BEHAVIORS OF SOLUTIONS FOR A SINGULAR PREY-PREDATOR MODEL AND ITS SHADOW SYSTEM

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ABSTRACT. We study the asymptotic behaviors and quenching of the solutions for a two-component system of reaction-diffusion equations modeling prey-predator interactions in an insular environment. First, we give a global existence result for the solutions to the corresponding shadow system. Then, by constructing some suitable Lyapunov functionals, we characterize the asymptotic behaviors of global solutions to the shadow system. Also, we give a finite time quenching result for the shadow system. Finally, some global existence results for the original reaction-diffusion system are given.

1. INTRODUCTION

In this work we are concerned with the following singular prey-predator reaction-diffusion system posed on a smooth and bounded domain $\Omega \subset \mathbb{R}^N$:

$$\begin{cases} B_t = d_b \Delta B + r_b B\left(1 - \dfrac{B}{K}\right) - \mu C, & x \in \Omega,\ t > 0, \\ C_t = d_c \Delta C + r_c C\left(1 - \mu \dfrac{C}{B}\right), & x \in \Omega,\ t > 0, \\ \dfrac{\partial B}{\partial \nu} = \dfrac{\partial C}{\partial \nu}(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ B(\cdot,0) = B_0 > 0, \quad C(\cdot,0) = C_0 \geq 0, & x \in \overline{\Omega}, \end{cases} \tag{1.1}$$

where $d_b, d_c, r_b, r_c, K, \mu$ are positive constants, while ν denotes the outward normal vector to the boundary $\partial\Omega$, and B_0, C_0 are continuous functions on $\overline{\Omega}$. The kinetic system associated to the above problem was proposed by Courchamp and Sugihara in [2] to study the prey-predator interactions within an isolated island. In that case B (resp. C) denotes the population density of birds (resp. cats). We also refer to Courchamp et al [1] for further applications of such a kinetic system of equations.

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The reaction-diffusion system (1.1) was considered by Gaucel in [6] and Gaucel and Langlais in [7].

Note that in (1.1) the prey population (birds) obeys a logistic growth with intrinsic growth rate r_b and carrying capacity K. The functional response of predation is given by the constant μ. The predator (cats) also follows a logistic dynamics with intrinsic growth rate r_c, but with a varying carrying capacity proportional to the density of prey, namely, B/μ. The parameters d_b, d_c denote the diffusivities of prey and predator, respectively.

The dynamical properties of the associated kinetic system

$$\begin{cases} B_t = r_b(1 - B/K)B - \mu C, \quad t > 0, \\ C_t = r_c(1 - \mu C/B)C, \quad t > 0, \end{cases} \tag{1.2}$$

has been studied completely in [7]. By introducing the function $P := C/B$, system (1.2) is reduced to the following ODE system:

$$\begin{cases} B_t = [r_b(1 - B/K) - \mu P]B, \quad t > 0, \\ P_t = [r_c - r_b + r_b B/K - \mu(r_c - 1)P]P, \quad t > 0. \end{cases} \tag{1.3}$$

From (1.3), the analysis of the dynamical properties of (1.2) in [7] can be carried out. Five regions in parameter space (r_b, r_c) are identified corresponding to different behavior. In particular, when $r_c > 1$ and $r_b > 1$, it is shown in [7] that the unique positive constant equilibrium of (1.2), namely, (B^*, C^*), is globally asymptotically stable for system (1.2) with initial data $(B_0, C_0) \in (0, \infty) \times (0, \infty)$. Herein we have set

$$B^* := K\left(1 - \frac{1}{r_b}\right) \text{ and } C^* = \frac{B^*}{\mu}. \tag{1.4}$$

This is done by showing that the corresponding positive equilibrium (B^*, P^*) with $P^* := C^*/B^* = 1/\mu$, is globally asymptotically stable for system (1.3).

On the other hand, when $r_c > 1$ and $r_b < 1$, $(0, P^{**})$ is globally asymptotically stable for system (1.3) with initial data $(B_0, P_0) \in (0, \infty) \times [0, \infty)$. Here we have set

$$P^{**} := \frac{r_c - r_b}{\mu(r_c - 1)}. \tag{1.5}$$

The above statement is equivalent to that solutions of system (1.2) converge to $(0, 0)$ as $t \to \infty$. Moreover when $r_c < 1$ the situation is much more complex. Finite time extinction of the populations as well as both global existence and finite time extinction of solutions may occur (depending on the initial data).

As far as the diffusion system is concerned, some finite time quenching as well as global existence results of solutions to system (1.1) with $d_b = d_c$ have been obtained by Gaucel and Langlais in [7]. Here by *quenching* we mean

$$\liminf_{t\uparrow T^-}\{\min_{\bar\Omega} B(\cdot,t)\} = 0 \text{ for some } T < \infty.$$

Indeed, when $d_b = d_c = d$ in (1.1), the equation for $P = C/B$ can be written as

$$P_t = d\Delta P + 2\frac{d}{B}\nabla B\cdot\nabla P + \left[r_c - r_b + r_b\frac{B}{K} - \mu(r_c - 1)P\right]P.$$

Therefore, the comparison principle can be employed to guarantee the quenching (i.e., blow-up for P) and global existence of solutions to the original system (1.1) with equal diffusion coefficients. Here, as usual, by *blow-up* it means

$$\limsup_{t\uparrow T^-}\{\max_{\bar\Omega} P(\cdot,t)\} = \infty \text{ for some } T < \infty.$$

In the equi-diffusional situation, Gaucel and Langlais in [7] make use of the above equation for $P = C/B$ and provide first results on the existence and non existence of global solutions of (1.1) depending upon the parameters r_b, r_c and the initial data. In particularly, they show that all solutions are globally defined as long as one of the following conditions is satisfied:

$$(0 < r_b < 1 \text{ and } r_c > 1) \text{ or } (r_b > 1 \text{ and } r_c > 1)\,.$$

Moreover, under certain conditions on the initial data, some solutions are globally defined when $r_c < 1$ and $r_b \geq 2 - r_c$. On the other hand, all solutions quench in finite time when $0 < r_b < r_c < 1$; and some solutions (for a class of initial data) quench in finite time when $r_c < 1$ and $r_b > r_c$.

Such an idea has also been used by Ducrot and Guo in [3] and by Ducrot and Langlais in [4, 5]. In [3], the authors study some properties of the quenching set and of the quenching rate for a simplified approximation of system (1.1) close to quenching and with identical diffusion rates on both species. More precisely, for the following simplified system of equations:

$$\begin{cases} B_t = B_{xx} - C, & x \in \mathbb{R},\, t > 0, \\ C_t = C_{xx} - r_c\dfrac{C^2}{B}, & x \in \mathbb{R},\, t > 0, \end{cases}$$

it is proved in [3] that the solutions are globally defined when $r_c \geq 1$, while finite time quenching always occurs when $0 < r_c < 1$. In the latter case the quenching set can be reduced to a single point. We refer to [3] for more details and also for some results on the quenching rate.

In [4], the authors study the spatial spread of the predator invasion for system (1.1) and derive refined results in the case where the vital dynamics of the prey is omitted ($r_b = 0$). Finally, in [5], the authors study the existence of a smooth continuation of the solutions of (1.1) after quenching in the equi-diffusional case $d_b = d_c$. Unfortunately, this idea cannot be applied to the more general situation where $d_b \neq d_c$. Indeed, in the non-equi-diffusional case, the P-equation exhibits a much more complex structure including an additional cross diffusion term (see (1.10)). This prevents us from the application of suitable comparison arguments.

In dealing with the case $d_b \neq d_c$, we consider the corresponding shadow system to the reaction-diffusion system (1.1). In fact, shadow systems are often used to approximate reaction-diffusion systems when one of the diffusion rates is large (see for instance [8]). The main purposes of this paper are to consider the corresponding shadow system to (1.1) and to give some information on the long time dynamics of solutions to (1.1) in the particular case where $d_b > d_c$.

Note that, in order to introduce the shadow system associated to (1.1), when the diffusion rate d_b becomes large, one expects that $B(x,t) \approx \xi(t)$, a spatially homogeneous function of the time t. Formally plugging $B(x,t) \equiv \xi(t)$ into (1.1) and integrating the first equation in (1.1) over Ω yields the following shadow system of (1.1) for (ξ, C):

$$
\begin{cases}
\xi_t = r_b \left(1 - \dfrac{\xi}{K}\right)\xi - \dfrac{\mu}{|\Omega|}\displaystyle\int_\Omega C\,dx, \quad t>0,\\
C_t = d_c \Delta C + r_c\left(1 - \mu\dfrac{C}{\xi}\right)C, \quad x\in\Omega,\ t>0,\\
\dfrac{\partial C}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t>0,\\
\xi(0) = \xi_0 := B_0 > 0, \quad C(\cdot,0) = C_0 \geq 0, \quad x \in \overline{\Omega}.
\end{cases}
\tag{1.6}
$$

It is easy to check that the corresponding non-singular locally Lipschitz continuous shadow system for (ξ, P) is given by

$$
\begin{cases}
\xi_t = \left\{r_b\left(1-\dfrac{\xi}{K}\right) - \dfrac{\mu}{|\Omega|}\displaystyle\int_\Omega P\,dx\right\}\xi, \quad t>0,\\
P_t = d_c\Delta P + \left[r_c - r_b + r_b\dfrac{\xi}{K} - \mu\Big(r_c P - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega P\,dx\Big)\right]P,\ x\in\Omega,\ t>0,\\
\dfrac{\partial P}{\partial \nu} = 0, \quad x\in\partial\Omega,\ t>0,\\
\xi(0) = \xi_0 = B_0 > 0, \quad P(\cdot,0) = P_0 := C_0/B_0 \geq 0, \quad x\in\overline{\Omega}.
\end{cases}
\tag{1.7}
$$

We shall now study the existence of globally defined solutions for the above shadow system as well as the asymptotic behavior of these solutions before coming to some global existence for the (one-dimensional) original system (1.1).

Let us first observe that $P(\cdot,t) \equiv 0$ for all time t when $P_0 \equiv 0$, so that the solution is globally defined and it satisfies, since $\xi_0 > 0$, the following properties:

$$\lim_{t\to\infty} \xi(t) = K \text{ and } P(\cdot,t) \equiv 0,\ \forall t \geq 0.$$

If $P_0 \geq 0$ and $P_0 \not\equiv 0$ then, by applying the strong maximum principle, the maximal solution of problem (1.7) satisfies $P(x,t) > 0$ for all $x \in \overline{\Omega}$ and for all $0 < t < T_{max}$, where $T_{max} > 0$ denotes the maximal existence time for the solution. Because of this property and up to a shift of the time with any small positive constant, we shall always assume in the sequel that $P_0(x) > 0$ for all $x \in \overline{\Omega}$.

Now our first theorem establishes the global existence of solutions to (1.7).

Theorem 1.1 (Global existence)**.** *Let $r_c \geq 1$. Then every solution of the shadow system* (1.7) *exists globally in time.*

Note that global solutions of (1.7) mentioned in the above theorem may be unbounded as time becomes large. However, when $r_c > 1$, the solutions are globally bounded in time and we have a uniform a priori estimate of the global solutions as follows.

Theorem 1.2 (Uniform bound)**.** *Suppose that $r_c > 1$. Let (ξ, P) be a solution of* (1.7) *with initial data (ξ_0, P_0) lying in a bounded subset $\mathcal{B}$ of $(0,\infty) \times L^\infty(\Omega)$. Then there exists a positive constant M independent of $(\xi_0, P_0) \in \mathcal{B}$ (depending only on the bound of $\mathcal{B}$) such that*

$$\|P(\cdot,t)\|_{L^\infty(\Omega)} \leq M,\ \forall\, t \geq 0.$$

Next, we consider the asymptotic behavior of global solutions to the shadow system (1.7) when $r_c > 1$. It is easy to see that, when $r_b > 1$, (ξ^*, P^*) is the unique positive constant equilibrium of (1.7), where $\xi^* := B^*$.

Depending on r_b, we have the following two different asymptotic behaviors for the shadow system (1.7).

Theorem 1.3 (Asymptotic behaviors of global solutions when $r_c > 1$)**.** *Suppose that $r_c > 1$. Let (ξ, P) be a solution of* (1.7)*. Then the following behaviors hold true:*

(i) *When* $r_b > 1$, *then, as* $t \to \infty$, *one has* $\xi(t) \to \xi^*$ *and* $P(\cdot,t) \to P^*$ *in* $L^\infty(\Omega)$.
(ii) *Suppose that* $r_b \le 1$. *Then, as* $t \to \infty$, $\xi(t) \to 0$ *and* $P(\cdot,t) \to P^{**}$ *in* $L^\infty(\Omega)$.

Now, we consider the limit case $r_c = 1$. As mentioned above, in this case, there is a possibility of infinite time blow-up. Our precise result is stated as follows.

Theorem 1.4 (Asymptotic behaviors of global solutions when $r_c = 1$)**.** *Suppose that* $r_c = 1$. *Let* (ξ, P) *be a solution of* (1.7). *Then the following two statements hold.*

(i) *If* $r_b > 1$, *then* $\xi(t) \to \xi^*$ *and* $P(\cdot,t) \to P^*$ *in* $L^\infty(\Omega)$ *as* $t \to \infty$.
(ii) *If* $r_b < 1$, *then* $\xi(t) \to 0$ *and* $\|P(\cdot,t)\|_{L^\infty(\Omega)} \to \infty$ *as* $t \to \infty$.

For $r_c < 1$, we obtain the following blow-up result for the solutions of the shadow system (1.7). Roughly speaking the solutions exhibit finite time blow-up behavior in the case where $r_b \le r_c$ for any initial data and where $r_b > r_c$ with a sufficiently large initial data P_0. Recalling that $\xi = C/P$, this blow-up result also provides a finite time quenching behavior for the component ξ of the shadow system (1.6).

Theorem 1.5 (Finite time blow-up)**.** *Suppose that* $r_c < 1$. *Then, for each initial data, solutions to system* (1.7) *blow up in finite time, if* $r_b \le r_c$. *Moreover, if* $r_b > r_c$, *then solutions to system* (1.7) *blow up in finite time for any initial data* (ξ_0, P_0) *satisfying*

$$\frac{1}{|\Omega|}\int_\Omega \ln P_0 \, dx > \ln P^{**}. \tag{1.8}$$

Here recall that P^{**} *is defined in* (1.5).

We now come back to the original system of equations (1.1). Putting $P = C/B$ in system (1.1), the function (B, P) satisfies

$$B_t = d_b \Delta B + r_b(1 - B/K)B - \mu P B, \tag{1.9}$$

$$\begin{aligned} P_t = {} & d_c \Delta P + (d_c - d_b)\frac{P}{B}\Delta B + 2\frac{d_c}{B}\nabla B \cdot \nabla P \\ & + \Big[r_c - r_b + \frac{r_b}{K}B - \mu(r_c - 1)P\Big]P, \end{aligned} \tag{1.10}$$

for $x \in \Omega$, $t > 0$, together with

$$\frac{\partial B}{\partial \nu} = \frac{\partial P}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t > 0, \tag{1.11}$$

$$B(\cdot,0) = B_0 > 0, \quad P(\cdot,0) = P_0 > 0, \quad x \in \overline{\Omega}. \tag{1.12}$$

Our next results are concerned with the global existence of solutions for the above system of equations. We shall focus on the one-dimensional case by assuming without loose of generality that Ω is a unit interval. This one-dimensional assumption will allow us to control the cross-diffusion terms appearing in the P-equation. In that context, we obtain the following theorems on the global existence of solutions for the original system.

Theorem 1.6 (Global existence and behavior of the solutions)**.** *Let $N = 1$ and $\Omega = (0, 1)$. Suppose that $r_c \geq 1$, $r_b > 1$, $2\pi^2 d_b + r_b \geq 2$, and $d_b \geq d_c$. Then every solution to system* (1.9)-(1.12) *exists and is bounded globally in time. Moreover, as $t \to \infty$, $B(\cdot, t) \to B^*$ and $P(\cdot, t) \to P^*$ in $L^\infty(\Omega)$.*

Here recall that B^* is defined in (1.4). One may also notice that B^* is positive only when the condition $r_b > 1$ is satisfied. Let us mention that we are not sure whether the condition $2\pi^2 d_b + r_b \geq 2$ arising in Theorem 1.6 is optimal or not. In fact, the global existence of the solutions can be proved to hold true for any $r_b > 0$ as in the following theorem.

Theorem 1.7 (Global existence)**.** *Suppose that $r_c \geq 1$, $d_b \geq d_c$ and $N = 1$. Then every solution to system* (1.9)-(1.12) *exists globally in time. Moreover, P is bounded in $L^1(\Omega)$ globally in time as long as the condition $r_c > 1$ holds true.*

In the equi-diffusional setting $d_b = d_c$, the above theorem extends the global existence results derived by Gaucel and Langlais in [7] to the case $r_c = 1$ and also to the case where $r_b = 1$ and $r_c > 1$. As for $r_c < 1$, we conjecture that quenching may occur for the general non-equi-diffusional case. We leave it as an open problem. However, some indications in this direction are given in Theorem 1.5 for the shadow system, that roughly speaking corresponds to a non-equi-diffusional case with $d_b >> 1$.

The rest of this paper is organized as follows. In §2, we study the global existence for the shadow system (1.7) when $r_c \geq 1$ and provide a proof of Theorem 1.1. Also, Theorem 1.2 on an a priori estimate for the global solutions is proved. Then, in §3, we study the asymptotic behaviors of global solutions to the shadow system (1.7) when $r_c > 1$. By constructing some suitable Lyapunov functionals, we give the proof of Theorem 1.3. In §4, applying Kaplan's convexity argument ([9]), we prove a blow-up result (Theorem 1.5) for the shadow system (1.7) when $r_c < 1$. Next, we study the critical case $r_c = 1$ and prove Theorem 1.4 in §5 for the shadow system (1.7). Finally, in §6, we consider the original system (1.9)-(1.12) and prove Theorems 1.6-1.7.

2. Global existence for the shadow system (1.7) when $r_c \geq 1$

In this section, we shall study the shadow system (1.7) when $r_c \geq 1$. We shall give the proofs of Theorems 1.1 and 1.2. Let us first prove Theorem 1.1.

Proof of Theorem 1.1. Let (ξ, P) be a solution of (1.7). Consider $w = P^{q/2}$ for $q \geq 1$ and compute

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega w^2\,dx = \frac{d}{dt}\int_\Omega P^q\,dx \\
= \;& q\int_\Omega P^{q-1}\Big\{d_c\Delta P + \Big[r_c - r_b + r_b\frac{\xi}{K} - \mu\Big(r_cP - \frac{1}{|\Omega|}\int_\Omega P\,dx\Big)\Big]P\Big\}\,dx \\
= \;& -\frac{4d_c(q-1)}{q}\int_\Omega |\nabla w|^2\,dx + q\int_\Omega w^2\Big(r_c - r_b + r_b\frac{\xi}{K}\Big)\,dx - q\mu I,
\end{aligned}$$

where

$$I := r_c\int_\Omega P^{q+1}\,dx - \frac{1}{|\Omega|}\int_\Omega P^q\,dx\int_\Omega P\,dx.$$

By Jensen's inequality, one has

$$\int_\Omega P^{q+1}\,dx \geq r_c|\Omega|^{-1/q}\Big(\int_\Omega P^q\,dx\Big)^{1+1/q}.$$

Furthermore, by Hölder's inequality,

$$\Big(\frac{1}{|\Omega|}\int_\Omega P^q\,dx\Big)^{1/q} \geq \frac{1}{|\Omega|}\int_\Omega P\,dx.$$

Hence

$$I \geq (r_c - 1)\frac{1}{|\Omega|}\int_\Omega P\,dx\int_\Omega P^q\,dx \geq 0,$$

since $r_c \geq 1$.

On the other hand, from the equation for ξ in (1.7) we have

$$\xi' \leq r_b\xi(1 - \xi/K),$$

since $\xi, P \geq 0$. Hence the following estimate holds

$$0 \leq \xi(t) \leq \max\{\xi(0), K\}. \tag{2.1}$$

Combining (2.1) with the fact that $I \geq 0$, we conclude that

$$\frac{d}{dt}\int_\Omega w^2\,dx \leq M\int_\Omega w^2\,dx$$

for some positive constant M independent of P. It follows that

$$\int_\Omega P^q\,dx = \int_\Omega w^2\,dx \leq e^{Mt}, \quad 0 \leq t < T_{max},$$

for all $q \geq 1$, where T_{max} is the maximum existence time of the solution (ξ, P). Therefore, the standard L^q regularity theory for parabolic equations and the Sobolev embedding theorem can be applied to obtain that $T_{max} = \infty$. This proves the global existence of solution to (1.7). $\square$

Next, we prove the uniform boundedness of the global solution when $r_c > 1$.

Proof of Theorem 1.2. Suppose that

$$\mathcal{B} = \{(\xi_0, P_0) \in (0, \infty) \times L^\infty(\Omega) \mid 0 < \xi_0 \leq M_1,\ 0 \leq P_0(\cdot) \leq M_2\}$$

for some positive constants M_1, M_2. Integrating the equation for P in (1.7) and applying Jensen's inequality, we obtain the following differential inequality

$$\frac{d}{dt}\Big(\frac{1}{|\Omega|}\int_\Omega P\,dx\Big) \leq \Big(r_c - r_b + \frac{r_b}{K}\xi\Big)\Big(\frac{1}{|\Omega|}\int_\Omega P\,dx\Big) - \mu(r_c - 1)\Big(\frac{1}{|\Omega|}\int_\Omega P\,dx\Big)^2.$$

Combining this with (2.1), we have the following uniform bound

$$\frac{1}{|\Omega|}\int_\Omega P\,dx \leq M_3 := \max\Big\{M_2, \Big[(r_c - r_b) + r_b\frac{\max\{M_1, K\}}{K}\Big]/[\mu(r_c - 1)]\Big\},\ \forall\, t \geq 0.$$

Hence P satisfies the differential inequality

$$P_t \leq d_c\Delta P + \Big(r_c - r_b + \frac{r_b}{K}\max\{M_1, K\} + M_3\mu\Big)P - \mu r_c P^2.$$

It follows that P is bounded uniformly for $t \in [0, \infty)$. Hence the theorem is proved. $\square$

We remark that P may grows exponentially as $t \to \infty$, if $r_c = 1$.

3. Asymptotic behaviors for the shadow system (1.7) when $r_c > 1$

In this section, we discuss the time asymptotic behaviors of solutions to (1.7) when $r_c > 1$ and we prove Theorem 1.3. We prove the first part (i) that corresponds to the case $r_b > 1$ and then we turn to the proof of the second point (ii) that is concerned with the case $r_b \leq 1$. Throughout this section we assume that $r_c > 1$.

Proof of Theorem 1.3 (i). First, we introduce the following Lyapunov functional for a solution (ξ, P) of (1.7)

$$E[\xi, P](t) := \int_\Omega \Big\{\frac{r_b}{K}\Big(\xi - \xi^* - \xi^*\ln\frac{\xi}{\xi^*}\Big) + \mu\Big(P - P^* - P^*\ln\frac{P}{P^*}\Big)\Big\}\,dx.$$

We calculate

$$\begin{aligned}\frac{dE[\xi,P](t)}{dt} &= \int_\Omega \left\{ \frac{r_b}{K}\frac{\xi-\xi^*}{\xi}\xi_t + \mu\frac{P-P^*}{P}P_t \right\} dx \\ &= \int_\Omega \frac{r_b}{K}(\xi-\xi^*)\Big\{ r_b(1-\xi/K) - \frac{\mu}{|\Omega|}\int_\Omega P\,dx \Big\} dx \\ &\quad +\mu\int_\Omega (P-P^*)\Big\{ d_c\frac{\Delta P}{P} + \Big[r_c - r_b + \frac{r_b}{K}\xi - \mu\Big(r_c P - \frac{1}{|\Omega|}\int_\Omega P\,dx \Big)\Big]\Big\} dx.\end{aligned}$$

Integrating by parts and using

$$r_b\Big(1-\frac{\xi^*}{K}\Big) - \mu P^* = 0, \quad (r_c - r_b) + \frac{r_b}{K}\xi^* - \mu(r_c-1)P^* = 0,$$

we obtain

$$\begin{aligned}&\frac{dE[\xi,P](t)}{dt} \\ =\ & \int_\Omega \frac{r_b}{K}(\xi-\xi^*)\Big\{ -\frac{r_b}{K}(\xi-\xi^*) - \frac{\mu}{|\Omega|}\int_\Omega (P-P^*)\,dx \Big\} dx - d_c\mu P^*\int_\Omega \frac{|\nabla P|^2}{P^2}\,dx \\ &\quad +\mu\int_\Omega (P-P^*)\Big\{ \frac{r_b}{K}(\xi-\xi^*) - \mu\Big[r_c(P-P^*) - \frac{1}{|\Omega|}\int_\Omega (P-P^*)\,dx \Big]\Big\} dx \\ =\ & -d_c\mu P^*\int_\Omega \frac{|\nabla P|^2}{P^2}\,dx - \frac{r_b^2}{K^2}|\Omega|(\xi-\xi^*)^2 \\ &\quad -\mu^2 r_c\int_\Omega (P-P^*)^2\,dx + \frac{\mu^2}{|\Omega|}\Big(\int_\Omega (P-P^*)\,dx\Big)^2\end{aligned}$$

It then follows from Jensen's inequality that

$$\frac{dE[\xi,P](t)}{dt} \le -d_c\mu P^*\int_\Omega \frac{|\nabla P|^2}{P^2}\,dx - \frac{r_b^2}{K^2}|\Omega|(\xi-\xi^*)^2 - \mu^2(r_c-1)\int_\Omega (P-P^*)^2\,dx.$$

Notice that $dE[\xi,P](t)/dt = 0$ only if $\xi(t)=\xi^*$ and $P(\cdot,t)\equiv P^*$.

To show the convergence of (ξ,P), let $\{t_n\}$ be a sequence such that $t_n\to\infty$ as $n\to\infty$. Consider the sequence $\{(\xi_n,P_n)\}$ defined by

$$\xi_n(t) := \xi(t+t_n), \quad P_n(x,t) := P(x,t+t_n).$$

Recall (2.1) and Theorem 1.2. Applying the standard parabolic regularity theory, the sequence $\{(\xi_n,P_n)\}$ is uniformly bounded in $C^1([0,\infty))\times C^{2+\alpha,1+\alpha/2}(\bar\Omega\times[0,\infty))$ for any $\alpha\in(0,1)$. Hence, up to the extraction of subsequence, we may assume without loss of generality that $\xi_n\to\xi_\infty$ and $P_n\to P_\infty$ as $n\to\infty$ uniformly for some function (ξ_∞,P_∞) satisfying (1.7) for $t\in\mathbb{R}$. Moreover, we have

$$\lim_{n\to\infty} E[\xi,P](t+t_n) = \lim_{n\to\infty} E[\xi_n,P_n](t) = E[\xi_\infty,P_\infty](t) \quad \text{for each } t\in\mathbb{R}.$$

Therefore, we obtain

$$\frac{dE[\xi_\infty,P_\infty](t)}{dt} = 0 \quad \text{for each } t\in\mathbb{R}$$

and this implies that $\xi_\infty \equiv \xi^*$ and $P_\infty \equiv P^*$. Since the limit is independent of the choice of the sequence $\{t_n\}$, we conclude the theorem. $\square$

Next, we prove Theorem 1.3 (ii).

Proof of Theorem 1.3 (ii). When $r_b \le 1$, there is the following Lyapunov functional

$$\bar{E}[\xi, P] = \int_\Omega \left\{ \frac{r_b}{K}\xi + \mu\Big(P - P^{**} - P^{**}\ln\frac{P}{P^{**}}\Big) \right\} dx.$$

Indeed, similar to the proof of Theorem 1.3, we have

$$\begin{aligned} \frac{d\bar{E}[\xi,P](t)}{dt} \le & -d_c\mu P^{**}\int_\Omega \frac{|\nabla P|^2}{P^2}\,dx - \frac{r_b}{K}|\Omega|\Big[-\frac{r_b}{K}\xi - \frac{r_c(1-r_b)}{r_c-1}\Big]\xi \\ & -\mu^2(r_c-1)\int_\Omega (P-P^{**})^2\,dx. \end{aligned}$$

Then the theorem can be proved by a similar argument to that of Theorem 1.3. We omit the details here. $\square$

4. Finite time blow-up for the shadow system (1.7) when $r_c < 1$

This section is devoted to derive a blow-up result for the shadow system (1.7) when $r_c < 1$.

Proof of Theorem 1.5. Assume for contradiction that the solution (ξ, P) of (1.7) exists globally in time. Then the integral

$$g(t) := \frac{1}{|\Omega|}\int_\Omega \ln P\,dx$$

exists globally in time.

Using (1.7), we compute

$$\begin{aligned} |\Omega| g'(t) &= d_c\int_\Omega \frac{\Delta P}{P}\,dx + \int_\Omega \Big\{ r_c - r_b + \frac{r_b}{K}\xi - \mu\Big(r_cP - \frac{1}{|\Omega|}\int_\Omega P\,dx\Big)\Big\}\,dx \\ &= d_c\int \Big|\frac{\nabla P}{P}\Big|^2 dx + (r_c - r_b)|\Omega| + \frac{r_b}{K}\xi|\Omega| + \mu(1-r_c)\int_\Omega P\,dx \\ &\ge (r_c-r_b)|\Omega| + \mu(1-r_c)\int_\Omega P\,dx, \end{aligned}$$

since $\xi \ge 0$ for all $t \ge 0$. Hence

$$g'(t) \ge (r_c - r_b) + \mu(1-r_c)\int_\Omega P\,\frac{dx}{|\Omega|} = (r_c - r_b) + \mu(1-r_c)\int_\Omega \exp(\ln P)\,\frac{dx}{|\Omega|}.$$

Due to $r_c < 1$, it follows from Jensen's inequality that

$$g'(t) \geq (r_c - r_b) + \mu(1 - r_c)\exp\{g(t)\} \quad \text{for all } t \geq 0. \tag{4.1}$$

When $r_b \leq r_c < 1$, (4.1) implies that

$$e^{-g(t)}g'(t) \geq \mu(1 - r_c) \quad \text{for all } t \geq 0,$$

and an integration of the above inequality from 0 to $T > 0$ gives

$$e^{-g(0)} \geq \int_0^T e^{-g(t)}g'(t) \geq \mu(1 - r_c)T \to \infty \ \text{ as } T \to \infty,$$

a contradiction. Hence solutions to system (1.7) blow up in finite time for any initial data (ξ_0, P_0).

On the other hand, when $r_b > r_c$ and $r_c < 1$, suppose that $g(0) > \ln P^{**}$. Then, by (4.1), $g'(0) > 0$ and $g(t) > \ln P^{**}$ for all $t \geq 0$. Hence there is a positive constant δ such that $g'(t) \geq \delta e^{g(t)}$ for all $t \geq 0$. The same argument as above leads to a contradiction. Therefore, solutions to (1.7) with initial data (ξ_0, P_0) satisfying $g(0) > \ln P^{**}$ blow up in finite time. The theorem is proved. □

5. The critical case $r_c = 1$

In this section, we consider the critical case $r_c = 1$. The global existence of solutions to the shadow system (1.7) is guaranteed in Theorem 1.1.

Proof of Theorem 1.4. Let (ξ, P) be a solution of (1.7). First, we prove (i).

Note that the Lyapunov functional $E[\xi, P](t)$ defined in the proof of Theorem 1.3 is well-defined even if $r_b > 1 = r_c$. In fact, we have

$$\frac{dE[\xi, P](t)}{dt} \leq -d_c\mu P^* \int_\Omega \frac{|\nabla P|^2}{P^2}\,dx - \frac{r_b^2}{K^2}|\Omega|(\xi - \xi^*)^2.$$

Hence $E[\xi, P](t) \leq E[\xi, P](0)$ for all $t \geq 0$. In particular, $-\xi^* \log \xi$ is uniformly bounded for all $t \geq 0$. Therefore, there exists a positive constant δ such that

$$\xi(t) \geq \delta \quad \text{for all } t \geq 0. \tag{5.1}$$

Substituting (2.1) into (1.6), we obtain the differential inequality

$$C_t \leq d_c\Delta C + \Big(1 - \frac{C}{\max\{\xi(0), K\}}\Big)C.$$

Hence we have a uniform bound

$$C(x,t) \leq \max\{\|C_0\|_{L^\infty(\Omega)}, \max\{\xi(0), K\}\}$$

for all $x \in \Omega$ and $t \geq 0$. Combining this uniform bound with (5.1) and using $P = C/\xi$, we conclude that P is globally bounded.

Now, following the proof of Theorem 1.3, the limit function (ξ_∞, P_∞) satisfies $\xi_\infty = \xi^*$ and $|\nabla P_\infty| \equiv 0$. Hence P_∞ is a function of t only and it may depend on the choice of the sequence $\{t_n\}$. However, the limit (ξ_∞, P_∞) satisfies (1.7). We must have $P_\infty \equiv P^*$. Therefore, the case (i) of the theorem is proved.

Next, we prove (ii). From the proof of Theorem 1.5, we have $g'(t) \geq (1 - r_b)$ for all $t \geq 0$. This implies that $g(t) \to \infty$ as $t \to \infty$. On the other hand, by Jensen's inequality, we have

$$g(t) = \frac{1}{|\Omega|} \int_\Omega \ln P \, dx \leq \ln\Big(\frac{1}{|\Omega|}\int_\Omega P\, dx\Big).$$

Thus we conclude that $\int_\Omega P\, dx \to \infty$ as $t \to \infty$. In particular, $\|P(\cdot, t)\|_{L^\infty(\Omega)} \to \infty$ as $t \to \infty$. ¿From the first equation of (1.7), it is clear that $\xi(t) \to 0$ as $t \to \infty$. This proves (ii) and the proof of the theorem is thereby completed. □

6. Global existence for the original system when $N = 1$

In this section, we always assume that $d_b \geq d_c$ and $N = 1$. For $N = 1$, system (1.9)-(1.10) can be re-written as

$$B_t = d_b B_{xx} + r_b(1 - B/K)B - \mu P B, \tag{6.1}$$

$$\begin{aligned} P_t &= d_c P_{xx} - (d_b - d_c)\frac{P}{B}B_{xx} + 2\frac{d_c}{B}B_x P_x \\ &\quad + \Big[r_c - r_b + \frac{r_b}{K}B - \mu(r_c - 1)P\Big]P, \end{aligned} \tag{6.2}$$

Without loss of generality, we may assume $\Omega = (0, 1)$.

Let (B, P) be a solution of (1.9)-(1.12). We introduce the functional

$$\begin{aligned} E[B,P](t) \; &:= \; d_b \int_0^1 \Big(\frac{B_x}{B}\Big)^2 dx + \frac{d_b - d_c}{2r_c}\int_0^1 \Big(\frac{C_x}{C}\Big)^2 dx \\ &\quad + \frac{r_b}{K}\int_0^1 \Big(B - B^* - B^* \ln \frac{B}{B^*}\Big)dx + \mu \int_0^1 \Big(P - P^* - P^* \ln\frac{P}{P^*}\Big)dx, \end{aligned}$$

where $r_b > 1$, $P^* = 1/\mu$ and $B^* = K(1 - 1/r_b)$ as before.

We first prove the following lemma.

Lemma 6.1. *Suppose that $N = 1$, $r_b > 1$, $2\pi^2 d_b + r_b \geq 2$, $r_c \geq 1$ and $d_b \geq d_c$. Let (B, P) be a solution of* (1.9)-(1.12). *Then the functional $E[B, P](t)$ is decreasing in t.*

Proof. By a simple calculation, using (6.1)-(6.2), we obtain

$$\frac{d}{dt}\left\{\frac{r_b}{K}\int_0^1 \Big(B - B^* - B^* \ln\frac{B}{B^*}\Big)\,dx + \mu\int_0^1 \Big(P - P^* - P^* \ln\frac{P}{P^*}\Big)\,dx\right\} = I_1 + I_2 - I_3,$$

where

$$\begin{aligned}
I_1 &:= \frac{r_b d_b}{K}\int_0^1 B_{xx}\Big(1 - \frac{B^*}{B}\Big)\,dx + \mu d_c\int_0^1 P_{xx}\Big(1 - \frac{P^*}{P}\Big)\,dx,\\
I_2 &:= -\mu(d_b - d_c)\int_0^1 \frac{B_{xx}}{B}(P - P^*)\,dx + 2\mu d_c\int_0^1 \frac{B_x}{B}\,P_x\Big(1 - \frac{P^*}{P}\Big)\,dx,\\
I_3 &:= \int_0^1 \left\{\frac{r_b^2}{K^2}(B - B^*)^2 + \mu^2(r_c - 1)(P - P^*)^2\right\}dx.
\end{aligned}$$

Using $P^* = 1/\mu$, we may write $I_2 = J_1 + J_2$, where

$$\begin{aligned}
J_1 &:= -\mu(d_b - d_c)\int_0^1 \frac{B_{xx}}{B}\,P\,dx + 2\mu d_c\int_0^1 \frac{B_x}{B}\,P_x\,dx,\\
J_2 &:= (d_b - d_c)\int_0^1 \frac{B_{xx}}{B}\,dx - 2d_c\int_0^1 \frac{B_x}{B}\frac{P_x}{P}\,dx.
\end{aligned}$$

Integrating by parts, using $P^* = 1/\mu$ and $B^* = K(1 - 1/r_b)$, we compute

$$\begin{aligned}
I_1 + J_2 &= -[r_b d_b(1 - 1/r_b) + (d_c - d_b)]\int_0^1 \Big(\frac{B_x}{B}\Big)^2 dx\\
&\quad - d_c\int_0^1 \Big(\frac{P_x}{P}\Big)^2 dx - 2d_c\int_0^1 \frac{B_x}{B}\,\frac{P_x}{P}\,dx\\
&= -d_c\int_0^1 \Big(\frac{B_x}{B} + \frac{P_x}{P}\Big)^2 dx - (r_b - 2)\,d_b\int_0^1 \Big(\frac{B_x}{B}\Big)^2 dx.
\end{aligned}$$

Also,

$$J_1 = -\mu(d_b - d_c)\int_0^1 \Big(\frac{B_x}{B}\Big)^2 P\,dx + \mu(d_b + d_c)\int_0^1 \frac{B_x}{B}P_x\,dx.$$

On the other hand, since the function $u := \ln B$ satisfies

$$u_t = d_b(u_{xx} + u_x^2) + r_b\Big(1 - \frac{e^u}{K}\Big) - \mu P, \qquad u_x(0, t) = u_x(1, t) = 0, \tag{6.3}$$

we compute, applying integration by parts,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 \Big(\frac{B_x}{B}\Big)^2\,dx = \frac{1}{2}\frac{d}{dt}\int_0^1 \big((\ln B)_x\big)^2\,dx \\
= &\int_0^1 u_x (u_x)_t\,dx = -\int_0^1 u_{xx}u_t\,dx \\
= &-d_b\int_0^1 (u_{xx})^2\,dx - d_b\int_0^1 (u_x)^2\,u_{xx}\,dx - \int_0^1 u_{xx}\Big(r_b(1-\frac{e^u}{K}) - \mu P\Big)\,dx \\
= &-d_b\int_0^1 (u_{xx})^2\,dx - \frac{r_b}{K}\int_0^1 e^u (u_x)^2\,dx - \mu\int_0^1 u_x P_x\,dx,
\end{aligned}
$$

where we have used the fact (only for $N=1$)

$$d_b\int_0^1 (u_x)^2\,u_{xx}\,dx = d_b\Big[\frac{u_x^3}{3}\Big]\Big|_0^1 = 0.$$

Thus, we conclude

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_0^1 \Big(\frac{B_x}{B}\Big)^2\,dx \;=\; &-d_b\int_0^1 \{(\ln B)_{xx}\}^2\,dx \\
&-\frac{r_b}{K}\int_0^1 B\Big(\frac{B_x}{B}\Big)^2\,dx - \mu\int_0^1 \frac{B_x}{B}P_x\,dx.
\end{aligned} \tag{6.4}
$$

Next, let us introduce the function $v=\ln C$. Then v satisfies

$$v_t = d_c(v_{xx}+v_x^2) + r_c(1-\mu P).$$

A similar argument as before yields

$$\frac{1}{2}\frac{d}{dt}\int_0^1 (v_x)^2\,dx = -d_c\int_0^1 v_{xx}^2\,dx - r_c\mu\int_0^1 v_x P_x\,dx. \tag{6.5}$$

By substituting $P=e^{v-u}$ and $P_x=(v_x-u_x)P$ into (6.5), we get

$$\frac{1}{2}\frac{d}{dt}\int_0^1 (v_x)^2\,dx = -d_c\int_0^1 v_{xx}^2\,dx - r_c\mu\int_0^1 P(v_x)^2\,dx + r_c\mu\int_0^1 Pu_x\Big(\frac{P_x}{P}+u_x\Big)\,dx.$$

Hence we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_0^1 \Big(\frac{C_x}{C}\Big)^2\,dx \;=\; &-d_c\int_0^1 \{(\ln C)_{xx}\}^2\,dx - r_c\mu\int_0^1 P\Big(\frac{C_x}{C}\Big)^2\,dx \\
&+r_c\mu\int_0^1 P\Big(\frac{B_x}{B}\Big)^2\,dx + r_c\mu\int_0^1 \frac{B_x}{B}P_x\,dx.
\end{aligned} \tag{6.6}
$$

Combining (6.4) and (6.6), we conclude that

$$\begin{aligned}\frac{dE[B,P](t)}{dt} &= -d_c\int_0^1\Big(\frac{B_x}{B}+\frac{P_x}{P}\Big)^2dx-(r_b-2)\,d_b\int_0^1\Big(\frac{B_x}{B}\Big)^2dx\\&\quad -2d_b\Big\{d_b\int_0^1\{(\ln B)_{xx}\}^2dx+\frac{r_b}{K}\int_0^1 B\Big(\frac{B_x}{B}\Big)^2dx\Big\}\\&\quad -\frac{d_b-d_c}{r_c}\Big\{\int_0^1 d_c\{(\ln C)_{xx}\}^2dx+r_c\mu\int_0^1\Big(\frac{C_x}{C}\Big)^2P\,dx\Big\}\\&\quad -\int_0^1\Big\{\frac{r_b^2}{K^2}(B-B^*)^2+\mu^2(r_c-1)(P-P^*)^2\Big\}dx.\end{aligned}$$

Now we apply the Poincaré inequality to the functions $(\ln B)_x$ and $(\ln C)_x$ to conclude

$$\begin{aligned}\frac{d}{dt}E[B,P] &\le -(2\pi^2d_b+r_b-2)d_b\int_0^1\Big(\frac{B_x}{B}\Big)^2dx-\frac{\pi^2(d_b-d_c)d_c}{r_c}\int_0^1\Big(\frac{C_x}{C}\Big)^2dx\\&\quad -\int_0^1\Big\{\frac{r_b^2}{K^2}(B-B^*)^2+\mu^2(r_c-1)(P-P^*)^2\Big\}dx\le 0,\end{aligned}$$

if $r_b>1$, $2\pi^2d_b+r_b\ge 2$, $r_c\ge 1$ and $d_b\ge d_c$. This proves the lemma. □

Proof of Theorem 1.6. By Lemma 6.1, due to $r_b>1$, $2\pi^2d_b+r_b\ge 2$, $r_c\ge 1$ and $d_b\ge d_c$, $E[B,P](t)\le E[B,P](0)$ for all $t\ge 0$ as long as the solution exists. Hence we have

$$\begin{aligned}& d_b\int_0^1\Big(\frac{B_x}{B}\Big)^2(x,t)\,dx+\frac{d_b-d_c}{2r_c}\Big(\frac{C_x}{C}\Big)^2(x,t)\,dx\\&+\frac{r_b}{K}\int_0^1\Big(B-B^*-B^*\ln\frac{B}{B^*}\Big)(x,t)\,dx+\mu\int_0^1\Big(P-P^*-P^*\ln\frac{P}{P^*}\Big)(x,t)\,dx\\ \le\ & d_b\int_0^1\Big(\frac{\partial_xB_0}{B_0}\Big)^2dx+\frac{d_b-d_c}{2r_c}\Big(\frac{\partial_xC_0}{C_0}\Big)^2dx\\&+\frac{r_b}{K}\int_0^1\Big(B_0-B_0^*-B^*\ln\frac{B_0}{B^*}\Big)dx+\mu\int_0^1(P_0-P^*-P^*\ln\frac{P_0}{P^*})\,dx\end{aligned}$$

for all $t\ge 0$ as long as the solution exists. Therefore, by the convexity of the functions

$$B\mapsto B-B^*-B^*\ln\frac{B}{B^*},\quad\text{and}\quad P\mapsto P-P^*-P^*\ln\frac{P}{P^*},$$

we obtain the boundedness for

$$\|B\|_{L^1(0,1)}-B^*-B^*\ln(\|B\|_{L^1(0,1)}/B^*),\ \|P\|_{L^1(0,1)}-P^*-P^*\ln(\|P\|_{L^1(0,1)}/P^*)$$

for all $t\ge 0$ as long as the solution exists. In particular, there exists a positive constant M depending on B_0 and P_0 such that

$$\|P(\cdot,t)\|_{L^1(0,1)}\le M,\ \|B(\cdot,t)\|_{L^1(0,1)}\le M,\ \Big\|\frac{B_x(\cdot,t)}{B(\cdot,t)}\Big\|_{L^2(0,1)}\le M$$

for all $t \geq 0$ as long as the solution exists. Hence the equation (6.3) has the form $u_t = d_b u_{xx} + g(x,t)$ with $\|g(\cdot,t)\|_{L^1(0,1)} \leq M$ holds for all $t \geq 0$ as long as the solution exists.

In order to apply the standard bootstrap argument (as that in [10]), we first derive the L^2 boundedness for u as follows. To this end, we first note from the boundedness of $E[B,P](t)$ that

$$-\int_0^1 \ln B\, dx \leq M$$

for all $t \geq 0$ as long as the solution exists.

Also, by the maximum principle, we have the estimate

$$B(x,t) \leq B_\infty := \max\{\|B_0\|_\infty, K\}, \qquad x \in [0,1], \quad t \geq 0. \tag{6.7}$$

This yields the uniform boundedness of the mean value $\int_0^1 \ln B\, dx$ for all $t \geq 0$. Combining these with the Poincaré-Wirtinger inequality

$$\pi^2 \| \ln B - \int_0^1 \ln B\, dx\|_{L^2(0,1)} \leq \|(\ln B)_x\|_{L^2(0,1)},$$

we obtain the uniform boundedness of $\|u(\cdot,t)\|_{L^2(0,1)} = \|\ln B(\cdot,t)\|_{L^2(0,1)}$ for $t \geq 0$.

Since we are considering the problem for $N = 1$, the standard bootstrap argument gives us a uniform L^∞ bound for $u = \ln B$ in $[0,1] \times [0,\infty)$.

More precisely, applying the standard semi-group interpolation inequality with $q \geq 2$:

$$\|e^{t\Delta}\phi\|_{L^r(0,1)} \leq M(q,r) \max\{1, t^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{r})}\} \|\phi\|_{L^q(0,1)}, \quad 1 \leq q \leq r \leq \infty,$$

gives us $\|u(\cdot,t)\|_{L^r(0,1)} \leq M$ for some constant $M > 0$ as long as

$$\frac{1}{2}\Big(\frac{2}{q} - \frac{1}{r}\Big) < 1 \quad \text{and} \quad q \leq r \leq \infty.$$

Hence the L^∞ bound for $u = \ln B$ in $[0,1] \times [0,\infty)$ is obtained. This means that, in particular, there exists a constant $\varepsilon > 0$ such that $B \geq \varepsilon$ for all $x \in [0,1]$ and $t \geq 0$, which implies the global existence of bounded solution (B,P) to (1.9)-(1.12).

Finally, the same argument as the proof for Theorem 1.3, we can prove that $B(\cdot,t) \to B^*$ and $P(\cdot,t) \to P^*$ in $L^\infty(\Omega)$ as $t \to \infty$, by utilizing the above Lyapunov functional $E[B,P]$. This completes the proof of Theorem 1.6. □

The global existence of the solution, which may be unbounded as $t \to \infty$, can be proved without the assumption $2\pi^2 d_b + r_b \geq 2$.

Proof of Theorem 1.7. As is proved in Lemma 6.1, we also have (6.4) and (6.6). Combining these two identities with

$$\frac{d}{dt}\int_0^1 P\,dx = (d_c - d_b)\int_0^1 \Big(\frac{B_x}{B}\Big)^2 P\,dx + (d_b + d_c)\int_0^1 \frac{B_x}{B}P_x\,dx + \int_0^1 \Big[r_c - r_b + \frac{r_b}{K}B - \mu(r_c-1)P\Big]P\,dx,$$

we obtain the following equality

$$\begin{aligned} &\frac{d}{dt}\int_0^1 \Big\{\mu P + d_b\Big(\frac{B_x}{B}\Big)^2 + \frac{d_b - d_c}{2r_c}\Big(\frac{C_x}{C}\Big)^2\Big\}\,dx \\ = \;& -2d_b\Big\{d_b\int_0^1 \{(\log B)_{xx}\}^2\,dx + \frac{r_b}{K}\int_0^1 B\Big(\frac{B_x}{B}\Big)^2\,dx\Big\} \\ & -\frac{d_b - d_c}{r_c}\Big\{\int_0^1 d_c\{(\log C)_{xx}\}^2\,dx + r_c\mu\int_0^1 \Big(\frac{C_x}{C}\Big)^2 P\,dx\Big\} \\ & +\mu\int_0^1 \Big[r_c - r_b + \frac{r_b}{K}B - \mu(r_c-1)P\Big]P\,dx. \end{aligned}$$

Now we apply the Poincaré inequality to conclude

$$\begin{aligned} &\frac{d}{dt}\int_0^1 \Big\{\mu P + d_b\Big(\frac{B_x}{B}\Big)^2 + \frac{d_b - d_c}{2r_c}\Big(\frac{C_x}{C}\Big)^2\Big\}\,dx \\ \le\; & -2\pi^2 d_b^2\int_0^1 \Big(\frac{B_x}{B}\Big)^2\,dx - \frac{\pi^2(d_b - d_c)d_c}{r_c}\int_0^1 \Big(\frac{C_x}{C}\Big)^2\,dx \\ & +\mu\int_0^1 \Big[r_c - r_b + \frac{r_b}{K}B - \mu(r_c-1)P\Big]P\,dx. \end{aligned}$$

By Jensen's inequality, we have

$$\begin{aligned} &\frac{d}{dt}\int_0^1 \Big\{\mu P + d_b\Big(\frac{B_x}{B}\Big)^2 + \frac{d_b - d_c}{2r_c}\Big(\frac{C_x}{C}\Big)^2\Big\}\,dx \\ \le\; & -2\pi^2 d_b^2\int_0^1 \Big(\frac{B_x}{B}\Big)^2\,dx - \frac{\pi^2(d_b - d_c)d_c}{r_c}\int_0^1 \Big(\frac{C_x}{C}\Big)^2\,dx \\ & +\mu\Big[r_c - r_b + \frac{r_b}{K}B_\infty\Big]\int_0^1 P\,dx - \mu^2(r_c-1)\Big(\int_0^1 P\,dx\Big)^2, \end{aligned} \tag{6.8}$$

where B_∞ is defined in (6.7).

Set $\rho := r_c - r_b + \frac{r_b}{K}B_\infty$. If $\rho \le 0$, then it is clear that the solution exists globally. ¿From now on we assume that $\rho > 0$. Since $r_c \ge 1$ and $d_b \ge d_c$, it follows from (6.8) that

$$\begin{aligned} &\frac{d}{dt}\int_0^1 \Big\{\mu P + d_b\Big(\frac{B_x}{B}\Big)^2 + \frac{d_b - d_c}{2r_c}\Big(\frac{C_x}{C}\Big)^2\Big\}\,dx \\ \le\; & \rho\int_0^1 \Big\{\mu P + d_b\Big(\frac{B_x}{B}\Big)^2 + \frac{d_b - d_c}{2r_c}\Big(\frac{C_x}{C}\Big)^2\Big\}\,dx. \end{aligned}$$

This gives us the $L^1(0,1)$ boundedness for P and $(B_x/B)^2$ in any finite time interval. On the other hand, by integrating (6.3), we obtain

$$\frac{d}{dt}\Big\{-\int_0^1 \log B\,dx\Big\} = -d_b\int_0^1 \Big(\frac{B_x}{B}\Big)^2 dx + \int_0^1 r_b\Big(1-\frac{B}{K}\Big)\,dx - \mu\int_0^1 P\,dx,$$

which yields a bound for $\|\log B(\cdot,t)\|_{L^1(0,1)}$, since B is bounded from above. Note that these bounds depend on the time interval and it may tend to infinity as the interval becomes unbounded. Repeating the same argument as the proof of Theorem 1.6 in each finite time interval, it follows that the solution (B,P) of (1.9)-(1.12) exists in any bounded time interval and so the global existence is proved for any $r_c \geq 1$.

Now, we suppose that $r_c > 1$. Then from the identity

$$-(r_c-1)\lambda^2 + (\rho+\rho^2)\lambda = -(r_c-1)\left[\lambda - \frac{\rho(1+\rho)}{2(r_c-1)}\right]^2 + \frac{\rho^2(1+\rho)^2}{4(r_c-1)},$$

it follows from (6.8) that

$$\begin{aligned} &\frac{d}{dt}\int_0^1 \Big\{\mu P + d_b\Big(\frac{B_x}{B}\Big)^2 + \frac{d_b-d_c}{2r_c}\Big(\frac{C_x}{C}\Big)^2\Big\}\,dx \\ \leq\ & \frac{\rho^2(1+\rho)^2}{4(r_c-1)} - \min\{2\pi^2 d_b, 2\pi^2 d_c, \rho^2\}\int_0^1 \Big\{\mu P + d_b\Big(\frac{B_x}{B}\Big)^2 + \frac{d_b-d_c}{2r_c}\Big(\frac{C_x}{C}\Big)^2\Big\}\,dx. \end{aligned}$$

Therefore, if $r_c > 1$ and $d_b \geq d_c$, we have the uniform boundedness of $L^1(0,1)$-norm for P and $(B_x/B)^2$ for all $t \geq 0$. Thus, we conclude the proof of Theorem 1.7. □

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