) follow easily from (6.10), (6.12) and the lower bound $r(t) \geq 2/M_1$ in (2.40). Thus all we need to prove is (6.13).

Now we define a sequence $0 < t_1 < t_2 < t_3 < \cdots \to T$ by

$$t_k = (1 - 2^{-k})T \quad (k = 1, 2, 3, \ldots).$$

Then the following holds:

**Lemma 6.5.** *The estimate* (6.13) *holds if*

$$\limsup_{k \to \infty} \frac{l(t_k)}{\sqrt{T - t_k}} < \infty. \tag{6.14}$$

*Proof.* By (2.38), $l(t) \leq l(t_k) + 2M(t - t_k)$ for $t \in [t_k, T)$, where $M := M_2 \cot \psi_{\min}$. Hence

$$\frac{l(t)}{\sqrt{T-t}} \leq \frac{l(t_k) + 2M(t-t_k)}{\sqrt{T-t}} \leq \frac{l(t_k) + 2M(T-t_k)}{\sqrt{T-t_{k+1}}} = \sqrt{2}\, \frac{l(t_k) + 2M(T-t_k)}{\sqrt{T-t_k}}$$

for $t \in [t_k, t_{k+1}],\ k = 1, 2, 3, \ldots$. It follows that

$$\limsup_{t \to T} \frac{l(t)}{\sqrt{T-t}} \leq \sqrt{2} \limsup_{k \to \infty} \frac{l(t_k)}{\sqrt{T-t_k}} < \infty.$$

The lemma is proved. □

Now we are ready to prove Proposition 6.2. We shall first prove it under the assumption that $u(x,t)$ is concave, and then prove it without this assumption.

*Proof of Proposition 6.2 (the concave case).* We assume that $u(x,t)$ is concave for all $t \in [0, T)$. By Lemma 6.5, it suffices to prove (6.14).

Suppose that (6.14) does not hold. Then, by a shift of indices, there exists a sequence $\{k_j + 1\}$ with $k_j \to \infty$ as $j \to \infty$ such that

$$\frac{l(t_{k_j+1})}{\sqrt{T - t_{k_j+1}}} \to \infty \quad \text{as } j \to \infty. \tag{6.15}$$

Without loss of generality, we may assume that

$$\frac{l(t_{k_j})}{\sqrt{T-t_{k_j}}} \leq \frac{l(t_{k_j+1})}{\sqrt{T-t_{k_j+1}}} \qquad (j = 1, 2, 3, \ldots).$$

Then, since $T - t_{k+1} = (T - t_k)/2$, we have

$$l(t_{k_j+1}) \geq l(t_{k_j})/\sqrt{2} \qquad (j = 1, 2, 3, \ldots). \tag{6.16}$$

On the other hand, by the same argument as in the proof of Lemma 6.5,

$$\sqrt{2}\, \frac{l(t_{k_j}) + 2M(T - t_{k_j})}{\sqrt{T - t_{k_j}}} \geq \frac{l(t_{k_j+1})}{\sqrt{T - t_{k_j+1}}}.$$

Hence we have

$$\frac{l(t_{k_j})}{\sqrt{T - t_{k_j}}} \to \infty \quad \text{as } j \to \infty. \tag{6.17}$$

Moreover, we have

$$l(t_{k_j}) \geq l(t_{k_j+1})/2 \quad \text{for large } j. \tag{6.18}$$

Indeed, by (6.17), it holds

$$l(t_{k_j}) \geq 2M\sqrt{T - t_{k_j}} \geq 2M(T - t_{k_j})$$

for all $j$ large enough. Combining with (2.38), we deduce that

$$l(t_{k_j+1}) \leq l(t_{k_j}) + 2M(t_{k_j+1} - t_{k_j}) \leq 2l(t_{k_j})$$

for all $j$ large enough. Hence (6.18) follows.

Now, for each $t \in [0,T)$, we define $r_i(t)$, $i = 0,1,2,3$, by

$$l_-(t) = r_0(t) < r_1(t) < r_2(t) < r_3(t) = l_+(t),$$

$$\int_{r_0(t)}^{r_1(t)} u(x,t)dx = \frac{\mu}{2}A(t), \qquad \int_{r_2(t)}^{r_3(t)} u(x,t)dx = \frac{1-\mu}{2}A(t),$$

where $\mu := \psi_-/(\psi_+ + \psi_-) \in (0,1)$. By Lemma 6.4, we have

$$\frac{r_1(t)-r_0(t)}{l(t)} \le \omega', \quad \frac{r_2(t)-r_1(t)}{l(t)} \le \omega, \quad \frac{r_3(t)-r_2(t)}{l(t)} \le \omega' \quad \text{for } t \in [0,T), \tag{6.19}$$

for some constants $\omega, \omega' \in (0, 1/\sqrt{2})$ that depend only on $\mu$. Also, by (2.38),

$$[\, l_-(t_{k_j+1}),\, l_+(t_{k_j+1})\,] \subset [\, l_-(t_{k_j}) - M(T-t_{k_j}),\, l_+(t_{k_j}) + M(T-t_{k_j})\,], \tag{6.20}$$

where $M := M_2 \cot \psi_{\min}$. Replacing $\{t_{k_j}\}$ by its subsequence if necessary, we may assume without loss of generality that the center of the interval $[l_-(t_{k_j+1}),\, l_+(t_{k_j+1})]$ lies on the left-hand side of the center of $[r_1(t_{k_j}), r_2(t_{k_j})]$ for all $j$, or on the right-hand side for all $j$. In what follows we assume the former, as the latter case can be treated in the same way. Thus

$$l_-(t_{k_j+1}) + l_+(t_{k_j+1}) \le r_1(t_{k_j}) + r_2(t_{k_j}) \quad (j = 1,2,3,\ldots). \tag{6.21}$$

Combining (6.21) with (6.16) and (6.19), we obtain

$$l_-(t_{k_j+1}) \le r_1^* - \frac{\sigma \ell^*}{2},$$

where $r_1^* := r_1(t_{k_j})$, $\ell^* := l(t_{k_j})$ and $\sigma := 1/\sqrt{2} - \omega$. Furthermore, combining (6.21) with (6.16), (6.17), (6.19) and (6.20), we get

$$l_+(t_{k_j+1}) \ge r_1^* + \sigma' \ell^* + o(\ell^*),$$

where $\sigma' := 1/\sqrt{2} - \omega'$. Hence, by (2.38),

$$l_-(t) \le r_1^* - \frac{\sigma \ell^*}{2} + o(\ell^*), \quad l_+(t) \ge r_1^* + \sigma' \ell^* + o(\ell^*) \quad \text{for } t \in [t_{k_j},\, t_{k_j+1}]. \tag{6.22}$$

Since $u(x,t)$ is concave, the first inequality in (6.22) implies

$$u_x(r_1^*, t) \le \frac{h(t)}{r_1^* - l_-(t)} \le \frac{2h(t)}{\sigma \ell^* + o(\ell^*)} \quad \text{for } t \in [t_{k_j},\, t_{k_j+1}].$$

Combining this and the fact that $h(t) = O\big(\sqrt{T-t}\big) = O\big(\sqrt{T-t_{k_j}}\big)$, we obtain

$$u_x(r_1^*, t) \le K \frac{\sqrt{T-t_{k_j}}}{l(t_{k_j})} := \varepsilon_j \quad \text{for } t \in [t_{k_j},\, t_{k_j+1}] \tag{6.23}$$

for some constant $K > 0$ and for all large $j$. By (6.17), we have $\varepsilon_j \to 0$ $(j \to \infty)$.

Now observe that

$$\begin{aligned}\frac{d}{dt}\int_{l_-(t)}^{r_1^*} u(x,t)dx &= \int_{l_-(t)}^{r_1^*}\Big(\frac{u_{xx}}{1+u_x^2}+c\sqrt{1+u_x^2}\Big)dx\\ &= \arctan u_x(r_1^*,t)-\psi_- + O(\ell^*)\\ &\le \varepsilon_j - \psi_- + O(\ell^*), \quad t\in[t_{k_j}, t_{k_j+1}].\end{aligned}$$

Since $\varepsilon_j \to 0$ as $j\to\infty$, we obtain

$$\begin{aligned}\int_{l_-(t_{k_j})}^{r_1^*} u(x,t_{k_j})dx - \int_{l_-(t_{k_j+1})}^{r_1^*} u(x,t_{k_j+1})dx &= \psi_-(t_{k_j+1}-t_{k_j}) + o(t_{k_j+1}-t_{k_j})\\ &= \frac{\psi_-}{2}(T-t_{k_j}) + o(T-t_{k_j+1}).\end{aligned}$$

On the other hand, by the definition of the point $r_1^* = r_1(t_{k_j})$ and (6.10), we have

$$\int_{l_-(t_{k_j})}^{r_1^*} u(x,t_{k_j})dx = \frac{\mu}{2}A(t_{k_j}) = \frac{\psi_-}{2}(T-t_{k_j}) + o\big(T-t_{k_j}\big).$$

Combining these estimates, we obtain

$$\int_{l_-(t_{k_j+1})}^{r_1^*} u(x,t_{k_j+1})dx = o\big(T-t_{k_j+1}\big) = o\big(A(t_{k_j+1})\big). \tag{6.24}$$

Therefore, by the concavity of $u(x,t_{k_j+1})$ and Lemma 6.3, along with (6.16) and (6.18),

$$r_1^* - l_-(t_{k_j+1}) = o\big(l(t_{k_j+1})\big) = o\big(l(t_{k_j})\big).$$

On the other hand, the first inequality in (6.22) implies

$$r_1^* - l_-(t_{k_j+1}) \ge \frac{\sigma}{2}\, l(t_{k_j}) + o\big(l(t_{k_j})\big),$$

which is a contradiction. This contradiction proves (6.14), and the proof of Proposition 6.2 is complete under the assumption that $u$ is concave. □

*Proof of Proposition 6.2 (the general case).* The proof for the general case is almost the same as above, with only a minor modification. Instead of concavity, we shall use the property given in Lemma 4.3. Note also that the assertions (6.15)-(6.17) hold without the concavity assumption.

The strategy is, as before, to derive a contradiction from the assumption (6.15). We define $r_0(t), r_1(t), r_2(t), r_3(t)$ exactly the same way. However, instead of the estimate (6.19), we shall derive

$$\frac{r_1(t_{k_j})-\xi_-(t_{k_j})}{l(t_{k_j})} \le \omega' + o(1), \quad \frac{r_2(t_{k_j})-r_1(t_{k_j})}{l(t_{k_j})} \le \omega, \quad \frac{\xi_+(t_{k_j})-r_2(t_{k_j})}{l(t_{k_j})} \le \omega' + o(1), \tag{6.25}$$

where $o(1)$ denotes a term that tends to 0 as $j\to\infty$ and $\xi_\pm$ are defined in Lemma 4.3. For this, we may assume without loss of generality that $\xi_-(t_{k_j}) < r_1(t_{k_j}) < r_2(t_{k_j}) < \xi_+(t_{k_j})$. Otherwise, there is nothing to prove.

By Lemma 4.3, we have

$$h(t) \geq u(\xi_\pm(t), t) \geq \pm[l_\pm(t) - \xi_\pm(t)]\delta_0. \tag{6.26}$$

Then, using (6.11) and (6.17), it follows from (6.26) that

$$|\xi_\pm(t_{k_j}) - l_\pm(t_{k_j})| \leq \delta_0^{-1} h(t_{k_j}) \leq C\frac{\sqrt{T - t_{k_j}}}{l(t_{k_j})} l(t_{k_j}) = o\big(l(t_{k_j})\big).$$

Hence

$$\eta(t_{k_j}) := \xi_-(t_{k_j}) - l_-(t_{k_j}) + l_+(t_{k_j}) - \xi_+(t_{k_j}) = o\big(l(t_{k_j})\big). \tag{6.27}$$

Also, by (6.26) and the concavity of $u(\cdot, t)$ in $[\xi_-(t), \xi_+(t)]$,

$$\begin{aligned} A(t_{k_j}) &\geq \int_{\xi_-(t_{k_j})}^{\xi_+(t_{k_j})} u(x,t)dx \\ &\geq \frac{1}{2}[u(\xi_+(t_{k_j}), t_{k_j}) + u(\xi_-(t_{k_j}), t_{k_j})] \cdot [\xi_+(t_{k_j}) - \xi_-(t_{k_j})] \\ &\geq \frac{1}{2}\delta_0 \eta(t_{k_j})[l(t_{k_j}) - \eta(t_{k_j})]. \end{aligned} \tag{6.28}$$

Now, Lemma 2.1 implies that

$$\left| \int_{l_\pm(t_{k_j})}^{\xi_\pm(t_{k_j})} u(x, t_{k_j})dx \right| \leq \frac{1}{2} M_1 [\xi_\pm(t_{k_j}) - l_\pm(t_{k_j})]^2 \leq \frac{1}{2} M_1 \eta^2(t_{k_j}).$$

Therefore, using (6.27) and (6.28), we obtain

$$\left| \int_{l_\pm(t_{k_j})}^{\xi_\pm(t_{k_j})} u(x, t_{k_j})dx \right| \leq \frac{1}{\delta_0} \frac{A(t_{k_j})}{l(t_{k_j}) - \eta(t_{k_j})} M_1 \eta(t_{k_j}) = o\big(A(t_{k_j})\big).$$

Hence, applying Lemma 6.4 to the interval $[\xi_-(t_{k_j}),\ \xi_+(t_{k_j})]$, we obtain (6.25). Similarly, the same estimate as (6.22), with $l_\pm$ being replaced by $\xi_\pm$, can be derived. The same estimate as (6.23) also follows from the inequality $u_x(r_1^*, t) \leq h(t)/(r_1^* - \xi_-(t))$. Then, using (6.25) instead of (6.19) and arguing as in the concave case with only a minor modification, we obtain the estimate (6.24), hence a contradiction. This completes the proof of Proposition 6.2. □

To proceed further, we first derive the following estimates from Proposition 6.2.

**Lemma 6.6.** *Let* $(u, l_\pm)$ *be a solution of type* (C) *and* $(w, p, q)$ *be the corresponding solution of* (1.17)-(1.20). *Then there is a constant* $C > 0$ *such that*

$$0 \leq w(z,s),\ |w_z(z,s)|,\ |w_{zz}(z,s)| \leq C \quad \text{for } p(s) \leq z \leq q(s),\ s \geq s_0, \tag{6.29}$$

$$-C \leq p(s) < q(s) \leq C \quad \text{for } s \geq s_0. \tag{6.30}$$

$$|p'(s)|,\ |q'(s)| \leq C \quad \text{for } s \geq s_0. \tag{6.31}$$

*Proof.* By (6.11), we obtain

$$0 \leq w(z,s) = \frac{u(x,t)}{\sqrt{2(T-t)}} \leq \frac{h(t)}{\sqrt{2(T-t)}} \leq C_1$$

for some positive constant $C_1$. Note that $w_z(z,s) = u_x(x,t)$. Hence, by Lemma 2.1, $|w_z(z,s)| = |u_x(x,t)| \leq M_1$. By Corollary 3.13 and recall that $d(t)$ is defined in Lemma 2.2, there exists $t^* \in (0,T)$ such that $d(t) = \min_{\tau\in[0,t]} h(\tau)$ for $t \in [t^*,T)$. Due to $h(t) \to 0$ as $t \nearrow T$, it follows from Lemma 2.1 and (2.3) that there exists $C_2 > 0$ such that

$$\frac{-C_2}{\min_{\tau\in[0,t]} h(\tau)} \leq u_{xx}(x,t) \leq C_2 \quad \text{for all } x \in [l_-(t), l_+(t)],\ t \in [t^*,T). \tag{6.32}$$

For each $t \in [t^*,T)$, there exists $t_m \in [0,t]$ such that $h(t_m) = \min_{\tau\in[0,t]} h(\tau)$. By (6.9) and (6.32) that there exists a positive constant $C_5$ such that

$$|w_{zz}(z,s)| \;=\; \sqrt{2(T-t)}\,|u_{xx}(x,t)| \leq C_2\frac{\sqrt{2(T-t)}}{h(t_m)} \leq C_2\|r\|_{L^\infty}\frac{\sqrt{2(T-t)}}{l(t_m)} \leq C_5$$

for all $z \in [p(s),q(s)]$ and $s \geq -\frac{1}{2}\ln(T-t^*)$. This completes the proof of (6.29).

Finally, for (6.30), note that $l_+(T) = l_-(T) = 0$ by assumption. It follows from (6.13) and (2.39) that

$$\frac{l_+(t)}{\sqrt{T-t}} \;=\; \frac{l(t)+l_-(t)}{\sqrt{T-t}} \leq \frac{l(t)+(c/\sin\psi_-)(T-t)}{\sqrt{T-t}} \leq C_4 \quad \text{for all } t \in [0,T)$$

for some positive constant $C_4$. This implies that $q(s) \leq C$ for all $s \geq s_0$. Similarly, we can derive that $p(s) \geq -C$ for all $s \geq s_0$. For (6.31), differentiating $l_-(t) = \sqrt{2(T-t)}p(s)$ in $t$ and using the identity (2.37), we have

$$p'(s) = p(s) + \sqrt{2(T-t)}\cot\psi_- u_t(l_-(t),t).$$

By the equation (1.1) and the identity $w_{zz}(z,s) = \sqrt{2(T-t)}u_{xx}(x,t)$,

$$p'(s) = p(s) + \frac{c}{\sin\psi_-}\sqrt{2(T-t)} + w_{zz}(z,s)\frac{\cos^2\psi_-}{\tan\psi_-}$$

Hence, using (6.29) and (6.30) we see that $|p'(s)| \leq C$ for some constant $C > 0$. Similarly, we can derive that $|q'(s)| \leq C$ for some constant $C > 0$. Hence we have completed the proof of Lemma 6.6. □

The next lemma follows immediately from (6.9).

**Lemma 6.7.** *There is a constant $k_1 > 0$ such that $q(s) - p(s) \geq k_1$ for all $s \geq s_0$.*

Now, we are ready to give a proof of Theorem 1.7. Since the proof is standard, we only outline it here.

*Proof of Theorem 1.7.* First, we introduce the Lyapunov functional borrowed from [28]:

$$E[w(\cdot,s)] := \int_{p(s)}^{q(s)} \exp\left\{-\left(\frac{z^2+w^2(z,s)}{2}\right)\right\}\sqrt{1+w_z^2(z,s)}\,dz.$$

By using (1.18) and (1.19), we compute that

$$\frac{d}{ds}E[w(\cdot,s)] = -\int_{p(s)}^{q(s)} w_s^2(z,s)\exp\left\{-\left(\frac{z^2+w^2(z,s)}{2}\right)\right\}[1+w_z^2(z,s)]^{-1/2}dz + \mathcal{J}, \tag{6.33}$$

where

$$\begin{aligned}\mathcal{J} &= q'(s)e^{-q^2(s)/2}\cos\psi_+ - p'(s)e^{-p^2(s)/2}\cos\psi_- \\ &\quad + \sqrt{2}ce^{-s}\int_{p(s)}^{q(s)}\exp\left\{-\left(\frac{z^2+w^2(z,s)}{2}\right)\right\}w_s(z,s)dz.\end{aligned}$$

In fact, a direct calculation gives

$$\begin{aligned}\frac{d}{ds}E[w(\cdot,s)] &= \int_{p(s)}^{q(s)}\left[e^{-(z^2+w^2)/2}\sqrt{1+w_z^2}(-ww_s) + e^{-(z^2+w^2)/2}(1+w_z^2)^{-1/2}w_zw_{zs}\right]dz \\ &\quad + e^{-q^2(s)/2}\sqrt{1+\tan^2\psi_+}\,q'(s) - e^{-p^2(s)/2}\sqrt{1+\tan^2\psi_-}\,p'(s).\end{aligned}$$

Then, by an integration by parts, we have

$$\begin{aligned}&\int_{p(s)}^{q(s)} e^{-(z^2+w^2)/2}(1+w_z^2)^{-1/2}w_zw_{zs}dz \\ &= e^{-(z^2+w^2)/2}(1+w_z^2)^{-1/2}w_zw_s\Big|_{p(s)}^{q(s)} - \int_{p(s)}^{q(s)} w_s\left[e^{-(z^2+w^2)/2}(1+w_z^2)^{-1/2}w_z\right]_z dz.\end{aligned}$$

We compute that

$$\left[e^{-(z^2+w^2)/2}(1+w_z^2)^{-1/2}w_z\right]_z = e^{-(z^2+w^2)/2}(1+w_z^2)^{-1/2}\left(\frac{w_{zz}}{1+w_z^2} - zw_z - ww_z^2\right).$$

Differentiating (1.18) with respect to $s$, we deduce that

$$w_s(p(s),s) = -\tan\psi_-\,p'(s), \quad w_s(q(s),s) = \tan\psi_+\,q'(s).$$

Hence we obtain

$$\begin{aligned}\frac{d}{ds}E[w(\cdot,s)] &= -\int_{p(s)}^{q(s)} e^{-(z^2+w^2)/2}(1+w_z^2)^{-1/2}w_s\left(\frac{w_{zz}}{1+w_z^2} - zw_z + w\right)dz \\ &\quad + q'(s)e^{-q^2(s)/2}\cos\psi_+ - p'(s)e^{-p^2(s)/2}\cos\psi_-\end{aligned}$$

and so (6.33) follows by using (1.17).

From Lemma 6.6 we see that

$$\sup_{s_0<s<\infty}\left|\int_{s_0}^{s}\mathcal{J}(\tau)d\tau\right| \le C < +\infty.$$

Therefore, for any given real numbers $a, b$ with $b > a \ge s_0$, one can derive that

$$\int_a^b\int_{p(s)}^{q(s)} w_s^2(z,s)\exp\left\{-\left(\frac{z^2+w^2(z,s)}{2}\right)\right\}[1+w_z^2(z,s)]^{-1/2}dzds \le C \tag{6.34}$$

for some finite constant $C > 0$ independent of $a, b$.

In order to prove Theorem 1.7, it suffices to show that, for any sequence $s_n \nearrow +\infty$, the sequence $\{w(z,s_n), p(s_n), q(s_n)\}$ has a subsequence that converges to $(\varphi, \bar{p}, \bar{q})$ as $n \to \infty$. To do so, we define

$$w_n(z,s) := w(z, s+s_n), \quad p_n(s) := p(s+s_n), \quad q_n(s) := q(s+s_n).$$

We then convert the free boundary into a fixed boundary by the transformation

$$\hat{w}(\zeta,s) := \frac{w(p(s)+\zeta(q(s)-p(s)),s)}{q(s)-p(s)}, \quad \hat{w}_n(\zeta,s) := \frac{w(p_n(s)+\zeta(q_n(s)-p_n(s)),s)}{q_n(s)-p_n(s)}.$$

Then $\hat{w}(\zeta,s)$ satisfies

$$\begin{aligned}\hat{w}_s \;=\; & \frac{1}{(q(s)-p(s))^2}\frac{\hat{w}_{\zeta\zeta}}{1+\hat{w}_\zeta^2} + \left[\frac{(1-\zeta)p'(s)+\zeta q'(s)}{q(s)-p(s)} - \frac{(1-\zeta)p(s)+\zeta q(s)}{q(s)-p(s)}\right]\hat{w}_\zeta \\ & + \Big(1 - \frac{q'(s)-p'(s)}{q(s)-p(s)}\Big)\hat{w} + \frac{\sqrt{2}ce^{-s}}{q(s)-p(s)}\sqrt{1+\hat{w}_\zeta^2}, \quad \zeta \in (0,1),\ s \geq s_0\end{aligned}$$

with $\hat{w}(0,s) = \hat{w}(1,s) = 0$, $\hat{w}_z(0,s) = \tan\psi_-, \hat{w}_z(1,s) = -\tan\psi_+$ for $s \geq s_0$. The same equation and the same boundary conditions are satisfied by $\hat{w}_n$ except that the term $e^{-s}$ on the right-hand side is replaced by $e^{-(s+s_n)}$. By Lemma 6.6 and Lemma 6.7, the above equation is uniformly parabolic and its coefficients are uniformly bounded. Thus we can apply parabolic $L^p$ estimates and the Sobolev embedding theorem to conclude that $\|\hat{w}_n\|_{C^{1+\alpha,(1+\alpha)/2}([0,1]\times[0,1])} \leq C$ for some positive constant $C$. Furthermore, by the interior Schauder estimates, for any $0 < \varepsilon < 1/2$ we have $\|\hat{w}_n\|_{C^{2+\alpha,1+\alpha/2}([\varepsilon,1-\varepsilon]\times[0,1])} \leq C_\varepsilon$ for some positive constant $C_\varepsilon$ independent of $n$. Thus, there exists $\hat{w}^* \in C^{2,1}((0,1)\times[0,1]) \cap C^{1,1/2}([0,1]\times[0,1])$ and a subsequence of $\{\hat{w}_n\}$, still denoted by $\{\hat{w}_n\}$, such that

$$\hat{w}_n \to \hat{w}^* \quad \text{in} \quad C^{2,1}((0,1)\times[0,1]) \cap C^{1,1/2}([0,1]\times[0,1]) \quad \text{as} \quad n \to \infty. \tag{6.35}$$

By (6.30), (6.31) and the Ascoli-Arzelà theorem, we may assume without loss of generality that, $p_n \to p^*, q_n \to q^*$ in $C([0,1])$ as $n \to \infty$. Clearly, $\hat{w}^*(\zeta,s)$ satisfies

$$\begin{aligned}\hat{w}_s^* =& \frac{1}{(q^*-p^*)^2}\frac{\hat{w}^*_{\zeta\zeta}}{1+(\hat{w}^*_\zeta)^2} + \left[\frac{(1-\zeta)p^{*\prime}+\zeta q^{*\prime}}{q^*-p^*} - \frac{(1-\zeta)p^*+\zeta q^*}{q^*-p^*}\right]\hat{w}^*_\zeta \\ & + \Big(1 - \frac{q^{*\prime}-p^{*\prime}}{q^*-p^*}\Big)\hat{w}^*, \quad \zeta \in (0,1),\ s \in [0,1]\end{aligned}$$

and the same boundary conditions as above. Therefore, if we set

$$w^*(z,s) := (q^*(s)-p^*(s))\ \hat{w}^*\Big(\frac{z-p^*(s)}{q^*(s)-p^*(s)}, s\Big),$$

then, $w^*$ satisfies

$$w^*_s = \frac{w^*_{zz}}{1+(w^*_z)^2} - zw^*_z + w^*, \quad z \in (p^*(s), q^*(s)),\ s \in [0,1] \tag{6.36}$$

$$w^*(p^*(s), s) = w^*(q^*(s), s) = 0, \quad s \in [0,1], \tag{6.37}$$

$$w^*_z(p^*(s), s) = \tan\psi_-, \quad w^*_z(q^*(s), s) = -\tan\psi_+, \quad s \in [0,1], \tag{6.38}$$

We next prove $w^*_s \equiv 0$ , $p^*(s) \equiv \bar{p}$ and $q^*(s) \equiv \bar{q}$ for $z \in (p^*(s), q^*(s))$, $s \in [0,1]$. For this, it follows from (6.34) that

$$\begin{aligned} &\int_0^1 \int_{p(s+s_n)}^{q(s+s_n)} w_s^2(z, s+s_n) \exp\left\{-\left(\frac{z^2 + w^2(z, s+s_n)}{2}\right)\right\} [1 + w_z^2(z, s+s_n)]^{-1/2} dzds \\ \le &\int_{s_n}^{\infty} \int_{p(s)}^{q(s)} w_s^2(z,s) \exp\left\{-\left(\frac{z^2 + w^2(z,s)}{2}\right)\right\} [1 + w_z^2(z,s)]^{-1/2} dzds. \end{aligned}$$

Letting $n \to \infty$ and recalling (6.34) give

$$\int_0^1 \int_{p^*(s)}^{q^*(s)} (w^*_s)^2(z,s) \exp\left\{-\left(\frac{z^2 + (w^*)^2(z,s)}{2}\right)\right\} [1 + (w^*_z)^2(z,s)]^{-1/2} dzds = 0,$$

which implies $w^*_s \equiv 0$ on $(p^*(s), q^*(s)) \times [0,1]$. Thus $w^*$ is a stationary solution of (6.36)-(6.38). Since the solution of (1.21)-(1.23) is unique (see [9]), we see that $p^*(s) \equiv \bar{p}$, $q^*(s) \equiv \bar{q}$ and $w^*(z,s) \equiv \varphi(z)$. This implies, in particular, that $\{(w(z, s_n), p(s_n), q(s_n))\}$ converges to $(\varphi(z), \bar{p}, \bar{q})$ up to a subsequence. Furthermore, the limit is independent of the choice of $\{s_n\}$. Therefore, we have $(w(z,s), p(s), q(s)) \to (\varphi(z), \bar{p}, \bar{q})$ as $s \to \infty$. The proof of Theorem 1.7 is complete. □

**Remark 6.1.** If we use the higher-order boundary derivative estimates in Section 7, then we easily see that the convergence in (6.35) actually takes place in $C^{2,1}([0,1] \times [0,1])$, or even in $C^\infty$. More precisely, let $v(y,\tau)$ be the function defined just below (7.9) (which represents the angle). Then $v_y$ represents a normalized curvature, and we have

$$v_y = \frac{L(t)}{\sqrt{T-t}} \, \frac{w_{zz}}{(1+w_z^2)^{3/2}}.$$

By Lemma 6.6, we see that $v_y$ is uniformly bounded as $\tau \to \infty$ (or, equivalently, as $s \to \infty$). Thus $\|v(\cdot,\tau)\|_{C^\alpha([0,1])}$ remains bounded as $\tau \to \infty$ for any $1/2 < \alpha < 1$. Consequently, by Lemma 7.1 (ii), $\|v(\cdot,\tau)\|_{C^{k+1+\alpha}([0,1])}$ remains bounded as $\tau \to \infty$ for any $k \in \mathbb{N}$. As is easily seen, this implies that $\|\hat{w}(\cdot,s)\|_{C^{k+2+\alpha}([0,1])}$ remains bounded for all large $s$. Hence the convergence in (6.35) takes place in $C^{2m,m}([0,1] \times [0,1])$ for any $m \in \mathbb{N}$. Combining this observation with the latter argument in the proof of Theorem 1.7 above, we see that

$$\hat{w}(\zeta, s) \to \frac{\varphi(\bar{p} + \zeta(\bar{q} - \bar{p}))}{\bar{q} - \bar{p}} \quad \text{in} \quad C^{2m}([0,1]) \quad \text{as} \quad s \to \infty. \tag{6.39}$$

for any $m \in \mathbb{N}$.

Finally, we give a proof of Theorem 1.4.

*Proof of Theorem 1.4.* Differentiating (1.21) yields $\eta' = z(1+\varphi_z^2)\eta$, where $\eta := \varphi_{zz}/(1+\varphi_z^2)$. Therefore $\eta$ does not change sign. Furthermore,

$$\int_{\bar{p}}^{\bar{q}} \eta(z)\,dz = -(\psi_+ + \psi_-) < 0;$$

hence $\eta$ is strictly negative on $\bar{p} \le z \le \bar{q}$, which implies

$$\varphi_{zz} < 0 \quad \text{on} \quad [\bar{p}, \bar{q}]. \tag{6.40}$$

By (6.39) (with $m = 1$) and (6.40), we have $\hat{w}_{\zeta\zeta} < 0$ for all $\zeta \in [0,1]$ and all sufficiently large $s$, which implies $w_{zz} < 0$ for $z \in [p(s), q(s)]$ and for all sufficiently large $s$. This implies the concavity of $u$ for $t$ sufficiently close to $T$, and the proof of the theorem is complete. □

## 7. Appendix

In this appendix, we shall provide a proof of the local existence and uniqueness theorem for (P) and discuss continuous dependence of solutions on the initial data. As before, the initial data $u^0$ is taken from the following class:

$$u^0(x) > 0 \ \text{ for } x \in (l_-^0, l_+^0)\ ,\ u^0(l_\pm^0) = 0,\ u_x^0(l_\pm^0) = \mp \tan\psi_\pm,\ u^0 \in h^{1+\alpha}([l_-^0, l_+^0]), \tag{7.1}$$

where $h^{1+\alpha}$ denotes the "little Hölder space" of exponent $1+\alpha$ with $0 < \alpha < 1$. Here, the little Hölder spaces on a closed bounded interval $I \subset \mathbb{R}$ are defined by

$$h^\alpha(I) := \Big\{ f \in C^\alpha(I) \,|\, \lim_{\delta \to 0} \sup_{x,y \in I, |x-y| \le \delta} \frac{|f(x) - f(y)|}{|x-y|^\alpha} = 0 \Big\},$$
$$h^{k+\alpha}(I) := \{ f \in C^k(I) \,|\, f^{(k)} \in h^\alpha(I) \},$$

where $k \in \mathbb{N}$ and $0 < \alpha < 1$. The space $h^\alpha(I)$ can be characterized as the closure of $C^\beta(I)$ in the space $C^\alpha(I)$, where $\beta \in (\alpha, \infty]$ is arbitrary (see, e.g., Proposition 0.2.1 of [32]). We also define

$$h_0^\alpha(I) := \{ f \in h^\alpha(I) \mid f(x) = 0 \ \text{ for } \ x \in \partial I \}.$$

It is not difficult to see that

$$h_0^\alpha(I) = \overline{C_c^\beta(I)}^{C^\alpha(I)} \tag{7.2}$$

for any $\beta \in (\alpha, \infty]$, where, with a slight abuse of the notation, $C_c^\beta(I)$ denotes the set of $C^\beta$ functions on the bounded closed interval $I$ that vanish in some neighborhood of $\partial I$, and $\overline{X}^{C^\alpha}$ denotes the closure of the set $X$ in the $C^\alpha$ topology (see Remark 7.1 below).

The existence theorem for the problem (P) with $c = 0$ was proved in [8] for initial data $u^0 \in C^{1+\alpha}, 0 < \alpha < 1$. Their method is first to normalize the interval $[l_-(t), l_+(t)]$ by the coordinate change $(x - l_-(t))/((l_+(t) - l_-(t)))$, thereby transforming (P) into a quasilinear parabolic equation on a fixed interval. Then they use a rather delicate optimal regularity theory for parabolic operators in order to obtain sufficient estimates for contraction mapping argument. This makes their proof technically highly involved and not easy to follow. Furthermore, they did not prove continuous dependence of solutions on the initial data.

In the present paper we take a totally different approach. We first rewrite (P) as an equation for the angle variable $\theta = \arctan u_x$ and use the normalized arclength parameter as the space variable. The problem (P) will then be converted into a simple semilinear problem, which makes it possible to apply a more elementary and well-documented general theory for semilinear problems.

We assume $\alpha \in (1/2, 1)$ for technical reasons. This makes our assumption slightly stronger than that in [8], but our approach has a great advantage in that our proof is more elementary and more self-contained; furthermore, the continuous dependence result follows almost for free. We also note that, unlike [8], we are adopting the little Hölder space $h^{1+\alpha}$ instead of $C^{1+\alpha}$, as it is a more natural space for discussing parabolic initial value problems.

**Remark 7.1.** While the assertion $h^\alpha(I) = \overline{C^\beta(I)}^{C^\alpha(I)}$ can be found in many textbooks, we could not find a precise reference for (7.2). For the convenience of the reader, let us give an outline of the proof of (7.2). Let $f$ be any element of $h_0^\alpha(I)$, and let $\tilde{f}$ denote the 0-extension of $f$ outside $I$. Clearly $\tilde{f} \in h^\alpha(\mathbb{R})$. Now, for each $\lambda > 1$, we define

$$f^\lambda(x) := \tilde{f}(x_0 + \lambda(x - x_0))|_I,$$

where $x_0$ is an arbitrary interior point of $I$, which we fix. Then we have $f^\lambda \in h_c^\alpha(I)$, where, again with a slight abuse of the notation, $h_c^\alpha(I)$ denotes the set of $h^\alpha(I)$ functions that vanish in some neighborhood of $\partial I$. By the definition of $h^\alpha$, we easily see that $f^\lambda$ depends on $\lambda$ continuously in the topology of $C^\alpha(I)$. Thus, letting $\lambda \downarrow 1$, we have $f^\lambda \to f$ in $h_0^\alpha(I)$, which implies that $h_c^\alpha(I)$ is dense in $h_0^\alpha(I)$. Next let $g$ be an arbitrary element of $h_c^\alpha(I)$. Fix $\rho \in C_c^\infty(\mathbb{R})$ with $\int_{\mathbb{R}} \rho(x)dx = 1$, and set $\rho^\varepsilon(x) := \varepsilon^{-1}\rho(\varepsilon^{-1}x)$. Then $\rho^\varepsilon * g \in C_c^\infty(I)$ for all sufficiently small $\varepsilon > 0$. Furthermore, as shown in [32], we have $\rho^\varepsilon * g \to g$ as $\varepsilon \downarrow 0$ in the $C^\alpha$ topology. This means that $C_c^\infty(I)$ is dense in $h_c^\alpha(I)$, hence in $h_0^\alpha(I)$. This proves (7.2).

7.1. **Local existence.** Let us begin with the local existence result.

**Theorem 7.1.** *Assume* (7.1) *for some* $l_-^0 < l_+^0$ *with* $\alpha \in (1/2, 1)$. *Assume also that*

$$\|u^0\|_{C^{1+\alpha}([l_-^0, l_+^0])} + (l_+^0 - l_-^0) \le N \tag{7.3}$$

*for some constant* $N > 0$. *Then there exist* $T_1$ *and* $N_1 > 0$ *depending only on* $N$ *such that the problem* (P) *has a unique classical solution*

$$u \in C^{1+\alpha,(1+\alpha)/2}(\overline{D}_{T_1}) \cap C^{2+\alpha,1+\alpha/2}(D_{T_1}), \quad l_\pm \in C([0,T_1]) \cap C^{1+\alpha/2}((0,T_1]),$$

*where* $D_{T_1} := \{(x,t) \mid l_-(t) \le x \le l_+(t),\ 0 < t \le T_1\}$, *and that this solution satisfies*

$$\|u(\cdot,t)\|_{C^{1+\alpha}([l_-(t),l_+(t)])} + |l_+(t)| + |l_-(t)| \le N_1, \qquad t \in [0,T_1].$$

*Furthermore,* $u(x,t)$ *is* $C^\infty$ *in* $D_{T_1}$ *and* $l_\pm(t)$ *is* $C^\infty$ *in* $(0,T_1]$, *and for any* $T_0 \in (0,T_1)$ *and* $k \in \mathbb{N}$, *there exists a constant* $N_2$ *depending only on* $N, k$ *and* $T_0$ *such that*

$$\|u\|_{C^{k+2+\alpha,(k+1+\alpha)/2}(D_{T_1}\setminus D_{T_0})} + \|l_+\|_{C^{(k+1+\alpha)/2}([T_0,T_1])} + \|l_-\|_{C^{(k+1+\alpha)/2}([T_0,T_1])} \le N_2. \tag{7.4}$$

We shall prove the above theorem by converting the quasilinear free boundary problem (P) into a semilinear problem with a fixed boundary and applying standard results for abstract semilinear parabolic equations. First note that the function $\theta(x,t) := \arctan u_x(x,t)$ satisfies

$$\begin{aligned} \theta_t &= (\cos^2\theta)\,\theta_{xx} + c(\sin\theta)\,\theta_x, && x \in (l_-(t), l_+(t)),\ t > 0,\\ \theta &= \mp\psi_\pm, && x = l_\pm(t),\ t > 0. \end{aligned}$$

Now we introduce the arclength parameter $s$ by

$$s(x,t) = \int_{l_-(t)}^{x} \sqrt{1 + u_{\tilde{x}}^2(\tilde{x},t)}\, d\tilde{x}$$

and define a function $\Theta(s,t)$ by $\Theta(s(x,t),t) = \theta(x,t)$. Hereafter in order not to get confusion we put tilde on the dummy variable in the integral to distinguish it from the variable. Then $\Theta$ satisfies

$$\begin{aligned} &\Theta_t = \Theta_{ss} + \Big\{c(\Theta - \psi_- - \cot\psi_-) - \cot\psi_- \Theta_s(0,t) + \int_0^s \Theta_s^2\, d\tilde{s}\Big\}\Theta_s,\ s \in (0, L(t)),\ t > 0,\\ &\Theta(0,t) = \psi_-,\ \Theta(L(t),t) = -\psi_+, \qquad t > 0 \end{aligned}$$

along with the following free boundary condition:

$$L'(t) = \cot\psi_+(\Theta_s(L(t),t) + c) + \cot\psi_-(\Theta_s(0,t) + c) + c(\psi_+ + \psi_-) - \int_0^{L(t)} \Theta_s^2\, ds,\ t > 0,$$

where $L(t)$ is the length of the curve defined by (1.7). Here we have used the identity

$$\Theta_s(s,t) = \frac{u_{xx}}{(1+u_x^2)^{3/2}} \tag{7.5}$$

and the equation

$$l_-'(t) = -\frac{\Theta_s(0,t) + c}{\sin\psi_-}, \quad t > 0. \tag{7.6}$$

The relation (7.6) is obtained by differentiating (1.2) with respect to $t$, and then using (1.1) and (1.3) along with the identity (7.5).

Once the solution $(\Theta, L)$ of the above free boundary problem is given, one can obtain the value of $l_-(t)$ by integrating (7.6) with $l_-(0) = l_-^0$. Then the function $s(x,t)$ can be recovered from the identity

$$x = l_-(t) + \int_0^{s(x,t)} \cos\Theta(\tilde{s},t)\,d\tilde{s}. \tag{7.7}$$

In view of these and the relation $u_x(x,t)dx = \sin\Theta(s,t)ds$, one can recover the original solution $(u, l_\pm)$ of (P) by

$$\begin{cases} u(x,t) = \displaystyle\int_0^{s(x,t)} \sin\Theta(\tilde{s},t)\,d\tilde{s}, \\ l_-(t) = l_-^0 - \displaystyle\int_0^t \frac{\Theta_s(0,\tilde{t}) + c}{\sin\psi_-}\,d\tilde{t}, \\ l_+(t) = l_-(t) + \displaystyle\int_0^{L(t)} \cos\Theta(s,t)\,ds. \end{cases} \tag{7.8}$$

We further rewrite the above free boundary problem into a problem on a fixed interval $[0,1]$ by using the transformation

$$\tau = \int_0^t \frac{d\tilde{t}}{L^2(\tilde{t})}, \qquad y = \frac{s}{L(t)}. \tag{7.9}$$

(Here the variable $y$ represents a normalized arclength parameter and has nothing to do with the $y$ axis.) Then the above problem is reduced to the following problem for $v(y,\tau) = \Theta(s,t)$, $\eta(\tau) = \ln L(t)$:

$$\begin{cases} v_\tau = v_{yy} + \{P(y,\tau,v,\eta) + Q(\tau,v,\eta)y\}v_y, \quad y \in (0,1),\ \tau > 0, \\ \eta' = Q(\tau,v,\eta), \quad \tau > 0, \\ v(0,\tau) = \psi_-,\ v(1,\tau) = -\psi_+, \quad \tau > 0, \end{cases} \tag{7.10}$$

where

$$\begin{aligned} P(y,\tau,v,\eta) &= ce^{\eta(\tau)}(v - \psi_- - \cot\psi_-) - (\cot\psi_-)v_y(0,\tau) + \int_0^y v_y^2\,dy, \\ Q(\tau,v,\eta) &= (\cot\psi_+)(v_y(1,\tau) + ce^{\eta(\tau)}) + (\cot\psi_-)(v_y(0,\tau) + ce^{\eta(\tau)}) \\ &\quad + ce^{\eta(\tau)}(\psi_+ + \psi_-) - \int_0^1 v_y^2\,dy. \end{aligned}$$

The initial conditions for (7.10) are given by

$$v^0(y) := v(y,0) = \arctan\{u_x^0(\zeta^{-1}(y))\}, \qquad \eta^0 := \eta(0) = \ln L(0), \tag{7.11}$$

where $\zeta^{-1}$ is the inverse function of

$$y = \zeta(x) = \frac{1}{L(0)} \int_{l_-^0}^x \sqrt{1 + (u_x^0)^2}\,dx. \tag{7.12}$$

By (7.9), we have

$$t = t(\tau) := \int_0^\tau e^{2\eta(\tau)}\,d\tau, \quad s = s(y,\tau) := ye^{\eta(\tau)}, \qquad \Theta(s(y,\tau), t(\tau)) = v(y,\tau).$$

This and (7.8) imply that, once the solution $(v,\eta)$ of (7.10) is obtained, one can recover the original solution $(u,l_\pm)$ of (P) as follows:

$$\begin{cases} u(x,t) = \displaystyle\int_0^{y(x,t)} e^{\eta(\tau(t))} \sin v(y,\tau(t))\,dy, \\ l_-(t) = l_-^0 - \displaystyle\int_0^{\tau(t)} \frac{e^{\eta(\tau)}\,v_y(0,\tau) + ce^{2\eta(\tau)}}{\sin\psi_-}\,d\tau, \\ l_+(t) = l_-(t) + \displaystyle\int_0^1 e^{\eta(\tau(t))} \cos v(y,\tau(t))\,dy, \end{cases} \tag{7.13}$$

where $y(x,t)$ and $\tau(t)$ are defined through the relations:

$$t = \int_0^{\tau(t)} e^{2\eta(\tau)}\,d\tau, \quad x = l_-(t) + \int_0^{y(x,t)} e^{\eta(\tau(t))} \cos v(y,\tau(t))\,dy, \tag{7.14}$$

which follows from (7.7). Note that $(v,\eta)$ determine $(\tau(t),l_\pm(t))$ first, and $(y(x,t),u(x,t))$ is determined later from these ones.

To prove Theorem 7.1, it suffices to establish the local existence, uniqueness and regularity results for the problem (7.10)-(7.11). We first introduce the following weighted space:

$$C_{[\beta]}((0,\mathcal{T}]\,;\,Y) := \{U \in C((0,\mathcal{T}])\,;\,Y) \mid \sup_{\tau\in(0,\mathcal{T})} \|\tau^\beta U(\tau)\|_Y < \infty\},$$

where $Y$ is any Banach space.

The following lemma gives existence and regularity of solutions of the normalized problem (7.10)-(7.11).

**Lemma 7.1.** *Assume $v^0 \in h^\alpha([0,1])$ for some $\alpha \in (1/2,1)$ and $v^0(0) = \psi_-, v^0(1) = -\psi_+$. Suppose that*

$$\|v^0\|_{C^\alpha([0,1])} + |\eta^0| \le N \tag{7.15}$$

*for some constant $N > 0$. Then there exist positive constants $\mathcal{T}_1, N_1$ depending only on $N$ such that*

(i) *a unique solution $(v,\eta) \in C_{[\frac{1-\alpha}{2}]}((0,\mathcal{T}_1]\,;\,C^1([0,1]) \times \mathbb{R})$ to the problem* (7.10)-(7.11) *exists and it satisfies*

$$\|v(\cdot,\tau)\|_{C^\alpha([0,1])} + |\eta(\tau)| \le N_1, \quad \tau^{\frac{1-\alpha}{2}}\|v(\tau)\|_{C^1([0,1])} \le N_1 \qquad \tau \in [0,\mathcal{T}_1]. \tag{7.16}$$

*Moreover, it is a classical solution:*

$$v \in C^{\alpha,\alpha/2}([0,1]\times[0,\mathcal{T}_1]) \cap C^{2+\alpha,1+\alpha/2}_{loc}([0,1]\times(0,\mathcal{T}_1]),$$
$$\eta \in C([0,\mathcal{T}_1]) \cap C^{1+\alpha/2}_{loc}((0,\mathcal{T}_1]).$$

(ii) *For any $\mathcal{T}_0 \in (0,\mathcal{T}_1)$ and $k \in \mathbb{N}$, there exists a constant $N_2$ depending only on $N,k,\mathcal{T}_0$ such that*

$$\|v\|_{C^{k+1+\alpha,(k+1+\alpha)/2}([0,1]\times[\mathcal{T}_0,\mathcal{T}_1])} + \|\eta\|_{C^{(k+1+\alpha)/2}([\mathcal{T}_0,\mathcal{T}_1])} \le N_2.$$

(iii) *If* $(v,\eta),(\bar v,\bar\eta)$ *are solutions with initial data* $(v^0,\eta^0),(\bar v^0,\bar\eta^0)$ *satisfying* (7.15), *respectively, then there exists a positive constant* $N_3$ *(depending only on* $N$*) such that for any* $\tau\in(0,\mathcal{T}_1]$,

$$\|v(\cdot,\tau)-\bar v(\cdot,\tau)\|_{C^\alpha([0,1])}+\tau^{\frac{1-\alpha}{2}}\|v_y(\cdot,\tau)-\bar v_y(\cdot,\tau)\|_{C([0,1])}+|\eta(\tau)-\bar\eta(\tau)|$$
$$\le N_3(\|v^0-\bar v^0\|_{C^\alpha([0,1])}+|\eta^0-\bar\eta^0|).$$

This lemma shall be proved in the framework of an abstract parabolic equation in the following form:

$$\begin{cases}\dfrac{dU}{d\tau}=AU+F(U), & \tau>0,\\ U(0)=U^0.\end{cases}\tag{7.17}$$

Here $A$ denotes the generator of an analytic semigroup on a Banach space $X$ with domain $\mathcal{D}(A)$ and $F:X_\beta\to X$ is a function satisfying the condition (7.20) below, where $\beta\in(0,1)$, and $X_\beta$ denotes the usual interpolation space between $X_0:=X$ and $X_1:=\mathcal{D}(A)$; namely

$$X_\beta:=(X,\mathcal{D}(A))_{\beta,\infty}=\{w\in X\mid \limsup_{t\searrow 0} t^{1-\beta}\|Ae^{tA}w\|_X<\infty\}.$$

In what follows, we shall consider the problem (7.17) in the interpolation space $X_\delta$ with $0<\delta<\beta$, and look for a solution that belongs to $C_{[\beta-\delta]}((0,\mathcal{T}_1]\,;X_\beta)\cap C([0,\mathcal{T}_1]\,;X_\delta)$. Here, by a solution, we mean a mild solution, namely a function $U(\tau)$ satisfying

$$U(\tau)=e^{\tau A}U^0+\int_0^\tau e^{(\tau-\sigma)A}F\big(U(\sigma)\big)\,d\sigma.\tag{7.18}$$

**Proposition 7.2.** *Let* $p>1$, $1>\beta>\delta>0$ *be constants satisfying*

$$p(\beta-\delta)<1-\delta,\tag{7.19}$$

*and let* $F:X_\beta\to X_0$ *be a map satisfying*

$$\|F(U_1)-F(U_2)\|_{X_0}\le C_F\|U_1-U_2\|_{X_\beta}\,(\|U_1\|_{X_\beta}^{p-1}+\|U_2\|_{X_\beta}^{p-1}+1)\tag{7.20}$$

*for some constant* $C_F>0$. *Then, for each* $N>0$, *there exist* $\mathcal{T}_1,N_1,N_3>0$ *depending only on* $A,F$ *and* $N$ *such that the following* (i),(ii),(iii) *hold.*

(i) *For any* $U^0\in X_\delta$ *with* $\|U^0\|_{X_\delta}\le N$, *there exists a unique mild solution* $U\in C_{[\beta-\delta]}((0,\mathcal{T}_1]\,;X_\beta)$ *to the problem* (7.18) *and this unique solution satisfies*

$$\|U\|_{C_{[\beta-\delta]}((0,\mathcal{T}_1]\,;X_\beta)}\le N_1,\qquad \sup_{\tau\in[0,\mathcal{T}_1]}\|U(\tau)\|_{X_\delta}\le N_1.\tag{7.21}$$

(ii) *Let* $U(\tau)$ *be as in* (i). *Then* $U(\tau)\to U^0$ *in* $X_\delta$ *as* $\tau\to 0$ *if and only if* $U^0\in\overline{\mathcal{D}(A)}^{X_\delta}$.

(iii) *If* $U,V$ *are mild solutions with initial data* $U^0,V^0\in\overline{\mathcal{D}(A)}^{X_\delta}$ *satisfying*

$$\|U^0\|_{X_\delta},\ \|V^0\|_{X_\delta}\le N,$$

*then*

$$\|U(\tau)-V(\tau)\|_{X_\delta}+\tau^{\beta-\delta}\|U(\tau)-V(\tau)\|_{X_\beta}\le N_3\|U^0-V^0\|_{X_\delta},\quad \tau\in(0,\mathcal{T}_1].$$

*If, in addition,* $U^0,V^0\in X_\beta$ *and satisfy* $\|U^0\|_{X_\beta},\|V^0\|_{X_\beta}\le N$*, then*

$$\|U(\tau)-V(\tau)\|_{X_\beta}\le N_3\|U^0-V^0\|_{X_\beta},\quad \tau\in[0,\mathcal{T}_1].$$

(iv) *If* $U^0\in\overline{\mathcal{D}(A)}^{X_\delta}$*, then* $U$ *is a classical solution that belongs to* $C^1((0,\mathcal{T}_1];X)\cap C((0,\mathcal{T}_1];\mathcal{D}(A))$*. Furthermore, for any* $\gamma\in(0,1)$ *and* $\mathcal{T}_0\in(0,\mathcal{T}_1)$*, it holds that* $U\in C^{1+\gamma}([\mathcal{T}_0,\mathcal{T}_1];X)\cap C^\gamma([\mathcal{T}_0,\mathcal{T}_1];\mathcal{D}(A))$*, and that there exists a constant* $N_2>0$ *depending only on* $A$,$F$, $N$, $\gamma$, $\mathcal{T}_0$ *such that* $\|U\|_{C^{1+\gamma}([\mathcal{T}_0,\mathcal{T}_1]);X)}+\|U\|_{C^\gamma([\mathcal{T}_0,\mathcal{T}_1]);\mathcal{D}(A))}\le N_2$.

*Outline of the proof.* The existence statement (i) follows by applying the contraction mapping theorem to the integral equation (7.18) in the space $C_{[\beta-\delta]}((0,\mathcal{T}_1]\,;X_\beta)$. The argument is rather standard; see Theorem 7.1.5 (ii) of [32]. (In [32], the author assumes $U^0\in\overline{\mathcal{D}(A)}^{X_\delta}$ and applies the contraction argument in the space $C_{[\beta-\delta]}((0,\mathcal{T}_1]\,;X_\beta)\cap C([0,\mathcal{T}_1]\,;X_\delta)$ instead of $C_{[\beta-\delta]}((0,\mathcal{T}_1]\,;X_\beta)$, but the basic arguments are the same.) For the reader's convenience, let us give a brief outline of this argument. We begin with the estimates (7.21). Let $\lambda\in[\delta,\beta]$. Then, by (7.18) and (7.20) (with $U_1=U,U_2=0$), we have

$$\begin{aligned}
\tau^{\lambda-\delta}\|U(\tau)\|_{X_\lambda}&\le\tau^{\lambda-\delta}\|e^{\tau A}U^0\|_{X_\lambda}+\tau^{\lambda-\delta}\int_0^\tau\|e^{(\tau-\sigma)A}F(U(\sigma))\|_{X_\lambda}\,d\sigma\\
&\le C_{A,\lambda,\delta}\|U^0\|_{X_\delta}+C_{A,\lambda,0}\tau^{\lambda-\delta}\int_0^\tau(\tau-\sigma)^{-\lambda}\|F(U(\sigma))\|_{X_0}\,d\sigma\\
&\le C_{A,\lambda,\delta}\|U^0\|_{X_\delta}+C_{A,\lambda,0}\,C_F\tau^{\lambda-\delta}\int_0^\tau(\tau-\sigma)^{-\lambda}\big(\|U(\sigma)\|^p_{X_\beta}+\|U(\sigma)\|_{X_\beta}\big)\,d\sigma\\
&\le C_{A,\lambda,\delta}N+C_{A,\lambda,0}\,C_F\tau^{\lambda-\delta}\Big(K^p\int_0^\tau(\tau-\sigma)^{-\lambda}\sigma^{-p(\beta-\delta)}\,d\sigma\\
&\qquad\qquad+K\int_0^\tau(\tau-\sigma)^{-\lambda}\sigma^{-(\beta-\delta)}\,d\sigma\Big)\\
&=C_{A,\lambda,\delta}N+C_{A,\lambda,0}C_F(K^p+K)O(\tau^{-\delta-p(\beta-\delta)}),
\end{aligned}$$

where

$$C_{A,\alpha_1,\alpha_2}:=\sup_{\tau\in[0,\mathcal{T}_1]}\tau^{\alpha_1-\alpha_2}\|e^{\tau A}\|_{L(X_{\alpha_1},X_{\alpha_2})},\qquad K:=\sup_{\tau\in(0,\mathcal{T}_1]}\|U\|_{C_{[\beta-\delta]}(0,\mathcal{T}_1]:X_\beta)}$$

for $1\ge\alpha_2>\alpha_2\ge0$. By setting $\lambda=\beta$ and $\lambda=\delta$, and using (7.19), we obtain the first and the second estimates of (7.21), respectively. Again by a similar argument, we can show that the operator on the right-hand side of (7.18) defines a contraction map in $C_{[\beta-\delta]}((0,\mathcal{T}_1];X_\beta)$ provided that $\mathcal{T}_1$ is chosen small enough. This proves the assertion (i).

Next, by the fact that $U(\tau)$ belongs to the space $C_{[\beta-\delta]}((0,\mathcal{T}_1]\,;X_\beta)$, it is easily seen that the term $\int_0^\tau e^{(\tau-\sigma)A}F\big(U(\sigma)\big)\,d\sigma$ tends to 0 in $X_\delta$ as $\tau\to0$. Hence $U(\tau)\to U^0$ as $\tau\to0$ is equivalent to $e^{\tau A}U^0 \to U^0$ as $\tau \to 0$. Therefore, the statement (ii) follows immediately from the well-known property of linear analytic semigroups; see, e.g., Proposition 2.1.4 (i) of [32]. The statement (iii) also follows from the standard contraction argument; see, e.g., Theorem 7.1.5 (i)-(ii) of [32].

The statement (iv) can be shown by using the expression (7.18) along with the following well-known property of analytic semigroup, namely, for any $\mathcal{T}_1 > 0$, there exists a positive constant $C$ such that

$$\tau^{\gamma}\|e^{\tau A}w\|_{X_\gamma} \le C\|w\|_X, \qquad 0<\tau<\mathcal{T}_1, \tag{7.22}$$

$$\tau^{1+\gamma}\|Ae^{\tau A}w\|_{X_\gamma} \le C\|w\|_X, \qquad 0<\tau<\mathcal{T}_1. \tag{7.23}$$

To be more precise, from (7.22) and (7.23), we obtain

$$\Big\|\int_0^\tau e^{(\tau-\sigma)A}F\big(U(\sigma)\big)\,d\sigma\Big\|_{C^{1-\gamma}([0,\mathcal{T}_1];X_\gamma)} \le C\|F(U(\cdot))\|_{L^\infty(0,\mathcal{T}_1;X_0)}.$$

(See Proposition 4.2.1 of [32] for details.) Combining this, (7.20) and (7.21) (with $U_1 = U, U_2 = 0$) yields

$$\begin{aligned}\Big\|\int_0^\tau e^{(\tau-\sigma)A}F\big(U(\sigma)\big)\,d\sigma\Big\|_{C^{1-\gamma}_{[\beta-\delta]}((0,\mathcal{T}_1];X_\gamma)} &\le C\|F(U(\cdot))\|_{C_{[\beta-\delta]}((0,\mathcal{T}_1];X_0)},\\ &\le CC_F\Big(\|(U(\cdot))\|^p_{C_{[\beta-\delta]}((0,\mathcal{T}_1];X_\beta)} + \|(U(\cdot))\|_{C_{[\beta-\delta]}((0,\mathcal{T}_1];X_\beta)}\Big) \le CC_F(N_1^p+N_1),\end{aligned}$$

where

$$C^{1-\gamma}_{[\beta-\delta]}((0,\mathcal{T}]\,;Y) := \{U\in C^{1-\gamma}((0,\mathcal{T}])\,;Y) \mid \|\tau^{\beta-\delta}U(\tau)\|_{C^{1-\gamma}_{loc}((0,\mathcal{T}]\,;Y)} < \infty\}.$$

Similarly, by using (7.22), we obtain

$$\|e^{\tau A}U^0\|_{C^{1-\gamma}([\mathcal{T}_0/2,\mathcal{T}_1];X_\gamma)} \le C\mathcal{T}_0^{-(1-\delta)}\|U^0\|_{X_\delta}.$$

Consequently, there exists a constant $\tilde{C} > 0$ that depends on $A, F, \gamma$ such that

$$\|U\|_{C^{1-\gamma}([\mathcal{T}_0/2,\mathcal{T}_1];X_\gamma)} \le \tilde{C}\big(N\mathcal{T}_0^{-(1-\delta)} + \mathcal{T}_0^{-(\beta-\delta)}(N_1^p+N_1)\big). \tag{7.24}$$

Therefore, $U \in C^{1-\gamma}([\mathcal{T}_0/2,\mathcal{T}_1];X_\gamma)$ and the map $\tau \mapsto F(U(\tau))$ belongs to $C^{1-\gamma}([\mathcal{T}_0,\mathcal{T}_1];X)$, since the map $\tau \to e^{\tau\Delta}U^0$ belongs to $C^\infty((0,\mathcal{T}_1];\mathcal{D}(A))$. By applying Theorem 4.3.1 of [32] to the mild solution of

$$U(\tau) = e^{(\tau-\mathcal{T}_0/2)}U(\mathcal{T}_0/2) + \int_{\mathcal{T}_0/2}^\tau e^{\tau-\sigma}F(U(\sigma))\,d\sigma, \qquad \mathcal{T}_0/2 \le \tau \le \mathcal{T}_1,$$

we see that $U$ is a classical solution belonging to $C^{1+\gamma}([\mathcal{T}_0,\mathcal{T}_1];X)\cap C^\gamma([\mathcal{T}_0,\mathcal{T}_1];\mathcal{D}(A))$ such that the following estimate holds:

$$\|U\|_{C^{1+\gamma}([\mathcal{T}_0,\mathcal{T}_1]);X)} + \|U\|_{C^{\gamma}([\mathcal{T}_0,\mathcal{T}_1]);\mathcal{D}(A))} \le N\{\|F\|_{C^\gamma([\mathcal{T}_0/2,\mathcal{T}_1]);X)} + \|U(\mathcal{T}_0/2)\|_{\mathcal{D}(A)}\}$$

for some constant $N > 0$ that depends on $A, F, \gamma$ . Combining this and (7.24), we proves (iv). Thereby the proof of the proposition is complete. $\square$

**Remark 7.2.** As one sees in Theorem 7.1.5 (iii) of [32] or Theorem 51.29 (ii) of [37], one can show the existence of the solution $U$ under a weaker assumption $\delta + p(\beta - \delta) = 1$, but the estimates (7.21) do not hold in general. See also Proposition 7.2 (i)-(iii) of [37] for a similar result.

**Remark 7.3.** The idea of using a contraction argument in a time-weighted space such as $C_{[\beta]}((0,\mathcal{T}_1]\,;X_\beta)$ for an abstract semilinear parabolic problem like (7.17) goes back to the works of [19] and [20]. The same idea was re-discovered independently in the work of [39], from which this idea came to be widely known.

Now we go back to the problem (7.10)-(7.11).

*Proof of Lemma 7.1.* We define

$$\tilde{v}(y,\tau) := v(y,\tau) - (1-y)\psi_- + y\psi_+.$$

Then $\tilde{v}(0,\tau) = \tilde{v}(1,\tau) = 0$ and

$$\begin{cases} \tilde{v}_\tau = \tilde{v}_{yy} + (P(y,\tau,v,\tau) + Q(\tau,v,\tau)y)(\tilde{v}_y - \psi_- - \psi_+), \\ \eta'(\tau) = Q(\tau,v,\tau), \quad \tau > 0. \end{cases} \tag{7.25}$$

Now we rewrite (7.25) in the abstract form (7.17). Let

$$X = C_0([0,1]) \times \mathbb{R}, \quad \text{where} \quad C_0([0,1]) = \{\tilde{v} \in C([0,1]) \mid \tilde{v}(0) = \tilde{v}(1) = 0\},$$

and let $A$ denote the differential operator $U := (\tilde{v},\eta) \mapsto (d^2\tilde{v}/dy^2, 0)$ with domain

$$\mathcal{D}(A) = \{(\tilde{v},\eta) \in C^2([0,1]) \times \mathbb{R} \mid \tilde{v}(0) = \tilde{v}(1) = 0\}.$$

By Corollary 3.1.21 (ii) of [32] for $n = 1$, the operator $A : \mathcal{D}(A) \subset X \to X$ is sectorial in $X$ and thus generates an analytic semigroup on $X$. The operator $F : C^1([0,1]) \times \mathbb{R} \to X$ is given by the lower order terms on the right-hand side of (7.25).

Now, we set $\delta := \alpha/2$. Then we have $X_\delta = C_0^\alpha([0,1]) \times \mathbb{R}$ (Theorem 3.1.29 of [32]). Hence, by (7.2),

$$\overline{D(A)}^{X_\delta} = h_0^\alpha([0,1]) \times \mathbb{R}. \tag{7.26}$$

To check (7.20), we need $X_\beta \subset C^1([0,1]) \times \mathbb{R}$. Indeed, for each $\beta \in (1/2,1)$, we have $X_\beta = C_0^{2\beta}([0,1]) \times \mathbb{R}$ (Theorem 3.1.29 of [32]), which is contained in $C^1([0,1]) \times \mathbb{R}$. Then by direction computations, we see that $F$ satisfies the condition (7.20) if $p \geq 3$ and $\beta \in (1/2,1)$. Thus, (7.19) is equivalent to

$$\alpha > \frac{2(p\beta - 1)}{p-1}.$$

From the above inequality, we can see why we need the restriction $\alpha \in (1/2, 1)$. Indeed, when $\beta \to 1/2$, we have $2(p\beta - 1)/(p-1) \to (p-2)/(p-1)$. The lower bound of $\alpha$ is given by

$$\min_{p \geq 3} \left\{ \frac{p-2}{p-1} \right\} = \left. \frac{p-2}{p-1} \right|_{p=3} = \frac{1}{2}.$$

Thus, for given any $\alpha \in (1/2, 1)$, by choosing $p = 3$ and $\beta$ sufficiently close to $1/2$, we see that (7.19) holds, which allows us to apply Proposition 7.2(i) to establish the existence of the solution to (7.25) and the estimates (7.16) follows from (7.21). Proposition 7.2(ii) and (7.26) imply that $U(\tau) \to U^0$ as $\tau \to 0$. By applying Proposition 7.2 (i),(iii) to the problem (7.10), we prove the assertions of the first part of (i) and (iii) in Lemma 7.1.

Next, we prove that $(v, \eta)$ is a classical solution, thereby completing the proof of Lemma 7.1 (i) and also that of Lemma 7.1 (ii) for $k = 1$. By (7.24), the function $(v, \eta)$ is bounded in $C^{1-\alpha/2}([\mathcal{T}_0/2, \mathcal{T}_1]; C^\alpha[0,1]) \times C^{1-\alpha/2}([\mathcal{T}_0/2, \mathcal{T}_1])$. Since $\alpha/2 < 1 - \alpha/2$, we have

$$(v, \eta) \in C^{\alpha, \frac{\alpha}{2}}([0,1] \times [\mathcal{T}_0/2, \mathcal{T}_1]) \times C^{\frac{\alpha}{2}}([\mathcal{T}_0/2, \mathcal{T}_1]).$$

Finally, we prove Lemma 7.1 (i). By Proposition 7.2 (iv), $(v, \eta)$ is a classical solution of (7.10)-(7.11) such that

$$(v, \eta) \in C^\gamma([\mathcal{T}_0, \mathcal{T}_1]; C^2[0,1]) \times C^\gamma([\mathcal{T}_0, \mathcal{T}_1]) \cap C^{1+\gamma}([\mathcal{T}_0, \mathcal{T}_1]; C[0,1]) \times C^{1+\gamma}([\mathcal{T}_0, \mathcal{T}_1]).$$

Thus we obtain the boundedness of $\|(P + Qy)(\tilde{v}_y - \psi_- - \psi_+)\|_{C^{\alpha, \frac{\alpha}{2}}([0,1] \times [\mathcal{T}_0, \mathcal{T}_1])}$. Hence, by the standard Schauder theory (See Lemma 5.1.8 of [32]), Lemma 7.1 (ii) for $k = 1$ is proved. Note that $v \in C_{loc}^{2+\alpha, 1+\alpha/2}([0,1] \times (0, \mathcal{T}_1])$ and $\eta \in C([0, \mathcal{T}_1]) \cap C_{loc}^{1+\alpha/2}((0, \mathcal{T}_1])$ implies that $(P + Qy)(\tilde{v}_y - \psi_- - \psi_+) \in C_{loc}^{1+\alpha, 1/2+\alpha/2}([0,1] \times (0, \mathcal{T}_1])$. Thus by applying the standard parabolic estimates to (7.25), Lemma 7.1(ii) is proved for $k = 2$. Repeating the same procedure, we obtain the desired result for any $k \in \mathbb{N}$. $\square$

Now we are ready to prove the local existence theorem for the problem (P).

*Proof of Theorem 7.1.* Our strategy is to rewrite the problem (P) into the form (7.10) and apply Lemma 7.1, then use (7.13)-(7.14) to get the desired result.

The assumption (7.3) implies that $\|u^0\|_{C^{1+\alpha}([l_-^0, l_+^0])} + L(0) \leq N_0$ for some constant $N_0 > 0$ that depends only on $N$. This gives us the corresponding bound for $\|\theta(\cdot, 0)\|_{C^\alpha([l_-^0, l_+^0])}$, and also (7.7) yields a bound of $\|s(\cdot, 0)\|_{C^{1+\alpha}([l_-^0, l_+^0])}$ that depends on $N_0$. Therefore, we get a similar estimate for $\|\Theta(\cdot, 0)\|_{C^\alpha([0, L(0)])}$. Then the same estimate can be obtained for $v^0(y) = \Theta(L(0)y, 0)$ measured in the space $C^\alpha([0,1])$. Hence $\|v^0\|_{C^\alpha([0,1])} + \eta^0$ is bounded by a constant that depends only on $N$.

By (i) of Lemma 7.1, there exists a unique classical solution $(v, \eta)$ satisfying 7.1.(i), where $N_1$ and $\mathcal{T}_1$ depend only on $N$. We define $t$ and $T_1$ by the relation

$$t = \int_0^\tau \frac{d\omega}{L^2(\omega)}, \quad t \in [0, \mathcal{T}_1), \qquad \mathcal{T}_1 = \int_0^{T_1} \frac{d\omega}{L^2(\omega)}.$$

Next we show that $\eta(\tau) = \ln L(t)$ is bounded. Let $x = x_s \in (l_-^0, l_+^0)$ be the smallest critical point of $u^0$. Then

$$\frac{\tan \psi_-}{l(0)^\alpha} = \frac{u_x(l_-(t), t)}{l(0)^\alpha} \leq \frac{|u_x(x_s, 0) - u_x(l_-^0, 0)|}{(x_s - l_-^0)^\alpha} \leq \|u^0\|_{C^{1+\alpha}([l_-^0, l_+^0])} \leq N_0,$$

which implies

$$\eta(0) \geq \ln l(0) \geq \frac{1}{\alpha} \ln \Big( \frac{\tan \psi_-}{N_0} \Big). \tag{7.27}$$

By the second equation of (7.10) and (7.16), the function $\eta(\tau)$ is uniformly bounded:

$$|\eta(\tau) - \eta(0)| \leq \frac{2N_1}{1+\alpha} (\cot \psi_+ + \cot \psi_-) \mathcal{T}_1^{\frac{1+\alpha}{2}} + c e^{N_1} (\psi_+ + \psi_- + \cot \psi_+ + \cot \psi_-) \mathcal{T}_1 + \frac{N_1^2 \mathcal{T}_1^\alpha}{\alpha}.$$

This implies that $T_1$ depends only on $N$, since $N_1$ and $\mathcal{T}_1$ depend only on $N$.

Note that (7.14) implies that $y(\cdot, t) \in C^{1+\alpha}([l_-(t), l_+(t)])$. Here we used

$$y_x(x, t) = 1/(e^{\eta(\tau(t))} \cos v(y(x, t), \tau(t)))$$

and the fact

$$|v(y, \tau)| \leq \max\{\psi_+, \psi_-, \|v(\cdot, 0)\|_{L^\infty([0,1])}\} \quad \text{for } y \in [0, 1],\ \tau \in [0, \mathcal{T}_1],$$

which follows from the maximum principle. Therefore, by (7.16) and (7.13)-(7.14), we obtain the uniform estimates:

$$\|u(\cdot, t)\|_{C^{1+\alpha}([l_-(t), l_+(t)])} + L(t) \leq N_1, \quad t^{\frac{1-\alpha}{2}} \|u(\cdot, t)\|_{C^2([l_-(t), l_+(t)])} \leq N_1$$

for all $t \in [0, T_1]$. We also get the uniform boundedness of $l_\pm(t)$ in Theorem 7.1, and (7.4) of Theorem 7.1 on $D_{T_1} \setminus D_{T_0}$ from Lemma 7.1 (ii) , where $T_0$ is defined through the relation

$$\mathcal{T}_0 = \int_0^{T_0} \frac{d\omega}{L^2(\omega)}.$$

Finally, we shall prove (7.4). The regularity for $v$ in Lemma 7.1 (ii) and (7.14) imply that $y(x, t)$ is $C^{k+2+\alpha, (k+2+\alpha)/2}$. This together with (7.13) and the regularity of $(v, \eta)$ implies that $u(x, t)$ is $C^{k+2+\alpha, (k+1+\alpha)/2}$ and $l_\pm$ is $C^{(k+1+\alpha)/2}$ with the corresponding estimates. □

7.2. **Continuous dependence on the initial data.** In this subsection we discuss continuous dependence of solutions on the initial data. In what follows, $[0, \mathcal{T}(U^0))$ with $0 < \mathcal{T}(U^0) \leq \infty$, will denote the maximum time interval for the existence of the solution of the problem (7.17). First we discuss the continuous dependence of solutions on the initial data of the problem (7.17).

**Lemma 7.3.** *Under the assumption of Proposition 7.2, let $U$ and $U_n$ be solutions of the problem* (7.17) *with initial values $U^0$ and $U_n^0$, respectively, satisfying* $\lim_{n\to\infty}\|U^0-U_n^0\|_{X_\delta}=0$. *Then for any $\tau_*\in(0,\mathcal{T}(U^0))$, the following hold:*

(i) $\mathcal{T}(U_n^0)>\tau_*$ *for all $n$ sufficiently large;*

(ii) $\lim_{n\to\infty}\sup_{\tau\in[0,\tau_*]}\|U(\tau)-U_n(\tau)\|_{X_\delta}=0$.

(iii) *For any $\varepsilon\in(0,\tau_*)$ sufficiently small, there exists a constant $K>0$ independent of $n$ such that*

$$\sup_{\tau\in[0,\varepsilon]}\tau^{\beta-\delta}\|U(\tau)-U_n(\tau)\|_{X_\beta}\le K\quad\text{for all}\quad n. \tag{7.28}$$

*Moreover,*

$$\lim_{n\to\infty}\sup_{\tau\in[\varepsilon,\tau_*]}\|U(\tau)-U_n(\tau)\|_{X_\beta}=0. \tag{7.29}$$

*Proof of Lemma 7.3.* First we prove (i)-(ii) of this lemma. Define $N=2\sup_{\tau\in[0,\tau_*]}\|U(\tau)\|_{X_\delta}$, and let $\mathcal{T}_1$ denote the constant in Proposition 7.2 corresponding to the above $N$. Since $\|U^0\|_{X_\delta}\le N/2$, we have $\|U_n^0\|_{X_\delta}\le N$ for all sufficiently large $n$. Therefore, by Proposition 7.2 (i), (iii) , we have $\mathcal{T}(U_n^0)\ge\mathcal{T}_1$ for all sufficiently large $n$ and

$$\lim_{n\to\infty}\sup_{\tau\in[0,\mathcal{T}_1]}\|U(\tau)-U_n(\tau)\|_{X_\delta}=0. \tag{7.30}$$

Combining this and the fact that $\|U(\tau)\|_{X_\delta}\le N/2$ for $\tau\in[0,\mathcal{T}_1]$, we obtain

$$\|U_n(\tau)\|_{X_\delta}\le N,\qquad \tau\in[0,\mathcal{T}_1]. \tag{7.31}$$

Now denote by $k^*$ the largest integer satisfying $0<k^*\mathcal{T}_1\le\tau_*$. If $k^*=0$, then this means that $\mathcal{T}_1\ge\tau_*$, therefore the conclusion of Lemma 7.3 follows from (7.30). Next suppose that $k^*\ge1$. Then, by (7.30) and (7.31), we have

$$\|U_n(\mathcal{T}_1)\|_{X_\delta}\le N,\qquad \lim_{n\to\infty}\|U(\mathcal{T}_1)-U_n(\mathcal{T}_1)\|_{X_\delta}=0.$$

Hence by the same argument as above, we obtain $\mathcal{T}(U_n^0)\ge2\mathcal{T}_1$ and

$$\|U_n(\tau)\|_{X_\delta}\le N,\qquad \tau\in[\mathcal{T}_1,2\mathcal{T}_1],$$
$$\lim_{n\to\infty}\sup_{\tau\in[\mathcal{T}_1,2\mathcal{T}_1]}\|U(\tau)-U_n(\tau)\|_{X_\delta}=0.$$

Repeating the same argument, we see that, for sufficiently large $n$, $\mathcal{T}(U_0^n)\ge(k^*+1)\mathcal{T}_1$, and

$$\|U_n(\tau)\|_{X_\delta}\le N,\qquad \tau\in[k\mathcal{T}_1,(k+1)\mathcal{T}_1],$$
$$\lim_{n\to\infty}\sup_{\tau\in[k\mathcal{T}_1,(k+1)\mathcal{T}_1]}\|U(\tau)-U_n(\tau)\|_{X_\delta}=0 \tag{7.32}$$

for $k=0,1,2,\dots,k^*$. Since $\tau_*<(k+1)\mathcal{T}_1$, the conclusion (i)-(ii) of the lemma now follows from (7.32).

Now we shall prove the continuity property (iii). Note that (7.28) is also shown for $\varepsilon \in [0, \mathcal{T}_1]$ when we proved (7.30). By the first inequality in (iii) of Proposition 7.2, we also have $\lim_{n\to\infty} \|U(\mathcal{T}_1) - U_n(\mathcal{T}_1)\|_{X_\beta} = 0$. By the second inequality of Proposition 7.2 (iii), we can repeat the same inductive argument as above to get (7.29). □

Set $U = (\tilde{v}, \eta)$, $U_n = (\tilde{v}_n, \eta_n)$, $\delta = \alpha/2$ and choose $\beta$ satisfying (7.19) with $p = 3$. Then the following result follows from Lemma 7.3.

**Corollary 7.4.** *Let $(v, \eta)$ be a solution of (7.10)-(7.11), and $(v_n, \eta_n)$ be a sequence of solutions of (7.10)-(7.11) with initial data $(v_n^0, \eta_n^0)$ such that*

$$\lim_{n\to\infty} \eta_n^0 = \eta^0, \quad \lim_{n\to\infty} \|v_n^0 - v^0\|_{C^\alpha([0,1])} = 0. \tag{7.33}$$

*Then for any $\tau_* \in (0, \mathcal{T}(v^0, \eta^0))$, the inequality $\mathcal{T}(v_n^0, \eta_n^0) > \tau_*$ holds for all sufficiently large $n$. Furthermore, for any $\varepsilon \in (0, \tau_*)$ sufficiently small, there exists a constant $K > 0$ independent of $n$ such that*

$$\lim_{n\to\infty} \sup_{\tau\in[0,\tau_*]} \|v_n(\cdot,\tau) - v(\cdot,\tau)\|_{C^\alpha([0,1])} = \lim_{n\to\infty} \sup_{\tau\in[\varepsilon,\tau_*]} \|(v_n)_y(\cdot,\tau) - v_y(\cdot,\tau)\|_{C([0,1])} = 0 \tag{7.34}$$

$$\sup_{\tau\in[0,\varepsilon]} \left(\tau^{\frac{1-\alpha}{2}} \|(v_n)_y(\cdot,\tau) - v_y(\cdot,\tau)\|_{C([0,1])}\right) \le K, \tag{7.35}$$

$$\lim_{n\to\infty} \eta_n(\tau) = \eta(\tau). \tag{7.36}$$

Finally, we consider the continuous dependence in the original variable. In what follows, $[0, T(u^0, l_\pm^0))$ will denote the maximal time interval for the existence of the solution $(u, l_\pm)$ of $(P)$ for the initial data $(u^0, l_\pm^0)$.

**Theorem 7.2.** *Let the functions $u^0 \in C^2([l_-^0, l_+^0])$ and $u_n^0 \in C^2([l_{n,-}^0, l_{n,+}^0])$ satisfy*

$$u^0 > 0 \quad in \quad (l_-^0, l_+^0), \qquad u^0(l_\pm^0) = 0, \qquad u_x^0(l_\pm^0) = \mp \tan\psi_\pm, \tag{7.37}$$

$$u_n^0 > 0 \quad in \quad (l_{n,-}^0, l_{n,+}^0), \quad u_n^0(l_{n,\pm}^0) = 0, \quad (u_n^0)_x(l_{n,\pm}^0) = \mp \tan\psi_\pm \tag{7.38}$$

*for all $n \in \mathbb{N}$. Assume that*

$$\lim_{n\to\infty} l_{n,\pm}^0 = l_\pm^0, \qquad \lim_{n\to\infty} \|u_n^0 - u^0\|_{L^\infty(\mathbb{R})} = 0, \tag{7.39}$$

*where we set $u^0, u_n^0 \equiv 0$ outside of $[l_-^0, l_+^0]$ and $[l_{n,-}^0, l_{n,+}^0]$, respectively. Assume also that there exists a constant $N > 0$ such that*

$$\|u^0\|_{C^2([l_-^0, l_+^0])}, \|u_n^0\|_{C^2([l_{n,-}^0, l_{n,+}^0])} \le N \qquad (n = 1, 2, 3, \cdots). \tag{7.40}$$

*Then for any $t \in (0, T(u^0, l_\pm^0))$, it holds that $t < T(u_n^0, l_{n,\pm}^0)$ for sufficiently large $n$ and that*

$$\lim_{n\to\infty} l_{n,\pm}(t) = l_\pm(t), \qquad \lim_{n\to\infty} \|u_n(t) - u(t)\|_{L^\infty(\mathbb{R})} = 0. \tag{7.41}$$

*Proof.* By (7.40) and a compactness argument, we see that the convergence $u_n^0 \to u^0$ in (7.39) takes place in a certain stronger sense, which, in particular, implies the following convergence:

$$\eta_n^0 = \ln L_n(0) = \ln\Big(\int_{l_{n,-}^0}^{l_{n,+}^0} \sqrt{1+(u_n^0)_x^2(x)}\,dx\Big) \to \ln L(0) = \eta^0 \quad \text{as} \quad n \to \infty.$$

Next, by differentiating (7.12), we can show the boundedness of $\{\zeta_n^{-1}\}$ in $C^2([0,1])$. Thus, we can choose a subsequence, still denoted by $\{\zeta_n^{-1}\}$, such that $\lim_{n\to\infty} \|\zeta_n^{-1} - \zeta^{-1}\|_{C^1([0,1])} = 0$. From (7.11) and (7.12), we have

$$v_y^0(\zeta^{-1}(y)) = \frac{u_{xx}^0(\eta^{-1}(y))}{1+u_x^0(\zeta^{-1}(y))^2}\frac{1}{\zeta'(\zeta^{-1}(y))} = \frac{L(0)u_{xx}^0(x)}{(1+(u_x^0(x))^2)^{3/2}},$$

and (7.40) implies that there exists a constant $C_N > 0$ that depends on $N > 0$ such that

$$\|v^0\|_{C^1([0,1])}, \|v_n^0\|_{C^1([0,1])} \le C_N \quad (n = 1, 2, 3, \cdots).$$

Thus $\{v_n^0\}$ is relatively compact in $C^\alpha([0,1])$. Therefore, by (7.11)-(7.12) and (7.37)-(7.39), $v_n^0$ converges to $v^0$ in $C^\alpha([0,1])$ as $n \to \infty$. Hence we can apply Corollary 7.4 to get (7.34)-(7.36). Thus the assertion $l_{n,-}(t) \to l_-(t)$ in (7.41) follows from (7.13), (7.35), (7.36) and the inequality $(1-\alpha)/2 < 1$. The assertion for $l_{n,+}$ follows from that for $l_{n,-}$ along with (7.13), (7.34) and (7.36). Combining this, (7.13), (7.34) and (7.36), we get (7.41) for $u_n$ for any $t \in [0, T(u^0, l_\pm^0))$. □

## References


[1] B. L. Adams, S. Ta'asan, D. Kinderlehrer, I. Livshits, D. E. Mason, C.-T. Wu, W. W. Mullins, G. S. Rother, A. D. Rollett, D. M. Saylor, *Extracting grain boundary and surface energy measurement of triple junction geometry*, Interface Sci. **7** (1999), 321–338.

[2] B. L. Adams, D. Kinderlehrer, W. W. Mullins, A. D. Rollett, S. Ta'asan, *Extracting the relative grain boundary free energy and mobility function from the geometry of microstructures*, Scripta Materialia **38** (1998), 531–536.

[3] S. Angenent, *The zero set of a solution of a parabolic equation*, J. Reine Angew. Math. **380** (1988), 79–96.

[4] S. Angenent, *On the formation of singularities in the curve shortening flow*, J. Diff. Geometry **33** (1991), 601–633.

[5] D. Bonn, J. Eggers, J. Indekeu, J. Meunier, E. Rolley, *Wetting and spreading*, Rev. Mod. Phys. **81** (2009), 739–805.

[6] P. K. Brazhnik, *Exact solutions for the kinematic model of autowaves in two-dimensional excitable media*, Physica D, **94** (1996), 205–220.

[7] L. A. Caffarelli, J. L. Vazquez, *A free boundary problem for the heat equation arising in flame propagation*, Trans. Amer. Math. Soc. **347** (1995), 411–441.

[8] Y.-L. Chang, J.-S. Guo, Y. Kohsaka, *On a two-point free boundary problem for a quasilinear parabolic equation*, Asymptotic Analysis **34** (2003), 333–358.

[9] X. Chen, J.-S. Guo, *Motion by curvature of planar curves with end points moving freely on a line*, Math. Ann. **350** (2011), 277–311.

[10] H.-H. Chern, J.-S. Guo, C.-P. Lo, *The self-similar expanding curve for the curvature flow equation*, Proc. Amer. Math. Soc. **131** (2003), 3191–3201.

[11] M. Do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice-Hall, Englewood Cliffs, New Jercy, 1976.

[12] P.G. De Gennes, *Wetting: statics and dynamics*, Rev. Mod. Phys. **57** (1985), 827–863.

[13] P. G. de Gennes, F. Brochard-Wyart, D. Quéré, *Capillarity and Wetting Phenomena: Drops, Bubbles, Pearls, Waves*, Springer-Verlag, 2004.

[14] K. Deckelnick, C. M. Elliott, G. Richardson, *Long time asymptotics for forced curvature flow with applications to the motion of a superconducting vortex*, Nonlinearity **10** (1997), 655–678.

[15] E. DiBenedetto, *Degenerate Parabolic Equations*, Springer, New York, 1993.

[16] J. Escher, G. Simonett, *The volume preserving mean curvature flow near spheres*, Proc. Amer. Math. Soc. **126** (1998), 2789–2796.

[17] P. C. Fife, *Dynamics of Internal Layers and Diffusive Interfaces*, (CBMS-NSF Reg. Conf. Ser. Appl. Math. **53**, 1988), SIAM, Philadelphia.

[18] R. Finn, *Equilibrium Capillary Surfaces*, Springer-Verlag, New York, 1986.

[19] H. Fujita, T. Kato, *On the nonstationary Navier-Stokes system*, Rend. Sem. Mat. Univ. Padova **32** (1962), 243–260.

[20] H. Fujita, T. Kato, *On the Navier-Stokes initial value problem. I*, Arch. Rational Mech. Anal. **16** (1964), 269–315.

[21] H. Knüpfer, N. Masmoudi, *Darcy flow on a plate with prescribed contact angle – well-posedness and lubrication approximation*, preprint arXiv:1204.2278 [math.AP].

[22] M. Gage, R. S. Hamilton, *The heat equation shrinking convex plane curves*, J. Diff. Geom. **23** (1986), 69–96.

[23] M. A. Grayson, *The heat equation shrinks embedded plane curves to round points*, J. Diff. Geom. **26** (1987), 285–314.

[24] J.-S. Guo, B. Hu, *On a two-point free boundary problem*, Quarterly Appl. Math. **64** (2006), 413–431.

[25] M. Gurtin, *Thermomechanics of evolving phase boundaries in the plane*, Oxford Science Publication, London (1993).

[26] C. Herring, *Surface tension as a motivation for sintering*, The Physics of Powder Metallurgy, (Kingston, W., ed.), McGraw-Hill, New York (1951).

[27] C. Herring, *The use of classical macroscopic concepts in surface-energy problems*, Structure and Properties of Solid Surfaces, (Gomer, R. & Smith, S. eds), U. Chicago Press, Chicago (1952).

[28] G. Huisken, *Asymptotic behavior for singularities of the mean curvature flow*, J. Differential Geom. **31**(1990), 285–299.

[29] I. Kim, A. Mellet, *Liquid drops sliding down an inclined plane*, Trans. Amer. Math. Soc. **366** (2014), 6119–6150.

[30] D. Kinderlehrer, C. Liu, *Evolution of grain boundaries*, Math. Models Methods Appl. Sci. **11** (2001), 713–729.

[31] M. Koiso, B. Palmer, *Anisotropic capillary surfaces with wetting energy*, Calculus of Variations and Partial Differential Equations **29** (2007), 295–345.

[32] A. Lunardi, *Analytic Semigroups and Optimal Regularity in Parabolic Problems*, Birkhaäuser, 1995.

[33] H. Matano, *Nonincrease of the lap-number of a solution for a one-dimensional semi-linear parabolic equation*, J. Fac. Sci. Univ. Tokyo Sect. IA **29** (1982) 401–441.

[34] W. Mullins, *Solid surface morphologies governed by capillarity*, Metal Surface: Structure, Energetics and Kinetics. Amer. Soc. Metals., (1963) 17–66.

[35] H. Ninomiya, M. Taniguchi, *Traveling curved fronts of a mean curvature flow with constant driving force*. Free Boundary Problems: Theory and Applications I, Mathematical Sciences and Applications **13**, Gakuto International Series, (2000) 206–221, Gakkotosho, Tokyo.

[36] F. Otto, *Lubrication approximation with prescribed non-zero contact angle*, Comm. Partial Differential Equations **23** (1998), 2077–2164.

[37] P. Quittner, Ph. Souplet, *Superlinear Parabolic Problems.* Blow-up, Global Existence and Steady States, Birkhäuser Advanced Texts, Basler Lehrbücher.

[38] W. Ren, D. Hu, E. Weinan, *Continuum models for the contact line problem*, Physics of Fluids **22** (2010), 102103.

[39] F.B Weissler *Local existence and nonexistence for semilinear parabolic equations in $L^p$*, Indiana Univ. Math. J. **29** (1980) 79–102.

[40] W. L. Winterbottom, *Equilibrium shape of a small particle in contact with a foreign substrate*, Acta Metallurgica **15** (1967), 303–310.

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