ON A TWO-POINT FREE BOUNDARY PROBLEM

JONG-SHENQ GUO AND BEI HU

ABSTRACT. We study a two-point free boundary problem for a quasilinear parabolic equation. This problem arises in the model of flame propagation in combustion theory. It also arises in the study of the motion of interface moving with curvature in which the studied problem is confined in the conical region bounded by two straight lines and the interface has prescribed touching angles with these two straight lines. Depending on these two touching angles, there are three different cases, namely, area expanding, preserving, and shrinking cases. We first give a proof of the global existence in the expanding and preserving cases. Then the convergence to a line in the preserving case is derived. Finally, in the shrinking case, we show the finite time vanishing and the convergence of the solution to self-similar solution.

1. INTRODUCTION

We consider the following two-point free boundary problem (FBP):

$$(1.1) \qquad u_t = [a(u_x)]_x, \quad -\xi_1(t) < x < \xi_2(t), \ t > 0,$$

$$(1.2) \qquad u_x(-\xi_1(t), t) = \tan\alpha_1, \ u_x(\xi_2(t), t) = \tan\alpha_2, \ t > 0,$$

$$(1.3) \qquad u(-\xi_1(t), t) = \xi_1(t)\tan\beta_1, \ u(\xi_2(t), t) = \xi_2(t)\tan\beta_2, \ t > 0,$$

$$(1.4) \qquad u(x,0) = u_0(x), \quad -\xi_1(0) \le x \le \xi_2(0), \quad \xi_1(0) = \xi_{01}, \quad \xi_2(0) = \xi_{02},$$

where we assume that

$$(1.5) \qquad a \in C^2(-\infty, \infty), \qquad a(0) = 0, \quad a'(s) > 0 \quad \text{for} \quad s \in (-\infty, \infty),$$

β_i and α_i are given constants satisfying

$$(1.6) \qquad \beta_i \in [0, \pi/2), \quad i = 1, 2, \qquad \alpha_1 \in (-\beta_1, \pi/2), \qquad \alpha_2 \in (-\pi/2, \beta_2),$$

ξ_{01} and ξ_{02} are positive constants,

$$(1.7) \qquad u_0 \in C^{1+\alpha}[-\xi_{01}, \xi_{02}], \quad u_0(-\xi_{01}) = \xi_{01}\tan\beta_1, \quad u_0(\xi_{02}) = \xi_{01}\tan\beta_2,$$

$$(1.8) \qquad u_0(x) > 0 \quad \text{for} \quad \xi_{01} < x < \xi_{02}.$$

In this problem, u, ξ_1, ξ_2 are unknown functions to be found.

Figure 1 shows the configuration of our problem, where $\Gamma(t)$ represents the surface $z = u(x,t)$, and the free boundaries are located at $x = -\xi_1(t)$ and $x = \xi_2(t)$.
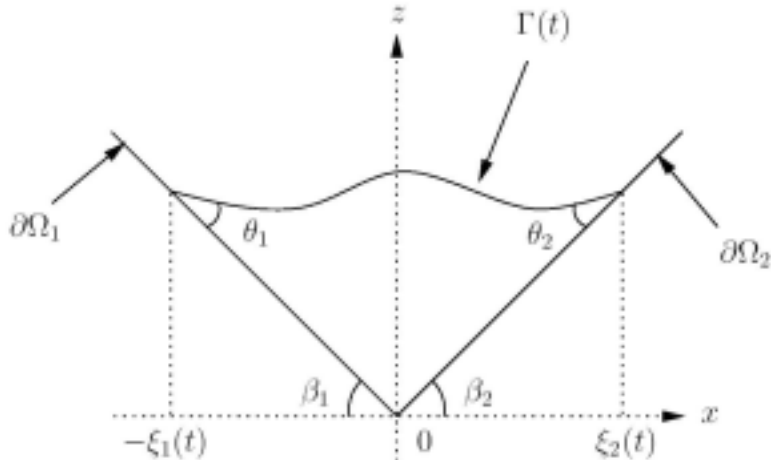
Figure 1: $\theta_1 = \alpha_1 + \beta_1 > 0$, $\theta_2 = \beta_2 - \alpha_2 > 0$.

This type of free boundary problem arises in various applications. For example, when $a(s) = s$ (the case of heat equation), (FBP) arises in the model of flame propagation in combustion theory (cf. [1, 9, 12] and the references cited therein). When $a(s) = \arctan(s)$ (the case of curvature flow equation), (FBP) arises in the study of the motion of interface moving with curvature in which the studied problem is confined in the conical region bounded by two straight lines and the interface has prescribed touching angles with these two straight lines (cf. [4] and the references cited therein).

There have been many interesting works on the so-called one-point free boundary problem (when $\beta_1 = 0$ and $\beta_2 = \pi/2$). For the case with $\alpha_1 = \pi/4$ and $\alpha_2 = 0$, we refer the readers to [10, 2, 12, 6]. We note that, by a reflection, this case is the symmetric case of our two-point free boundary problem with $\beta_1 = \beta_2 = 0$ and $-\alpha_2 = \alpha_1 = \pi/4$. For the case with $\alpha_1, \alpha_2 \in (0, \pi/2)$, we refer the readers to [11].

We now consider the general case (1.1)–(1.4). The local (in time) existence, uniqueness of a classical solution is proved in [3]:

Proposition 1.1 ([3]). *Let the assumptions (1.5)–(1.8) be in force. Then*

(i) there exists a $T > 0$ such that the classical solution exists and is unique in the interval $[0, T)$, where T depends on the $C^{1+\alpha}$ norm of the initial data u_0;

(ii) if $\xi_1(t) + \xi_2(t) \geq \delta > 0$ for $0 < t < T$, $\|u(\cdot, t)\|_{C^{1+\alpha}[-\xi_1(t), \xi_2(t)]} \leq C$ for $0 < t < T$, and the limit $\lim_{t \to T-0} \xi_i(t) := \xi_i(T)$ $(i = 1, 2)$ exists, then there exists a $T' > 0$ such that the classical solution obtained in (i) can be uniquely extended to the interval $[0, T + T')$;

(iii) furthermore, for any $c_0 > 0$, the solution obtained in (i) satisfies
$$(u, \xi_1, \xi_2) \in C^{2+\alpha, 1+\alpha/2}\{(x, t); -\xi_1(t) \leq x \leq \xi_2(t), \ c_0 \leq t < T\} \times C^{1+\alpha/2}[c_0, T) \times C^{1+\alpha/2}[c_0, T).$$

The regularity does not extends to $t = 0$, because only $C^{1+\alpha}$ initial data is assumed in [3]. If we assume initial data to be in the class $C^{2+\alpha}$, then the argument in [3] shows that the regularity also extends to $t = 0$.

Notice that if $\xi_i'(t)$ $(i = 1, 2)$ are bounded on $[0, T)$, then $\lim_{t \to T-0} \xi_i(t) = \xi_i(T)$ exists. It is also clear that $W^{2,\infty}$ estimates imply $C^{1+\alpha}$ estimates.

In view of Proposition 1.1 and the discussion above, the following a priori estimates

$$(1.9) \qquad \xi_1(t) + \xi_2(t) > \delta \quad \text{for} \quad t \in [0, T), \qquad (\delta > 0),$$

$$(1.10) \qquad |\xi_1'(t)| + |\xi_2'(t)| \leq C \quad \text{for} \quad t \in [c_0, T),$$

$$(1.11) \qquad \|u(\cdot, t)\|_{W^{2,\infty}[-\xi_1(t), \xi_2(t)]} \leq C \quad \text{for} \quad t \in [c_0, T)$$

will be sufficient for us to derive the existence and uniqueness of a classical solution for the system (1.1)–(1.4) in the interval $[0, T)$ and extends the solution to $[0, T + T')$ for some $T' > 0$.

Set $\gamma_1 = \tan \alpha_1$ and $\gamma_2 = \tan \alpha_2$. If $\xi_1(t) > 0$ and $\xi_2(t) > 0$ for $t \in [0, \tau]$, then by the maximum principle $u(x, t) > 0$ for $t \in [0, \tau]$; in this case, we can apply the maximum principle in the regions $\{(x, t); -\xi_1(t) < x < 0\}$ and $\{(x, t); 0 < x < \xi_2(t)\}$ respectively to obtain

$$(1.12) \qquad u(x, t) \geq \max[(\tan \beta_2)x, -(\tan \beta_1)x], \ t \in [0, \tau].$$

this is shown in Figure 1. Clearly, the area $D(t)$ of the region enclosed by the interface and the given two straight lines is given by (see Figure 1):

$$(1.13) \qquad D(t) = \int_{-\xi_1(t)}^{\xi_2(t)} u(y, t) dy - \frac{1}{2}\xi_1^2(t) \tan \beta_1 - \frac{1}{2}\xi_2^2(t) \tan \beta_2.$$

A simple computation shows that, for $t > 0$, (notice that $\xi_i(t)$ $(i = 1, 2)$ is continuously differentiable for $t > 0$, by Proposition 1.1),

$$(1.14) \qquad D'(t) = a(\gamma_2) - a(\gamma_1);$$

thus the area $D(t)$ is expanding in the case $\gamma_1 < \gamma_2$, preserved in the case $\gamma_1 = \gamma_2$, and shrinking in the case $\gamma_1 > \gamma_2$.

In the case $\gamma_1 < \gamma_2$, the area is expanding. In this area expanding case, it is proved in [3] that

(i) there exists a unique solution to the system (1.1)–(1.4) for all $0 < t < \infty$;

(ii) under the assumption that $\beta_1, \beta_2 \in [0, \pi/2)$, $-\beta_1 < \alpha_1 < \pi/2$ and $\alpha_1 < \alpha_2 < \beta_2$, there exists a unique forward self-similar solution to the system (1.1)–(1.3) of the form

$$(1.15) \qquad u(x, t) = \sqrt{2(t+1)}\varphi(\frac{x}{\sqrt{2(t+1)}});$$

(iii) in this expanding case, with appropriate assumptions on the data, the solution of (1.1)–(1.4) will converge to the forward self-similar solution; this is done in [3] through explicit and delicate comparison.

If $a(s) \equiv s$, then we have the heat equation. In this case, various properties of self-similar solutions are discussed in [9]. In particular,

(i) the exact forward self-similar solutions to the system (1.1)–(1.3) of the form (1.15) is studied in the expanding case;

(ii) the exact backward self-similar solutions to the system (1.1)–(1.3) of the form

$$u(x, t) = \sqrt{2(T-t)}\varphi((\frac{x}{\sqrt{2(T-t)}})$$

is studied in the shrinking case. For certain range of data, the uniqueness of the backward self-similar solutions is not valid.

The question of asymptotic behavior of the solution to the system (1.1)–(1.4) in the area-preserving and shrinking cases is not studied, even for the case of heat equation ($a(s) \equiv s$). The goal of this paper is to study these two cases. We begin with the following global existence of classical solution in the expanding and preserving cases. Note that the global existence for the expanding case has been established in [3], however, our global existence proof covers both the area-preserving and expanding cases.

Theorem 1.2. *Consider the expanding and preserving cases $\gamma_1 \leq \gamma_2$. Let the assumptions (1.5)–(1.8) be in force. Assume also that $u_0 \in C^{1+\alpha}$ for some $\alpha > 0$. Then there exists a unique classical solution for $0 < t < \infty$.*

In the case $\gamma_1 = \gamma_2$, the area is preserved. We have the following result for the asymptotic behavior, the result is new even for heat equation case.

Theorem 1.3. *Consider the area preserving case $\gamma_1 = \gamma_2$. Let the assumptions (1.5)–(1.8) be in force. Then there exists $d_1 > 0$, determined uniquely by the initial data, such that*

$$(1.16) \qquad\qquad u(x,t) \to \gamma x + d_1$$

uniformly in x as $t \to \infty$.

In the shrinking case when $\gamma_1 > \gamma_2$, the area should vanish in finite time. We shall establish the asymptotic behavior in this case (Theorems 1.4 and 1.5). The result is new even for the heat equation case.

Theorem 1.4. *Consider the area shrinking cases $\gamma_1 > \gamma_2$. Let the assumptions (1.5)–(1.8) be in force. We further assume*

$$(1.17) \qquad\qquad -\beta_1 \leq \alpha_2 < \alpha_1 \leq \beta_2.$$

Then

(i) there exists a unique classical solution $(u(x,t), \xi_1(t), \xi_2(t))$ to the system (1.1)–(1.4) for $0 \leq t < T$ with T given by

$$(1.18) \qquad\qquad T = D(0)/[a(\gamma_1) - a(\gamma_2)],$$

where

$$D(0) = \int_{-\xi_{10}}^{\xi_{20}} u_0(x)dx - \frac{1}{2}\xi_{10}^2 \tan\beta_1 - \frac{1}{2}\xi_{20}^2 \tan\beta_1.$$

(ii) $u \in C^{2+\alpha, 1+\alpha/2}(\Omega_T) \cap C^{0+1, 1/2}(\overline{\Omega}_T)$ and $\xi_1, \xi_2 \in C^{1+\alpha/2}(0, T) \cap C^{1/2}[0, T]$, where $\Omega_T = \{(x,t); -\xi_1(t) \leq x \leq \xi_2(t), 0 < t < T\}$.

(iii) $\lim_{t \to T-0} \|u(\cdot, t)\|_{L^\infty} = 0$ and $\xi_1(T) = \xi_2(T) = 0$.

To study the asymptotic behavior at the vanishing time T, we make a change of variables and define $(v(y,s), p(s), q(s))$ by

$$u(x,t) = \sqrt{2(T-t)}\, v(y,s), \quad y = \frac{x}{\sqrt{2(T-t)}}, \quad T-t = e^{-2s},$$

$$\xi_1(t) = \sqrt{2(T-t)}\, p(s), \quad \xi_2(t) = \sqrt{2(T-t)}\, q(s).$$

Then

(1.19) $$v_s = [a(v_y)]_y - yv_y + v, \quad -p(s) < y < q(s), \ s > s_0 := -\ln(T)/2,$$

(1.20) $$v_y(-p(s), s) = \gamma_1, \ v_y(q(s), s) = \gamma_2, \ s > s_0,$$

(1.21) $$v(-p(s), s) = p(s)\tan\beta_1, \ v(q(s), s) = q(s)\tan\beta_2, \ s > s_0.$$

The steady-state solutions (φ, p, q) of (1.19)–(1.21) are give by

(1.22) $$[a(\varphi')]' - y\varphi' + \varphi = 0, \quad -p < y < q, \quad p > 0, q > 0,$$

(1.23) $$\varphi'(-p) = \gamma_1, \quad \varphi'(q) = \gamma_2,$$

(1.24) $$\varphi(-p) = p\tan\beta_1, \quad \varphi(q) = q\tan\beta_2.$$

Theorem 1.5. *Let the assumptions (1.5)–(1.8) be in force. We further assume*

(1.25) $$-\beta_1 < \alpha_2 < \alpha_1 < \beta_2.$$

Let T be given as in (1.18) and (v, p, q) be defined as above. Then
  (i) for any initial data in $C^{1+\alpha}$, the corresponding ω-limit set is not empty;
  (ii) any ω-limit is a solution to (1.22)–(1.24);
  (ii) if we further assume $a(s)$ to be analytic, then $v(y, s)$ will converge to one of the ω-limit as $s \to \infty$.

Remark 1.1. *The backward self-similar solutions in the case $a(s) \equiv s$ were extensively studied in [9], where existence, non-uniqueness as well as the properties of solutions were discussed. In the case of general $a(s)$, Theorem 1.5 establishes the existence of such a backward self-similar solution under the additional assumption (1.25).*

This paper is organized as follows. In section 2, we derive some a priori estimates for solutions of (FBP). Also, we give a proof of the global existence of solutions of (FBP) in the area-preserving and expanding cases (Theorem 1.2). We then give a proof for the convergence of the solution to a line in the area-preserving case in section 3 (Theorem 1.3). In section 4, we turn to the shrinking case and give some preliminary results, where we also prove the existence result as well as finite time vanishing result (Theorem 1.4). Finally, we study in section 5 the asymptotic behavior of solution of (FBP) in the shrinking case and prove Theorem 1.5. The convergence result in section 5 depends also on the properties of the backward self-similar solutions derived in section 6.

2. Some a priori estimates

In this section we derive some a priori estimates for solutions of (FBP). For the estimates in this section, we shall always assume (1.5)–(1.8). Note that under these assumptions, the free boundary $x = \xi_i(t)$ $(i = 1, 2)$ and their first order derivatives are Hölder continuous for $t > 0$ (Proposition 1.1). These regularity properties of the free boundary should be used throughout this paper.

We first get the L^∞ bounds on u_x, which is a simple application of the comparison principle. We assume that the classical solution exists on $[0, \tau)$, for some $0 < \tau \leq +\infty$.

Lemma 2.1.

$$(2.1) \qquad\qquad |u_x(x,t)| \le C_1 \quad for \quad 0 < t < \tau,$$

where

$$(2.2) \qquad\qquad C_1 = \max(|\gamma_1|, |\gamma_2|, \|(u_0)_x\|_{L^\infty}).$$

Note that Lemma 2.1 implies that $a'(u_x(x,t))$ is uniformly bounded from above, and bounded from below by a positive constant. Thus the equation (1.1) becomes uniformly parabolic.

We next derive half-sided estimates for u_{xx}.

Lemma 2.2. *For any $c_0 > 0$, there exists a constant C such that*

$$(2.3) \qquad\qquad u_{xx} \le C \quad for \quad -\xi_1(t) < x < \xi_2(t),\ c_0 < t < \tau.$$

Proof. Differentiating boundary conditions (1.2), (1.3) at $x = \xi_2(t)$ in t, we obtain

$$(2.4) \qquad u_{xx}(\xi_2(t),t)\xi_2'(t) + u_{xt}(\xi_2(t),t) = 0, \quad u_x(\xi_2(t),t)\xi_2'(t) + u_t(\xi_2(t),t) = \xi_2'(t)\tan\beta_2.$$

Using the equation (1.1) and the boundary condition (1.2), we now get

$$
\begin{aligned}
u_{tx}(\xi_2(t),t) &= -u_{xx}(\xi_2(t),t)\xi_2'(t) \\
&= -\frac{u_t(\xi_2(t),t)}{a'(u_x(\xi_2(t),t))}\xi_2'(t) \\
&= -\frac{u_t(\xi_2(t),t)}{a'(u_x(\xi_2(t),t))}\left(\frac{u_t(\xi_2(t),t)}{\tan\beta_2 - u_x(\xi_2(t),t)}\right) \\
&= -\frac{1}{(\tan\beta_2 - \gamma_2)a'(\gamma_2)}u_t^2(\xi_2(t),t) \le 0.
\end{aligned}
$$

Similarly,

$$-u_{tx}(-\xi_1(t),t) = -\frac{1}{(\tan\beta_1 + \gamma_1)a'(\gamma_1)}u_t^2(\xi_1(t),t) \le 0.$$

It is clear that u_t satisfies

$$u_{tt} = a'(u_x)u_{txx} + a''(u_x)u_{tx}.$$

It is proved in [3] that $u_{xx}(x,t)$ is bounded for $t = c_0$ for small $c_0 > 0$, provided we assume $u_0 \in C^{1+\alpha}$ for some $\alpha \in (0,1)$. It follows that $u_t = a'(u_x)u_{xx}$ is bounded for $t = c_0$. Thus we can apply the maximum principle to u_t to conclude that u_t is bounded from above. This implies that $u_{xx} = u_t/a'(u_x)$ is bounded from above. $\square$

Remark 2.1. *If $u_0 \in C^2$ and satisfies the 2nd order compatibility condition at the boundary, then we can take $c_0 = 0$ in the above lemma.*

We next derive the lower bound for u_{xx}.

Lemma 2.3. *If*

$$(2.5) \qquad\qquad u(0,t) \ge m \quad for \quad 0 < t < \tau,$$

for some $m > 0$, then for any $c_0 > 0$, there exists C independent of m such that

$$(2.6) \qquad\qquad u_{xx} \ge -\frac{C}{m} \quad for \quad -\xi_1(t) < x < \xi_2(t),\ c_0 < t < \tau.$$

Proof. By changing the value of $a(p)$ outside $|p| > C_1$ (where C_1 is from (2.2) in Lemma 2.1) if necessary, we may assume without loss of generality that $l < a'(p) < 1/l$ for all $p \in (-\infty, \infty)$ for some $l > 0$ and $a \in C^2$. This implies that the inverse function of $a^{-1}(\cdot)$ is well defined on $(-\infty, \infty)$.

We choose w such that

$$a(w) = \frac{C}{m}(u - x \tan \beta_2) + a(\gamma_2).$$

If we take C such that $C + a(\gamma_2) \geq a(C_1)$, then $u_x(0, t) \leq w(0, t)$, $u_x(\xi_2(t), t) = \gamma_2 = w(\xi_2(t), t)$ for $0 < t < \tau$. Since u_{xx} is bounded for $t = c_0$ and $\tan \beta_2 - u_x(\xi_2(c_0), c_0) = \tan \beta_2 - \gamma_2 > 0$, there exists $\varepsilon > 0$ such that $\tan \beta_2 - u_x(x, c_0) > (\tan \beta_2 - \gamma_2)/2$ for $x \in [\xi_2(c_0) - \varepsilon, \xi_2(c_0)]$. Using the bounds on u_{xx} and u_x, we can choose C large enough so that

$$
\begin{aligned}
u_{xx}(x, c_0) \quad &> \quad -\frac{Cl}{2m}(\tan \beta_2 - \gamma_2) \\
&\geq \quad -\frac{C}{ma'(w)}(\tan \beta_2 - u_x(x, c_0)) \\
&= \quad w_x(x, c_0) \quad \text{for} \quad x \in [\xi_2(c_0) - \varepsilon, \xi_2(c_0)].
\end{aligned}
$$

Integrating this inequality over the interval $[x, \xi_2(c_0)]$, using also the boundary condition $u_x(\xi_2(c_0), c_0) = w(\xi_2(c_0), c_0) = \gamma_2$ we find that

$$u_x(x, c_0) \leq w(x, c_0) \quad \text{for} \quad \xi_2(c_0) - \varepsilon \leq x \leq \xi_2(c_0).$$

Since $u - x \tan \beta_2$ is positive on the interval $x \in [0, \xi_2(c_0) - \varepsilon]$, it is clear that the above inequality is valid for $0 < x < \xi_2(c_0) - \varepsilon$ if we choose the constant C in the definition of w to be large enough.

Now that the constant C in w is fixed. We can finally define the uniformly parabolic operator $\mathcal{L}$ (in the definition we fix on u_x, u_{xx} and w) by

$$\mathcal{L}[\varphi] = \varphi_t - [a(\varphi)]_{xx} - \frac{C}{ma'(w)} u_{xx}[a'(u_x) - a'(\varphi)].$$

Clearly

$$\mathcal{L}[u_x] = u_{xt} - [a(u_x)]_{xx} - \frac{C}{ma'(w)} u_{xx}[a'(u_x) - a'(u_x)] = 0,$$

and

$$
\begin{aligned}
\mathcal{L}[w] \quad &= \quad w_t - [a(w)]_{xx} - \frac{C}{ma'(w)} u_{xx}[a'(u_x) - a'(w)] \\
&= \quad \frac{C}{m}\left(\frac{1}{a'(w)} u_t - u_{xx}\right) - \frac{C}{ma'(w)} u_{xx}[a'(u_x) - a'(w)] \\
&= \quad \frac{C}{m}\left(\frac{1}{a'(w)} a'(u_x) u_{xx} - u_{xx}\right) - \frac{C}{ma'(w)} u_{xx}[a'(u_x) - a'(w)] \\
&= \quad 0.
\end{aligned}
$$

Thus by comparison

$$u_x < w \quad \text{for} \quad 0 < x < \xi_2(t), c_0 < t < \tau.$$

Thus

$$u_{xx}(\xi_2(t), t) \geq w_x(\xi_2(t), t) \geq -\frac{C}{ml}(\tan\beta_2 - \gamma_2) \quad \text{for} \quad c_0 < t < \tau.$$

One can similarly show that

$$u_x > a^{-1}\left(-\frac{C}{m}(u + x\tan\beta_1) + a(\gamma_1)\right) \quad \text{for} \quad -\xi_1(t) < x < 0, c_0 < t < \tau.$$

and obtain

$$u_{xx}(-\xi_1(t), t) \geq -\frac{C}{ml}(\gamma_1 + \tan\beta_1) \quad \text{for} \quad c_0 < t < \tau.$$

These estimates imply that (by using the equation)

$$(2.7) \qquad u_t(x, t) \geq -\frac{C^*}{m} \quad \text{for} \quad x = -\xi_1(t), \ x = \xi_2(t), \ c_0 < t < \tau.$$

Differentiating the equation (1.1) in t and applying the maximum principle to u_t, we obtain

$$u_t \geq -\frac{C^*}{m} \quad \text{for} \quad -\xi_1(t) < x < \xi_2(t), \ c_0 < t < \tau.$$

Using the equation (1.1) again, we conclude the lemma. $\qquad\square$

These estimates imply the classical global existence in the area-preserving and expanding cases as claimed in Theorem 1.2.

Proof of Theorem 1.2. We can choose any $\gamma \in [\gamma_1, \gamma_2]$ and $d > 0$ such that $u_0(x) \geq \gamma x + d$ for $x \in [-\xi_1(0), \xi_2(0)]$. Then by comparison, we immediately get

$$(2.8) \qquad u(x, t) \geq \gamma x + d \quad \text{for} \quad -\xi_1(t) < x < \xi_2(t).$$

By Lemma 2.1, u_x is uniformly bounded by C_1. By (2.8), $u(0, t) \geq d$. Thus if we choose $m = d$ in Lemma 2.3, then u_{xx} is also uniformly bounded for $t > c_0$, which in term implies that u_t is bounded for $t > c_0$ by using the equation (1.1). The estimate (2.8) also implies

$$(2.9) \qquad \min[\xi_1(t), \xi_2(t)] \geq \delta$$

for some $\delta > 0$. From (2.4), (1.2) and the uniform bound on u_t, we conclude

$$(2.10) \qquad \xi_2'(t) = \frac{u_t(\xi_2(t), t)}{\tan\beta_2 - \gamma_2}$$

is uniformly bounded for $t > t_0$. Similarly, $\xi_2'(t)$ is also bounded for $t > c_0$. Therefore $\xi_i(t)$ $(i = 1, 2)$ are bounded by $Ct + C$ and $u(x, t)$ is therefore also bounded by $Ct + C$. Using these bounds in Proposition 1.1, we find that the solution can be extended to all t. $\qquad\square$

Remark 2.2. *We proved that for the case $\gamma_1 \leq \gamma_2$, the solution is global, and*

$$(2.11) \qquad \sup_{c_0 < t < \infty} \{\|u(\cdot, t)\|_{W^{2,\infty}[-\xi_1(t),\xi_2(t)]} + \|u_t(\cdot, t)\|_{L^\infty[-\xi_1(t),\xi_2(t)]} + |\xi_1'(t)| + |\xi_2'(t)|\} \leq C.$$

This estimate is useful in the asymptotic study for the area-preserving case.

3. Asymptotic limit for the area-preserving case

For the case $\gamma_1 \geq \gamma_2$, we can choose any $\gamma \in [\gamma_2, \gamma_1]$ and $D > 0$ such that $u_0(x) \leq \gamma x + D$ for $x \in [-\xi_1(0), \xi_2(0)]$. Then by comparison, we immediately get

$$(3.1) \qquad u(x,t) \leq \gamma x + D \quad \text{for} \quad -\xi_1(t) < x < \xi_2(t).$$

Choosing $\gamma = \gamma_1$ and $\gamma = \gamma_2$ respectively, using also (1.3), we obtain

$$(3.2) \qquad \xi_1(t) \leq \frac{D}{\tan \beta_1 + \gamma_1}, \qquad \xi_2(t) \leq \frac{D}{\tan \beta_2 - \gamma_2}.$$

In particular, if $\gamma_1 = \gamma_2 := \gamma$, then we can combine (2.8) and (3.1):

$$(3.3) \qquad \gamma x + d \leq u(x,t) \leq \gamma x + D \text{ for } -\xi_1(t) < x < \xi_2(t), \ 0 < t < \infty.$$

By (2.11), we also have uniform bounds on u_x, u_{xx}, $\xi_i'(t)$ $(i = 1,2)$ for $t \in [c_0, \infty)$.

Proof of Theorem 1.3. For $A(s) := \displaystyle\int_0^s a(y)dy$ and

$$J(t) = \int_{-\xi_1(t)}^{\xi_2(t)} A(u_x(x,t))dx,$$

we have (using the boundary conditions)

$$
\begin{aligned}
J'(t) &= \int_{-\xi_1(t)}^{\xi_2(t)} a(u_x)u_{xt}dx + A(\gamma)\xi_2'(t) + A(\gamma)\xi_1'(t) \\
&= -\int_{-\xi_1(t)}^{\xi_2(t)} [a(u_x)]_x u_t dx + a(u_x)u_t \Big|_{-\xi_1(t)}^{\xi_2(t)} + A(\gamma)\xi_2'(t) + A(\gamma)\xi_1'(t) \\
&= -\int_{-\xi_1(t)}^{\xi_2(t)} u_t^2 dx + [a(\gamma)(\tan \beta_2 - \gamma) + A(\gamma)]\xi_2'(t) \\
&\quad + [-a(\gamma)(\tan \beta_1 + \gamma) + A(\gamma)]\xi_1'(t).
\end{aligned}
$$

Thus

$$J(t) + \int_{c_0}^t \int_{-\xi_1(t)}^{\xi_2(t)} u_t^2 dx \leq J(c_0) + C(\|\xi_1\|_{L^\infty[c_0,\infty)} + \|\xi_2\|_{L^\infty[c_0,\infty)}).$$

The right-hand of the above equality is uniformly bounded. Thus

$$(3.4) \qquad \int_{c_0}^\infty \int_{-\xi_1(t)}^{\xi_2(t)} u_t^2 dx < \infty.$$

This estimate, together with the compactness of the solution as $t \to \infty$ (subsequence of $u(x, t + t_j)$ converges in $C^{1+\alpha}$ in x up to the boundary and C^2 in the interior, $\xi_i(t_j) \to \xi_{i\infty}$ $(i = 1, 2)$), we immediately obtain the convergence on the subsequence. By following the standard argument as in [7], we find that the subsequential limit function must be a solution to the steady state equation $[a(u_x)]_x = 0$ with the boundary conditions (1.2), (1.3). It is clear that the only solutions to the steady state are lines of the type $\gamma x + d_1$ for some d_1.

With the $D(t)$ defined in (1.6), it is clear that $D'(t) = 0$ in the area preserving case. In particular, $D(t) \equiv D(0)$ for all $t > 0$. If we pass the limit along the convergent subsequence, we find

$$(3.5) \qquad \int_{-\xi_{1\infty}}^{\xi_{2\infty}} (\gamma x + d_1)dx - \frac{1}{2}\xi_{1\infty}^2 \tan\beta_1 - \frac{1}{2}\xi_{2\infty}^2 \tan\beta_2 = D(0).$$

From (1.3), we also have

$$(3.6) \qquad -\gamma\xi_{1\infty} + d_1 = \xi_{1\infty}\tan\beta_1, \qquad \gamma\xi_{2\infty} + d_1 = \xi_{2\infty}\tan\beta_2.$$

It is clear that (3.5) and (3.6) uniquely determine d_1, $\xi_{1\infty}$ and $\xi_{2\infty}$. As in [7], the uniqueness of the limit $(d_1, \xi_{1\infty}, \xi_{2\infty})$ implies that $u(x,t)$, $\xi_i(t)$ $(i = 1, 2)$ all converge uniformly as $t \to \infty$. □

Remark 3.1. *Let $b = \lim_{t\to\infty}[\xi_1(t) + \xi_2(t)]$. The convergence theorem of the standard parabolic theory gives the exponential convergence rate up to the first eigenvalue (see [5] for the convergence rate for the linearized problem, the uniform convergence of the nonlinear problem, together with the convergence rate for the linearized problem, imply the convergence rate for the nonlinear problem). More precisely, for any $\lambda \in (0, a'(\gamma)\pi^2/b^2)$, $\lim_{t\to\infty} e^{\lambda t}|u(x,t) - \gamma x - d_1| = 0$.*

4. Preliminary for the shrinking case

The assumption $\gamma_1 > \gamma_2$ means that there is a negative total heat flux through the boundary and therefore u should decrease overall.

Notice that the estimates (3.1) and (3.2) were proved for both the area preserving and shrinking cases. These bounds are valid as long as the solution exists:

$$u(x,t) \le \gamma x + D, \quad \xi_1(t) \le C, \quad \xi_2(t) \le C.$$

In this section we assume that $[0, T)$ is the maximal existence interval for a classical solution. In the Lemmas 4.1–4.3 in this section, we only need to assume that the initial data u_0 is $C^{1+\alpha}$ and we do not need the condition (1.17).

We first prove the Hölder estimates for $u(x,t)$ in t direction. We extend $u(x,t)$ to all $x \in (-\infty, \infty)$ by defining

$$u(x,t) = \begin{cases} -x\tan\beta_1 & \text{for} \quad x < -\xi_1(t), \\ x\tan\beta_2 & \text{for} \quad x > \xi_2(t). \end{cases}$$

Lemma 4.1. *The extended function $u(x,t)$ satisfies, for any $t_1, t_2 \in [0, T)$,*

$$(4.1) \qquad |u(x,t_1) - u(x,t_2)| \le C|t_1 - t_2|^{1/2}.$$

Proof. The extended function satisfies the equation (1.1) both in and outside the region $\{(x,t), \ x \in (-\xi_1(t), \xi_2(t))\}$. There is a jump on the first order x-derivative across the boundary but $|u_x|$ is uniformly bounded by $\max\{C_1, \tan\beta_1, \tan\beta_2\}$ for all $x \in (-\infty, \infty)$. And u satisfies in the distribution sense the equation

$$u_t = \frac{\partial}{\partial x}\Big[a(u_x) + \{a(\gamma_2) - a(\tan\beta_2)\}H(x - \xi_2(t)) - \{a(\gamma_1) - a(-\tan\beta_1)\}H(x + \xi_1(t))\Big],$$

where $H(t)$ is the Heaviside function. It follows that, for any x_1, x_2 and $t_1 < t_2$,

$$\left| \int_{x_1}^{x_2} \Big(u(x, t_1) - u(x, t_2) \Big) dx \right| \le \int_{t_1}^{t_2} \left| \int_{x_1}^{x_2} u_t(x, \tau) dx \right| d\tau \le C|t_1 - t_2|.$$

For any given x^* and t_1, t_2, we choose x_1, x_2 such that $x_1 < x^* < x_2$ and $|x_1 - x_2| = |t_1 - t_2|^{1/2}$, then

$$
\begin{aligned}
&|u(x^*, t_1) - u(x^*, t_2)| \\
\le\ & \frac{1}{|x_1 - x_2|} \left| \int_{x_1}^{x_2} \Big(u(x, t_1) - u(x, t_2) \Big) dx \right| + \|u_x\|_{L^\infty} |x_1 - x_2| \\
\le\ & C|t_1 - t_2|^{1/2}.
\end{aligned}
$$

This completes the proof. $\square$

If we replace the positiveness assumption on $u(0, t)$ by certain assumptions on $u(x_0, t)$, we can repeat the proof of Lemma 2.3 to conclude :

Lemma 4.2. *If*

$$u(x_0, t) - \max[(\tan \beta_2) x_0, -(\tan \beta_1) x_0] > m \quad for \quad 0 < t < \tau,$$

for some $m > 0$, then for any $c_0 > 0$, there exists C independent of m such that

$$(4.2) \qquad u_{xx} \ge -\frac{C}{m} \quad for \quad -\xi_1(t) < x < \xi_2(t), \ c_0 < t < \tau.$$

For a positive solution, we have

Lemma 4.3. *If $u(x, t) > 0$ for $t \in [0, \tau]$, then*

$$u(x, t) > \max[(\tan \beta_2) x, -(\tan \beta_1) x], \ x \in (-\xi_1(t), \xi_2(t))$$

for $t \in [0, \tau]$.

Proof. Using the boundary condition, we can compare the solution with the function $(\tan \beta_2) x$ and $-(\tan \beta_1) x$ respectively to conclude the result. $\square$

Next we give a sufficient condition for the solution u to remain positive.

If we have the additional assumption (1.17), then we claim that u is always positive as follows.

Lemma 4.4. *Suppose that $[0, T)$ is the maximal existence interval for the classical solution and $T < \infty$. Under the additional assumption (1.17), we have $u(x, t) > 0$ for $0 \le t < T$.*

Proof. Let $[0, T^*)$ be the maximal interval on which $u(x, t)$ remains positive. Then both ξ_1 and ξ_2 are positive on this interval. We claim that $T^* = T$. In fact, if $T^* < T$, then $u(x, t)$ is C^2 for $t \in [0, T^*]$ and $\xi_1(T^*) + \xi_2(T^*) > 0$. Lemma 4.3 implies that $u(x, T^*)$ is positive for $x \ne 0$. Therefore we must have $u(0, T^*) = 0$. If $-\xi_1(T^*) < 0 < \xi_2(T^*)$, then $u(x, t)$ reaches the minimum at an interior point $(0, T^*)$ and therefore $u \equiv 0$ and we get a contradiction. If $\xi_1(T^*) = 0$, then $\xi_2(T^*) > 0$. Using Lemma 4.3 and applying strong maximum principle to $u - (\tan \beta_2) x$, we obtain $u_x(0, T^*) - \tan \beta_2 > 0$, i.e., $\tan \alpha_1 > \tan \beta_2$. This contradicts (1.17). We can similarly get a contradiction if $\xi_2(T^*) = 0$. Thus we proved $T^* = T$, and $u(x, t) > 0$ for $0 \le t < T$. $\square$

We now prove that the positive solution must vanish at $t = T$.

Lemma 4.5. *Suppose that $[0, T)$ is the maximal existence interval for the classical solution and $T < \infty$. Under the additional assumption (1.17), we have $\xi_1(t) > 0$ and $\xi_2(t) > 0$ for all $t \in [0, T)$. Moreover, $\xi_1(T) = \xi_2(T) = 0$ and $u(0, T) = 0$.*

Proof: Note that the assumption (1.17) implies that $\beta_1 + \beta_2 > 0$. Hence it follows from Lemma 4.3 that $\xi_i(t) \geq 0$ for all $t < T$ for $i = 1, 2$. Using the condition (1.17) and following the same proof as in Lemma 4.4, we can show that $\xi_1(t) > 0$ and $\xi_2(t) > 0$ for all $t \in [0, T)$.

Since u_t is bounded above by Lemma 2.2, $\xi_i'(t) \leq C$ $(i = 1, 2)$ (cf. (2.10)); the function $\xi_i(t) - Ct$ is therefore monotonically decreasing and the limit $\lim_{t \to T_-} (\xi_i(t) - Ct)$ exists. Therefore the limit $\lim_{t \to T_-} \xi_i(t)$ exists and we define it to be $\xi_i(T)$. With this definition, it is clear that $\xi_i(t)$ is continuous on $[0, T]$. If $\xi_1(T) + \xi_2(T) > 0$, then for $x_0 = [\xi_1(T) + \xi_2(T)]/2$, we have $u(x_0, T) > \max[(\tan \beta_2)x_0, -(\tan \beta_1)x_0]$, by Lemma 4.3 and the strong maximum principle. Now we can apply Lemma 4.2 to conclude that u_{xx} remains bounded near $t = T$ and therefore the classical solution can be further extended beyond $t = T$. This contradicts our assumption that $[0, T)$ is the maximal existence interval. Thus we must have $\xi_1(T) + \xi_2(T) = 0$ and the lemma is proved.   □

We are now ready to prove Theorem 1.4.

Proof of Theorem 1.4. In view of Lemmas 4.1–4.5, the parts (ii) and (iii) of Theorem 1.4 are already established if we can establish (i), namely, for the maximal existence interval $[0, T)$, T is finite and is given in (1.18). Let us denote the right-hand side of (1.18) to be T^*, i.e., $T^* = D(0)/[a(\gamma_1) - a(\gamma_2)]$ and we want to show $T = T^*$. It is clear from (1.14) that T^* is the unique time such that

$$(4.3) \qquad\qquad D(t) > 0 \quad \text{for} \quad 0 < t < T^*, D(T^*) = 0.$$

By Lemma 4.4, $u(x, t) > 0$ on the maximal existence interval $[0, T)$. Then using Lemma 4.3, we find that $D(t)$ defined (1.13) satisfies

$$D(t) > 0 \quad \text{for} \quad 0 \leq t < T.$$

By Lemma 4.5, $D(T) = 0$. By comparing this with (4.3), we conclude that $T = T^*$. This finishes the proof.   □

The rest of this section will be devoted to deriving additional estimates needed for the proof of asymptotic expansion. If we replace (1.17) with (1.25), we can also derive the estimate on $\xi_j(t)$, $j = 1, 2$.

Lemma 4.6. *Suppose that $[0, T)$ is the maximal existence interval for the classical solution, where T is given by (1.18). Under the further assumption (1.25)*

$$(4.4) \qquad\qquad 0 < \xi_j(t) < C\sqrt{T - t} \quad \text{for} \quad j = 1, 2, \ t < T.$$

Proof. First, it follows from Lemma 4.5 that $\xi_j(t) > 0$ for $j = 1, 2$, $t < T$.
Integrating the inequality (2.3), we obtain

$$\gamma_2 - C(\xi_2(t) - x) \leq u_x(x, t) \leq \gamma_1 + C(x + \xi_1(t)) \quad \text{for} \quad -\xi_1(t) < x < \xi_2(t), t < T.$$

Integrating this inequality again, we get

$$(4.5) \qquad u(x,t) \leq \begin{cases} u(0,t) + \gamma_1 x + C\left(\frac{1}{2}x^2 + x\xi_1(t)\right) & \text{for} \quad 0 < x < \xi_2(t), \\ u(0,t) + \gamma_2 x + C\left(\frac{1}{2}x^2 - x\xi_2(t)\right) & \text{for} \quad -\xi_1(t) < x < 0, \end{cases}$$

Thus

$$(4.6) \qquad \int_{-\xi_1(t)}^{\xi_2(t)} u(x,t)dx \leq [\xi_1(t) + \xi_2(t)]u(0,t) + \frac{\gamma_1}{2}\xi_2^2(t) - \frac{\gamma_2}{2}\xi_1^2(t) + \widetilde{C}[\xi_1^3(t) + \xi_2^3(t)].$$

Substituting this estimate into the expression for $D(t)$ in (1.13), we derive

$$(4.7) \qquad \begin{aligned} D(t) + \tfrac{1}{2}(\tan\beta_1 + \gamma_2)\xi_1^2(t) + \tfrac{1}{2}(\tan\beta_2 - \gamma_1)\xi_2^2(t) \\ \leq [\xi_1(t) + \xi_2(t)]u(0,t) + \widetilde{C}[\xi_1^3(t) + \xi_2^3(t)]. \end{aligned}$$

Notice that the assumption (1.25) implies that the coefficient $(\tan\beta_1 + \gamma_2)$ and $(\tan\beta_2 - \gamma_1)$ are positive. Since $D(t) > 0$ for $t < T$, (4.7) implies, for $r(t) = \sqrt{\xi_1^2(t) + \xi_2^2(t)}$

$$r^2(t)[1 - C^* r(t)] \leq Cr(t)u(0,t).$$

Choosing t to be sufficiently close to T such that $1 - C^* r(t) \leq 1/2$, then we obtain

$$r(t) \leq Cu(0,t) \leq C\sqrt{T-t}.$$

The last inequality was obtained from the Hölder continuity in t for $u(0,t)$ (Lemma 4.1). This completes the proof. $\square$

We finally prove

Lemma 4.7. *Under the assumptions of Lemma 4.5, there exists $c > 0$ such that*

$$(4.8) \qquad u(0,t) \geq c\sqrt{T-t}.$$

Proof. Using Lemma 4.6 in (4.7), we obtain

$$[a(\gamma_1) - a(\gamma_2)](T-t) = D(t) \leq C\sqrt{T-t}\, u(0,t) + C(T-t)^{3/2},$$

from which the lemma follows. $\square$

5. Asymptotic behavior for the shrinking case

In this section we want to study the asymptotic behavior as $t \to T$. Throughout this section we shall assume that (1.25) is satisfied. Let

$$u(x,t) = \sqrt{2(T-t)}\, v(y,s), \quad y = \frac{x}{\sqrt{2(T-t)}}, \quad T-t = e^{-2s},$$

$$\xi_1(t) = \sqrt{2(T-t)}\, p(s), \quad \xi_2(t) = \sqrt{2(T-t)}\, q(s).$$

Then

$$(5.1) \qquad v_s = [a(v_y)]_y - yv_y + v, \quad -p(s) < y < q(s), \ s > s_0 := -\ln(T)/2,$$

$$(5.2) \qquad v_y(-p(s),s) = \gamma_1, \ v_y(q(s),s) = \gamma_2, \ s > s_0,$$

$$(5.3) \qquad v(-p(s),s) = p(s)\tan\beta_1, \ v(q(s),s) = q(s)\tan\beta_2, \ s > s_0.$$

Under the assumption (1.25), it is proved in section 4 that u is positive. It is clear that

$$p(s) > 0, \ q(s) > 0.$$

Combining the lemmas from section 4, we have

Lemma 5.1. *For a positive solution, we have*

(5.4) $$\max[(\tan \beta_2)y, -(\tan \beta_1)y] \le v(y,s) \le C, \quad |v_y(y,s)| \le C,$$

and

(5.5) $$0 < p(s) \le C, \quad 0 < q(s) \le C.$$

Proof. (5.5) is an immediate result from Lemma 4.6. The gradient bounds is an immediate result from the gradient bound for u. Lemma 4.3 gives the lower bound estimates for $v(y,s)$.

The estimate $u(0,t) \le C\sqrt{T-t}$ implies that $v(0,s) \le C$. Since $p(s), q(s)$ are bounded,

$$v(y,s) \le v(0,s) + \|v_y\|_{L^\infty} \le C.$$

The lemma is proved.   □

Next, we prove the following non-degeneracy lemma.

Lemma 5.2. *There exists $c > 0$ such that*

(5.6) $$p(s) > c, \ q(s) > c.$$

Proof. Using Lemma 4.7, we obtain, for some $c_1 > 0$,

$$v(0,s) \ge c_1.$$

Using this estimates and the gradient bounds on v, we obtain

$$v(y,s) \ge \frac{c_1}{2} \quad \text{for} \quad |y| < c_2,$$

for some $c_2 > 0$. Now the lemma follows from the boundary condition.   □

Using Lemma 5.2 and interior parabolic estimates, all derivatives of $v(y,s)$ are bounded at $y = 0$. We now derive the following estimate for the second order derivative.

Lemma 5.3. *There exists a constant C such that*

(5.7) $$-C \le v_{yy}(y,s) \le Ce^{-s} \quad \text{for} \quad -p(s) \le y \le q(s), s_0 < s < \infty.$$

Proof. If we rewrite the estimate in Lemma 2.3, we obtain

$$v_{yy} \le \sqrt{2}Ce^{-s}.$$

Using Lemmas 2.3 and 4.7, we get

$$u_{xx}(x,t) \ge -\frac{C}{\sqrt{T-t}}.$$

Rewriting this in y variable, we get $v_{yy} \ge -C$.   □

Combining these lemmas and using the boundary conditions, we obtain

Lemma 5.4.

(5.8) $$|p'(s)| + |q'(s)| \le C \quad \text{for} \quad s_0 < s < \infty.$$

Proof: Since v_s is bounded, we use the boundary condition to write $p'(s)$ and $q'(s)$ in terms of v_s to conclude the lemma. $\square$

If we let

$$D^*(s) = \int_{-p(s)}^{q(s)} v(y,s)dy - \frac{1}{2}p_1^2(s)\tan\beta_1 - \frac{1}{2}p_2^2(s)\tan\beta_2,$$

Then

$$(5.9) \qquad D^*(s) = \frac{1}{2(T-t)}D(t) = \frac{1}{2}[a(\gamma_1) - a(\gamma_2)] \quad \text{for} \quad s_0 < s < \infty.$$

We next derive energy estimates.

Following [13] (see also [8]), we define

$$(5.10) \qquad E(s) = \int_{-p(s)}^{q(s)} \Phi\Big(y, v(y,s), v_y(y,s)\Big)dy,$$

where $\Phi = \Phi(y,v,w)$ is to be determined. Then

$$(5.11) \qquad \frac{d}{ds}E(s) = J_0 + J_1 + J_2,$$

where

$$
\begin{aligned}
J_0 &= -\int_{-p(s)}^{q(s)} \frac{1}{a'(v_y(y,s))}\Phi_{ww}(y,v(y,s),v_y(y,s))|v_s|^2(y,s)dy, \\
J_1 &= \Phi(q(s), q(s)\tan\beta_2, \gamma_2)q'(s) + \Phi(-p(s), p(s)\tan\beta_1, \gamma_1)p'(s) \\
&\quad + \Phi_w(q(s), q(s)\tan\beta_2, \gamma_2)(\tan\beta_2 - \gamma_2)q'(s) \\
&\quad - \Phi_w(-p(s), p(s)\tan\beta_1, \gamma_1)(\tan\beta_1 + \gamma_1)p'(s), \\
J_2 &= \int_{-p(s)}^{q(s)} \left\{\Phi_v - \Phi_{wy} - \Phi_{wv}v_y - \Phi_{ww}\left[\frac{yv_y - v}{a'(v_y)}\right]\right\}v_s(y,s)dy \\
&\equiv \int_{-p(s)}^{q(s)} K\Big(y, v(y,s), v_y(y,s)\Big)v_s(y,s)dy.
\end{aligned}
$$

Let

$$\Phi(y,v,w) = \int_0^w (w-\sigma)P(y,v,\sigma)d\sigma - \int_0^v \frac{\mu}{a'(0)}P(y,\mu,0)d\mu.$$

Then

$$\Phi_{ww}(y,v,w) = P(y,v,w),$$

and,

$$
\begin{aligned}
K(y,v,w) &= \int_0^w \left\{ -\sigma\frac{\partial P}{\partial v}(y,v,\sigma) - \frac{\partial P}{\partial y}(y,v,\sigma) \right. \\
&\quad \left. -\frac{\partial}{\partial\sigma}\left[P(y,v,\sigma)\frac{y\sigma - v}{a'(\sigma)}\right] \right\}d\sigma.
\end{aligned}
$$

We want to construct P so that the braces $\{\cdots\}$ in the above vanishes. Let $P(y, v, w) = \exp(Q(y, v, w))$, then this is equivalent to

$$(5.12) \qquad wQ_v(y, v, w) + Q_y(y, v, w) + \frac{yw - v}{a'(w)}Q_w(y, v, w) + \frac{\partial}{\partial w}\left\{\frac{yw - v}{a'(w)}\right\} = 0.$$

This linear PDE is solved through characteristics.

We let $\psi(\eta; y, v, w)$ be defined as the solution of the *conjugate* problem with the terminal condition:

$$(5.13) \qquad a'(\psi_\eta)\psi_{\eta\eta} - \eta\psi_\eta + \psi = 0, \quad \eta \in (-C^*, y), \quad y \leq C^*$$

$$(5.14) \qquad \psi(y; y, v, w) = v, \quad \psi_\eta(y; y, v, w) = w,$$

where we assume that $p(s) \leq C^*$, $q(s) \leq C^*$ for all $s > s_0$. Since $a'(s) \in (l, 1/l)$ for some $l > 0$, the solution of this ODE is well defined. It is clear that for bounded v and w, the solution ψ and its derivative ψ_η are uniformly bounded.

Differentiating (5.14) in y, we obtain $\psi_\eta(y; y, v, w) + \psi_y(y; y, v, w) = 0$, and hence

$$\psi_y(y; y, v, w) = -\psi_\eta(y; y, v, w) = -w.$$

Similarly, since $\psi_{\eta\eta}(y; y, v, w) + \psi_{\eta y}(y; y, v, w) = 0$,

$$\begin{aligned} \psi_{y\eta}(y; y, v, w) &= -\psi_{\eta\eta}(y; y, v, w) \\ &= \frac{-y\psi_\eta(y; y, v, w) + \psi(y; y, v, w)}{a'(\psi_\eta(y; y, v, w))} = \frac{-yw + v}{a'(w)}. \end{aligned}$$

Next, differentiating (5.14) in v and w, respectively, we find that

$$\psi_v(y; y, v, w) = 1, \quad \psi_{v\eta}(y; y, v, w) = 0, \quad \psi_w(y; y, v, w) = 0, \quad \psi_{w\eta}(y; y, v, w) = 1.$$

Thus the functions $\psi_y(\eta; y, v, w)$ and $-w\psi_v(\eta; y, v, w) + \dfrac{-yw + v}{a'(w)}\psi_w(\eta; y, v, w)$ and their first derivatives in η agree at $\eta = y$.

Differentiating (5.13) with respect to the parameters y, v and w, using also the uniqueness of the solution of the corresponding ODE, we derive

$$(5.15) \qquad \psi_y(\eta; y, v, w) = -w\psi_v(\eta; y, v, w) + \frac{-yw + v}{a'(w)}\psi_w(\xi; y, v, w).$$

Let $R(y, v, w) = \dfrac{\partial}{\partial w}\left\{\dfrac{yw - v}{a'(w)}\right\}$ and define

$$(5.16) \qquad Q(y, v, w) = -\int_0^y R(\eta, \psi(\eta, y, v, w), \psi_\eta(\eta, y, v, w))d\eta.$$

Then a direct computation shows that (using also (5.15))

$$wQ_v(y, v, w) + Q_y(y, v, w) + \frac{yw - v}{a'(w)}Q_w(y, v, w) + \frac{\partial}{\partial w}\left\{\frac{yw - v}{a'(w)}\right\}$$

$$= wQ_v(y, v, w) + \frac{yw - v}{a'(w)}Q_w(y, v, w) - \int_0^y (R_v\psi_y + R_w\psi_{\eta y})d\eta$$

$$= -\int_0^y \left\{w(R_v\psi_v + R_w\psi_{\eta v}) + \frac{yw - v}{a'(w)}(R_v\psi_w \right.$$

$$\left. + R_w\psi_{\eta w}) + (R_v\psi_y + R_w\psi_{\eta y})\right\}d\eta$$

$$= -\int_0^y \left\{R_v \cdot \left[w\psi_v + \frac{yw - v}{a'(w)}\psi_w + \psi_y\right]\right\}d\eta$$

$$- \int_0^y \left\{R_w \cdot \left[w\psi_{\eta v} + \frac{yw - v}{a'(w)}\psi_{\eta w} + \psi_{\eta y}\right]\right\}d\eta.$$

The first term of the above expressions vanishes by (5.15) and the second term also vanishes by differentiating (5.15) in η.

Substituting this equality into the expression for K, we get

$$K(y, v, w) \equiv 0,$$

and hence

$$J_2 = 0.$$

Using the bounds on $p(s)$ and $q(s)$, we obtain that

$$(5.17) \qquad \sup_{s_0 < s < \infty}\left|\int_{s_0}^s J_1(\tau)d\tau\right| \le C < \infty.$$

Thus we proved:

Theorem 5.5.

$$(5.18) \qquad \int_{s_0}^\infty \int_{-p(s)}^{q(s)} \frac{1}{a'(v_y(y, s))}P(y, v(y, s), v_y(y, s))|v_s|^2(y, s)ds < \infty.$$

It is clear that $l \le a'(s) \le 1/l$ for some $l > 0$ and therefore

$$c_0 \le \frac{P(y, v(y, s), v_y(y, s))}{a'(v_y(y, s))} \le C \quad \text{for} \quad s_0 < s < \infty$$

for some $0 < c_0 < C < \infty$.

Proof of Theorem 1.5 (i) (ii). We follow the standard procedure in [7], which was already used in the proof of Theorem 1.3.

Theorem 5.5, together with the estimates on the derivatives of the solution established in sections 4 and 5, immediately imply that a subsequence will convergence to the ω-limit which is a stationary solution of (5.1). The v_{yy} estimates ensures that the boundary conditions remain in force after taking the limit. $\square$

We call a stationary solution of (5.1)-(5.3) a backward self-similar solution. Note that in general we do not have the uniqueness of backward self-similar solutions. For example, in the heat equation case, there are at least two backward self-similar solutions if $\tan^{-1} G_c < \alpha_2 <$

$-\beta_1 < \alpha_1 \leq \beta_2$, where $G_c < -\tan \beta_1$ is a constant depending only on α_1 and β_1. Also, there is at least one backward self-similar solution if (1.25) holds (cf. Theorem 4.1 in [9]).

Although the ODE solution may not be unique in general, there can only be finitely many solutions in the case that $a(s)$ is analytic. We will prove this fact in section 6. This fact will enable us to establish the following theorem.

Proof of Theorem 1.5 (iii). Suppose that both φ_1 and φ_2 are the ω-limit. The estimates in the previous sections imply that φ_i $(i = 1, 2)$ are well-defined in a small neighborhood of 0. If $\varphi_1 \equiv \varphi_2$ in a small neighborhood of 0, then it is clear that $\varphi_1 \equiv \varphi_2$ everywhere, by uniqueness of the ODE. Thus if the ω-limit is not unique, then there must exists a small δ such that $\varphi_1(\delta) \neq \varphi_2(\delta)$. For any number η between $\varphi_1(\delta)$ and $\varphi_2(\delta)$, we can choose $s_j \to \infty$ such that $v(\delta, s_j) = \eta$. Using the compactness and the Lyapunov function we find that $v(y, s + s_j)$ will converge to the ODE solution which is a ω-limit with value η at $y = \delta$. Thus we obtain a continuum of ω-limit, which is a contradiction to the result in the next section. □

6. The backward self-similar solution

The backward self-similar solution in the case of heat equation $(a(s) \equiv s)$ has been studied in [9].

For the general case, the limit φ will satisfy:

$$[a(\varphi')]' - y\varphi' + \varphi = 0, -p < y < q, \quad p > 0, q > 0, \tag{6.1}$$

$$\varphi'(-p) = \gamma_1, \quad \varphi'(q) = \gamma_2, \tag{6.2}$$

$$\varphi(-p) = p \tan \beta_1, \quad \varphi(q) = q \tan \beta_2. \tag{6.3}$$

Lemma 6.1. *Under the assumption (1.25), there exists $\varepsilon_0 > 0$ such that (6.1)–(6.3) has no solution for $p \in [0, \varepsilon_0]$.*

Proof. We let $\varphi(y, p)$ be the solution of

$$[a(\varphi_y)]_y - y\varphi_y + \varphi = 0, \ y > -p, \tag{6.4}$$

$$\varphi_y(y, p) = \gamma_1 \quad \text{for} \quad y = -p, \tag{6.5}$$

$$\varphi(y, p) = p \tan \beta_1 \quad \text{for} \quad y = -p. \tag{6.6}$$

It is clear that if $p = 0$, then $\varphi(y, 0) = \gamma_1 y$ is the unique solution of (6.4)–(6.6) and therefore the second equality in (6.2) cannot be satisfied. For $p > 0$, let $G(y, p) = y\varphi_y(y, p) - \varphi(y, p)$, then

$$G_y = \frac{y}{a'(\varphi_y(y, p))} G, \quad G(-p, p) = -p(\gamma_1 + \tan \beta_1).$$

Thus

$$G(y, p) = -p(\gamma_1 + \tan \beta_1) \exp \int_{-p}^{y} \frac{\tau}{a'(\varphi_y(\tau, p))} d\tau < 0, \tag{6.7}$$

and

$$\lim_{y \to +\infty} G(y, p) = -\infty. \tag{6.8}$$

In particular, $\varphi_{yy}(y, p) = G/a'(\varphi_y) < 0$, and $\varphi_y(y, p) < \gamma_1$ for $y > -p$. Thus if a solution of (6.1)–(6.3) exists, then

$$q \tan \beta_2 = \varphi(q, p) \leq \varphi(-p, p) + \gamma_1(q + p) = p \tan \beta_1 + \gamma_1(q + p),$$

which implies that $q(\tan \beta_2 - \gamma_1) \leq p(\tan \beta_1 + \gamma_1)$. From (6.7) and the relationship $\varphi_{yy}(y, p) = G/a'(\varphi_y)$, we also obtain the estimates for φ_{yy}. Combining all these estimates, we obtain

$$\gamma_1 - \gamma_2 = -\int_{-p}^{q} \varphi_{yy}(\tau, p) d\tau \leq (p + q) \max_{\tau \in [-p, q]} (-\varphi_{yy}(\tau, p)) \leq C \varepsilon_0^2 \quad \text{for} \quad p \in (0, \varepsilon_0].$$

This is a contradiction if ε_0 is small.   □

We next prove:

Lemma 6.2. *In addition to the assumption (1.25), we assume that $a(s)$ is analytic in s for $s \in [\gamma_2, \gamma_1]$. Then there are at most finitely many solutions of (p, q) satisfying (6.1)–(6.3).*

Proof. For $p > \varepsilon_0$, the $-\varphi_{yy}(y, p)$ is bounded from below by a positive constant. Thus we can uniquely solve $q = q(p) > -p$ such that

$$\varphi_y(q(p), p) \equiv \gamma_2.$$

The estimates (6.8) implies that $q(p)$ is bounded from above. It is also clear that $\gamma_2 \leq \varphi_y \leq \gamma_1$ for $-p \leq p \leq q(p)$.

Define, for $p \geq \varepsilon_0$,

$$(6.9) \qquad\qquad\qquad K(p) := \varphi(q(p), p) - q(p) \tan \beta_2.$$

The above discussion implies that $K(p)$ is well defined for all $p \geq \varepsilon_0$.

If p_0 is a zero for $K(p)$, we want to show p_0 is an isolated zero for K.

If $a(s)$ is analytic for $s \in [\gamma_2, \gamma_1]$, then K is analytic. Therefore the zeros of K can not have a limit point unless $K(p)$ is identically zero. Thus if p_0 is not an isolated zero for K, then $K(p) \equiv 0$ for $p \geq \varepsilon_0$. However, Lemma 6.1 implies that $K(\varepsilon_0) \neq 0$. Thus we obtain a contradiction and we proved that the zeros of K are isolated.

If p is sufficiently large, then it is clear from (6.7) that $q(p) < 0$ and thus $K(p) > 0$. Since $K(p)$ can only have finitely many isolated zeros on a bounded interval, the lemma is proved.   □

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References

[1] J. D. Buckmaster and G. S. S. Ludford, Theory of Laminar Flames, Cambridge University Press, Cambride, 1982.

[2] L. A. Caffarelli and J. L. Vazquez, A free boundary problem for the heat equation arising in flame propagation, *Trans. Amer. Math. Soc.* 347 (1995), 411-441.

[3] Y.-L. Chang, J.-S. Guo, and Y. Kohsaka, On a two-point free boundary problem for a quasilinear parabolic equation, *Asymptotic Analysis* 34 (2003), 333-358.

[4] H.-H. Chern, J.-S. Guo and C.-P. Lo, The self-similar expanding curve for the curvature flow equation, *Proc. Amer. Math. Soc.* 131 (2003), 3191-3201.

[5] A. Friedman, Partial Differential Equations of Parabolic Type, Prentice-Hall, Englewood Cliffs, NJ, 1964.

[6] V. A. Galaktionov, J. Hulshof and J. L. Vazquez, Extinction and focusing behaviour of spherical and annular flames described by a free boundary problem, *J. Math. Pures Appl.* 76 (1997), 563-608.

[7] Y. Giga and R. V. Kohn, Asymptotic self-similar blowup of semilinear heat equations, *Comm. Pure and Applied Math.* 38 (1985), 297-319.

[8] J.-S. Guo, and B. Hu, Quenching profile for a quasilinear parabolic equation, *Quarterly of Applied Math.* 58 (2000), 613-626.

[9] J.-S. Guo and Y. Kohsaka, Self-similar solutions of two-point free boundary problem for heat equation, *Nonlinear Diffusive Systems and Related Topics*, RIMS Kokyuroku 1258, Research Institute for Mathematical Sciences, Kyoto University, April, 2002, pp.94-107.

[10] D. Hilhorst and J. Hulshof, A free boundary focusing problem, *Proc. Amer. Math. Soc.* 121 (1994), 1193-1202.

[11] Y. Kohsaka, Free boundary problem for quasilinear parabolic equation with fixed angle of contact to a boundary, *Nonlinear Analysis* 45 (2001), 865-894.

[12] J. L. Vazquez, The free boundary problem for the heat equation with fixed gradient condition, Free boundary problems, theory and applications, Zakopane, Poland (1995), Pitman Res. Notes in Math. Ser. 363, 277-302.

[13] T. I. Zelenjak, Stabilization of solutions of boundary value problems for a second order parabolic equation with one space variable, *Differential Equations* 4 (1968), 17-22.

DEPARTMENT OF MATHEMATICS, NATIONAL TAIWAN NORMAL UNIVERSITY, S-4 TING CHOU ROAD, TAIPEI 117, TAIWAN
   E-mail address: `jsguo@math.ntnu.edu.tw`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NOTRE DAME, ROOM 255, HURLEY, NOTRE DAME, INDIANA 46556
   E-mail address: `b1hu@nd.edu`