DYNAMICS FOR A TWO-SPECIES COMPETITION-DIFFUSION MODEL WITH TWO FREE BOUNDARIES

JONG-SHENQ GUO AND CHANG-HONG WU

Abstract. To understand the spreading and interaction of two-competing species, we study the dynamics for a two-species competition-diffusion model with two free boundaries. Here, the two free boundaries which describe the spreading fronts of two competing species, respectively, may intersect each other. Our result shows, there exists a critical value such that the superior competitor always spreads successfully if its territory size is above this constant at some time. Otherwise, the superior competitor can be wiped out by the inferior competitor. Moreover, if the inferior competitor spreads not fast enough such that the superior competitor can catch up with it, the inferior competitor will be wiped out eventually and then a spreading-vanishing trichotomy is established. We also provide some characterization of the spreading-vanishing trichotomy via some parameters of the model. On the other hand, when the superior competitor spreads successfully but with a sufficiently low speed, the inferior competitor can also spread successfully even the superior species is much stronger than the weaker one. It means that the inferior competitor can survive if the superior species cannot catch up with it.

1. Introduction

The spreading or invasion phenomenon of multiple competing species is an important factor to understand the complexity of ecology. Mathematically, there has been tremendous studies concerned with the existence of positive traveling wave solutions connecting different constant equilibria [6, 14, 16, 19, 20, 27, 29] and the asymptotic spreading speed associated with the Cauchy problem [23, 24, 31]. Recently, a different approach proposed by Du and Lin [10] models the spreading phenomenon for a single species by assuming the spreading front as a free boundary, where the key assumption is that the population density vanishes at the front and the mechanism of spreading is determined by the spatial population gradient at the front. A mathematical deduction is to consider the population loss near the spreading front and the Allee effect is taken into account [2]. More results for more general models have been obtained in, for example, [7, 8, 9, 12, 13, 18, 21, 28] and references cited therein.

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Following such approach, there are different biological considerations to two-species Lotka-Volterra type competition models. In [11], the authors consider that an invasive species exists initially in a ball and invades into the environment, while the resident species distributes in the whole space $\mathbb{R}^N$. In [15, 33], the two weak-competition species are assumed to spread along the same free boundary. Similar works but for two-species Lotka-Volterra type predator-prey models can be found in [30, 34]. We also refer to much earlier works [26, 25] in which the environment is assumed to be a bounded domain. For traveling wave solutions of free boundary problems, see [4, 5, 32] for examples.

Based on these works, we may ask: if two species u, v spread only at the same direction but with different free boundaries, then what the dynamics can be. More precisely, we envision that two species initially occupy the intervals $[0, s_1^0]$ and $[0, s_2^0]$, respectively. Also, they only move to the right and their territories expand to $[0, s_1(t)]$ and $[0, s_2(t)]$, respectively, at time t. We ask: does the superior competitor always wipe out the inferior one if it establishes persistent populations? If not, how is it possible for weaker species to survive? For this, we shall look for the unknown (u, v, s_1, s_2) satisfying the following free boundary problem **(P)**:

$$u_t = d_1 u_{xx} + r_1 u(1 - u - kv), \quad 0 < x < s_1(t),\ t > 0, \tag{1.1}$$

$$v_t = d_2 v_{xx} + r_2 v(1 - v - hu), \quad 0 < x < s_2(t),\ t > 0, \tag{1.2}$$

$$u_x(0,t) = v_x(0,t) = 0, \quad t > 0, \tag{1.3}$$

$$u \equiv 0 \quad \text{for all } x \geq s_1(t) \text{ and } t > 0; \quad v \equiv 0 \quad \text{for all } x \geq s_2(t) \text{ and } t > 0, \tag{1.4}$$

$$s_1'(t) = -\mu_1 u_x(s_1(t), t),\ t > 0; \quad s_2'(t) = -\mu_2 v_x(s_2(t), t),\ t > 0, \tag{1.5}$$

$$s_1(0) = s_1^0,\ s_2(0) = s_2^0,\ u(x,0) = u_0(x),\ v(x,0) = v_0(x) \text{ for } x \in [0, \infty), \tag{1.6}$$

where $u(x,t)$ and $v(x,t)$ represent the population densities of two competing species at the position x and time t; d_1, d_2 are diffusion rates of species u, v; r_1, r_2 are net birth rates of species u, v; h, k are competition coefficients of species u, v; the parameters μ_1 and μ_2 measure the intention to spread into new territories of u, v, respectively. All the parameters are positive and the initial data (u_0, v_0, s_1^0, s_2^0) satisfy

$$\begin{cases} s_1^0 > 0,\ s_2^0 > 0,\ u_0 \in C^2[0, s_1^0],\ v_0 \in C^2[0, s_2^0],\ u_0'(0) = v_0'(0) = 0, \\ u_0(x) > 0 \text{ for } x \in [0, s_1^0),\ u_0(x) = 0 \text{ for } x \geq s_1^0, \\ v_0(x) > 0 \text{ for } x \in [0, s_2^0),\ v_0(x) = 0 \text{ for } x \geq s_2^0. \end{cases} \tag{1.7}$$

Notice that the free boundaries $x = s_1(t)$ and $x = s_2(t)$ may intersect each other at some time. Also, the derivatives of u, v at the free boundary will be considered as left derivatives.

We now describe the main results of this paper as follows.

Theorem 1 (Global existence and uniqueness)**.** *The problem* **(P)** *admits a unique global in time solution* (u, v, s_1, s_2) *with* $s_1, s_2 \in C^{1+\alpha/2}([0,\infty))$ *and*

$$u \in C^{2,1}(\Omega_1) \cap C^{1+\alpha,(1+\alpha)/2}(\overline{\Omega}_1),\ \ v \in C^{2,1}(\Omega_2) \cap C^{1+\alpha,(1+\alpha)/2}(\overline{\Omega}_2),$$

where $\alpha \in (0,1)$ *is arbitrary and*

$$\Omega_j := \{(x,t) : 0 \le x \le s_j(t),\ t > 0\},\ j = 1, 2.$$

Furthermore,

$$0 < u(x,t) \le K_1 := \max\{1, \|u_0\|_{L^\infty}\}, \quad x \in [0, s_1(t)),\ t \ge 0, \tag{1.8}$$

$$0 < v(x,t) \le K_2 := \max\{1, \|v_0\|_{L^\infty}\}, \quad x \in [0, s_2(t)),\ t \ge 0, \tag{1.9}$$

$$0 < s_1'(t) \le 2\mu_1 K_1 \max\left\{ \sqrt{\frac{r_1}{2d_1}},\ \frac{4}{3},\ \frac{-4}{3}\left(\min_{x\in[0,s_1^0]} u_0'(x) \right) \right\},\ t > 0, \tag{1.10}$$

$$0 < s_2'(t) \le 2\mu_2 K_2 \max\left\{ \sqrt{\frac{r_2}{2d_2}},\ \frac{4}{3},\ \frac{-4}{3}\left(\min_{x\in[0,s_2^0]} v_0'(x) \right) \right\},\ t > 0. \tag{1.11}$$

Due to (1.10) and (1.11), the limits

$$s_{1,\infty} := \lim_{t\to\infty} s_1(t), \quad s_{2,\infty} := \lim_{t\to\infty} s_2(t)$$

are well-defined such that $s_{i,\infty} \le \infty$, $i = 1, 2$. As in [11, 15, 30, 34], we see that the dynamics of **(P)** strongly depends on their territory sizes. To describe the spreading and vanishing phenomena, we define

- The species u (resp., v) *vanishes eventually* if $s_{1,\infty} < +\infty$ (resp., $s_{2,\infty} < +\infty$) and
$$\lim_{t\to+\infty} \|u(\cdot,t)\|_{C([0,s_1(t)])} = 0 \quad (\text{resp., } \lim_{t\to+\infty} \|v(\cdot,t)\|_{C([0,s_2(t)])} = 0);$$
- The species u (resp., v) *spreads successfully* if $s_{1,\infty} = +\infty$ (resp., $s_{2,\infty} = +\infty$) and the species u (resp., v) *persists* in the sense that there exist $\varepsilon > 0$ and two continuous curves $x = l_\pm(t)$ such that $l_+(t) - l_-(t) \ge \varepsilon$ for all large t and $u(x,t) \ge \varepsilon$ (resp., $v(x,t) \ge \varepsilon$) for all $x \in [l_-(t), l_+(t)]$ and for all large t.

In this paper, we always assume

(H) $0 < k < 1 < h$ (so that u is a superior competitor and v is an inferior competitor).

We introduce the following three quantities:

$$s_* := \frac{\pi}{2}\sqrt{\frac{d_1}{r_1}}, \quad s^* := \frac{\pi}{2}\sqrt{\frac{d_1}{r_1}}\frac{1}{\sqrt{1-k}}, \quad s^{**} := \frac{\pi}{2}\sqrt{\frac{d_2}{r_2}}.$$

Note that $s_* < s^*$.

Our next result is to determine the dynamics of **(P)** via their asymptotical territory sizes $s_{i,\infty}$, $i = 1, 2$.

Theorem 2. *Assume* **(H)***. Then the followings hold:*

(i) *If $s_{1,\infty} \le s_*$, then the species u vanishes eventually. In this case, if $s_{2,\infty} \le s^{**}$, the species v vanishes eventually; if $s_{2,\infty} > s^{**}$, v spreads successfully and*

$$\lim_{t\to\infty} v(\cdot,t) = 1 \quad \text{uniformly for any bounded subset of } [0,\infty). \tag{1.12}$$

(ii) *If $s_{1,\infty} \in (s_*, s^*]$, then u vanishes eventually and v spreads successfully with behavior* (1.12).

(iii) *If $s_{1,\infty} > s^*$, then u spreads successfully. Furthermore,*

$$\liminf_{t\to\infty} u(\cdot,t) \ge 1 - k\rho_2 \quad \text{uniformly for any bounded subset of } [0,\infty),$$

where $\rho_2 := \limsup_{t\to\infty} \|v(\cdot,t)\|_{C[0,s_2(t)]} \in [0,1]$.

Theorem 2 shows that the inferior competitor may win the competition if the the territory of the superior species does not cross over $[0,s^*]$. However, u is always unbeatable if its territory exceeds $[0,s^*]$. A natural question arises: does the weaker species v always die out eventually if u spreads successfully?

Intuitively, if the superior competitor spreads faster enough than the inferior competitor, the inferior competitor would have no chance to survive eventually even its initial populations and initial habitat sizes are large. In this situation, it is impossible that both two species spread successfully. Thus, the spreading and vanishing trichotomy is established.

Before stating the trichotomy result, we recall a result of [2] to characterize the trichotomy region:

Proposition 1 (Propositions 2.1 and 2.2 of [2])**.** *For any given $a > 0$, $b > 0$, $d > 0$ and $c \in [0, 2\sqrt{ad}\,)$, the problem*

$$cU' = dU'' + U(aU - b) \text{ in } (0,\infty),\ U(0) = 0,\ U(\infty) = \frac{b}{a} \tag{1.13}$$

has a unique positive solution $U = U_c$ and $U_c'(\cdot) > 0$ in $[0,\infty)$. Moreover, the followings hold:

(i) *$U_{c_1}'(0) > U_{c_2}'(0)$ and $U_{c_1}(x) > U_{c_2}(x)$ for all $x > 0$ if $0 \le c_1 < c_2 < 2\sqrt{ad}$.*

(ii) *For each $\mu > 0$, there exists a unique $c_0 = c_0(a,b,d,\mu) \in (0, 2\sqrt{ad})$ such that $\mu U_{c_0}'(0) = c_0$ and*

$$\lim_{\frac{a\mu}{bd}\to 0} \frac{c_0}{\sqrt{ad}}\frac{bd}{a\mu} = \frac{1}{\sqrt{3}}, \quad \lim_{\frac{a\mu}{bd}\to\infty} \frac{c_0}{\sqrt{ad}} = 2. \tag{1.14}$$

(iii) *c_0 is strictly increasing in a and μ, respectively, and is strictly decreasing in b.*

For every $d_i > 0$, $r_i > 0$ $(i = 1,2)$, $0 < k < 1 < h$, define

$$\mathcal{A} := \{(\mu_1,\mu_2) \in (0,\infty)\times(0,\infty) : c_0(r_1(1-k), r_1, d_1, \mu_1) > c_0(r_2, r_2, d_2, \mu_2)\}.$$

By Proposition 1, $\mathcal{A}$ is non-empty. Indeed, by (1.14),

$$\lim_{\mu_1\to\infty} c_0(r_1(1-k), r_1, d_1, \mu_1) = 2\sqrt{d_1 r_1(1-k)}, \quad \lim_{\mu_2\to 0} c_0(r_2, r_2, d_2, \mu_2) = 0,$$

there exist $\mu_1^* > 0$ and $\mu_2^* > 0$ such that $[\mu_1^*, \infty) \times (0, \mu_2^*] \subset \mathcal{A}$.

Theorem 3. *Assume* **(H)** *and $d_i > 0$, $r_i > 0$ are given, $i = 1, 2$. Suppose that $(\mu_1, \mu_2) \in \mathcal{A}$ and $s_{1,\infty} > s^*$. Then u spreads successfully and v vanishes eventually. In this case, we have*

$$\lim_{t\to\infty} u(\cdot, t) = 1 \quad \textit{uniformly for any bounded subset of } [0, \infty). \tag{1.15}$$

Theorem 3 together with Theorem 2 imply that we have the spreading and vanishing trichotomy, namely, both species vanish eventually, u vanishes eventually and v spreads successfully, or, u spreads successfully and v vanishes eventually, when $(\mu_1, \mu_2) \in \mathcal{A}$. More precisely, we have

Corollary 1 (spreading and vanishing trichotomy)**.** *Assume* **(H)** *and $d_i > 0$, $r_i > 0$ are given, $i = 1, 2$. If $(\mu_1, \mu_2) \in \mathcal{A}$, then the dynamics of* **(P)** *satisfies the following trichotomy:*

(i) *both two species vanish eventually: $s_{1,\infty} \le s_*$ and $s_{2,\infty} \le s^{**}$,*
(ii) *u vanishes eventually and v spreads successfully: $s_{1,\infty} \le s^*$,*
(iii) *u spreads successfully and v vanishes eventually.*

Remark 1. *In the vanishing cases in Cor. 1, the upper bounds of $s_{1,\infty}, s_{2,\infty}$ can be given as in Parts (i)(ii). These bounds depend only on the parameters in the system. However, for Part (iii), there does not exist an upper estimate for $s_{2\infty}$ depending only on the parameters in the system. It also depends on the initial data.*

Next, we characterize the set $\mathcal{A}$ as follows.

Theorem 4 (characterization of the set $\mathcal{A}$)**.** *Assume* **(H)** *and $d_i > 0$, $r_i > 0$ are given, $i = 1, 2$. Then there exist a strictly increasing C^1 function $\Lambda(\cdot)$ with $\Lambda(0^+) = 0$ and two positive constants ν_1 and ν_2 satisfying*

$$\begin{aligned}
&\Lambda : (0, \infty) \to (0, \nu_1) \quad \textit{with } \Lambda(\infty) = \nu_1 \textit{ if } \quad \sqrt{r_1 d_1(1-k)} > \sqrt{r_2 d_2};\\
&\Lambda : (0, \infty) \to (0, \infty) \quad \textit{with } \Lambda(\infty) = \infty \textit{ if } \quad \sqrt{r_1 d_1(1-k)} = \sqrt{r_2 d_2};\\
&\Lambda : (0, \nu_2) \to (0, \infty) \quad \textit{with } \Lambda(\nu_2^-) = \infty \textit{ if } \quad \sqrt{r_1 d_1(1-k)} < \sqrt{r_2 d_2}
\end{aligned}$$

such that the following hold:

- *If $\sqrt{r_1 d_1(1-k)} \ge \sqrt{r_2 d_2}$, then*

$$(\mu_1, \mu_2) \in \mathcal{A} \Longleftrightarrow \mu_1 > \Lambda(\mu_2), \quad \mu_2 \in (0, \infty);$$

- *If $\sqrt{r_1 d_1(1-k)} < \sqrt{r_2 d_2}$, then*

$$(\mu_1, \mu_2) \in \mathcal{A} \Longleftrightarrow \mu_1 > \Lambda(\mu_2), \quad \mu_2 \in (0, \nu_2).$$

Theorem 4 helps us to understand more about the sufficient condition for which spreading-vanishing trichotomy (given in Corollary 1) holds via the parameters μ_1 and μ_2. It shows that, roughly speaking, the inferior competitor cannot spread successfully if μ_2, the intention of v to spread, is too small.

Our final result provides some conditions for which both species can spread successfully.

Theorem 5. *Assume* **(H)**. *Given d_1, μ_2, r_i, $i = 1, 2$, u_0 and v_0 with $s_1^0 < s_2^0$ and $(v_0)'(x) \leq 0$ for all $x \in [s_1^0, s_2^0]$. Suppose that $s_{1,\infty} > s^*$ (e.g., $s_1^0 > s^*$). Then there exists $\bar{d} > 0$ depending on $d_1, \mu_2, r_1, r_2, u_0$ and v_0 such that if $d_2 > \bar{d}$, then both two species spread successfully as long as*

$$\mu_1 \leq \bar{\mu} \quad \textit{and } s_2^0 - s_1^0 > 2\pi \left[\sqrt{\frac{r_2}{d_2}\left(1 - \frac{\bar{d}}{d_2}\right)} \right]^{-1}, \tag{1.16}$$

for some positive constant $\bar{\mu}$ depending only on d_2 and $\bar{d}$.

Theorem 5 shows that if the superior competitor spreads too slow to catch up with the inferior competitor, it may leave enough space for the inferior competitor to survive.

The rest of the paper is organized as follows. In Section 2, we prove the global existence and uniqueness of solution to **(P)**. Although the problem **(P)** is related to some recent works (e.g., [5, 11, 15, 25, 30, 34]), it seems that their arguments in the proof of the existence and uniqueness of solution cannot be applied directly to our problem. In fact, since in our case the two free boundaries may intersect each other at some time, it leads to that these two free boundaries may not be straightened locally into two cylindrical domains at the same time. Thus our proof here becomes more complicated than those of the above-mentioned related works. In Section 3, we first recall some fundamental results from [2, 10] and give a basic estimate which shall be used to derive the main results of this paper. Then we determine the dynamics of **(P)** via $s_{i,\infty}$, $i = 1, 2$, and give proofs of Theorems 2, 3, 4 and 5. Also, some sufficient conditions for spreading and vanishing via the initial data are presented. Finally, in Section 4 we shall give a brief discussion with some future direction of studies.

2. Existence and uniqueness

In this section, we shall deal with global existence and uniqueness of solutions to the free boundary problem **(P)**. For the local existence, we shall consider the following problem with a more general nonlinearity:

$$
(2.1)\qquad \begin{cases} u_t = d_1 u_{xx} + f(u,v), \quad 0 < x < s_1(t),\ t > 0, \\ v_t = d_2 v_{xx} + g(u,v), \quad 0 < x < s_2(t),\ t > 0, \\ u_x(0,t) = v_x(0,t) = 0, \quad t > 0, \\ u \equiv 0 \quad \text{if } x \geq s_1(t) \text{ and } t > 0; \quad v \equiv 0 \quad \text{if } x \geq s_2(t) \text{ and } t > 0, \\ s_1'(t) = -\mu_1 u_x(s_1(t),t),\ t > 0; \quad s_2'(t) = -\mu_2 v_x(s_2(t),t),\ t > 0, \\ s_1(0) = s_1^0,\ s_2(0) = s_2^0,\ u(x,0) = u_0(x),\ v(x,0) = v_0(x) \text{ for } x \in [0,\infty), \end{cases}
$$

where the initial data satisfies (1.7), and the nonlinearities satisfy

(A) f and g are locally Lipschitz continuous for $u, v \in [0,\infty)$ with

$$
f(0,v) = 0 = g(u,0) \ \text{ for } u, v \geq 0.
$$

Our first goal is to establish the local existence result for (2.1):

Proposition 2 (Local existence). *Assume* (1.7), **(A)** *and* $\alpha \in (0,1)$. *Suppose that*

$$
(2.2)\qquad \|u_0\|_{C^2[0,s_1^0]} + \|v_0\|_{C^2[0,s_2^0]} + s_1^0 + s_2^0 \leq M
$$

for some $M > 0$. *Then there exists* $T_0 \in (0,\infty)$ *and* $M_0 > 0$ *depending only on* α, M *and the local Lipschitz constants of* f, g *such that the problem* (2.1) *admits a unique solution*

$$
(u,v,s_1,s_2) \in C^{1+\alpha,(1+\alpha)/2}(D^1_{T_0}) \times C^{1+\alpha,(1+\alpha)/2}(D^2_{T_0}) \times C^{1+\alpha/2}[0,T_0] \times C^{1+\alpha/2}[0,T_0]
$$

satisfying

$$
(2.3)\qquad \|u\|_{C^{1+\alpha,(1+\alpha)/2}(D^1_{T_0})} + \|v\|_{C^{1+\alpha,(1+\alpha)/2}(D^2_{T_0})} + \sum_{i=1}^{2} \|s_i\|_{C^{1+\alpha/2}[0,T_0]} \leq M_0,
$$

where $D^i_{T_0} := \{(x,t) : 0 \leq x \leq s_i(t),\ t \in [0,T_0]\}$ *for* $i = 1, 2$.

Our strategy of the proof of Proposition 2 is as follows: for a given small constant $T > 0$ we introduce the function spaces

$$
\Sigma_i := \{s \in C^1[0,T] :\ s(0) = s_i^0,\ s'(0) = s_i^*,\ \|s' - s_i^*\|_{C[0,T]} \leq 1\} \quad i = 1, 2,
$$

where $s_1^* := -\mu_1 u_0'(s_1^0)$ and $s_2^* := -\mu_2 v_0'(s_2^0)$. Given $(\hat{s}_1, \hat{s}_2) \in \Sigma_1 \times \Sigma_2$, we consider the following problem with variable fixed domains:

$$
(2.4)\qquad \begin{cases} u_t = d_1 u_{xx} + f(u,v), \quad 0 < x < \hat{s}_1(t),\ t > 0, \\ v_t = d_2 v_{xx} + g(u,v), \quad 0 < x < \hat{s}_2(t),\ t > 0, \\ u_x(0,t) = v_x(0,t) = 0, \quad t > 0, \\ u \equiv 0 \quad \text{if } x \geq \hat{s}_1(t) \text{ and } t > 0; \quad v \equiv 0 \quad \text{if } x \geq \hat{s}_2(t) \text{ and } t > 0, \\ \hat{s}_1(0) = s_1^0,\ \hat{s}_2(0) = s_2^0,\ u(x,0) = u_0(x),\ v(x,0) = v_0(x) \text{ for } x \in [0,\infty). \end{cases}
$$

Then the proof of Proposition 2 can be carried out in two steps:

- **Step 1.** For any given $(\hat{s}_1, \hat{s}_2) \in \Sigma_1 \times \Sigma_2$ there exists small $\tau_1 \in (0, \infty)$ such that the problem (2.4) has a unique solution $(\hat{u}, \hat{v})$ for $t \in [0, \tau_1]$.
- **Step 2.** Define the following two mappings:

$$\mathcal{F}_i(\hat{s}_i)(t) := s_i^0 - \mu_1 \int_0^t \varphi_{i,x}(\hat{s}_i(\tau), \tau) d\tau, \quad i = 1, 2,$$

  where $\varphi_1 = \hat{u}$ and $\varphi_2 = \hat{v}$. Then we show that $\mathcal{F} := (\mathcal{F}_1, \mathcal{F}_2)$ defined on $\Sigma_1 \times \Sigma_2$ admits a unique fixed point using the contraction mapping theorem.

Combining **Step 1** and **Step 2**, we see the problem (2.1) admits a solution, so does the problem **(P)**.

We shall divide our discussion into three subsections.

2.1. **The local existence and uniqueness for** (2.4)**.** In this subsection, we study the problem (2.4) with given $(\hat{s}_1, \hat{s}_2) \in \Sigma_1 \times \Sigma_2$.

Lemma 2.1. *Assume* (1.7), (2.2) *and* $\alpha \in (0,1)$. *Then there exist* $M_1 > 0$ *and* $\tau_1 \in (0, \infty)$ *depending only on* M, α *and the local Lipschitz constants of* f, g *such that the problem* (2.4) *has a unique solution* $(\hat{u}, \hat{v})$ *for* $t \in [0, \tau_1]$ *satisfying*

$$\|\hat{u}\|_{C^{1+\alpha,(1+\alpha)/2}(\hat{D}^1_{\tau_1})} + \|\hat{v}\|_{C^{1+\alpha,(1+\alpha)/2}(\hat{D}^2_{\tau_1})} \le M_1, \tag{2.5}$$

where $\hat{D}^i_{\tau_1} := \{(x,t) : 0 \le x \le \hat{s}_i(t),\ t \in [0, \tau_1]\}$, $i = 1, 2$.

Proof. For any given $\hat{s}_i(t) \in \Sigma_i$ for $i = 1, 2$, we first straighten the given boundary $x = \hat{s}_1(t)$ into a flat boundary by the transformation $y = x/\hat{s}_1(t)$. Also, let

$$U(y,t) := u(x,t), \quad V(y,t) := v(x,t), \quad \eta(t) := \frac{\hat{s}_2(t)}{\hat{s}_1(t)}.$$

Then (U, V) satisfies the following problem:

$$\begin{cases} U_t = \dfrac{d_1}{(\hat{s}_1(t))^2} U_{yy} + \dfrac{\hat{s}_1'(t) y}{\hat{s}_1(t)} U_y + f(U,V),\ 0 < y < 1,\ t > 0, \\ V_t = \dfrac{d_2}{(\hat{s}_1(t))^2} V_{yy} + \dfrac{\hat{s}_1'(t) y}{\hat{s}_1(t)} V_y + g(U,V),\ 0 < y < \eta(t),\ t > 0, \\ U_y(0,t) = V_y(0,t) = 0, \quad t > 0, \\ U \equiv 0 \quad \text{if } x \ge 1,\ t > 0; \quad V \equiv 0 \quad \text{if } x \ge \eta(t),\ t > 0, \\ \eta(0) = s_2^0/s_1^0,\ (U,V)(y,0) = (U^0, V^0)(y) = (u^0, v^0)(s_1^0 y),\ y \in [0, \infty). \end{cases} \tag{2.6}$$

Next, we introduce the function spaces

$$\begin{aligned} X_T^1 &:= \{U \in C([0,\infty)\times[0,T]) : U(y,0) = U^0(y),\ U \equiv 0 \text{ if } y \geq 1,\ t \in [0,T], \\ &\quad \|U - U^0\|_{C([0,\infty)\times[0,T])} \leq 1\}; \\ X_T^2 &:= \{V \in C([0,\infty)\times[0,T]) : V(y,0) = V^0(y),\ V \equiv 0 \text{ if } y \geq \eta(t),\ t \in [0,T], \\ &\quad \|V - V^0\|_{C([0,\infty)\times[0,T])} \leq 1\}. \end{aligned}$$

Given $(\hat{U},\hat{V}) \in X_T^1 \times X_T^2$. Since there exist $T_1 \in (0,T)$, $c_1 > 0$ which depend only on M and the local Lipschitz constants of f, g such that

$$\frac{1}{c_1} \leq \hat{s}_1(t) \leq c_1,\ |\hat{s}_1'(t)/\hat{s}(t)| \leq c_1,\ t \in [0,T_1],\ \|f\|_{L^\infty([0,\infty)\times[0,T_1])} \leq c_1,$$

we can apply the standard parabolic L^p theory and the Sobolev embedding theorem (see [17, 22]) to deduce that the system

$$\begin{cases} U_t = \dfrac{d_1 U_{yy}}{(\hat{s}_1(t))^2} + \dfrac{\hat{s}_1'(t)y}{\hat{s}_1(t)} U_y + f(\hat{U},\hat{V}), \quad 0 < y < 1,\ t > 0, \\ U_y(0,t) = 0 = U(1,t), \quad t > 0, \\ U(y,0) = U^0(y), \quad 0 \leq y \leq 1. \end{cases} \tag{2.7}$$

has a unique solution $U \in C^{1+\alpha,(1+\alpha)/2}([0,1]\times[0,T_1])$ with

$$\|U\|_{C^{1+\alpha,(1+\alpha)/2}([0,1]\times[0,T_1])} \leq C_1, \tag{2.8}$$

where the constant C_1 depends only on α, M and the local Lipschitz constants of f, g.

Let us now turn to the following problem:

$$\begin{cases} V_t = \dfrac{d_2 V_{yy}}{(\hat{s}_1(t))^2} + \dfrac{\hat{s}_1'(t)y}{\hat{s}_1(t)} V_y + g(\hat{U},\hat{V}),\ 0 < y < \eta(t),\ t > 0, \\ V_y(0,t) = 0 = V(\eta(t),t), \quad t > 0, \\ V(y,0) = V^0(y), \quad 0 \leq y \leq \eta(0). \end{cases} \tag{2.9}$$

As before, we can straighten the given boundary $y = \eta(t)$. Then, again, the standard parabolic L^p theory and the Sobolev embedding theorem (see [17, 22]) give a unique solution $V \in C^{1+\alpha,(1+\alpha)/2}(R_{T_2})$ of the problem (2.9) for some $0 < T_2 \leq T_1$, satisfying

$$\|V\|_{C^{1+\alpha,(1+\alpha)/2}(R_{T_2})} \leq C_2, \tag{2.10}$$

where constants T_2, C_2 depend only on α, M and the local Lipschitz constants of f, g, and

$$R_{T_2} := \{(y,t) : 0 \leq y \leq \eta(t),\ 0 \leq t \leq T_2\}.$$

From the above discussions, we are able to define the mapping W on $X_T^1 \times X_T^2$ such that

$$W(\hat{U},\hat{V}) = (U,V).$$

Then one can prove that W has a unique fixed point as long as $T \in (0,1)$ small enough using the contraction mapping theorem. To do so, we first prove that W maps $X_T^1 \times X_T^2$ into itself for small T. For this, we set

$$\hat{R}_T := \{(y,t) : \eta(t) \le y \le \eta(0),\ t \in [0,T]\} \quad \text{(note that } \hat{R}_T \text{ may be empty).}$$

By setting $\overline{V}^0(y,t) := V^0(y)$ on $\hat{R}_T$, we can derive

$$\|\overline{V}^0\|_{C(\hat{R}_T)} \le TM^2 \sup_{t\in[0,T]} |\eta'(t)| \tag{2.11}$$

where M is given in (2.2). Indeed, using the mean value theorem twice, we have

$$\begin{aligned} |V^0(y)| &= |v^0(s_1^0 y)| = |v^0(s_1^0 y) - v^0(s_1^0\eta(0))| \quad (\text{since } v^0(s_1^0\eta(0)) = 0) \\ &\le s_1^0 \|v_x^0\|_{C[0,s_2^0]} |y - \eta(0)| \le s_1^0 \|v_x^0\|_{C[0,s_2^0]} |\eta(t) - \eta(0)| \\ &\le M^2 T \sup_{t\in[0,T]} |\eta'(t)| \end{aligned}$$

for all $\eta(t) \le y \le \eta(0)$. Hence (2.11) holds.

Using (2.8), (2.10) and (2.11), there exists $C_3 > 0$ depending only on α, M and the local Lipschitz constants of f, g such that

$$\begin{aligned} & \|U - U^0\|_{C([0,\infty)\times[0,T])} + \|V - V^0\|_{C([0,\infty)\times[0,T])} \\ = \;& \|U - U^0\|_{C([0,1]\times[0,T])} + \max\{\|V - V^0\|_{C(R_T)},\ \|\overline{V}^0\|_{C(\hat{R}_T)}\} \\ \le \;& T^{\frac{1+\alpha}{2}} \|U - U^0\|_{C^{0,(1+\alpha)/2}([0,1]\times[0,T])} + T^{\frac{\alpha}{2}} \|V - V^0\|_{C^{0,\alpha/2}(R_T)} + TM^2 \sup_{t\in[0,T]} |\eta'(t)| \\ \le \;& C_3 T^{\frac{\alpha}{2}} \quad (\text{choosing } T < \min\{1, T_2\}). \end{aligned}$$

Thus, W maps $X_T^1 \times X_T^2$ into itself as long as $0 < T < \min\{1, T_2, C_3^{-2/\alpha}\}$.

On the other hand, for any $(\hat{U}_i, \hat{V}_i)$, we can define U_i and V_i as the solution of (2.7) and (2.9) respectively, for $t \in [0,T]$, i.e., $W(\hat{U}_i, \hat{V}_i) = (U_i, V_i)$, $i = 1, 2$. Note that (U_1, V_1) and (U_2, V_2) are defined in the same domain. Thus, by setting

$$\tilde{U} := U_1 - U_2, \quad \tilde{V} := V_1 - V_2,$$

we obtain the following system:

$$\begin{cases} \tilde{U}_t = \dfrac{d_1}{(\hat{s}_1(t))^2}\tilde{U}_{yy} + \dfrac{\hat{s}_1'(t)y}{\hat{s}_1(t)}\tilde{U}_y + f(\hat{U}_1, \hat{V}_1) - f(\hat{U}_2, \hat{V}_2),\ 0 < y < 1,\ t > 0, \\ \tilde{V}_t = \dfrac{d_2}{(\hat{s}_1(t))^2}\tilde{V}_{yy} + \dfrac{\hat{s}_1'(t)y}{\hat{s}_1(t)}\tilde{V}_y + g(\hat{U}_1, \hat{V}_1) - g(\hat{U}_2, \hat{V}_2),\ 0 < y < \eta(t),\ t > 0, \\ \tilde{U}_y(0,t) = \tilde{V}_y(0,t) = 0, \quad t > 0, \\ \tilde{U} \equiv 0 \quad \text{if } y \ge 1,\ t > 0; \quad \tilde{V} \equiv 0 \quad \text{if } y \ge \eta(t),\ t > 0, \\ \eta(0) = s_2^0/s_1^0,\ (\tilde{U}, \tilde{V})(y,0) = (0,0),\ y \in [0,\infty). \end{cases}$$

By L^p estimates and the Sobolev embedding theorem we have, for some large p,

$$\begin{aligned}\|\tilde{U}\|_{C^{1+\alpha,(1+\alpha)/2}([0,1]\times[0,T_2])} &\leq C_4\|\tilde{U}\|_{W_p^{2,1}((0,1)\times(0,T_2))}\\ &\leq C_5\|f(\hat{U}_1,\hat{V}_1)-f(\hat{U}_2,\hat{V}_2)\|_{L^p([0,1]\times[0,T_2])}\\ &\leq C_6(\|\hat{U}_1-\hat{U}_2\|_{C([0,\infty]\times[0,T_2])}+\|\hat{V}_1-\hat{V}_2\|_{C([0,\infty)\times[0,T_2])}),\end{aligned}$$

for some C_6 depending on α, M and the local Lipschitz constants of f,g. Thus, we obtain

$$\|U_1-U_2\|_{C([0,\infty)\times[0,T_2])}\leq C_6T_2^{\frac{1+\alpha}{2}}(\|\hat{U}_1-\hat{U}_2\|_{C([0,\infty)\times[0,T_2])}+\|\hat{V}_1-\hat{V}_2\|_{C([0,\infty)\times[0,T_2])}).$$

Similarly, we have (by straightening the boundary $y=\eta(t)$),

$$\|V_1-V_2\|_{C([0,\infty)\times[0,T_2])}\leq C_7T_2^{\frac{1+\alpha}{2}}(\|\hat{U}_1-\hat{U}_2\|_{C([0,\infty)\times[0,T_2])}+\|\hat{V}_1-\hat{V}_2\|_{C([0,\infty)\times[0,T_2])}).$$

for some C_7 depending on α, M and the local Lipschitz constants of f,g. Combining the above two estimates, we have

$$\begin{aligned}&\|U_1-U_2\|_{C([0,\infty)\times[0,T_2])}+\|V_1-V_2\|_{C([0,\infty)\times[0,T_2])}\\ \leq\ & C_8T_2^{\frac{1+\alpha}{2}}\left[\|\hat{U}_1-\hat{U}_2\|_{C([0,\infty)\times[0,T_2])}+\|\hat{V}_1-\hat{V}_2\|_{C([0,\infty)\times[0,T_2])}\right]\end{aligned}$$

for some C_8 depending on α, M and the local Lipschitz constants of f,g. Thus, by choosing

$$0<T<\min\{1,T_2,C_3^{-\frac{2}{\alpha}},C_7^{-\frac{2}{1+\alpha}}\},$$

we see that W forms a contraction mapping. Applying the contraction mapping theorem, W has a unique fixed point (still denoted by (U,V)). Thus, the problem (2.4) has a unique solution $(\hat{u},\hat{v})$ for $t\in[0,\tau_1]$ with $\hat{u}(x,t):=U(y,t)$, $\hat{v}(x,t):=V(y,t)$ and $y=x/\hat{s}_1(t)$. Moreover, (2.5) follows from (2.8) and (2.10). This completes the proof of Lemma 2.1. □

2.2. **Proof of Proposition 2.** For any given $\hat{s}_i\in\Sigma_i$, $i=1,2$, due to Lemma 2.1, one can introduce the map $\mathcal{F}:(\hat{s}_1,\hat{s}_2)\longmapsto(\bar{s}_1,\bar{s}_2)$ satisfying

$$\bar{s}_i(t):=s_i^0-\mu_i\int_0^t\varphi_{i,x}(\hat{s}_i(\tau),\tau)d\tau,\quad t\in[0,\tau_1],\quad i=1,2, \tag{2.12}$$

where $\varphi_1=\hat{u}$, $\varphi_2=\hat{v}$ and $(\hat{u},\hat{v})$ is the solution of the problem (2.4) for $t\in[0,\tau_1]$. Note that

$$\bar{s}_i'(t)=-\mu_i\varphi_{i,x}(\hat{s}_i(t),t)\in C^{\alpha/2}[0,\tau_1],\ i=1,2. \tag{2.13}$$

From (2.5), there exists $M_2>0$ depending only on M, α and the local Lipschitz constants of f,g such that

$$\sum_{i=1}^{2}\|\bar{s}_i'\|_{C^{\alpha/2}[0,\tau_1]}\leq M_2. \tag{2.14}$$

It follows from (2.14) that

$$\sum_{i=1}^{2}\|\bar{s}_i'-s_i^*\|_{C[0,\tau_1]}\leq M_2\tau_1^{\alpha/2}.$$

Hence $\mathcal{F}$ maps $\Sigma_1 \times \Sigma_2$ into itself as long as $\tau_1 \in (0, M_2^{-2/\alpha})$.

To apply the contraction mapping theorem, we define $(\hat{u}^s, \hat{v}^s)$ and $(\hat{u}^\sigma, \hat{v}^\sigma)$ as solutions of (2.4) for $t \in [0,T]$ corresponding to the given boundaries $(\hat{s}_1, \hat{s}_2)$ and $(\hat{\sigma}_1, \hat{\sigma}_2)$ in $\Sigma_1 \times \Sigma_2$, respectively. For convenience, we set

$$\gamma_-^i(t) := \min\{\hat{s}_i(t), \hat{\sigma}_i(t)\}, \quad \gamma_+^i(t) := \max\{\hat{s}_i(t), \hat{\sigma}_i(t)\},\ i = 1, 2.$$

Then we have the following estimate.

Lemma 2.2. *There holds*

$$\|\hat{u}^s - \hat{u}^\sigma\|_{C(\Gamma_T^1)} + \|\hat{v}^s - \hat{v}^\sigma\|_{C(\Gamma_T^2)} \le C^* \sum_{i=1}^{2} \|\hat{s}_i - \hat{\sigma}_i\|_{C[0,T]}, \quad t \in [0,T], \tag{2.15}$$

where $\Gamma_T^i := \{(x,t) : 0 \le x \le \gamma_-^i(t),\ t \in [0,T]\}$, $i = 1, 2$, *and* C^* *is a positive constant depending only on* M, α *and the local Lipschitz constants of* f, g.

Proof. We set

$$U(x,t) := \hat{u}^s(x,t) - \hat{u}^\sigma(x,t), \quad V(x,t) := \hat{v}^s(x,t) - \hat{v}^\sigma(x,t).$$

By direct computations, (U,V) satisfies

$$\begin{cases} U_t = d_1 U_{xx} + f(\hat{u}^s, \hat{v}^s) - f(\hat{u}^\sigma, \hat{v}^\sigma),\ 0 < x < \gamma_-^1(t),\ t \in [0,T], \\ V_t = d_2 V_{xx} + g(\hat{u}^s, \hat{v}^s) - g(\hat{u}^\sigma, \hat{v}^\sigma),\ 0 < x < \gamma_-^2(t),\ t \in [0,T], \\ U_x(0,t) = V_x(0,t) = 0, \quad t \in [0,T], \\ U \equiv 0 \text{ for } x \ge \gamma_+^1(t);\ V \equiv 0 \text{ for } x \ge \gamma_+^2(t),\ t \in [0,T], \\ U(x,0) = 0,\ x \in [0, s_1^0], \quad V(x,0) = 0,\ x \in [0, s_2^0]. \end{cases} \tag{2.16}$$

In order to derive (2.15), we need to estimate $U(\gamma_-^1(t),t)$ and $V(\gamma_-^2(t),t)$ first. To do so, we observe that

$$|U(\gamma_-^1(t),t)| = \begin{cases} |\hat{u}^s(\hat{\sigma}_1(t),t)| & \text{if } \gamma_-^1(t) = \hat{\sigma}_1(t), \\ |\hat{u}^\sigma(\hat{s}_1(t),t)| & \text{if } \gamma_-^1(t) = \hat{s}_1(t). \end{cases}$$

Also, using $\hat{u}^s(\hat{s}_1(t),t) = 0 = \hat{u}^\sigma(\hat{\sigma}_1(t),t)$, the mean value theorem yields that

$$|U(\gamma_-^1(t),t)| \le M_1 \|\hat{s}_1 - \hat{\sigma}_1\|_{C[0,T]} \quad \text{for all } t \in [0,T], \tag{2.17}$$

where $M_1 > 0$ is given by (2.5). Similarly, we have

$$|V(\gamma_-^2(t),t)| \le M_1 \|\hat{s}_2 - \hat{\sigma}_2\|_{C[0,T]} \quad \text{for all } t \in [0,T]. \tag{2.18}$$

From (2.16), (2.17) and (2.18), applying the maximum principle we conclude that

$$\begin{cases} |U(x,t)| \le M_1 \|\hat{s}_1 - \hat{\sigma}_1\|_{C[0,T]} + M_3 \displaystyle\int_0^t \max_{x \in [0,\gamma_-^1(\tau)]} \{|U| + |V|\}(x,\tau) d\tau \text{ in } \Gamma_T^1, \\ |V(x,t)| \le M_1 \|\hat{s}_2 - \hat{\sigma}_2\|_{C[0,T]} + M_3 \displaystyle\int_0^t \max_{x \in [0,\gamma_-^2(\tau)]} \{|U| + |V|\}(x,\tau) d\tau \text{ in } \Gamma_T^2 \end{cases} \tag{2.19}$$

for some constant $M_3 > 0$ depending on the local Lipschitz constants of f, g.

Next, let

$$J(t) := \max_{x\in[0,\gamma^1_-(t)]} |U(x,t)| + \max_{x\in[0,\gamma^2_-(t)]} |V(x,t)|.$$

Then we can derive the following estimate:

$$\max_{x\in[0,\gamma^1_-(t)]}\{(|U|+|V|)(x,t)\} \le M_1\|\hat{s}_2-\hat{\sigma}_2\|_{C[0,T]} + J(t),\ t\in[0,T]. \tag{2.20}$$

To obtain (2.20), we observe that for all $t\in[0,T]$,

$$\max_{x\in[0,\gamma^1_-(t)]}\{(|U|+|V|)(x,t)\} \le J(t) + \max_{x\in[\gamma^2_-(t),\gamma^2_+(t)]} |V(x,t)|. \tag{2.21}$$

Note that

$$|V(x,t)| = \begin{cases} |\hat{v}^s(x,t)| & \text{for all } x\in[\gamma^2_-(t),\gamma^2_+(t)] \text{ if } \gamma^2_-(t)=\hat{\sigma}_2(t), \\ |\hat{v}^\sigma(x,t)| & \text{for all } x\in[\gamma^2_-(t),\gamma^2_+(t)] \text{ if } \gamma^2_-(t)=\hat{s}_2(t), \end{cases}$$

by the mean value theorem (as in deriving the estimate (2.17)), we have

$$\max_{x\in[\gamma^2_-(t),\gamma^2_+(t)]} |V(x,t)| \le M_1\|\hat{s}_2-\hat{\sigma}_2\|_{C[0,T]} \quad \text{for all } t\in[0,T]. \tag{2.22}$$

Combining (2.21) and (2.22), we arrive at (2.20).

Similarly, one can obtain

$$\max_{x\in[0,\gamma^2_-(t)]}\{(|U|+|V|)(x,t)\} \le M_1\|\hat{s}_1-\hat{\sigma}_1\|_{C[0,T]} + J(t),\ t\in[0,T]. \tag{2.23}$$

Due to (2.20) and (2.23), the inequalities (2.19) can be reduced into

$$J(t) \le M_1(1+M_3T)\sum_{i=1}^{2}\|\hat{s}_i-\hat{\sigma}_i\|_{C[0,T]} + 2M_3\int_0^t J(\tau)d\tau \quad \text{for } t\in[0,T].$$

By the Gronwall's inequality, (2.15) follows. This completes the proof of Lemma 2.2. □

We are ready to show Proposition 2.

Proof of Proposition 2. To apply the contraction mapping theorem, it suffices to show the contraction of $\mathcal{F}$. If necessary we choose τ_1 smaller such that

$$\|\hat{s}_i\|_{C[0,\tau_1]},\ \|\hat{\sigma}_i\|_{C[0,\tau_1]} \ge \frac{s_i^0}{2},\ i=1,2. \tag{2.24}$$

We now prove there exists $C'>0$ depending only on α, M and the local Lipschitz constants of f, g such that

$$\sum_{i=1}^{2}\|\bar{s}_i'-\bar{\sigma}_i'\|_{C[0,T]} \le C'\sum_{i=1}^{2}\|\hat{s}_i-\hat{\sigma}_i\|_{C[0,T]} \tag{2.25}$$

as long as $T>0$ small enough, where $(\bar{\sigma}_1,\bar{\sigma}_2)$ is defined similarly as in (2.12).

To do so, we set

$$U^s(y,t) := \hat{u}^s(x,t), \quad V^s(y,t) := \hat{v}^s(x,t), \quad y = \frac{x}{\hat{s}_1(t)}, \quad \eta(t) := \frac{\hat{s}_2(t)}{\hat{s}_1(t)},$$

we see that (U^s, V^s) satisfies (2.6). Similarly, by setting

$$U^\sigma(y,t) := \hat{u}^\sigma(x,t), \quad V^\sigma(y,t) := \hat{v}^\sigma(x,t), \quad y = \frac{x}{\hat{\sigma}_1(t)}, \quad \xi(t) := \frac{\hat{\sigma}_2(t)}{\hat{\sigma}_1(t)},$$

we obtain that (U^σ, V^σ) satisfies (2.6) with $\hat{s}_1(t)$ and $\eta(t)$ replaced by $\hat{\sigma}_1(t)$ and $\xi(t)$, respectively.

Also, we introduce

$$\gamma_-(t) := \min\{\eta(t), \xi(t)\},\ \gamma_+(t) := \max\{\eta(t), \xi(t)\},\ i = 1, 2,$$
$$P(y,t) := U^s(y,t) - U^\sigma(y,t), \quad Q(y,t) := V^s(y,t) - V^\sigma(y,t).$$

By direct computations, (P, Q) satisfies

$$\begin{cases} P_t = \dfrac{d_1 P_{yy}}{(\hat{s}_1(t))^2} + \dfrac{\hat{s}_1'(t) y P_y}{\hat{s}_1(t)} + d_1 B_1(t) U^\sigma_{yy} + y B_2(t) U^\sigma_y + F(y,t),\ 0 < y < 1,\ t \in [0,T], \\ Q_t = \dfrac{d_2 Q_{yy}}{(\hat{s}_1(t))^2} + \dfrac{\hat{s}_1'(t) y Q_y}{\hat{s}_1(t)} + d_2 B_1(t) V^\sigma_{yy} + y B_2(t) V^\sigma_y + G(y,t),\ 0 < y < \gamma_-(t),\ t \in [0,T], \\ P_y(0,t) = Q_y(0,t) = 0, \quad t \in [0,T], \\ P \equiv 0 \text{ for } y \ge 1;\ Q \equiv 0 \text{ for } y \ge \gamma_+(t),\ t \in [0,T], \\ P(y,0) = 0, \quad Q(y,0) = 0,\ y \in [0,\infty), \end{cases}$$

where $T \in (0, \tau_1)$ is given and

$$\begin{aligned} B_1(t) &:= \frac{1}{(\hat{s}_1(t))^2} - \frac{1}{(\hat{\sigma}_1(t))^2}, \quad B_2(t) := \frac{\hat{s}_1'(t)}{\hat{s}_1(t)} - \frac{\hat{\sigma}_1'(t)}{\hat{\sigma}_1(t)}; \\ F(y,t) &:= f(U^s, V^s) - f(U^\sigma, V^\sigma), \quad G(y,t) := g(U^s, V^s) - g(U^\sigma, V^\sigma). \end{aligned} \tag{2.26}$$

In the following we shall estimate $\|P\|_{C(\Gamma_{1T})} + \|Q\|_{C(\Gamma_{2T})}$, where

$$\Gamma_{1T} := \{(y,t) : 0 \le y \le 1,\ t \in [0,T]\}, \quad \Gamma_{2T} := \{(y,t) : 0 \le y \le \gamma_-(t),\ t \in [0,T]\}.$$

By Lemma 2.2 and (2.5), for each $(y,t) \in \Gamma_{1T}$, without loss of generality, we assume $\hat{s}_1(t) \le \hat{\sigma}_1(t)$, then

$$\begin{aligned} |P(y,t)| &\le |\hat{u}^s(y\hat{s}_1(t), t) - \hat{u}^\sigma(y\hat{s}_1(t), t)| + |\hat{u}^\sigma(y\hat{s}_1(t), t) - \hat{u}^\sigma(y\hat{\sigma}_1(t), t)| \\ &\le C^* \sum_{i=1}^{2} \|\hat{s}_i - \hat{\sigma}_i\|_{C[0,T]} + \sup_{t \in [0,T]} \|\hat{u}^\sigma_x(\cdot, t)\|_{C[0,1]} \|\hat{s}_1 - \hat{\sigma}_1\|_{C[0,T]} \\ &\le M' \sum_{i=1}^{2} \|\hat{s}_i - \hat{\sigma}_i\|_{C[0,T]} \end{aligned}$$

for some $M' > 0$. Thus, we have

$$\|P\|_{C(\Gamma_{1T})} \le M' \sum_{i=1}^{2} \|\hat{s}_i - \hat{\sigma}_i\|_{C[0,T]} \tag{2.27}$$

Similarly,

$$\|Q\|_{C(\Gamma_{2T})} \le M'' \sum_{i=1}^{2} \|\hat{s}_i - \hat{\sigma}_i\|_{C[0,T]} \tag{2.28}$$

for some $M'' > 0$.

We are ready to prove (2.25). From (2.13), we see that

$$\begin{aligned} |\bar{s}_1'(t) - \bar{\sigma}_1'(t)| &\le \mu_1 \left| \frac{U_y^s(1,t)}{\hat{s}_1(t)} - \frac{U_y^\sigma(1,t)}{\hat{\sigma}_1(t)} \right| \\ &\le \mu_1 \left[\frac{\|P_y\|_{C(\Gamma_{1T})}}{\hat{s}_1(t)} + \frac{\|\hat{s}_1 - \hat{\sigma}_1\|_{C[0,T]} \|U_y^\sigma\|_{C(\Gamma_{1T})}}{\hat{s}_1(t)\hat{\sigma}_1(t)} \right]. \end{aligned}$$

Then using L^p estimate and the Sobolev embedding theorem,

$$\|P_y\|_{C(\Gamma_{1T})} \le C_6 \left[\sup_{t\in[0,T]} \sum_{i=1}^{2} |B_i(t)| + \|P\|_{C(\Gamma_{1T})} + \|Q\|_{C(\Gamma_{2T})} \right],$$

where $B_i(t)$ is given by (2.26) $(i = 1, 2)$ and the constant $C_6 > 0$ depending only on α, M and the local Lipschitz constants of f, g. Also, by (2.5) and (2.24),

$$\frac{\|U_y^\sigma\|_{C(\Gamma_{1T})}}{\hat{s}_1(t)\hat{\sigma}_1(t)} \le C_7$$

for some $C_7 > 0$ depending only on α, M and the local Lipschitz constants of f, g. Thus, we are led to

$$|\bar{s}_1'(t) - \bar{\sigma}_1'(t)| \le C_8 \left[\sup_{t\in[0,T]} \sum_{i=1}^{2} |B_i(t)| + \|P\|_{C(\Gamma_{1T})} + \|Q\|_{C(\Gamma_{2T})} + \|\hat{s}_1 - \hat{\sigma}_1\|_{C[0,T]} \right]$$

From (2.26), (2.27) and (2.28), there exists $C_8 > 0$ depending only on α, M and the local Lipschitz constants of f, g such that

$$|\bar{s}_1'(t) - \bar{\sigma}_1'(t)| \le C_8 \sum_{i=1}^{2} \|\hat{s}_i - \hat{\sigma}_i\|_{C[0,T]}.$$

Similarly, we can derive

$$|\bar{s}_2'(t) - \bar{\sigma}_2'(t)| \le C_9 \sum_{i=1}^{2} \|\hat{s}_i - \hat{\sigma}_i\|_{C[0,T]},$$

where $C_9 > 0$ depending only on α, M and the local Lipschitz constants of f, g. Thus, (2.25) follows. On the other hand, since $\hat{s}_i(0) = \hat{\sigma}_i(0) = s_i^0$, $i = 1, 2$, it follows that

$$\|\hat{s}_i - \hat{\sigma}_i\|_{C[0,T]} \le T\|\hat{s}_i' - \hat{\sigma}_i'\|_{C[0,T]},\ i = 1, 2.$$

Together with (2.25), we see that $\mathcal{F}$ is a contraction mapping as long as $T > 0$ small enough. By the contraction mapping theorem, the problem (2.1) admits a unique solution. Moreover, (2.3) follows from (2.8), (2.10) and (2.14). This completes the proof of Proposition 2. $\square$

2.3. **Proof of Theorem 1.** To prove Theorem 1, we first derive some a priori estimates for solutions of **(P)**.

Lemma 2.3 (A priori estimates)**.** *Let (u, v, s_1, s_2) be a solution of* **(P)** *for $t \in [0, T]$ for some $T > 0$. Then $u > 0$ for $x \in [0, s_1(t)),\ t \in [0, T]$ and $v > 0$ for $x \in [0, s_2(t)),\ t \in [0, T]$. Moreover, the estimates* (1.8)*,* (1.9)*,* (1.10) *and* (1.11) *hold for $t \in [0, T]$.*

Proof. The strong maximum principle yields that $u > 0$ for $x \in [0, s_1(t)),\ t \in [0, T]$ and $v > 0$ for $x \in [0, s_2(t)),\ t \in [0, T]$. Thus, we see from (1.3) that $u_x(s_1(t), t) < 0$ and $v_x(s_2(t), t) < 0$ for $t \in (0, T]$. By (1.5), $s_i'(t) > 0$ for $t \in (0, T]$ and $i = 1, 2$.

To derive upper bound of u, we consider $\bar{u} = \bar{u}(t)$, the solution of $u' = r_1 u(1 - u)$ with the initial data $\bar{u}(0) = \|u_0\|_{L^\infty}$. By the standard comparison principle, we have $u(x, t) \le \bar{u}(t) \le K_1$ for all $x \in [0, s(t)],\ t \in [0, T]$. Similarly, we can derive the upper bound estimate for v.

Finally, by exactly the same argument of [15, Lemma 2.2], we can prove (1.10) and (1.11). We omit the detailed proof here. Then Lemma 2.3 follows. □

We are ready to give a proof of Theorem 1 as follows.

Proof of Theorem 1. By Propositions 2, we have the local existence and uniqueness of the $C^{1+\alpha,(1+\alpha)/2}$ solution to the problem **(P)**. Furthermore, note that $u, v \in C^{\alpha,\alpha/2}$ in $\{(x, t) : x \in [0, \infty),\ t \in [0, T_0]\}$. By the Schauder's estimates, we see that the solution is actually in classical sense.

Next, we shall prove that the solution can be extended to all $t > 0$. For this, we define the maximal existence time of the solution by $T_{\max} > 0$. By the same argument of [10], one can show $T_{\max} = \infty$. For reader's convenience, we repeat the proof here. Indeed, using a contradiction argument we assume that $T_{\max} < \infty$. By Lemma 2.3, we can find a constant $K > 0$ independent of $T_{\max}$ such that $0 \le u(x, t), v(x, t), s_1'(t), s_2'(t) \le K$ for all $x \in [0, s(t)]$ and $t \in [0, T_{\max})$. In particular,

$$s_i^0 \le s_i(t) \le s_i^0 + Kt \quad \text{for all } t \in [0, T_{\max}) \text{ and } i = 1, 2.$$

Choosing $\epsilon \in (0, T_{\max})$, from the standard regularity theory we see that there exists $M > 0$ depending only on ϵ, K such that

$$\|u(\cdot, t)\|_{C^2[0,s_1(t)]},\ \|v(\cdot, t)\|_{C^2[0,s_2(t)]} \le M \quad \forall\ t \in [\epsilon, T_{\max}).$$

By Proposition 2, there is a $\tau > 0$ depending only on K and M such that the solution of **(P)** with any initial time $t \in [\epsilon, T_{\max})$ can be uniquely extended to the interval $[t, t + \tau)$. Then we reach a contradiction with the definition of $T_{\max}$, since the solution with the initial time $T_{\max} - \tau/2$ can be uniquely extended to the time $T_{\max} + \tau/2$. It follows that $T_{\max} = \infty$. Thus, we complete the proof of Theorem 1. □

3. Proofs of main theorems

In this section, we shall give proofs of our main theorems stated in Section 1. First, we give some known results to be used later. The next two propositions can be found in [10, 13].

Proposition 3 (Theorem 3.3 of [10] and Theorem 1.2 of [13]). *Let (w,h) be a solution of*

$$\begin{cases} w_t = dw_{xx} + w(a-bw), \ 0<x<h(t), \ t>0, \\ w_x(0,t)=0, \ w(h(t),t)=0, \ t>0, \\ h'(t) = -\mu w_x(h(t),t), \ t>0, \\ h(0)=h_0, \ w(x,0)=w_0(x), \ 0<x<h_0, \end{cases} \tag{3.1}$$

where $h_0>0$, $w_0 \in C^2[0,h_0]$ and $w_0(x)>0=w_0'(0)=0=w_0(h_0)$ for $x\in[0,h_0)$. Then the following hold:

(i) (*Spreading-vanishing dichotomy*) *Either*

$$\lim_{t\to\infty} h(t)=\infty, \quad \lim_{t\to\infty} w(x,t)=\frac{a}{b}$$

uniformly in any bounded subset of $[0,\infty)$ or

$$\lim_{t\to\infty} h(t) \le \frac{\pi}{2}\sqrt{\frac{d}{a}}, \quad \lim_{t\to\infty} \|w(\cdot,t)\|_{C[0,h(t)]}=0.$$

(ii) *When $\lim_{t\to\infty} h(t)=\infty$, $h(t)/t \to c_0(a,b,d,\mu)$ as $t\to\infty$ and*

$$\lim_{t\to\infty} \sup_{x\in[0,h(t)]} |w(x,t)-U_{c_0}(h(t)-x)|=0,$$

where c_0 and U_{c_0} are defined in Proposition 1.

Proposition 4 (Lemma 3.5 of [10]). *Assume that $\sigma\in C^1[0,T]$ and $\bar{w}\in C(\overline{D^\sigma}_T)\cap C^{2,1}(D^\sigma_T)$, where $D^\sigma_T := \{(x,t)\in\mathbb{R}^2 : \ 0<x<\sigma(t), \ 0<t\le T\}$ and*

$$\begin{cases} \bar{w}_t \ge d\bar{w}_{xx} + \bar{w}(a-bw), \ 0<x<\sigma(t), \ t>0, \\ \bar{w}_x(0,t)\le 0, \ \bar{w}(\sigma(t),t)=0, \ t>0, \\ \sigma'(t) \ge -\mu \bar{w}_x(\sigma(t),t), \ t>0. \end{cases}$$

If $h_0\le\sigma(0)$ and $w_0(x)\le\bar{w}(x,0)$ for all $x\in[0,h_0]$, then the solution (w,h) of (3.1) satisfies $h(t)\le\sigma(t)$ for all $t\in(0,T]$ and $w(x,t)\le\bar{w}(x,t)$ for $0\le x\le h(t)$, $0\le t\le T$.

Remark 2. *We call $(\bar{w},\sigma)$ defined in Proposition 4 a super-solution of (3.1). A sub-solution can be defined if we reverse all the inequalities in Proposition 4 (also replacing the interval $[0,h_0]$ by $[0,\sigma(0)]$).*

The strategy of the proof in the following lemma is similar to the one in [12] (see also [11, 15]). For reader's convenience, we give a proof here.

Lemma 3.1. *Let* (u, v, s_1, s_2) *be a solution of* **(P)**. *If* $s_{1,\infty} < +\infty$ (resp., $s_{2,\infty} < +\infty$), *then there exists* $C > 0$ *independent of* t *such that*

$$\|u\|_{C^{1+\alpha,(1+\alpha)/2}([0,s_1(t)]\times[1,\infty))} + \|s_1'\|_{C^{\alpha/2}[1,\infty)} \leq C, \tag{3.2}$$
$$(\text{resp., } \|v\|_{C^{1+\alpha,(1+\alpha)/2}([0,s_2(t)]\times[1,\infty))} + \|s_2'\|_{C^{\alpha/2}[1,\infty)} \leq C).$$

In particular, $\lim_{t\to\infty} s_1'(t) = 0$ (resp., $\lim_{t\to\infty} s_2'(t) = 0$).

Proof. We only deal with the case that $s_{1,\infty} < +\infty$, since the proof of the other case is similar. To straighten the free boundary $x = s_1(t)$, we perform the following transformations

$$y := \frac{x}{s_1(t)}, \quad (U, V)(y, t) := (u, v)(x, t), \quad \eta(t) := \frac{s_2(t)}{s_1(t)}. \tag{3.3}$$

Then (U, V) satisfies the system (2.6) without hat sign. By using L^p estimate and the Sobolev's embedding theorem we can conclude that

$$\|U\|_{C^{1+\alpha,(1+\alpha)/2}([0,1]\times[1,\infty))} \leq C'$$

for some $C' > 0$. Also, by (1.6), there exists a positive constant C'' such that

$$\|s_1'\|_{C^{\alpha/2}[1,\infty)} \leq C''. \tag{3.4}$$

Thus, (3.2) follows. Moreover, since $s_{1,\infty} < +\infty$, by (3.4), we easily obtain $\lim_{t\to\infty} s_1'(t) = 0$. The same argument can apply to the case that $s_{2,\infty} < +\infty$. This completes the proof of Lemma 3.1. $\square$

In order to prove Theorem 2, we prepare the following lemmas.

Lemma 3.2. *Suppose that*

$$\limsup_{t\to\infty} \|u(\cdot, t)\|_{C[0,s_1(t)]} := \rho_1, \quad \limsup_{t\to\infty} \|v(\cdot, t)\|_{C[0,s_2(t)]} := \rho_2.$$

Then $\rho_i \leq 1$ *for* $i = 1, 2$. *Moreover, the followings hold:*

(i) $s_{1,\infty} = +\infty$ *and*

$$\liminf_{t\to\infty} u(x, t) \geq 1 - k\rho_2 \quad \textit{uniformly for any bounded subset of } [0, \infty) \tag{3.5}$$

as long as

$$s_{1,\infty} > \frac{\pi}{2}\sqrt{\frac{d_1}{r_1}}\frac{1}{\sqrt{1 - k\rho_2}} := \bar{s}_1. \tag{3.6}$$

In particular, $s_{1,\infty} = +\infty$ *if* $s_{1,\infty} > s^*$.

(ii) *If* $1 - h\rho_1 > 0$, *then* $s_{2,\infty} = +\infty$ *and* $\liminf_{t\to\infty} v(x, t) \geq 1 - h\rho_1$ *uniformly for any bounded subset of* $[0, \infty)$ *as long as*

$$s_{2,\infty} > \frac{\pi}{2}\sqrt{\frac{d_2}{r_2}}\frac{1}{\sqrt{1 - h\rho_1}} := \bar{s}_2.$$

Proof. First, we consider $w = w(t)$ as the solution of $w' = rw(1-w)$ with $r := \max\{r_1, r_2\}$ and the initial data $w(0) = \max\{\|u_0\|_{L^\infty}, \|v_0\|_{L^\infty}\}$. By the standard comparison principle, we see that $\rho_i \leq 1$ for $i = 1, 2$. In particular, $\bar{s}_1$ is well-defined because $k\rho_2 < 1$.

Since the proof of (i) and (ii) are similar, we only deal with (i). By (3.6), there exists a sufficiently small $\varepsilon > 0$ such that

$$s_{1,\infty} > \frac{\pi}{2}\sqrt{\frac{d_1}{r_1}}\frac{1}{\sqrt{1-k(\rho_2+\varepsilon)}} := \bar{s}_{1,\varepsilon}.$$

Since $\limsup_{t\to\infty} \|v(\cdot,t)\|_{C[0,s_2(t)]} := \rho_2$, there exists $T \gg 1$ such that $s_1(T) > \bar{s}_{1,\varepsilon}$ and $v \leq \rho_2 + \varepsilon$ for all $x \in [0,\infty)$ and $t \geq T$. This implies that

$$u_t \geq d_1 u_{xx} + r_1 u[1 - k(\rho_2+\varepsilon) - u], \quad x \in [0, s_1(t)],\ t \geq T.$$

Hence (u, s_1) is a super-solution of

$$\begin{cases} w_t = d_1 w_{xx} + r_1 w[1-k(\rho_2+\varepsilon) - w],\ 0 < x < \sigma(t),\ t > T, \\ w_x(0,t) = 0,\ w(\sigma(t),t) = 0,\ t > T, \\ \sigma'(t) = -\mu_1 w_x(\sigma(t),t),\ t > T, \\ \sigma(T) := s_1(T),\ w(x,T) = u(x,T),\ x \in [0,\sigma(T)], \end{cases}$$

Since $\sigma(T) := s_1(T) > \bar{s}_{1,\varepsilon}$, Propositions 3 and 4 yield that $s_{1,\infty} \geq \sigma(\infty) = \infty$ and

$$\liminf_{t\to\infty} u(x,t) \geq \lim_{t\to\infty} w(x,t) = 1 - k(\rho_2+\varepsilon)$$

uniformly for any bounded subset of $[0,\infty)$. Note that $\varepsilon > 0$ is arbitrary, (3.5) follows. Moreover, since $s^* \geq \bar{s}_1$, it follows that $s_{1,\infty} = +\infty$ if $s_{1,\infty} > s^*$. This completes the proof of Lemma 3.2. □

Note that $\bar{s}_1 = s_*$ if $\rho_2 = 0$ and $\bar{s}_2 = s^{**}$ if $\rho_1 = 0$.

Lemma 3.3. (i) *If $s_{1,\infty} \leq s_*$, then $\lim_{t\to\infty} \|u(\cdot,t)\|_{C[0,s_1(t)]} = 0$.* (ii) *If $s_{2,\infty} \leq s^{**}$, then $\lim_{t\to\infty} \|v(\cdot,t)\|_{C[0,s_1(t)]} = 0$.*

Proof. We now prove (i). Choose $l \in [s_{1,\infty}, s_*]$. Let $\bar{u}$ be the unique solution for $u_t = d_1 u_{xx} + r_1 u(1-u)$, $(x,t) \in (0,l) \times (0,+\infty)$ with the boundary condition $u_x(0,t) = u(l,t) = 0$ for $t > 0$ and the initial data

$$u(x,0) = \begin{cases} u_0(x) & \text{if } x \in [0, s_0], \\ 0 & \text{if } x \in [s_0, l]. \end{cases}$$

Then it is well known that $\lim_{t\to+\infty} \|\bar{u}(\cdot,t)\|_{C([0,l])} = 0$ since $l \leq s_*$ (see, for example, [3, Proposition 3.3]). By comparing $\bar{u}$ with u over $\{(x,t) : 0 \leq x \leq s_1(t),\ t \geq 0\}$, we obtain $0 \leq u \leq \bar{u}$ and so $\lim_{t\to+\infty} \|u(\cdot,t)\|_{C([0,s_1(t)])} = 0$. The same argument applies to (ii). Thus, we complete the proof of Lemma 3.3. □

Lemma 3.4. (i) *Suppose that* $s_{1,\infty} < \infty$*. If* $s_1(t) \le s_2(t)$ *for all large* t*, then*

$$\lim_{t\to\infty} \|u(\cdot,t)\|_{C[0,s_1(t)]} = 0.$$

(ii) *Suppose that* $s_{2,\infty} < \infty$*. If* $s_2(t) \le s_1(t)$ *for all large* t*, then*

$$\lim_{t\to\infty} \|v(\cdot,t)\|_{C[0,s_2(t)]} = 0.$$

Proof. It suffices to deal with (i) since the same argument can be applied to (ii). To prove (i), we shall modify a proof of [11]. For contradiction we assume that $\limsup_{t\to\infty} \|u(\cdot,t)\|_{C([0,s_1(t)])} > 0$. Then we can find a sequence $\{(x_n,t_n)\}$ with $x_n \in [0, s_1(t_n))$ and $\lim_{n\to\infty} t_n = \infty$ such that $u(x_n,t_n) \to \kappa$ as $t \to \infty$ for some $\kappa > 0$. Up to a subsequence, we may assume that $\lim_{n\to\infty} x_n = \bar{x}$. We now show that $\bar{x} \ne s_{1,\infty}$. For contradiction, if $\bar{x} = s_{1,\infty}$, then by the mean value theorem and using that $s_{1,\infty} < \infty$, we have $\xi_n \in (x_n, s_1(t_n))$ such that

$$u_x(\xi_n,t_n) = \frac{u(x_n,t_n) - u(s_1(t_n),t_n)}{x_n - s_1(t_n)} = \frac{u(x_n,t_n)}{x_n - s_1(t_n)} \to \infty \quad \text{as } n \to \infty,$$

which contradicts Lemma 3.1. Thus, we must have that $\bar{x} \in [0, s_{1,\infty})$.

Since $s_{1,\infty} < \infty$, we can use the same transformation as in (3.3) to obtain the system (2.6) without hat sign. We now consider

$$\hat{u}_n(y,t) := U(y, t+t_n), \quad \hat{v}_n(y,t) := V(y,t+t_n) \quad \text{for } y \in [0,1] \text{ and } t \in [0,1].$$

Since $s_1(t) \le s_2(t)$ for all large t, we have $\eta(t) \ge 1$ for all large t. Similar to the proof of Lemma 3.1, we have

$$\|V\|_{C^{1+\alpha,(1+\alpha)/2}([0,1]\times[1,\infty))} \le M \quad \text{for some positive constant } M.$$

Here we use $[0,1] \subset [0,\eta(t)]$ for all large t.

Together with (3.2), we obtain

$$\|U\|_{C^{1+\alpha,(1+\alpha)/2}([0,1]\times[1,\infty))} + \|V\|_{C^{1+\alpha,(1+\alpha)/2}([0,1]\times[1,\infty))} \le M' \tag{3.7}$$

for some positive constant M'.

By (3.7) and $\lim_{n\to\infty} s_1'(t_n) = 0$ (Lemma 3.1), we have (up to a subsequence)

$$(\hat{u}_n, \hat{u}_n)(y,t) \to (u^*, v^*)(y,t) \quad \text{in } C^{1,1/2}([0,1]\times[0,1]) \text{ as } n \to \infty, \tag{3.8}$$

where $u^*(\bar{x}/s_{1,\infty}, 0) = \kappa > 0$ and

$$\begin{cases} u_t^* = d_1[s_{1,\infty}]^{-2} u_{yy}^* + r_1 u^*(1 - u^* - kv^*), \ y \in (0,1), \ t \in (0,1), \\ u_y^*(0,t) = u^*(1,t) = 0, \ t \in (0,1). \end{cases} \tag{3.9}$$

Then the strong maximum principle implies that $u^* > 0$ over $\{(y,t) : y \in (0,1), \ t \in (0,1)\}$. By Hopf's Lemma, there exists $\theta > 0$ such that $u_y^*(1,t) \le -\theta$ for all $t \in (1/4, 1)$. Combining (3.8) and (1.5),

$$s_1'(t_n + \frac{1}{2}) = -\mu_1 u_x(s_1(t_n + \frac{1}{2}), t_n + \frac{1}{2}) = -\mu_1 \frac{\hat{u}_{n,y}(1,1/2)}{s_1(t_n + 1/2)} \ge \frac{\theta\mu_1}{2s_{1,\infty}} \quad \text{for all large } n.$$

This contradicts Lemma 3.1. Hence we must have $\limsup_{t\to\infty}\|u(\cdot,t)\|_{C([0,s_1(t)])}=0$ and then the proof of Lemma 3.4 is completed. □

Lemma 3.5. *Suppose that $s_{1,\infty}\in(s_*,s^*]$. Then $s_1(t)-s_2(t)$ changes sign only finitely many times. Furthermore, $s_{2,\infty}=\infty$ and*

$$\lim_{t\to\infty}\|u(\cdot,t)\|_{C[0,s_1(t)]}=0,\quad \lim_{t\to\infty}v(\cdot,t)=1\quad \textit{locally uniformly for } x\in[0,\infty).$$

Proof. We first show that

$$s_{2,\infty}>s^{**}. \tag{3.10}$$

If (3.10) does not hold, then Lemma 3.3(ii) implies that $\lim_{t\to\infty}\|v(\cdot,t)\|_{C[0,s_2(t)]}=0$. Applying Lemma 3.2(i) with $\rho_2=0$, we have $s_{1,\infty}=\infty$, a contradiction to that $s_{1,\infty}\le s^*$. Thus, we obtain (3.10).

We next use a contradiction argument to prove that $s_1(t)-s_2(t)$ changes sign only finitely many times. Assume that it changes sign infinitely many times, then we have $s_{1,\infty}=s_{2,\infty}<\infty$. If we can prove that $\lim_{t\to\infty}\|u(\cdot,t)\|_{C[0,s_1(t)]}=0$, then using (3.10) and Lemma 3.2(ii) with $\rho_1=0$ we obtain $s_{2,\infty}=\infty$. This leads a contradiction to that $s_{2,\infty}<\infty$. Hence $s_1(t)-s_2(t)$ must change sign only finitely many times.

To prove that $\lim_{t\to\infty}\|u(\cdot,t)\|_{C[0,s_1(t)]}=0$, we shall modify the proof of Lemma 3.4. For contradiction we assume that $\limsup_{t\to\infty}\|u(\cdot,t)\|_{C([0,s_1(t)])}>0$. Then we can choose a sequence $\{(x_n,t_n)\}$ with $x_n\in[0,s_1(t_n))$ and $\lim_{n\to\infty}t_n=\infty$ such that $u(x_n,t_n)\to\beta$ as $t\to\infty$ for some $\beta>0$ and $\lim_{n\to\infty}x_n=\bar{x}$ (up to a subsequence). As in the proof of Lemma 3.4, we have that $\bar{x}\in[0,s_{1,\infty})$.

Again, using the transformation as in (3.3) we have the system (2.6) without hat sign. We now consider

$$\hat{u}_n(y,t):=U(y,t+t_n),\quad \hat{v}_n(y,t):=V(y,t+t_n)\quad \text{for } y\in[0,\gamma_n] \text{ and } t\in[0,1],$$

where $\gamma_n:=\min\{1,\min_{t\in[t_n,t_n+1]}\eta(t)\}$ and $\eta(t)$ is defined in (3.3). Note that $s_{1,\infty}=s_{2,\infty}$, we see that $\lim_{n\to\infty}\gamma_n=1$.

Since $s_{1,\infty}=s_{2,\infty}<\infty$, by Lemma 3.1,

$$\|\hat{u}_n\|_{C^{1+\alpha,(1+\alpha)/2}([0,\gamma_n]\times[0,1])}+\|\hat{v}_n\|_{C^{1+\alpha,(1+\alpha)/2}([0,\gamma_n]\times[0,1])}\le M' \tag{3.11}$$

for some positive constant M' independent of n.

Using (3.11), $\lim_{n\to\infty}\gamma_n=1$ and $\lim_{n\to\infty}s_1'(t_n)=0$, we have (up to a subsequence)

$$(\hat{u}_n,\hat{u}_n)(y,t)\to(u^*,v^*)(y,t)\quad \text{in } C^{1,1/2}([0,1]\times[0,1]) \text{ as } n\to\infty,$$

where $u^*(\bar{x}/s_{1,\infty},0)=\beta>0$ and (u^*,v^*) satisfies the same system (3.9). Again, as in the proof of Lemma 3.4, using the strong maximum principle and Hopf's Lemma we can derive

$$s_1'(t_n+\frac{1}{2})\geq\delta \quad \text{for some } \delta>0 \text{ and for all large } n.$$

This contradicts Lemma 3.1. Hence $\limsup_{t\to\infty}\|u(\cdot,t)\|_{C([0,s_1(t)])}=0$.

Therefore, $s_1(t)-s_2(t)$ changes sign only finitely many times. Then we see that either $s_1(t)\leq s_2(t)$ for all large t or $s_1(t)\geq s_2(t)$ for all large t. In fact, the latter case cannot happen. Otherwise, by Lemma 3.4(ii) and Lemma 3.2(i) (with $\rho_2=0$), we see that $s_{1,\infty}=\infty$, a contradiction. Thus, we have $s_1(t)\leq s_2(t)$ for all large t. Consequently, Lemma 3.5 follows from Lemma 3.4(i) and Lemma 3.2(ii) (with $\rho_1=0$). □

Now, we are ready to give a proof of Theorem 2.

Proof of Theorem 2. For (i), the vanishing of u follows from Lemma 3.3(i). Moreover, by Lemma 3.2(ii) with $\rho_1=0$, we see that v spreads successfully and satisfies (1.12) if $s_{2,\infty}>s^{**}$. When $s_{2,\infty}\leq s^{**}$, the vanishing of v follows from Lemma 3.3(ii). Part (ii) follows from Lemma 3.5 immediately. By Lemma 3.2(i), part (iii) holds. Hence we complete the proof of Theorem 2. □

To prove Theorem 3, we need the following lemma.

Lemma 3.6. *Suppose that $s_{1,\infty}=\infty$ and c_0 is defined in Proposition 1. Then*

$$c_0(r_1(1-k),r_1,d_1,\mu_1)\leq\liminf_{t\to\infty}\frac{s_1(t)}{t}\leq\limsup_{t\to\infty}\frac{s_1(t)}{t}\leq c_0(r_1,r_1,d_1,\mu_1). \tag{3.12}$$

Moreover, for each $0<\hat{c}<c_0(r_1(1-k),r_1,d_1,\mu_1)$,

$$\liminf_{t\to\infty}\left[\min_{x\in[0,\hat{c}t]}u(x,t)\right]\geq 1-k. \tag{3.13}$$

Proof. It is easy to check that (u,s_1) forms a subsolution of

$$\begin{cases}\bar{w}_t=d_1\bar{w}_{xx}+r_1\bar{w}(1-\bar{w}),\ 0<x<\bar{h}(t),\ t>0,\\ \bar{w}_x(0,t)=0,\ \bar{w}(\bar{h}(t),t)=0,\ t>0,\\ \bar{h}'(t)=-\mu_1\bar{w}_x(\bar{h}(t),t),\ t>0,\\ \bar{h}(0)=s_0,\ \bar{w}(x,0)=u_0(x),\ 0<x<s_0,\end{cases}$$

By Proposition 4, $\bar{h}(t)\geq s_1(t)$ for all t, which implies that $\bar{h}(\infty)=\infty$. Thus, from Proposition 3(ii) we see that $\bar{h}(t)/t\to c_0(r_1,r_1,d_1,\mu_1)$ as $t\to\infty$. Consequently, we have

$$\limsup_{t\to\infty}\frac{s_1(t)}{t}\leq c_0(r_1,r_1,d_1,\mu_1).$$

To derive the lower bound estimate in (3.12), we choose any small $\epsilon > 0$ and $T(\epsilon) \gg 1$ such that

$$v(x,t) \le 1+\epsilon \quad \text{for all } x \in [0,\infty) \text{ and } t \ge T(\epsilon); \tag{3.14}$$

$$s_1(T(\epsilon)) > \frac{\pi}{2}\sqrt{\frac{d_1}{r_1[1-k(1+\epsilon)]}}. \tag{3.15}$$

Then from (3.14) it is easy to check that (u, s_1) forms a supersolution of

$$\begin{cases} w_t = d_1 w_{xx} + r_1 w[1-k(1+\epsilon)-w], \ 0 < x < h(t), \ t > T(\epsilon), \\ w_x(0,t) = 0, \ w(h(t),t) = 0, \ t > T(\epsilon), \\ h'(t) = -\mu_1 w_x(h(t),t), \ t > T(\epsilon), \\ w(x, T(\epsilon)) = u(x, T(\epsilon)), \ 0 < x < h(T(\epsilon)) := s_1(T(\epsilon)), \end{cases}$$

Using (3.15), we see that $h(\infty) = \infty$. From Proposition 3(ii) it follows that

$$\frac{h(t)}{t} \to c^*(\epsilon) := c_0(r_1[1-k(1+\epsilon)], r_1, d_1, \mu_1) \quad \text{as } t \to \infty,$$
$$w(x,t) \ge U_{c^*(\epsilon)}(h(t)-x) - \varepsilon \quad \text{for all } x \in [0, h(t)] \text{ and } t \gg 1,$$

By Proposition 4, we have $\liminf_{t\to\infty}[s_1(t)/t] \ge c^*(\epsilon)$ and for each $0 < \hat{c} < c^*(\epsilon)$,

$$\min_{x\in[0,\,\hat{c}\,t]} u(x,t) \ge U_{c^*(\epsilon)}(h(t) - \hat{c}t) - \varepsilon \quad \text{for all } t \gg 1$$

Note that $U_{c^*(\epsilon)}(h(t) - \hat{c}t) \to 1 - k(1+\epsilon)$ as $t \to \infty$ since $0 < \hat{c} < c^*(\epsilon)$. Thus, by taking $\epsilon \to 0$, we obtain the lower bound estimates in (3.12) and (3.13). This completes the proof of Lemma 3.6. □

Similarly, we have the following result.

Lemma 3.7. *It holds that*

$$\limsup_{t\to\infty} \frac{s_2(t)}{t} \le c_0(r_2, r_2, d_2, \mu_2),$$

where c_0 is defined in Proposition 1.

We are ready to prove Theorem 3.

Proof of Theorem 3. We shall divide our proof into two parts:

(a) $s_{1,\infty} = \infty$ and $s_{2,\infty} < \infty$;

(b) $\lim_{t\to+\infty} \|v(\cdot,t)\|_{C([0,s_2(t)])} = 0$ and $\lim_{t\to\infty} u(\cdot,t) = 1$ uniformly for any bounded subset of $[0,\infty)$.

For (a), since $s_{1,\infty} > s^*$, by Theorem 2 we have $s_{1,\infty} = \infty$. To prove $s_{2,\infty} < \infty$, we argue by contradiction and assume that $s_{2,\infty} = \infty$. Since $(\mu_1, \mu_2) \in \mathcal{A}$, by Lemmas 3.6 and 3.7, there exists $T \gg 1$ and a constant $\hat{c}$ such that

$$c_* =: c_0(r_2, r_2, d_2, \mu_2) < \hat{c} < c_0(r_1(1-k), r_1, d_1, \mu_1) := c^*,$$
$$s_2(t) < \hat{c}t < s_1(t) \quad \text{for all } t \ge T.$$

As in [15], we shall apply an iteration scheme. For this, we define two sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ as follows:

$$a_1 = 1-k,\ b_1 = 1, \quad b_{n+1} := 1-ha_n,\ a_{n+1} := 1-kb_{n+1}, \quad n\in\mathbb{N}.$$

Since $h>1>k>0$, it is not hard to see that there exists $N\in\mathbb{N}$ such that $a_N\in[1/h,1)$.

We shall prove that

$$\liminf_{t\to\infty}\left[\min_{x\in[0,\hat{c}t]} u(x,t)\right] \geq a_N \in [1/h,1). \tag{3.16}$$

First, by Lemma 3.6, we have

$$\liminf_{t\to\infty}\left[\min_{x\in[0,\hat{c}t]} u(x,t)\right] \geq a_1.$$

Thus, if $N=1$, then (3.16) follows.

Assume that $N>1$, i.e., $a_1<1/h$. Then for each small $\varepsilon>0$, there exists $T_1\gg 1$ such that $u\geq a_1-\varepsilon$ for $x\in[0,\hat{c}t]$ and $t\geq T_1$. Without loss of generality, we may also assume that $\hat{c}t>s_2(t)$ for all $t\geq T_1$. Then we have

$$v_t = d_2v_{xx}+r_2v(1-v-hu) \leq d_2v_{xx}+r_2v[1-v-h(a_1-\varepsilon)]$$

for $x\in[0,s_2(t)]$ and $t\geq T_1$. Let V be the solution of

$$\frac{dV}{dt} = r_2V[1-h(a_1-\varepsilon)-V],\ t\geq T_1, \quad V(T_1) = \|v(\cdot,T_1)\|_{L^\infty([0,\infty))}.$$

Thus, by comparing v and V, we conclude that (using that $\varepsilon>0$ is arbitrary small)

$$\limsup_{t\to\infty}\|v(\cdot,t)\|_{C[0,\infty)} \leq b_2. \tag{3.17}$$

We now use the same argument in the proof of Lemma 3.6 to derive

$$\liminf_{t\to\infty}\left[\min_{x\in[0,\hat{c}t]} u(x,t)\right] \geq a_2. \tag{3.18}$$

Using $s_{1,\infty}=\infty$ and (3.17), there exists $T_2>T_1$ such that

$$s_1(T_2) > \frac{\pi}{2}\sqrt{\frac{d_1}{r_1[1-k(b_2+\varepsilon)]}}, \tag{3.19}$$

$$v(x,t)\leq b_2+\varepsilon \quad \text{for all } x\in[0,\infty) \text{ and } t\geq T_2. \tag{3.20}$$

It follows from (3.20) that (u,s_1) forms a supersolution of

$$\begin{cases} w_t = d_1w_{xx}+r_1w[1-k(b_2+\varepsilon)-w],\ 0<x<\gamma(t),\ t>T_2, \\ w_x(0,t)=0,\ w(\gamma(t),t)=0,\ t>T_2, \\ \gamma'(t) = -\mu_1w_x(\gamma(t),t),\ t>T_2, \\ w(x,T_2)=u(x,T_2),\ 0<x<\gamma(T_2):=s_1(T_2), \end{cases}$$

Thus, Proposition 4 gives us

$$\gamma(t)\leq s_1(t) \ \text{ and } \ w(x,t)\leq u(x,t) \text{ for } x\in[0,\gamma(t)],\ t\geq T_2.$$

On the other hand, because of (3.19), Proposition 3 implies that

$$\gamma(t)/t \to c_0(r_1[1-k(b_2+\varepsilon)], r_1, d_1, \mu_1) \text{ as } t \to \infty.$$

Moreover, by the monotonicity of c_0, we have (if necessary, we choose ε smaller)

$$c_0(r_1[1-k(b_2+\varepsilon)], r_1, d_1, \mu_1) > c_0(r_1(1-k), r_1, d_1, \mu_1) > \hat{c},$$

which implies that

$$\max_{x\in[0,\hat{c}t]} \left|w(x,t) - [1-k(b_2+\varepsilon)]\right| = \max_{x\in[0,\hat{c}t]} \left|w(x,t) - (a_2 - k\varepsilon)\right| \to 0 \text{ as } t \to \infty.$$

Note that that $u(x,t) \geq w(x,t)$ for all $x \in [0, \hat{c}t]$ and $t \geq T_2$. Hence (3.18) follows since ε can be arbitrary small.

By repeating the above process, we obtain (3.16). Without loss of generality we may assume that $a_N > 1/h$. Otherwise, we can replace $a_1 = 1-k$ by $a_1 = 1-k-\epsilon$ for sufficiently small $\epsilon > 0$ such that $a_n \neq 1/h$ for all n. Hence it follows from (3.16) that there exists $T \gg 1$ such that

$$v_t = d_2 v_{xx} + r_2 v(1 - v - hu) \leq d_2 v_{xx}$$

over $\{(x,t) : x \in [0, s_2(t)],\ t \geq T\}$. By comparing (v, s_2) and (ϕ, σ), where

$$\begin{cases} \phi_t = d_2 \phi_{xx},\ x \in (0, \sigma(t)),\ t \geq T, \\ \phi_x(0,t) = 0 = \phi(\sigma(t), t),\ t \geq T, \\ \sigma'(t) = -\mu_2 \phi_x(\sigma(t), t),\ t \geq T, \\ \phi(x,T) = v(x,T),\ x \in [0, \sigma(T)], \quad \sigma(T) = s_2(T), \end{cases}$$

we have $s_2(t) \leq \sigma(t)$ for all $t \geq T$. It is well known that $\sigma(\infty) < \infty$. Hence we obtain $s_{2,\infty} < \infty$, a contradiction. Consequently, (a) follows.

Also, since $s_{1,\infty} = \infty$, we can apply Lemma 3.4(ii) to conclude that

$$\lim_{t\to+\infty} \|v(\cdot, t)\|_{C([0, s_2(t)])} = 0.$$

Finally, Lemma 3.2(i) with $\rho_2 = 0$ implies (b). Hence the proof of Theorem 3 is complete. $\square$

Before proving Theorem 4, we establish the following lemma.

Lemma 3.8. *Let U_{c_0} be the positive solution of (1.13). Suppose that c_0 is thought of as the function of μ (other parameters are fixed). Then c_0 is C^1 in μ and*

$$\frac{\partial c_0(\mu)}{\partial \mu} = \frac{U'_{c_0(\mu)}(0)}{1 - \mu \frac{\partial U'_{c_0(\mu)}(0)}{\partial c_0}} > 0.$$

Proof. Recall Proposition 1,

$$c_0(\mu) = \mu U'_{c_0(\mu)}(0), \tag{3.21}$$

where $U'_{c_0}(0)$ is strictly decreasing in c_0 and c_0 is strictly increasing in μ. By the standard ODE theory, we see that $U'_{c_0}(0)$ is C^1 in c_0. Thus, by differentiating (3.21) with respect to μ, we obtain that c_0 is C^1 in μ and

$$\frac{\partial c_0}{\partial \mu} = U'_{c_0}(0) + \mu \frac{\partial U'_{c_0}(0)}{\partial c_0} \frac{\partial c_0}{\partial \mu}.$$

Since $\frac{\partial U'_{c_0}(0)}{\partial c_0} < 0$, we have

$$\frac{\partial c_0(\mu)}{\partial \mu} = \frac{U'_{c_0(\mu)}(0)}{1 - \mu \frac{\partial U'_{c_0(\mu)}(0)}{\partial c_0}} > 0.$$

This completes the proof of Lemma 3.8. □

Proof of Theorem 4. We shall apply the Implicit Function Theorem to show the existence of $\Lambda(\cdot)$. Also, using Proposition 1 we can complete the proof of Theorem 4.

For convenience, we set

$$\begin{aligned} &c^*(\mu_1) := c_0(r_1(1-k), r_1, d_1, \mu_1), \quad c_*(\mu_2) := c_0(r_2, r_2, d_2, \mu_2), \\ &F(\mu_1, \mu_2) := c^*(\mu_1) - c_*(\mu_2). \end{aligned}$$

Due to Lemma 3.8, we have $F \in C^1((0,\infty) \times (0,\infty))$ and $\frac{\partial F}{\partial \mu_1} = \frac{\partial c^*}{\partial \mu_1} > 0$ for $\mu_1 \in (0,\infty)$.

For $\sqrt{r_1 d_1 (1-k)} > \sqrt{r_2 d_2}$, by Proposition 1, we have

$$\begin{aligned} 0 = c^*(0^+) < c^*(\cdot) < c^*(\infty) = 2\sqrt{r_1 d_1 (1-k)}, \\ 0 = c_*(0^+) < c_*(\cdot) < c_*(\infty) = 2\sqrt{r_2 d_2}. \end{aligned}$$

It follows that for each $\hat{\mu}_2 > 0$, there exists a unique $\hat{\mu}_1 > 0$ such that $F(\hat{\mu}_1, \hat{\mu}_2) = 0$. Moreover, there exists a unique ν_1 such that $c^*(\nu_1) = 2\sqrt{r_2 d_2}$. By the monotonicity of $c^*(\cdot)$, we have $c^*(\cdot) > 2\sqrt{r_2 d_2}$ on (ν_1, ∞). It follows that

$$\{(\mu_1, \mu_2) \in (0,\infty) \times (0,\infty) : F(\mu_1, \mu_2) = 0\} \subset (0, \nu_1) \times (0, \infty).$$

By the Implicit Function Theorem, there exists a C^1 function Λ defined for $\mu_2 \in (0,\infty)$ such that $F(\Lambda(\mu_2), \mu_2) = 0$. Moreover, by Lemma 3.8,

$$\Lambda'(\cdot) = -\frac{\partial F}{\partial \mu_2} \Big/ \frac{\partial F}{\partial \mu_1} = \frac{\partial c_*}{\partial \mu_2} \Big/ \frac{\partial c^*}{\partial \mu_1} > 0.$$

It follows that $\Lambda(\infty)$ exists. We now prove that $\Lambda(\infty) = \nu_1$. Note that $c_*(\mu_2) \uparrow 2\sqrt{r_2 d_2}$ as $\mu_2 \uparrow \infty$. It follows that

$$0 = F(\Lambda(\infty), \infty) = c^*(\Lambda(\infty)) - 2\sqrt{r_2 d_2}.$$

By the definition of ν_1, we obtain $\Lambda(\infty) = \nu_1$. Hence we have proved the existence of Λ. Also, using $\frac{\partial F}{\partial \mu_1} > 0$ for $\mu_1 \in (0,\infty)$ and

$$\mathcal{A} = \{(\mu_1, \mu_2) \in (0,\infty) \times (0,\infty) : F(\mu_1, \mu_2) > 0\},$$

we see that

$$(\mu_1, \mu_2) \in \mathcal{A} \Longleftrightarrow \mu_1 > \Lambda(\mu_2), \quad \mu_2 \in (0, \infty).$$

The same argument as above can be applied to the case that $\sqrt{r_1 d_1(1-k)} \leq \sqrt{r_2 d_2}$. We omit the detailed proof here and then Theorem 4 follows. □

Corollary 2. *Assume* **(H)**. *Let* (u, v, s_1, s_2) *be a solution of* **(P)**. *Then the followings hold.*

(a) *If* $s_1^0 < s_*$ *and* $\|u_0\|_{L^\infty}$ *is small enough, then* u *vanishes eventually. When* u *vanishes eventually, the following hold:*
  - (a-1) *if* $s_2^0 < s^{**}$, *then* v *also vanishes eventually as long as* $\|v_0\|_{L^\infty}$ *is small enough;*
  - (a-2) *if* $s_2^0 < s^{**}$, *then* v *spreads successfully as long as* $\|v_0\|_{L^\infty}$ *is large enough;*
  - (a-3) *if* $s_2^0 \geq s^{**}$, *then* v *always spreads successfully regardless of its initial population.*

(b) *Given* d_i, r_i, $i = 1, 2$. *Suppose that* $s_1^0 > s^*$. *Then* u *spreads successfully and* v *vanishes eventually as long as*

$$\mu_1 > \Lambda(\mu_2), \quad \mu_2 \in (0, \infty) \quad \text{if } \sqrt{r_1 d_1(1-k)} \geq \sqrt{r_2 d_2}.$$
$$\mu_1 > \Lambda(\mu_2), \quad \mu_2 \in (0, \nu_2) \quad \text{if } \sqrt{r_1 d_1(1-k)} < \sqrt{r_2 d_2},$$

regardless of their initial population size, where ν_2 *and* Λ *are defined in Theorem 4.*

Proof. The proof of (a) can be done by the similar argument of [10, Lemma 3.7, Lemma 3.8]. We do not repeat it here again. For (b), note that $s_1^0 > s^*$ implies that u spreads successfully. Then by Theorem 3 and Theorem 4, we obtain (b). This completes the proof of Corollary 2. □

To prove Theorem 5, we need show the monotonicity of the profile $v(\cdot, t)$ nearby the free boundary $x = s_2(t)$. The idea is to apply a reflection argument as follows.

Lemma 3.9. *Suppose that* $s_1(t) < s_2(t)$ *for* $t \in [0, \tau_1]$ *and* $\eta(t) := [s_1(t) + s_2(t)]/2$. *Then* $v_x(x, t) < 0$ *for all* $x \in [\eta(t), s_2(t)]$ *and for all* $t \in (0, \tau_1]$ *as long as* $(v_0)'(x) \leq 0$ *for all* $x \in [s_1^0, s_2^0]$.

Proof. For given $\tau \in (0, \tau_1]$ and $L \in [\eta(\tau), s_2(\tau))$, we consider

$$D_\tau := \{(x, t) : 2L - s_2(t) < x < s_2(t),\ t \in (\tau^*, \tau]\},$$

where $\tau^* := 0$ if $L \leq s_2^0$; while $\tau^* := s_2^{-1}(L)$ if $L > s_2^0$.

Using $s_i'(t) > 0$ for $i = 1, 2$, we have

$$2L - s_2(t) \geq 2\eta(\tau) - s_2(t) = s_1(\tau) + s_2(\tau) - s_2(t) \geq s_1(\tau) \geq s_1(t), \quad t \in (\tau^*, \tau].$$

Thus, $u = 0$ over D_τ and so v satisfies

$$v_t = d_2 v_{xx} + r_2 v(1 - v), \quad (x, t) \in D_\tau. \tag{3.22}$$

We now set

$$V(x,t) := v(x,t) - v(2L-x,t) \tag{3.23}$$

defined on $D'_\tau := \{(x,t) : L < x < s_2(t),\ t \in (\tau^*, \tau]\}$. Note that $(x,t) \in D'_\tau$ implies that $(x,t), (2L-x,t) \in D_\tau$. Hence, using (3.22) it gives us

$$V_t = d_2 V_{xx} + c(x,t)V, \quad (x,t) \in D'_\tau,$$

for some function c which is bounded in D'_τ. Note that $V(L,t) = 0$ and $V(s_2(t),t) = -v(s_1(t),t) < 0$ for $t \in (\tau^*, \tau]$. Note that, when $\tau^* = 0$, we have $V(x,0) \le 0$ for $x \in [L, s_2^0]$, since $(v_0)'(x) \le 0$ for $x \in [s_1^0, s_2^0]$. On the other hand, when $\tau^* > 0$, we have $L = s_2(\tau^*)$. Then we can apply the strong maximum principle to conclude that $V < 0$ over D'_τ. Furthermore, due to $V(L,\tau) = 0$ and Hopf's Lemma, $V_x(L,\tau) < 0$. It follows from (3.23) that $v_x(L,\tau) = V_x(L,\tau)/2 < 0$. Note that $v_x(s_2(\tau),\tau) < 0$ for all $\tau \in (0, \tau_1]$. Thus the proof is complete. ☐

Before we start to prove Theorem 5, we explain the idea behind the proof and how Lemma 3.9 is applied here. To prove the persistence of v, our strategy is to construct a suitable subsolution defined on some suitable region $\mathcal{D}$. Note that we have $v_x(\eta(t),t) < 0$ (Lemma 3.9), where $\eta(t) := [s_1(t)+s_2(t)]/2$. It is natural to consider the region $\mathcal{D} := \{(x,t) : \eta(t) \le x \le \eta(t)+L,\ t \ge 0\}$ for some $L > 0$ and choose a subsolution with spatial population gradient attaining zero at the left boundary $x = \eta(t)$, which allow us to compare u with the subsolution on $\mathcal{D}$.

Proof of Theorem 5. Given μ_2, d_1, r_1, r_2, u_0 and v_0, we choose

$$\mu_1 \le \mu_2 \quad \text{and } d_2 \ge \min\left\{\frac{9r_2}{32}, \frac{d_1 r_2}{r_1}\right\} := \hat{d}. \tag{3.24}$$

Also, set

$$\Lambda := 2\mu_2 \max\{K_1, K_2\} \max\left\{\sqrt{\frac{r_1}{2d_1}},\ \frac{4}{3},\ \frac{-4}{3}\left(\min_{x\in[0,s_1^0]} u_0'(x)\right),\ \frac{-4}{3}\left(\min_{x\in[0,s_2^0]} v_0'(x)\right)\right\},$$

where K_1 and K_2 are defined in (1.8) and (1.9), respectively. Using (3.24), (1.10) and (1.11), we easily obtain

$$\Lambda \ge \eta'(t) := \frac{s_1'(t) + s_2'(t)}{2}, \quad t > 0. \tag{3.25}$$

We now introduce the variable $y = x - \eta(t)$ and $\hat{v}(y,t) = v(x,t)$. Then

$$\eta(t) \le x \le s_2(t) \iff 0 \le y \le \sigma(t) := \frac{s_2(t) - s_1(t)}{2}.$$

Note that $\hat{v}$ satisfies

$$\hat{v}_t = d_2 \hat{v}_{yy} + \eta'(t)\hat{v}_y + r_2(1-\hat{v})\hat{v},\ y \in (0, \sigma(t)),\ t > 0,$$
$$\hat{v}(\sigma(t),t) = 0,\ \hat{v}(0,t) = v(\eta(t),t) > 0,\ t > 0,$$

as long as $\sigma(t) > 0$.

Set $\bar{d} := \max\{\Lambda^2/(4r_2), \hat{d}\}$. Then for any $d_2 > \bar{d}$, we set

$$l^* := \frac{\pi}{2}\left[\sqrt{\frac{r_2}{d_2}\left(1-\frac{\bar{d}}{d_2}\right)}\right]^{-1} > 0.$$

For each $d_2 > \bar{d}$, by the condition $s_2^0 - s_1^0 > 4l^*$ (see (1.16)) we can choose

$$l \in (l^*, \frac{s_2^0 - s_1^0}{4}) \tag{3.26}$$

and consider the function

$$w(y) := e^{-\frac{\Lambda}{2d_2}y}\cos\frac{\pi y}{2l}.$$

It is easy to check that w satisfies

$$w'' + \frac{\Lambda}{d_2}w' + \lambda w = 0, \quad y \in (-l, l), \quad w(\pm l) = 0,$$

where

$$\lambda = (\frac{\pi}{2l})^2 + (\frac{\Lambda}{2d_2})^2 < \frac{r_2}{d_2} \quad \text{(using } l > l^* \text{ and the definition of } \bar{d}). \tag{3.27}$$

Moreover, there exists $l_0 \in (0, l)$ such that $w'(-l_0) = 0$, $w'(y) > 0$ if $y \in (-l, -l_0)$ and $w'(y) < 0$ if $y \in (-l_0, l)$. Let $w^*(y) := w(y - l_0)$. Then we have

$$(w^*)'(0) = 0, \quad (w^*)'(y) < 0 \quad \text{for } y \in (0, l + l_0). \tag{3.28}$$

To finish the proof of Theorem 5, it suffices to show

$$\sigma(t) \geq l + l_0,\ \forall\, t \geq 0, \quad \hat{v} \geq \delta w^*,\ \forall\, y \in [0, l + l_0],\ t \geq 0, \tag{3.29}$$

for some small $\delta > 0$ under the condition

$$\sigma(0) > 2l, \quad d_2 > \bar{d}, \quad \mu_1 \leq \bar{\mu},$$

where $\bar{\mu} > 0$ depending on d_2 and $\bar{d}$ will be determined later.

To do so, we first choose $\delta > 0$ small enough such that

$$\hat{v}(y, 0) > \delta w^*(y) \quad \text{for all } y \in [0, l + l_0]. \tag{3.30}$$

Note that it can be done because $\sigma(0) > 2l > l + l_0$ (using (3.26)). Due to (3.25), (3.28) and (3.27) (if necessary we choose δ smaller), we have

$$\begin{aligned} & d_2(w^*)'' + \eta'(t)(w^*)' + r_2(1 - \delta w^*)w^* \\ = \ & -[\Lambda - \eta'(t)](w^*)' + w^*(r_2 - d_2\lambda - r_2\delta w^*) \geq 0,\ y \in (0, l + l_0). \end{aligned} \tag{3.31}$$

For such fixed $\delta > 0$, we choose

$$\bar{\mu} := \min\left\{\frac{\mu_2 \delta \pi}{4M^* l}\exp\left\{-\frac{\Lambda l}{2d_2}\right\}, \mu_2\right\},$$

where

$$M^* := K_1 \max\left\{\sqrt{\frac{r_1}{2d_1}},\ \frac{4}{3},\ \frac{-4}{3}\left(\min_{x\in[0,s_1^0]} u_0'(x)\right)\right\}$$

We now prove (3.29). From (3.30), we see that $\hat{v}(y,t) > \delta w^*(y)$ for $y \in [0, l+l_0]$ and $\sigma(t) > l + l_0$ for all small $t > 0$. For contradiction, we assume that there exists $T^* > 0$ such that $\sigma(T^*) = l + l_0$ and $\sigma(t) > l + l_0$ for $t \in [0, T^*)$. Then we have $\sigma'(T^*) \le 0$ and so

$$-\mu_2 \hat{v}_y(\sigma(T^*), T^*) = s_2'(T^*) \le s_1'(T^*) \le 2\mu_1 M^*, \tag{3.32}$$

where the last inequality follows from (1.10). Next, we introduce

$$Q(y,t) := \hat{v}(y,t) - \delta w^*(y).$$

From (3.31), it follows that

$$Q_t - d_2 Q_{yy} + \eta'(t) Q_y + \gamma(x,t) w \le 0 \quad \text{for } y \in (0, l+l_0),\ t \in (0, T^*),$$

for some bounded function γ. Also, we have

$$\begin{aligned}
&Q(y,0) > 0, \quad y \in [0, l+l_0] \text{ (by (3.30))},\\
&Q_y(0,t) = \hat{v}_y(0,t) - \delta (w^*)'(0) = v_y(\eta(t),t) < 0, \quad t \in [0,T^*] \text{ (by Lemma 3.9)},\\
&Q(l+l_0,t) = \hat{v}(l+l_0,t) - \delta(w^*)(l+l_0) = \hat{v}(l+l_0,t) \ge 0, \quad t \in [0,T^*].
\end{aligned}$$

Thus, we can apply the strong maximum principle and Hopf's Lemma to conclude that $Q_y(\sigma(T^*), T^*) < 0$. This implies

$$-\mu_2 \hat{v}_y(\sigma(T^*), T^*) > -\mu_2 \delta (w^*)'(l+l_0) = \frac{\mu_2 \delta \pi}{2l} \exp\left\{-\frac{\Lambda l}{2d_2}\right\}.$$

Together with (3.32), it leads to a contradiction to $\mu_1 \le \bar{\mu}$. Hence (3.29) follows and then the proof of Theorem 5 is completed. □

4. Discussion

In this paper, we consider a free boundary problem which describe the spreading of two competing species in a one-dimensional habitat. We assume that u is a superior competitor occupying the interval $[0, s_1(t)]$, while v is an inferior competitor with the territory $[0, s_2(t)]$ at time t. Here, the two free boundaries $x = s_i(t)$, $i = 1, 2$, differently from the previous works, may intersect each other. They are used to describe the spreading fronts of two competing species, respectively. Our goal is to investigate its dynamics. Due to the fact that two free boundaries may intersect each other, it seems very difficult to understand the whole dynamics of this model.

In comparing to the Cauchy problem, our model shows that (under **(H)**) the superior competitor is not always the winner. If the superior competitor's territory size cannot cross some critical value, it can lose the competition, while if its territory is above this critical value, then spreading occurs. This result is consistent with the one in [11]. An interesting phenomenon appearing in our model is that when spreading of the superior competitor occurs, our model shows the weaker species does not necessarily die out eventually over their territory. In fact, if the superior competitor spreads too slow to catch up with the inferior competitor, it may leave enough space for the inferior competitor to establish persistent population.

From the modeling point of view, the real case should be the case of two-dimensional habitat. Mathematically, the 1d case is the simplest case to do the analysis (for example, the existence and uniqueness issue). For the higher dimensional case, our approach still works in a radially symmetric setting, i.e., the habitat and the solution are assumed to be radially symmetric. Then (1.5) becomes

$$s_1'(t) = -\mu_1 u_r(s_1(t),t),\ t>0; \quad s_2'(t) = -\mu_2 v_r(s_2(t),t),\ t>0,$$

where $r := |x|$ and $u = u(r,t), v = v(r,t)$. For general non-symmetric case, the Stefan condition (1.5) can be replaced by the condition (cf. [8])

$$\Phi_t = \mu \nabla_x u \cdot \nabla_x \Phi$$

if the free boundary is represented by $\Gamma(t) = \{x \in \mathbb{R}^N : \Phi(x,t) = 0\}$ for some suitable function Φ. We leave this general higher dimensional case as a future study.

On the other hand, the condition (1.3) means that no flux can across the left boundary. This condition is equivalent to the (radial) symmetric case in 1d, if we consider the following general setting:

$$\begin{aligned}
&u_t = d_1 u_{xx} + r_1 u(1-u-kv), \quad s_1^-(t) < x < s_1^+(t),\ t>0,\\
&v_t = d_2 v_{xx} + r_2 v(1-v-hu), \quad s_2^-(t) < x < s_2^+(t),\ t>0,\\
&u \equiv 0 \quad \text{for } x \notin (s_1^-(t), s_1^+(t)),\ t>0; \quad v \equiv 0 \quad \text{for } x \notin (s_2^-(t), s_2^+(t)),\ t>0,\\
&(s_1^{\pm})'(t) = -\mu_1 u_x(s_1^{\pm}(t),t),\ t>0; \quad (s_2^{\pm})'(t) = -\mu_2 v_x(s_2^{\pm}(t),t),\ t>0.
\end{aligned}$$

Indeed, our analysis works for this general case. However, it would increase the complexity of our presentation. For simplicity, we only treat the symmetric case in this paper. We leave the general case to the reader.

For the issue of spreading speed, if one species vanishing eventually, the model can be thought of as the single species model of Du and Lin [10]. Thus, the spreading speed of the species that spreads successfully can be understood as in [10]. If both two species spread successfully, it would be interesting to characterize their spreading speed (we only have some rough estimates). We leave this issue for the future study. We also refer to [1] for the asymptotic behaviour of moving interfaces for some free boundary problems.

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