DEAD-CORE AT TIME INFINITY FOR A HEAT EQUATION WITH STRONG ABSORPTION

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Dedicated to Prof. Masayasu Mimura on the occasion of his sixty-fifth birthday

Abstract. We study an initial boundary value problem for a heat equation with strong absorption. We first prove that the solution of this problem stays positive for any finite time and converges to the unique steady state for a large class of initial data. This gives an example in which the dead-core is developed in infinite time. Then some estimates of the dead-core rate(s) are derived. Finally, we provide the uniformly exponential rate of convergence of the solution to the unique steady state.

1. Introduction

We study the following initial boundary value problem (P) for the heat equation with strong absorption:

$$u_t = u_{xx} - u^p,\ 0 < x < 1,\ t > 0, \tag{1.1}$$

$$u_x(0,t) = 0,\ u(1,t) = k_p,\ t > 0, \tag{1.2}$$

$$u(x,0) = u_0(x),\ 0 \le x \le 1, \tag{1.3}$$

where $p \in (0,1)$, $k_p := [2\alpha(2\alpha-1)]^{-\alpha}$, $\alpha := 1/(1-p)$, and u_0 is a smooth function defined on $[0,1]$ such that

$$u_0'(0) = 0,\ u_0(1) = k_p,\ u_0'(x) \ge 0,\ U(x) < u_0(x) \le k_p \ \text{ for } x \in [0,1). \tag{1.4}$$

We note that the constant k_p is chosen so that the unique steady state $U(x) := k_p x^{2\alpha}$ of (1.1)-(1.2) is positive for $x \ne 0$ and $U(0) = 0$.

Problem (P) arises in the modelling of an isothermal reaction-diffusion process [1, 10] and a description of thermal energy transport in plasma [8, 6]. In the first example, the solution u of (P) represents the concentration of the reactant which is injected with a fixed amount on the boundary $x = \pm 1$ (after a symmetric reflection), and p is the order of reaction.

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It is trivial that, for any u_0 as above, problem (P) admits a unique global classical solution. Also, it follows from the strong maximum principle that $u > U$ and $u_x > 0$ in $(0, 1) \times (0, \infty)$.

The problem (P) with general boundary values (i.e., any $k > 0$) has been studied extensively. We refer the reader to a recent work of one of the authors and Souplet [4] and the references cited therein. Recall that the region where $u = 0$ is called the *dead-core*, the first time when u reaches zero is called the *dead-core time* and the rate of convergence to zero in time is called the *dead-core rate*. In [4], we studied the case when the dead-core is developed in a finite time. In [4], it is proved that the finite time dead-core rate is always non-self-similar. Indeed, it is shown in [5] that there can be infinitely many different finite time dead-core rates depending on the initial data.

By taking the special constant k_p, we shall show that the solution of (P) is always positive for all $t > 0$ and tends to the unique steady state U uniformly as $t \to \infty$. In particular, we have $u(0, t) \to 0$ as $t \to \infty$. This means that the dead-core occurs at time infinity.

A natural question arises, namely, how the solution u tends to U. In particular, we shall investigate the dead-core rate, i.e., the exact convergence rate of $u(0, t)$ to zero as $t \to \infty$. For some related works, we refer the reader to [2, 3, 9]. We note that there is a singularity in the sense that the reaction rate u^{p-1} tends to infinity when u tends to zero. This causes a certain difficulty in dealing with the problem (P).

This paper is organized as follows. We first study some properties of the solution of (P) in §2. In particular, we prove that the dead-core is developed at time infinity. In §3, some properties of the associated steady states to (1.1) are given and some further properties of the solution of (P) in terms of these steady states are also derived. Section 4 is devoted to the spectrum analysis of the linearized operator around the unique steady state U and the related approximated operators to this linearized operator. Then, in §5, we give some estimates for the dead-core rate(s). Unfortunately, we are unable to derive the exact dead-core rate. We suspect that the dead-core rate might depend on the initial data. We leave this important question as an open problem. Finally, the uniformly exponential rate of convergence of u to U over the whole domain as $t \to \infty$ is given in §6.

2. Dead-core at Time Infinity

In this section, we shall study some basic properties of the solution u of (P). First, we have the following result of positivity of u. This also implies that the dead-core can only be developed at time infinity.

Theorem 2.1. *We have $u > 0$ for all $0 \le x \le 1$ and $t > 0$.*

Proof. For contradiction, we may assume that

$$T := \sup\{\tau > 0 \mid u(x, t) > 0 \ \ \forall (x, t) \in [0, 1] \times [0, \tau]\} < \infty.$$

By the maximum principle, we have $u > U$ in $(0, 1) \times [0, T]$. In particular,

$$u(1/2, t) > U(1/2) \ \ \forall t \in [0, T]. \tag{2.1}$$

Let $\{u_n\}_{n \ge 1}$ be a sequence of functions defined on $[0, 1]$ such that

$$u_n'' = u_n^p \quad \text{on } [0, 1]; \quad u_n(0) = 0, \ u_n'(0) = 1/n.$$

It is easy to see that $u_n \geq u_{n+1} \geq U$ on $[0,1]$ for all $n \geq 1$. Furthermore, $u_n \to U$ uniformly on $[0,1]$ as $n \to \infty$. It follows from (2.1) that $u(1/2,t) > U_N(1/2)$ for all $t \in [0,T]$ for some sufficiently large N. By choosing N larger (if necessary), we also have

$$u_0(x) > U_N(x) \ \ \forall x \in [0,1/2].$$

It follows from the maximum principle that $u \geq u_N$ on $[0,1/2]\times[0,T]$. Since $u(0,T) = 0$, we obtain that $u_x(0,T) \geq u_N'(0) > 0$, a contradiction. Hence the theorem is proved. $\square$

The next theorem shows that u converges to the unique steady state U as $t \to \infty$. As a consequence, the dead-core does occur at time infinity.

Theorem 2.2. *There holds $u(x,t) \to U(x)$ uniformly for $x \in [0,1]$ as $t \to \infty$.*

Proof. First, we show that u, u_x, u_t are bounded on $[0,1]\times[0,\infty)$. Indeed, the boundedness of u follows from the maximum principle. Since the function $v := u_t$ satisfies

$$\begin{aligned} &v_t = v_{xx} - pu^{p-1}v, \ 0 < x < 1, \ t > 0, \\ &v_x(0,t) = 0, \ v(1,t) = 0, \ t > 0, \\ &v(x,0) = u_0''(x) - u_0^p(x), \ 0 \leq x \leq 1. \end{aligned}$$

It follows from the maximum principle that v (and so u_t) is bounded on $[0,1]\times[0,\infty)$. Now, from (1.1) we see that u_{xx} is bounded on $[0,1]\times[0,\infty)$. Consequently, u_x is also bounded, since $u_x(0,t) = 0$ for all $t > 0$.

Now, we take any sequence $\{t_j\}$ with $t_j \to \infty$ as $j \to \infty$. We define $u_j(x,t) := u(x,t+t_j)$ for any $j \in \mathbb{N}$. From the boundedness of u and u_x it follows that $\{u_j\}$ is uniformly bounded and equi-continuous on $[0,1]\times[0,\infty)$. It follows from the Arzela-Ascoli Theorem that there exists a subsequence, still denoted by u_j, such that $u_j \to w$ uniformly on $[0,1]$ as $j \to \infty$ for some function w satisfying

$$\begin{aligned} &w_t = w_{xx} - w^p, \ 0 < x < 1, \ t > 0, \\ &w_x(0,t) = 0, \ w(1,t) = k_p, \ t > 0. \end{aligned}$$

We claim that $w_t \equiv 0$. To do this, we introduce the energy functional

$$E(t) := \frac{1}{2}\int_0^1 u_x^2 dx + \frac{1}{p+1}\int_0^1 u^{p+1} dx.$$

By a simple computation, we have

$$E'(t) = -\int_0^1 u_t^2 dx.$$

For any fixed $T > 0$, an integration yields

$$\int_0^T \int_0^1 u_t^2 dxdt = E(0) - E(T) \leq E(0) < \infty.$$

It follows that

$$\int_0^\infty \int_0^1 u_t^2 dxdt < \infty.$$

This implies that

$$\int_0^\infty \int_0^1 u_{j,t}^2 dxdt = \int_{t_j}^\infty \int_0^1 u_t^2 dxdt \to 0 \ \text{ as } j \to \infty.$$

On the other hand, for any $T > 0$, since $\{u_{j,t}\}_{j\in\mathbb{N}}$ is uniformly bounded in $L^2([0,1] \times [0,T])$, it follows that $u_{j,t}$ converges weakly to w_t in $L^2([0,1]\times[0,T])$. This implies that

$$\int_0^T \int_0^1 w_t^2 dxdt \le \liminf_{j\to\infty} \int_0^T \int_0^1 u_{j,t}^2 dxdt = 0.$$

Hence $w_t \equiv 0$ and so $w = U$.

Since the sequence $\{t_j\}$ is arbitrary, the theorem follows. □

The following theorem implies that the convergence of $u(0,t)$ to zero is at least exponentially fast.

Theorem 2.3. *There exist positive constants C and β such that*

$$0 < u(0,t) \le Ce^{-\beta t} \tag{2.2}$$

for all $t > 0$.

Proof. First, following an idea from [9], we derive the following estimate

$$\int_0^1 [u(x,t) - U(x)]^2 dx \le Ce^{-\gamma t} \tag{2.3}$$

for all $t > 0$ for some positive constants C and γ. To this end, we set $w = u - U$. Then w satisfies

$$\begin{aligned} &w_t = w_{xx} + U^p - u^p \le w_{xx},\ 0 < x < 1,\ t > 0,\\ &w_x(0,t) = 0 = w(1,t),\ t > 0. \end{aligned}$$

It then follows that

$$\int_0^1 ww_t dx \le \int_0^1 ww_{xx} dx.$$

Using an integration by parts and applying the Poincaré Inequality, we get

$$\frac{1}{2}\frac{d}{dt}\int_0^1 w^2 dx \le -\int_0^1 w_x^2 dx \le -\frac{\pi^2}{4}\int_0^1 w^2 dx.$$

Hence (2.3) follows with $\gamma = \pi^2/2$.

By a comparison, it suffices to consider the case when $u_0(x) \equiv k_p$. Recall that $u_x > 0$ on $(0,1)\times(0,\infty)$. It implies that

$$u(x,t) \ge u(0,t) \ge U(x) = k_p x^{2\alpha}\ \ \forall x \in [0, h(t)], \tag{2.4}$$

where $h(t) := [u(0,t)/k_p]^{1/(2\alpha)} \le 1$ for $t > 0$. Then it follows from (2.3) and (2.4) that

$$\begin{aligned} Ce^{-\gamma t} &\ge \int_0^1 [u(x,t) - U(x)]^2 dx\\ &\ge \int_0^{h(t)} [u(0,t) - U(x)]^2 dx\\ &= \int_0^{h(t)} k_p^2 [h(t)^{2\alpha} - x^{2\alpha}]^2 dx\\ &= k_p^2 h(t)^{4\alpha+1} \int_0^1 (1 - s^{2\alpha})^2 ds, \end{aligned}$$

by a change of variable $s := x/h(t)$.

Hence the theorem follows by taking $\beta = 2\alpha\gamma/(4\alpha+1)$. $\square$

3. Relations of the Solution to Steady States

Now, for any $\eta \geq 0$, let U_η be the solution of

$$u'' = u^p,\ u > 0\ \ \forall y > 0;\quad u(0) = \eta,\ u'(0) = 0. \tag{3.1}$$

In particular, $U_0(y) = U(y) = k_p y^{2\alpha}$ for $y \geq 0$. Note that, by a re-scaling, we have

$$U_\eta(y) = \eta U_1(\eta^{(p-1)/2}y)\ \ \forall \eta > 0. \tag{3.2}$$

Also, by a simple comparison, we have $U_{\eta_1} > U_{\eta_2}$ if $\eta_1 > \eta_2 \geq 0$. Moreover, $U_\eta \to U_0$ as $\eta \to 0^+$.

Concerning the asymptotic behavior of U_η as $\eta \to 0^+$, we recall from [5] that

Lemma 3.1. *As $\eta \to 0^+$,*

$$U_\eta(x) = U_0(x) + a\eta^{(1-p)/2}x^{2\alpha-1}(1 + o(1))$$

for any $x > 0$, where a is a positive constant.

In the sequel, for convenience we denote $\sigma(t) := u(0,t)$. The proof of the following lemma is based on a zero number argument (see also Theorem 4.1 of [9]).

Lemma 3.2. *For all t sufficiently large, $\sigma(t)$ is strictly decreasing and*

$$u(x,t) < U_{\sigma(t)}(x)\ \ \textit{in } (0,1]. \tag{3.3}$$

Proof. Define $z_\eta(x,t) := u(x,t) - U_\eta(x)$. Then z_η satisfies

$$(z_\eta)_t = (z_\eta)_{xx} + c_\eta(x,t)z_\eta$$

for some function c_η. Since $z_\eta(1,t) < 0$ and $(z_\eta)_x(0,t) = 0$ for all $t > 0$, we see that the zero number $J_\eta(t)$ of z_η defined by $J_\eta(t) := \#\{x \in [0,1] \mid z_\eta(x,t) = 0\}$ is non-increasing in t.

We first claim that there exists $\eta^* > 0$ such that $J_\eta(1) = 1$ for all $\eta \in (0,\eta^*]$. Indeed, since $z_{0,x}(1,1) < 0$, there exists $\delta > 0$ such that $z_{0,x}(x,1) < 0$ for all $x \in [1-\delta,1]$. Since $z_{\eta,x}(x,1) \to z_{0,x}(x,1)$ uniformly on $[0,1]$ as $\eta \to 0^+$, there is $\eta_0 > 0$ such that

$$z_{\eta,x}(x,1) < 0\ \ \forall x \in [1-\delta,1]\ \ \forall \eta \in (0,\eta_0]. \tag{3.4}$$

On the other hand, since $u(x,1) > U(x)$ on $[0,1-\delta]$ and $U_\eta \to U$ uniformly on $[0,1-\delta]$ as $\eta \to 0^+$, there exists an $\eta^* \in (0,\eta_0)$ such that

$$z_\eta(x,1) > 0\ \ \forall x \in [0,1-\delta]\ \ \forall \eta \in (0,\eta^*]. \tag{3.5}$$

Recall that $z_\eta(1,1) < 0$ for all $\eta > 0$. We conclude from (3.4) and (3.5) that $J_\eta(1) = 1$ for all $\eta \in (0,\eta^*]$.

Next, we fix any $\eta \in (0,\eta^*]$. Note that $J_\eta(t) \leq 1$ for all $t > 1$. We claim that $\sigma(t_0) > \eta$, if $J_\eta(t_0) = 1$ for some $t_0 > 1$. For contradiction, we suppose that $\sigma(t_0) \leq \eta$, i.e., $u(0,t_0) \leq U_\eta(0)$. Note that $u(1,t) < U_\eta(1)$ for all $t > 0$. If $u(0,t_0) = U_\eta(0)$, then $u(x,t_0) < U_\eta(x)$ for all $x \in (0,1]$, since $J_\eta(t_0) = 1$. Since $J_\eta(t) = 1$ for all $t \in [1,t_0]$, there exists $x(t) \in [0,1)$ such that $u(x(t),t) = U_\eta(x(t))$ and $u(x,t) < U_\eta(x)$ for $x \in (x(t),1]$ for each $t \in [1,t_0]$. By Hopf's Lemma, $u_x(0,t_0) < U_\eta'(0) = 0$, a contradiction. On the other hand, if $u(0,t_0) < U_\eta(0)$, then there exists $t^* \in (1,t_0)$ such that $u(0,s) < U_\eta(0)$ for all $s \in [t^*,t_0]$. Since $u(1,s) < U_\eta(1)$, we can find $x(s) \in (0,1)$

such that $u(x(s),s) = U_\eta(x(s))$ and $u(x,s) < U_\eta(x)$ for $x \neq x(s)$ for all $s \in [t^*, t_0]$. This is a contradiction to the maximum principle. This proves that $\sigma(t_0) > \eta$, if $J_\eta(t_0) = 1$ for some $t_0 > 1$.

Now, since $\sigma(t) \to 0$ as $t \to \infty$, there is t_1 sufficiently large such that $\sigma(t) \leq \eta^*$ for all $t \geq t_1$. Hence $J_{\sigma(t)}(t) = 0$ for all $t \geq t_1$. This implies that

$$u(x,t) < U_{\sigma(t)}(x) \quad \text{on } [0,1]$$

for all $t \geq t_1$. Therefore, (3.3) follows. Moreover, $J_{\sigma(t)}(s) = 0$ for all $s > t \geq t_1$. Then $u(x,s) < U_{\sigma(t)}(x)$ for $x \in [0,1]$. In particular,

$$\sigma(s) = u(0,s) < U_{\sigma(t)}(0) = \sigma(t)$$

and the lemma is proved. □

Indeed, we have the convergence of $u(x,t)$ to $U_{\sigma(t)}(x)$ near $x = 0$ as $t \to \infty$. To prove this, we make the following transformations:

$$u(x,t) := \sigma(t)\theta(\xi,\tau), \quad \xi := \sigma(t)^{(p-1)/2}x, \quad \tau := \int_0^t \sigma(s)^{p-1}ds. \tag{3.6}$$

Then it is easy to check that θ satisfies the equation

$$\theta_\tau = \theta_{\xi\xi} - \theta^p - g(\tau)\left(\theta - \frac{1-p}{2}\xi\theta_\xi\right), \tag{3.7}$$

where $g(\tau) := \sigma'(t)\sigma(t)^{-p}$. Also, $\theta(0,\tau) = 1$ and $\theta_\xi(0,\tau) = 0$ for all $\tau > 0$. Moreover, it follows from Lemma 3.2 and (3.2) that $\theta(\xi,\tau) < U_1(\xi)$.

We shall study the stabilization of the solution θ of (3.7). First, by considering the function

$$J(x,t) := \frac{1}{2}u_x^2 - Cu^{p+1}$$

for some positive constant C and applying a maximum principle (cf. p. 660 of [4]), we can also derive the following estimate

$$0 \leq u_x \leq Cu^{(p+1)/2} \;\forall x \in [0,1],\; t > 0, \tag{3.8}$$

where C is a positive constant. Consequently, by an integration, we deduce from (3.8) that

$$u(x,t) \leq [\sigma(t)^{(1-p)/2} + cx]^{2\alpha} \;\forall x \in [0,1],\; t > 0, \tag{3.9}$$

for some positive constant c.

Using (3.9), (3.6), and $u_x = \sigma^{(1+p)/2}\theta_\xi$, we obtain the following estimate for the solution θ of (3.7):

$$0 \leq \xi\theta_\xi(\xi,\tau),\; \theta(\xi,\tau) \leq C(1+\xi)^{2\alpha} \;\; \forall\, \xi \in [0, \sigma^{(p-1)/2}(t)],\; \tau > 0, \tag{3.10}$$

for some positive constant C.

Next, it follows from the Hopf Lemma that $u_{xx}(0,t) > 0$ and so $u_t(0,t) > -u^p(0,t)$ by (1.1). Hence $g(\tau) > -1$ for all $\tau > 0$. We conclude from Lemma 3.2 that $-1 < g(\tau) < 0$ for all $\tau \gg 1$. Note that

$$\int_0^\infty g(\tau)d\tau = -\infty.$$

Nevertheless, we have the following lemma.

Lemma 3.3. *There holds* $\lim_{\tau\to\infty} g(\tau) = 0$.

Proof. Otherwise, there is a sequence $\{\tau_n\} \to \infty$ such that $g(\tau_n) \to -\gamma$ as $n \to \infty$ for some constant $\gamma > 0$. By using (3.10) and the standard regularity theory, we can show that there is a subsequence, still denote it by $\{\tau_n\}$, such that

$$\theta(\xi, \tau + \tau_n) \to \tilde{\theta}(\xi, \tau) \ \text{ as } \ n \to \infty$$

uniformly on any compact subsets, where $\tilde{\theta}$ solves the equation

$$\tilde{\theta}_\tau = \tilde{\theta}_{\xi\xi} - \tilde{\theta}^p + \gamma(\tilde{\theta} - \frac{1-p}{2}\xi\tilde{\theta}_\xi), \ \xi > 0, \ \tau > 0, \tag{3.11}$$

with $\tilde{\theta}(0, \tau) = 1$ and $\tilde{\theta}_\xi(0, \tau) = 0$. Moreover, it is easily to check that $\tilde{\theta} \le U_1$ and $\tilde{\theta}_\xi \ge 0$.

Furthermore, it follows from the so-called energy argument (cf. the proof of Proposition 3.1 in [4]) that $\tilde{\theta}(\xi, \tau) \to V(\xi)$ as $\tau \to \infty$ for some V satisfying

$$\begin{aligned} &V'' - V^p + \gamma(V - \frac{1-p}{2}\xi V') = 0, \ \xi > 0, \\ &V'(0) = 0, \ V(0) = 1. \end{aligned}$$

Note that $V \le U_1$ and $V' \ge 0$. Set

$$W(y) := \Big(\frac{\gamma}{\alpha}\Big)^\alpha V(\sqrt{\frac{\alpha}{\gamma}}y).$$

Then W satisfies

$$\begin{aligned} &W'' - W^p + \alpha(W - \frac{1-p}{2}yW') = 0, \ y > 0, \\ &W'(0) = 0, \ W(0) = (\gamma/\alpha)^\alpha. \end{aligned}$$

Since $W > 0$, $W' \ge 0$ for $y > 0$, and $V \le U_1$ gives the polynomial boundedness of W, it follows from Proposition 3.3 of [4] that either $W = U$ or $W \equiv \alpha^{-\alpha}$. The first case is impossible, since $U(0) = 0$. The second case is also impossible, since θ is unbounded by Theorem 2.2. Hence the lemma follows. □

Again, by the standard limiting process with the estimate (3.10) and Lemma 3.3, for any given sequence $\{\tau_n\} \to \infty$, we can show that there is a limit $\tilde{\theta}$ satisfying

$$\begin{aligned} &\tilde{\theta}_\tau = \tilde{\theta}_{\xi\xi} - \tilde{\theta}^p, \ \xi > 0, \ \tau > 0, \\ &\tilde{\theta}(0, \tau) = 1, \ \tilde{\theta}_\xi(0, \tau) = 0, \end{aligned}$$

such that $\theta(\xi, \tau + \tau_n) \to \tilde{\theta}(\xi, \tau)$ as $n \to \infty$ uniformly on compact subsets. Since we also have

$$\tilde{\theta}(\xi, \tau) \le U_1(\xi), \ \tilde{\theta}(0, \tau) = U_1(0), \ \tilde{\theta}_\xi(0, \tau) = (U_1)_\xi(0),$$

the Hopf Lemma implies that $\tilde{\theta} \equiv U_1$. Since this limit is independent of the given sequence $\{\tau_n\}$, we see that $\theta(\xi, \tau) \to U_1(\xi)$ as $\tau \to \infty$ uniformly on any compact subsets. Returning to the original variables and using the relation (3.2), we thus have proved the following so-called inner expansion.

Theorem 3.4. *As $t \to \infty$, we have*

$$u(x, t) = U_{\sigma(t)}(x)(1 + o(1))$$

uniformly in $\{0 \le \sigma^{(p-1)/2}(t)x \le C\}$ for any positive constant C.

4. Spectrum Analysis

In this section, we shall study the following linearized operator

$$\mathcal{L}v := -v'' + \frac{b}{x^2}v,\ b := (2\alpha-1)(2\alpha-2)$$

which is from the linearization of (1.1) around the steady state U.

Consider the eigenvalue problem

$$\mathcal{L}\phi = \lambda\phi,\ 0 < x < 1; \quad \phi'(0) = 0,\ \phi(1) = 0. \tag{4.1}$$

We introduce the following Hilbert space and quantities:

$$\mathbb{H} := \{\phi \in H^1([0,1]) \mid \int_0^1 \frac{\phi^2(x)}{x^2}dx < \infty,\ \phi(1) = 0\},$$

$$J(\phi) := \int_0^1 \phi_x^2(x)dx + b\int_0^1 \frac{\phi^2(x)}{x^2}dx, \quad I(\phi) := \int_0^1 \phi^2(x)dx.$$

Then the principal eigenvalue λ^* of (4.1) can be characterized by

$$\lambda^* := \inf\{J(\phi)/I(\phi) \mid \phi \in \mathbb{H},\ I(\phi) > 0\}. \tag{4.2}$$

It is easy to see that $\lambda^* > b > 0$. Also, by taking a minimization sequence, we can show that this λ^* can be attained by a function $\phi^* \in \mathbb{H}$ which is the eigen-function of (4.1) such that

$$\phi^* > 0 \ \text{ in } (0,1), \qquad \int_0^1 (\phi^*(x))^2 dx = 1.$$

Note that $\phi^*(0) = 0$. It is also easy to see that

$$\phi^*(x) = dx^{2\alpha-1}(1 + o(1)) \ \text{ as } x \to 0 \tag{4.3}$$

for some positive constant d.

On the other hand, it is easily seen that for any $\varepsilon \in (0,1)$ there exists the principal eigen-pair $(\lambda_\varepsilon, \phi_\varepsilon)$ of the following eigenvalue problem[1]:

$$\mathcal{L}_\varepsilon\phi_\varepsilon = \lambda_\varepsilon\phi_\varepsilon,\ 0 < x < 1; \quad \phi_\varepsilon'(0) = \phi_\varepsilon(1) = 0 < \phi_\varepsilon(x)\ \forall x \in (0,1), \tag{4.4}$$

where

$$\mathcal{L}_\varepsilon v := -v'' + \frac{b(1-\varepsilon)}{x^2}\chi_{[\varepsilon,1]}(x)v$$

and χ is the indicator function. Note that ϕ_ε is only a C^1 function on $[0,1]$ and ϕ_ε'' has a jump discontinuity at $x = \varepsilon$.

Lemma 4.1. *There holds $\lambda_\varepsilon \to \lambda^*$ as $\varepsilon \to 0^+$.*

Proof. By the characterization of the principal eigenvalue λ_ε of (4.4) and $I(\phi^*) = 1$, we have

$$\lambda_\varepsilon \le J_\varepsilon(\phi^*),$$

where

$$J_\varepsilon(\phi) := \int_0^1 \phi_x^2(x)dx + b(1-\varepsilon)\int_\varepsilon^1 \frac{\phi^2(x)}{x^2}dx.$$

[1]This approximated eigenvalue problem was suggested by an anonymous referee which we would like to acknowledge here

It is clear that $J_\varepsilon(\phi^*) < J(\phi^*)$. Hence $\lambda_\varepsilon < \lambda^*$ for all $\varepsilon > 0$ and so

$$\limsup_{\varepsilon\to 0^+} \lambda_\varepsilon \leq \lambda^*. \tag{4.5}$$

On the other hand, we introduce a C^∞-function θ by $\theta(s) = 0$ for $s \leq 1/2$, $\theta(s) = 1$ for $s \geq 1$, and $\theta' \geq 0$ in $[1/2, 1]$. Let $\theta_\varepsilon(x) := \theta(x/\varepsilon)$ for any $\varepsilon \in (0, 1)$. Set $\tilde{\phi}_\varepsilon = \phi_\varepsilon$ in $[\varepsilon, 1]$ and $\tilde{\phi}_\varepsilon = \varepsilon$ in $[0, \varepsilon]$. Then for $\psi_\varepsilon := \theta_\varepsilon\tilde{\phi}_\varepsilon$ we have

$$J(\psi_\varepsilon) \leq J_\varepsilon(\phi_\varepsilon) + b\varepsilon \int_\varepsilon^1 \frac{\phi_\varepsilon^2(x)}{x^2}dx + \varepsilon\left(\int_{1/2}^1 (\theta')^2(s)ds + b\int_{1/2}^1 \frac{\theta^2(s)}{s^2}ds\right),$$

$$I(\psi_\varepsilon) = \int_\varepsilon^1 \phi_\varepsilon^2(x)dx + \varepsilon^3\int_{1/2}^1 \theta^2(s)ds.$$

Since $\lambda^* \leq J(\psi_\varepsilon)/I(\psi_\varepsilon)$ for all $\varepsilon \in (0, 1)$, we conclude that

$$\lambda^* \leq \liminf_{\varepsilon\to 0^+} \lambda_\varepsilon. \tag{4.6}$$

Therefore, the lemma follows by combining (4.5) and (4.6). □

5. Dead-core Rate Estimates

In this section, we shall give some estimates of the dead-core rate. First, the upper bound of dead-core rate can be derived from Theorem 2.3 that

$$\limsup_{t\to\infty} \frac{\ln \sigma(t)}{t} \leq -2\alpha \cdot \frac{\pi^2}{2(4\alpha + 1)}.$$

Next, we derive the following lower bound estimate for $u - U$.

Lemma 5.1. *There exists a small positive constant δ such that*

$$u(x, t) - U(x) \geq \delta e^{-\lambda^* t}\phi^*(x),\ x \in [0, 1],\ t > 1. \tag{5.1}$$

Proof. Write $w = u - U$. Then $w(0, t) > 0$, $w(1, t) = 0$, and w satisfies the equation

$$w_t = w_{xx} - \frac{b}{x^2}w + F(x, w), \tag{5.2}$$

where

$$F(x, w) := U^p - (w + U)^p + \frac{b}{x^2}w = \frac{1}{2}p(1 - p)\tilde{U}^{p-2}w^2, \tag{5.3}$$

for some $\tilde{U} \in (U, U + w)$. Note that $F \geq 0$. Set $\hat{w}(x, t) := \delta e^{-\lambda^* t}\phi^*(x)$, where δ is a positive constant to be determined later. Then

$$\hat{w}_t = \hat{w}_{xx} - \frac{b}{x^2}\hat{w},\ x \in (0, 1),\ t > 0,$$
$$\hat{w}(0, t) = 0,\ \hat{w}(1, t) = 0,\ t > 0.$$

Recall that $(\phi^*)'(1) < 0$. Also, note that $u_x(1, 1) - U'(1) < 0$, by the Hopf Lemma. By the continuity, there exist positive constants δ and η such that

$$u_x(x, 1) - U'(x) - \delta e^{-\lambda^*}(\phi^*)'(x) < 0 \tag{5.4}$$

for all $x \in [1 - \eta, 1]$. It follows from (5.4) that $w(x, 1) \geq \hat{w}(x, 1)$ for all $x \in [1 - \eta, 1]$. Using $u(\cdot, 1) > U(\cdot)$ in $[0, 1 - \eta]$ and by choosing smaller positive δ (if necessary), we obtain that $w(x,1) \geq \hat{w}(x,1)$ for all $x \in [0,1]$. Therefore, by the comparison principle, the estimate (5.1) follows. □

For the lower bound of dead-core rate, we recall from Lemmas 3.1 and 3.2 that for any $x > 0$:

$$u(x,t) \leq U_{\sigma(t)}(x) = U(x) + a\sigma^{(1-p)/2}(t)x^{2\alpha-1}(1+o(1)) \ \text{ as } \ t \to \infty. \tag{5.5}$$

On the other hand, by (5.1) and (4.3), we have

$$u(x(t),t) \geq U(x(t)) + d\delta e^{-2\alpha\lambda^* t}(1+o(1)) \ \text{ as } \ t \to \infty, \tag{5.6}$$

where $x(t) := e^{-\lambda^* t}$. Consequently, there exists a positive constants d_1 such that

$$e^{-\lambda^* t} \leq d_1 \sigma^{(1-p)/2}(t)(1+o(1)) \ \text{ as } \ t \to \infty.$$

Hence we obtain that

$$\sigma(t) \geq d_2 e^{-2\alpha\lambda^* t}(1+o(1)) \ \text{ as } \ t \to \infty \tag{5.7}$$

for some positive constant d_2. This implies that

$$\liminf_{t\to\infty} \frac{\ln \sigma(t)}{t} \geq -2\alpha\lambda^*.$$

6. Rate of Convergence

Recall the principal eigen-pair $(\lambda_\varepsilon, \phi_\varepsilon)$ of (4.4) for any $\varepsilon \in (0,1)$. Hereafter we shall fix the eigenfunction ϕ_ε so that

$$\phi_\varepsilon > 0 \ \text{ in } (0,1), \quad \int_0^1 \phi_\varepsilon^2(x)dx = 1.$$

Then it is clear that $\phi_\varepsilon \to \phi^*$ in $C^0([0,1])$ as $\varepsilon \to 0^+$. Then we have the following lemma for the upper bound of $u - U$.

Lemma 6.1. *For each $\varepsilon \in (0,1)$, there exist positive constants c_ε and t_ε such that*

$$u(x,t) - U(x) \leq c_\varepsilon e^{-\lambda_\varepsilon t}\phi_\varepsilon(x), \ x \in [0,1], \ t \geq t_\varepsilon. \tag{6.1}$$

Proof. Again, we set $w = u - U$. We first estimate F as follows. Since $\tilde{U} \in (U, U+w)$, we compute from (5.3) that

$$F(x,w) \leq \frac{1-p}{2}[U^{-1}w][pU^{p-1}w] = \frac{1-p}{2}[U^{-1}w]\left(\frac{b}{x^2}w\right).$$

By Theorem 2.2, there is t_ε sufficiently large such that

$$\frac{1-p}{2}[U^{-1}(x)w(x,t)] \leq \varepsilon \ \ \forall x \in [\varepsilon,1], \ t \geq t_\varepsilon.$$

Consequently, we obtain from (5.2) that w satisfies the following inequality

$$w_t \leq w_{xx} - \frac{b(1-\varepsilon)}{x^2}w \ \ \forall x \in [\varepsilon,1), \ t \geq t_\varepsilon. \tag{6.2}$$

Note that $w_x(0,t) = w(1,t) = 0$ for all $t > 0$. Since $u > U$, we have $w_t - w_{xx} \leq 0$ for all $x \in [0,1]$.

Now, set $\hat{w}(x,t) := c_\varepsilon e^{-\lambda_\varepsilon t}\phi_\varepsilon(x)$, where c_ε is a positive constant to be determined. Then

$$\hat{w}_t = \hat{w}_{xx} - \frac{b(1-\varepsilon)}{x^2}\chi_{[\varepsilon,1]}(x)\hat{w},\ x \in (0,1),\ t > 0,$$
$$\hat{w}_x(0,t) = 0,\ \hat{w}(1,t) = 0,\ t > 0.$$

Recall that $(\phi_\varepsilon)'(1) < 0$. Then by the continuity there exist a small positive constant η and a large positive constant c_ε such that

$$u_x(x,t_\varepsilon) - U'(x) - c_\varepsilon e^{-\lambda_\varepsilon t_\varepsilon}(\phi_\varepsilon)'(x) > 0 \ \ \forall x \in [1-\eta, 1]. \tag{6.3}$$

It follows from (6.3) that $w(x,t_\varepsilon) \leq \hat{w}(x,t_\varepsilon)$ for $x \in [1-\eta,1]$. Then, by choosing c_ε larger (if necessary), we obtain that $w(x,t_\varepsilon) \leq \hat{w}(x,t_\varepsilon)$ for $x \in [0,1]$. Therefore, the lemma follows by applying the comparison principle for weak solutions (cf. [7]). $\square$

Since $u > U$, we have the following uniformly exponential rate of convergence of u to U over the whole domain by using (5.1) and (6.1).

Theorem 6.2. *For each $\varepsilon > 0$, there exist positive constants d and d_ε such that*

$$\|u(\cdot,t) - U\|_{C^0([0,1])} \geq d e^{-\lambda^* t} \ \text{ for all } \ t > 1, \tag{6.4}$$

$$\|u(\cdot,t) - U\|_{C^0([0,1])} \leq d_\varepsilon e^{-\lambda_\varepsilon t} \ \text{ for all } \ t > t_\varepsilon. \tag{6.5}$$

Indeed, the constants d and d_ε in Theorem 6.2 can be taken as $d = \delta\phi^*(1/2)$ and $d_\varepsilon = c_\varepsilon\|\phi_\varepsilon\|_{C^0([0,1])}$. Notice that $\lambda_\varepsilon < \lambda^*$ for all $\varepsilon > 0$ and $\lambda_\varepsilon \to \lambda^*$ as $\varepsilon \to 0^+$.

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