THE SPREADING SPEED AND THE MINIMAL WAVE SPEED OF A PREDATOR-PREY SYSTEM WITH NONLOCAL DISPERSAL

ARNAUD DUCROT, JONG-SHENQ GUO, GUO LIN, AND SHUXIA PAN

Abstract. This paper is concerned with the propagation dynamics of a predator-prey system with nonlocal dispersal. We obtain a threshold phenomenon for the invasion of the predator into the habitat of the aborigine prey. It turns out that this threshold is the so-called spreading speed of the predator as well as the minimal wave speed of traveling wave solutions connecting the predator-free state to a nontrivial state.

1. Introduction

In recent years, many nonlocal dispersal models have been derived from material science and other fields to model long distance effects and nonadjacent interactions, see, e.g., Hopf [14], Fife [12] and Bates [4] for the physical background. The classical diffusion originated from Fick's law of diffusion formulated by certain elliptic operators could be thought as a limit case of the nonlocal dispersal and more particularly when the dispersal kernel is highly concentrated. However, in comparing with the classical diffusion model, there are many significant differences from the nonlocal dispersal in mathematical theory. For examples, the nonlocal dispersal model often admits lower regularity than the classical diffusion one [2] and there are plentiful propagation dynamics of nonlocal models due to the nonlocal effect [1]. Nonlocal models also arise in biology and more specifically in population dynamics to describe long distance dispersal of individuals. We refer to Lutscher et al [25] and the references cited therein.

In this paper, we investigate a predator-prey system in which both the prey and the predator populations are subject to long distance dispersal. The model we consider is posed for the unknown functions $(U, V) = (U, V)(x, t)$ with $t > 0$ and $x \in \mathbb{R}$ and reads as follows

$$
\begin{cases} \dfrac{\partial U}{\partial t}(x,t) = d_1\mathcal{N}_1[U(\cdot,t)](x) + F\left(U(x,t),V(x,t)\right) \\ \dfrac{\partial V}{\partial t}(x,t) = d_2\mathcal{N}_2[V(\cdot,t)](x) + G\left(U(x,t),V(x,t)\right), \end{cases} \tag{1.1}
$$

in which the functions $F, G : \mathbb{R}^2 \to \mathbb{R}$ are defined by

$$
F(U,V) := r_1U(1-U) - aUV, \quad G(U,V) := bUV - r_2V\left(1+\mu V\right).
$$

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In system (1.1), $U = U(x,t)$ (resp. $V = V(x,t)$) describes the density of the prey (resp. the predator) at time $t > 0$ and the spatial position $x \in \mathbb{R}$. The functions F and G describe the prey-predator interactions between these two populations and r_1, r_2, a, b, μ are positive constants. Here the dynamics of the prey population follows a logistic growth with a normalized (to one) carrying capacity and r_1 denotes its intrinsic growth rates. The parameter r_2 denotes the – density dependent – death rate for the predator. The parameter μ describes the intensity of the intra-specific competition in the predator population. Finally, the coupling constants a and b denote the predation rate and the biomass conversion rate, respectively.

In (1.1), the terms $d_1\mathcal{N}_1$ and $d_2\mathcal{N}_2$ describe the spatial dispersal of the prey and the predator, respectively. Here $d_1 > 0$ and $d_2 > 0$ are the diffusion coefficients and, for $i = 1, 2$, $\mathcal{N}_i$ is the linear nonlocal diffusion operators defined by

$$\mathcal{N}_i[\varphi](x) := (J_i * \varphi)(x) - \varphi(x) = \int_{\mathbb{R}} J_i(x-y)\varphi(y)dy - \varphi(x),$$

wherein J_1 and J_2 are probability kernel functions satisfying certain conditions to be specified later while the symbol $*$ denotes the convolution product with respect to the spatial variable, $x \in \mathbb{R}$.

As far as the propagation theory of predator-prey systems is concerned, one typical problem is to study the spatial invasion process of the predator when introduced into the habitat of a prey, see Fagan and Bishop [10], Owen and Lewis [27]. In this paper, we characterize the features of the predator invasion process by the asymptotic spreading and traveling wave solutions.

Although the study of nonlocal dispersal models has some well-known difficulties, it is possible to apply some abstract theory from dynamical systems to derive spreading properties for nonlocal models. For example, in Fang and Zhao [11], an abstract theory for monotone semiflows is developed. We also refer the reader to the earlier works by Weinberger [31], Lui [24], Weinberger et al. [32] and Liang and Zhao [20], and the references cited therein. In the aforementioned works, the minimal wave speed of traveling waves as well as the estimation on the spreading speed are addressed.

However, our predator-prey system (1.1) is a nonmonotone system and so that the theory related to monotone semiflows cannot be applied. Propagation for nonmonotone systems, and in particular for predator-prey problems, has been scarcely studied. We refer the reader to [21, 28, 30] for studies of predator invasion with usual linear diffusion, to [9] for a study of prey and predator co-invasion and to [8] for an other type of reaction-diffusion system.

In this paper, we firstly investigate initial value problem for system (1.1) to study the spreading speed of the predator in §2. More precisely, we study system (1.1) supplemented with the initial condition

$$U(x,0) = 1, \quad V(x,0) = v_0(x), \quad x \in \mathbb{R}, \tag{1.2}$$

wherein v_0 is some nonnegative continuous function with nonempty compact support. From the viewpoint of population dynamics, system (1.1) with (1.2) describes a situation where the predator population is introduced in an environment where the prey population is the aborigine with a density at its carrying capacity. And, we are interested in the persistence and spatial invasion of the predator population by using the notion of spreading speed as introduced by Aronson and Weinberger in [3] for scalar reaction-diffusion equations.

Before going to our first main result about predator invasion, coming back to system (1.1), we shall assume throughout this work the following parameter conditions:

$$\mu = 1, \quad b > r_2, \quad a(b - r_2) < r_1 r_2. \tag{1.3}$$

Note that the condition $\mu = 1$ is not restrictive but can be assumed using a suitable change of functions (μV replaced by V). Moreover, we shall also assume that the probability kernel functions J_i, for $i = 1, 2$, are suitable thin-tail kernels. In order to precise this meaning, let us state the following definition.

Definition 1.1. *Let $\overline{\lambda} \in (0, \infty]$ be given. We say that the kernel function $J : \mathbb{R} \to \mathbb{R}$ belongs to the class $\mathcal{T}(\overline{\lambda})$ if it satisfies the following properties:*

(J1) *The kernel J is nonnegative and continuous in $\mathbb{R}$;*

(J2) *it holds that*

$$\int_{\mathbb{R}} J(y)dy = 1 \text{ and } J(y) = J(-y) \text{ for all } y \in \mathbb{R};$$

(J3) *it holds that $\int_{\mathbb{R}} J(y)e^{\lambda y}dy < \infty$ for any $\lambda \in (0, \bar{\lambda})$ and*

$$\int_{\mathbb{R}} J(y)e^{\lambda y}dy \to \infty \text{ as } \lambda \uparrow \bar{\lambda}.$$

Now our main assumption for the kernel functions J_i, for $i = 1, 2$, reads as follows:

$$\text{for } i = 1, 2, \text{ there exists } \overline{\lambda}_i \in (0, \infty] \text{ such that } J_i \in \mathcal{T}\left(\overline{\lambda}_i\right). \tag{1.4}$$

Due to the above assumption, for J_2, we define the quantity

$$c^* := \inf_{0<\lambda<\overline{\lambda}_2} \frac{d_2\left[\int_{\mathbb{R}} J_2(y)e^{\lambda y}dy - 1\right] + b - r_2}{\lambda}. \tag{1.5}$$

Note that due to properties (J1)-(J3) (for J_2 and $\overline{\lambda} = \overline{\lambda}_2$), it is easy to see that c^* is well-defined and $c^* > 0$ since $b > r_2$ (see (1.3)).

We now state our main result on the spreading speed of the predator as follows.

Theorem 1.2. (Predator's spreading) *Under the assumption* (1.3)*, the constant c^*, defined in* (1.5)*, corresponds to the spreading speed of the predator for system* (1.1) *with initial data* (1.2) *as long as v_0 is a non-zero compactly supported continuous function with $0 \le v_0 \le b - r_2$. This means that the density of the predator $V = V(x, t)$ satisfies*

$$\lim_{t\to\infty} \sup_{|x|>ct} V(x, t) = 0 \text{ for any } c > c^*;$$

$$\liminf_{t\to\infty} \inf_{|x|<ct} V(x, t) > 0 \text{ for any given } c \in (0, c^*).$$

Theorem 1.2 is proved by deriving some delicate a priori estimates that are combined with known results on scalar equations with nonlocal dispersal. Our arguments strongly relies on the regularity of the solutions of (1.1) and more particularly their uniform continuity properties recalled below. For mathematical results on scalar equations with nonlocal dispersal, we refer the reader to, for examples, [16, 17, 19] and the references cited therein. We also refer to Coville et al [6, 7] and the references therein for results on traveling wave solutions for scalar equations with nonlocal dispersal and monostable nonlinearities.

Secondly, we study the existence and non-existence of traveling wave solutions to (1.1). Here a solution (U, V) to (1.1) is called a traveling wave solution of (1.1), if there exist a constant $c \in \mathbb{R}$, *the wave speed*, and a function pair (Φ, Ψ), *the wave profile*, such that $(U, V)(x, t) = (\Phi, \Psi)(\xi)$, $\xi := x + ct$, for any $(x, t) \in \mathbb{R} \times \mathbb{R}$ and with $\Phi > 0$ and $\Psi > 0$. Here we are interested in the traveling waves connecting the predator-free state $(1, 0)$ at $\xi = -\infty$ to a nontrivial state at $\xi = \infty$ in the sense that

$$\liminf_{\xi \to \infty} \Phi(\xi) > 0, \quad \liminf_{\xi \to \infty} \Psi(\xi) > 0. \tag{1.6}$$

In fact, here a "nontrivial state" at $\xi = \infty$ defined in the sense of (1.6) only means that the right tail of the traveling wave stays away from zero. The exact description of this nontrivial state is a difficult question because of the lack of comparison principle for the system and also because of the nonlocal dispersion. This is left as an open question. However, from the numerical simulations, one may expect that these nontrivial states are given by the unique co-existence state of the kinetic part, namely,

$$(u^*, v^*) := \left(\frac{\bar{a} + 1}{\bar{a}\bar{b} + 1}, \frac{\bar{b} - 1}{\bar{a}\bar{b} + 1} \right), \quad \bar{a} := \frac{a}{r_1},\ \bar{b} := \frac{b}{r_2}. \tag{1.7}$$

From a rigorous point of view, once the wave profile is known to converge at $\xi = +\infty$, then it converges to the above co-existence state (see Proposition 3.7).

In §3, we shall show that the minimal wave speed of traveling wave solutions to (1.1) is the same as the spreading speed obtained in §2 in a slightly weaker sense. More precisely, we have

Theorem 1.3. *Suppose, in addition to* (1.3)*, that* $ab < r_1 r_2$*. Then the following holds true:*

(i) *For any speed* $c > c^* > 0$ *system* (1.1) *admits a traveling wave solution connecting* $(1, 0)$ *at* $\xi = -\infty$ *to a nontrivial state at* $\xi = \infty$*.*
(ii) *If we furthermore assume that* J_2 *is compactly supported then* (1.1) *admits a traveling wave solution for* $c = c^*$*.*

Moreover, under the assumption (1.3)*, there is no traveling wave solution to* (1.1) *connecting* $(1, 0)$ *to a nontrivial state, in the sense of* (1.6)*, with speed* $c \in (0, c^*)$*.*

The existence of traveling wave solutions is proved by constructing some suitable upper and lower solutions with the help of Schauder's fixed point theorem (cf. e.g., [26, 15, 18, 23, 22, 5]). Although this method is quite standard now, one needs to choose the parameters in the formulations of those upper-lower-solutions carefully. Due to some technical difficulties, we need to assume the compactness of the support of J_2 for the existence of traveling wave with the critical speed c^*. On the other hand, the nonexistence of traveling wave solutions is proved by applying the theory of asymptotic spreading for scalar equations.

With some extra conditions, the constant c^* is actually the minimal wave speed to the system (1.1) in the usual sense as follows.

Corollary 1.4. *In addition to* (1.3)*, assume that* $ab < r_1 r_2$ *and* $d_2 < b - r_2$*, and that* J_2 *has a compact support. Then,* (1.1) *admits traveling wave solution connecting* $(1, 0)$ *to a nontrivial state with speed* $c \in \mathbb{R}$ *if and only if* $c \geq c^*$*.*

In view of Theorem 1.3, the above corollary simply means that under the set of assumptions stated above, the wave speed of any traveling waves is positive.

We continue this section by providing some numerical experiments of problem (1.1). These simulations allow us to observe the shape of the propagating profile of the wave solutions. In our numerical simulations, we take the kernel functions as follows

$$J_1(x) = \frac{1}{2}e^{-|x|}$$
$$J_2(x) = \frac{4}{3}(1-x^2) \text{ if } |x| < 1, \quad = 0 \text{ otherwise.}$$

The diffusion coefficient are set to be $d_1 = 1$ and $d_2 = \frac{3}{4}$. We fix $\mu = 1$, $r_1 = 1$, $r_2 = 1$, $a = 1$ and we vary the last parameter b. The initial population of the prey is identically equal to one in the whole domain and a compactly supported density of the predator is introduced on the right hand side of the domain. The corresponding numerical simulations of the system are presented in Figures 1-1 and 1-2. The different figures present the shape of the solution of the evolution problem (1.1) at a given time.

Here one may observe that after some time the solution takes the form of a traveling front. According to our spreading result above (see Theorem 1.2), this suggests that the solutions of (1.1) with compactly supported initial data for the prey approach a traveling wave front associated to the minimal wave speed in the large times. Moreover the wave profiles obtained from numerical simulations seems to be monotonic for small values of the parameter b. It becomes rapidly non-monotone when b is increased. This latter point is in sharp contrast with Fisher-KPP scalar equation, for which the traveling wave profiles are monotonic. In addition, the profile of these waves seems to converge – possibly with damped oscillations– to the positive equilibrium after invasion whatever the parameter sets. In other words, the nontrivial states, describing the tail of the traveling waves at $\xi = +\infty$, obtained in Theorem 1.3 and Corollary 1.4 seems to be the unique positive coexistence state (u^*, v^*). The rigorous proof of such a property is left as an open question. When the compactly supported disturbance of the predator is initially located at the center of the frame, spatial invasion of the predator arises in both sides of the domain. This situation is depicted in Figure 2 with the same choices of the parameters as above and with $b = 7$.

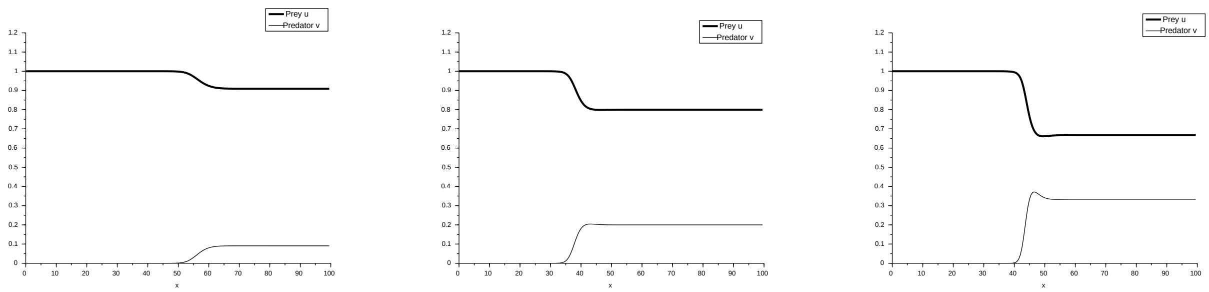

Figure 1-1: Traveling wave profiles for various different values for the parameter $b = 1.2; 1.5; 2$ (from left to right).

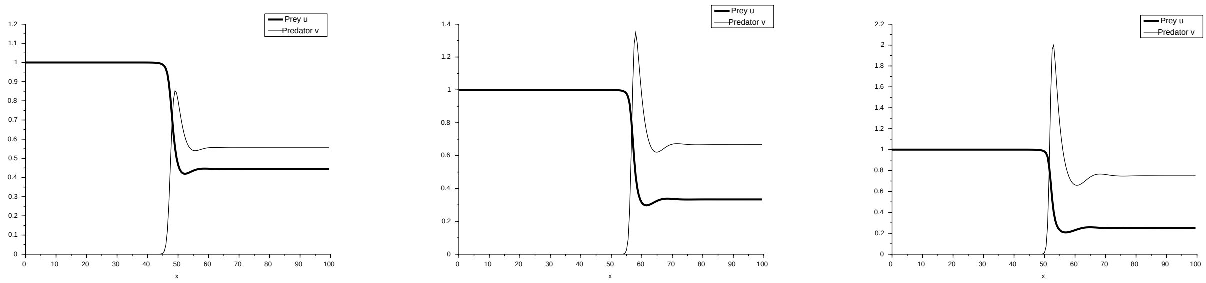

Figure 1-2: Traveling wave profiles for various different values for the parameter $b = 3.5; 5; 7$ (from left to right).

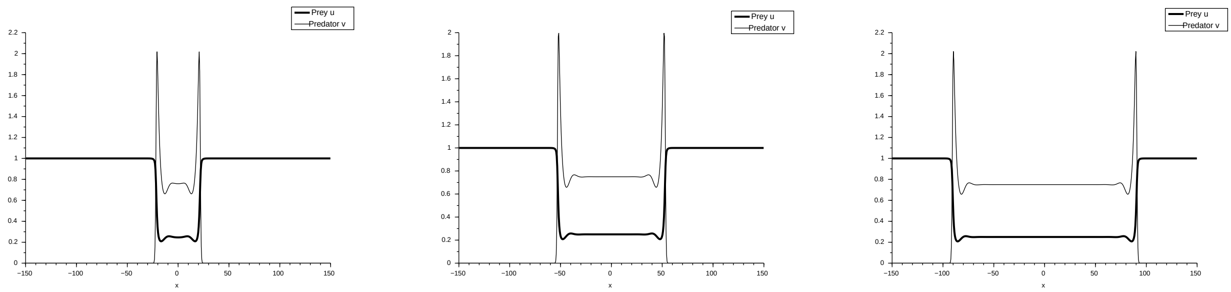

Figure 2: Screen shots of the solution at the three different times (increasing from left to right) with $b = 7$.

The rest of this paper is organized as follows. Section 2 is devoted to the proof of spreading speed of the predator. In particular, the proof of Theorem 1.2 is given. Then we study the traveling wave solutions of (1.1) connecting the predator-free state to a nontrivial state in §3, in which the proofs of Theorem 1.3 and Corollary 1.4 are carried out. The idea of the proof of Corollary 1.4 is from [13]. We suspect that the condition $d_2 < b - r_2$ may be removed. We leave it as an open problem. Finally, we add a remark on the relation between equations with and without nonlocal dispersal at end of this paper.

2. Spreading speed

In this section we shall prove Theorem 1.2. To that aim we denote by X the space of all uniformly continuous bounded functions defined in $\mathbb{R}$. It is a Banach space when endowed with the sup-norm. We also consider its positive cone, denoted by X^+, defined by

$$X^+ = \{w \in X : \ w(x) \geq 0, \ \forall x \in \mathbb{R}\}.$$

Furthermore, for any constant $d > 0$, we let

$$X_d = \{w \in X : 0 \leq w(x) \leq d, \forall x \in \mathbb{R}\}.$$

To derive the spreading speed of the predator for (1.1) and prove Theorem 1.2, we shall first recall some known result on the scalar logistic equation with nonlocal dispersion.

Let $d > 0$, $r > 0$ and $s > 0$ be given. Let $\overline{\lambda} \in (0, \infty]$ and a kernel J in the class $\mathcal{T}\left(\overline{\lambda}\right)$ be given. We consider the following nonlocal logistic equation

$$\begin{cases} \dfrac{\partial w(x,t)}{\partial t} = d\mathcal{N}[w(\cdot,t)](x) + rw(x,t)\left[s - w(x,t)\right], \ x \in \mathbb{R}, \ t > 0, \\ w(x,0) = \chi(x), \ x \in \mathbb{R}, \end{cases} \tag{2.1}$$

where $\mathcal{N}[w] := J * w - w$ and wherein the initial data $\chi \in X_s$ admits a nonempty compact support. When the scalar equation is concerned, Jin and Zhao [17] studied a periodic equation with nonlocal dispersal. Their results remain true with the above equation with constant coefficients. In particular, the above scalar nonlocal equation enjoys the following comparison principle, see [17, Theorem 2.3].

Proposition 2.1. *Let w be a solution of* (2.1) *with $w(\cdot,t) \in X_s$ for all $t > 0$ for a given $\chi \in X_s$. If $z(\cdot,0) \in X_s$ and $z(x,t)$ satisfies*

$$\begin{cases} \dfrac{\partial z(x,t)}{\partial t} \geq d\mathcal{N}[z(\cdot,t)](x) + rz(x,t)\left[s - z(x,t)\right], \ x \in \mathbb{R}, \ t > 0, \\ z(x,0) \geq \chi(x), \ x \in \mathbb{R}, \end{cases}$$

then $z(x,t) \geq w(x,t)$ for all $x \in \mathbb{R}$, $t > 0$. Similar result holds for the reverse inequality.

Next, we define the quantity $\bar{c}$ by

$$\bar{c} := \inf_{0<\lambda<\bar{\lambda}} \frac{d\left[\int_{\mathbb{R}} J(y)e^{\lambda y}dy - 1\right] + rs}{\lambda}.$$

Then $\bar{c}$ is well-defined and $\bar{c} > 0$ since $rs > 0$. Moreover, the quantity $\bar{c}$ corresponds to the spreading speed of the solution, w, to (2.1) as follows.

Proposition 2.2 ([17])**.** *Let w be a solution of* (2.1) *with $w(\cdot,t) \in X_s$ for all $t > 0$ for a given $\chi \in X_s$. Assume that χ has a nonempty compact support. Then we have*

$$\lim_{t\to\infty} \inf_{|x|<ct} w(x,t) = s \quad \text{for any } c \in (0,\bar{c}), \tag{2.2}$$

$$\lim_{t\to\infty} \sup_{|x|>ct} w(x,t) = 0 \quad \text{for any } c > \bar{c}. \tag{2.3}$$

Now observe that the linear operators $\mathcal{N}_i$, for $i = 1, 2$, are both continuous on X and, the semigroups $\{\exp(td_i\mathcal{N}_i)\}_{t\geq 0}$ generated by these operators are positive in the sense that

$$\exp(td_i\mathcal{N}_i)X^+ \subset X^+, \ \forall t \geq 0.$$

We refer the reader to [16, 19] for more details on these semigroups. From this and Cauchy Lipschitz theorem, problem (1.1) generated a maximal positive nonlinear semiflow, denoted by $\mathbb{S}$, on $X^+ \times X^+$. Although the comparison principle does not apply for system (1.1), it does apply to each equation separately using Proposition 2.1 above.

Using the positivity of $\mathbb{S}$, it easily follows, recalling (1.3), that the set X_B^2, defined by

$$X_B^2 := X_1 \times X_\alpha \text{ with } \alpha := \frac{b}{r_2} - 1,$$

is positively invariant with respect to the semiflow $\mathbb{S}$. In particular, this means that the initial value problem (1.1) with initial condition (1.2) in which $v_0 \in X_\alpha$ admits a unique globally defined solution (U,V) with

$$(U,V) \in C^1\left([0,\infty), X^2\right) \text{ and } (U,V)(\cdot,t) \in X_B^2 \ \forall\, t \geq 0.$$

In addition, since (U,V) is bounded from $[0,\infty)$ into X^2, it follows from (1.1) that the time derivative of (U,V) is also bounded from $[0,\infty)$ into X^2. Hence, for each given initial data in $v_0 \in X_\alpha$, the corresponding solution $(U,V) = (U,V)(x,t)$ of system (1.1) and (1.2) is uniformly continuous on $\mathbb{R} \times [0,\infty)$. This property shall be important for our next arguments and more precisely to prove Claim 2.3 below. If furthermore v_0 admits a nonempty compact support, then $V(t,x)>0$ for all $t>0$ and $x\in\mathbb{R}$.

Now we fix an initial data $v_0\in X_\alpha\setminus\{0\}$ with a compact support. We consider the corresponding solution, (U,V), of problem (1.1) and (1.2). Before proving our spreading speed result as stated in Theorem 1.2, observe that, due to $V(x,t)\le\alpha$, the function U satisfies

$$\frac{\partial U(x,t)}{\partial t}\ge d_1\mathcal{N}_1[U(\cdot,t)](x)+U[r_1-a\alpha-r_1U],\ x\in\mathbb{R},\ t>0.$$

Hence, recalling the third condition in (1.3) and $U(x,0)=1$, it follows from Proposition 2.1 that

$$U(x,t)\ge U_{\min}:=1-\frac{a}{r_1r_2}(b-r_2),\ \forall t\ge 0,\ x\in\mathbb{R}. \tag{2.4}$$

Then we are ready to give a proof of Theorem 1.2 as follows.

We split the proof into two parts. First, we prove that

$$\lim_{t\to\infty}\sup_{|x|\ge ct}V(x,t)=0,\ \forall c>c^*. \tag{2.5}$$

The proof of (2.5) directly follows from Propositions 2.1 and 2.2. Indeed, since $U\le 1$, the function V satisfies

$$\frac{\partial V(x,t)}{\partial t}\le d_2\mathcal{N}_2[V(\cdot,t)](x)+bV(x,t)-r_2V(x,t)[1+V(x,t)],\ x\in\mathbb{R},\ t>0.$$

Applying Proposition 2.1, we get $V(x,t)\le\overline{V}(x,t)$ for all $(x,t)\in\mathbb{R}\times[0,\infty)$, wherein the function $\overline{V}$ is defined as the solution of the following problem

$$\frac{\partial\overline{V}(x,t)}{\partial t}=d_2\mathcal{N}_2[\overline{V}(\cdot,t)](x)+\overline{V}(x,t)[b-r_2-r_2\overline{V}(x,t)],\ x\in\mathbb{R},\ t>0,$$

supplemented with the initial data $\overline{V}(x,0)=V(x,0)=v_0(x)$. Hence (2.5) follows from Property (2.3) in Proposition 2.2.

We now turn to the proof of the second part, namely,

$$\liminf_{t\to\infty}\inf_{|x|\le ct}V(x,t)>0,\ \forall c\in(0,c^*). \tag{2.6}$$

The proof of (2.6) is much more involved and that shall make crucially use of the uniform continuity of $V=V(x,t)$ on $\mathbb{R}\times[0,\infty)$.

To prove (2.6), we first observe that, using (2.4) and $U\le 1$, the function U satisfies

$$\frac{\partial U}{\partial t}(x,t)\ge d_1\mathcal{N}_1[U(\cdot,t)](x)+r_1U_{\min}(1-U)-aV,\ x\in\mathbb{R},\ t>0,$$

so that, since $U(x,0)=1$, the function $1-U(x,t)$ satisfies

$$1-U(x,t)\le\overline{u}(x,t):=a\int_0^t e^{-r_1U_{\min}(t-s)}\left\{\exp((t-s)d_1\mathcal{N}_1)[V(\cdot,s)]\right\}(x)ds$$

for all $x\in\mathbb{R}$, $t\ge 0$. Plugging this inequality into the V-equation in (1.1) yields

$$\frac{\partial V}{\partial t}(x,t)\ge d_2\mathcal{N}_2[V(\cdot,t)](x)+V(x,t)[b-r_2-r_2V(x,t)-b\overline{u}(x,t)],\ x\in\mathbb{R},\ t>0. \tag{2.7}$$

Next, let $\epsilon \in (0, c^*)$ be given. For this ϵ, let $\delta > 0$ small enough (smaller than $b - r_2$) be given such that

$$\inf_{0<\lambda<\overline{\lambda}_2} \frac{d_2\left[\int_{\mathbb{R}} J_2(y)e^{\lambda y}dy - 1\right] + (b - r_2 - \delta)}{\lambda} > c^* - \epsilon.$$

The following claim is the key to prove (2.6).

Claim 2.3. *There exist $M > 0$ and $\tau > 0$ large enough such that*

$$b\overline{u}(x,t) \leq \delta + MV(x,t),\ x \in \mathbb{R},\ t \geq \tau. \tag{2.8}$$

Suppose for a moment that Claim 2.3 holds true. Inserting (2.8) into (2.7) yields

$$\frac{\partial V}{\partial t}(x,t) \geq d_2\mathcal{N}_2[V(\cdot,t)](x) + V(x,t)\left[b - r_2 - \delta - (r_2 + M)V(x,t)\right],\ x \in \mathbb{R},\ t \geq \tau.$$

From the comparison principle one obtains that $V(x,t+\tau) \geq \underline{V}(x,t)$ for any $x \in \mathbb{R}$, $t \geq 0$, wherein the function $\underline{V}$ is a solution of the scalar logistic equation

$$\frac{\partial \underline{V}}{\partial t}(x,t) = d_2\mathcal{N}_2[\underline{V}(\cdot,t)](x) + \underline{V}(x,t)\left[b - r_2 - \delta - (r_2 + M)\underline{V}(x,t)\right],\ x \in \mathbb{R},\ t \geq \tau,$$

with $\underline{V}(\cdot,0)$ a nontrivial compactly supported function such that $\underline{V}(x,0) \leq V(x,\tau)$ for all $x \in \mathbb{R}$. Then, applying Proposition 2.2 to $\underline{V}$, one proves that

$$\liminf_{t\to\infty} \inf_{|x|\leq c^*-\epsilon} V(x,t) \geq \lim_{t\to\infty} \inf_{|x|\leq c^*-\epsilon} \underline{V}(x,t) = \frac{b - r_2 - \delta}{r_2 + M} > 0.$$

This completes the proof of (2.6), since $\epsilon > 0$ is as small as we want.

Therefore, to complete the proof of Theorem 1.2, it remains to prove Claim 2.3.

To that aim we need more detailed properties of the strongly positive semigroup $\{T(t)\}_{t\geq 0}$, where

$$T(t) := \exp\left(td_1\mathcal{N}_1\right),\ t \geq 0.$$

We refer to [19] and [16] for a detailed study of such a semigroup. According to [16], the fundamental solution of this semigroup, namely, the solution W of the problem

$$\frac{\partial w(x,t)}{\partial t} = d_1\{[J_1 * w(\cdot,t)](x) - w(x,t)\},\ t > 0,\quad \text{with } w(\cdot,0) = \delta_0,$$

wherein δ_0 denotes the Dirac mass at $x = 0$, can be decomposed as

$$W(x,t) = e^{-d_1t}\delta_0(x) + K(x,t),\ x \in \mathbb{R},\ t \geq 0,$$

where K is a nonnegative smooth function satisfying the estimate

$$\int_{\mathbb{R}} K(x,t)dx \leq 2,\ \forall t \geq 0. \tag{2.9}$$

Hence the semigroup $\{T(t)\}_{t\geq 0}$ can be expressed as

$$T(t)[\varphi](x) = e^{-d_1t}\varphi(x) + \int_{\mathbb{R}} K(x-y,t)\varphi(y)dy,\ \forall t \geq 0,\ \varphi \in X.$$

We are now ready to give a proof of Claim 2.3.

Proof of Claim 2.3. First, according to the above formula for $T(t)$, the function $b\overline{u}(x,t)$ can be decomposed as $b\overline{u}(x,t) = W_1(x,t) + W_2(x,t)$, where

$$W_1(x,t) := ab\int_0^t e^{-\beta_1(t-s)}V(x,s)ds, \quad \beta_1 := d_1 + r_1 U_{\min} > 0,$$

$$W_2(x,t) := ab\int_0^t \int_{\mathbb{R}} e^{-\beta_2(t-s)}K(x-y,t-s)V(y,s)dyds, \ \beta_2 := r_1 U_{\min} > 0.$$

Recalling that $\delta > 0$ is fixed. For this fixed δ, there exists $\tau > 0$ large enough such that

$$ab(b-r_2)\int_0^{t-\tau} e^{-\beta_1(t-s)}ds \le \frac{\delta}{4}, \ 2ab(b-r_2)\int_0^{t-\tau} e^{-\beta_2(t-s)}ds \le \frac{\delta}{4}, \ \forall\, t \ge \tau. \tag{2.10}$$

To prove Claim 2.3, it is sufficient to show that there exists some constant $M > 0$ such that, for any $t \ge \tau$ and $x \in \mathbb{R}$, one has

$$b\overline{u}(x,t) \ge \delta \ \Rightarrow \ b\overline{u}(x,t) \le \delta + MV(x,t).$$

To that aim, consider $t_0 \ge \tau$ and $x_0 \in \mathbb{R}$ with $b\overline{u}(x_0,t_0) \ge \delta$. First, since $V \le b - r_2$, one obtains using (2.9) that

$$\begin{aligned}\delta \le b\overline{u}(x_0,t_0) \ \le \ & ab(b-r_2)\int_0^{t_0-\tau} e^{-\beta_1(t_0-s)}ds + 2ab(b-r_2)\int_0^{t_0-\tau} e^{-\beta_2(t_0-s)}ds \\ & + ab\int_{t_0-\tau}^{t_0} e^{-\beta_1(t_0-s)}V(x_0,s)ds \\ & + ab\int_{t_0-\tau}^{t_0}\int_{\mathbb{R}} e^{-\beta_2(t_0-s)}K(x_0-y,t_0-s)V(y,s)dyds.\end{aligned} \tag{2.11}$$

This, together with (2.10), yields

$$ab\int_0^{\tau} e^{-\beta_1 l}V(x_0,t_0-l)dl + ab\int_0^{\tau}\int_{\mathbb{R}} e^{-\beta_2 l}K(y,l)V(x_0-y,t_0-l)dydl \ge \frac{\delta}{2}.$$

Next, choose $R > 0$ large enough such that

$$ab(b-r_2)\int_0^{\tau}\int_{|y|\ge R} e^{-\beta_2 l}K(y,l)dydl \le \frac{\delta}{4}.$$

Using again $V \le b - r_2$, one obtains that

$$ab\int_0^{\tau} e^{-\beta_1 l}V(x_0,t_0-l)dl + ab\int_0^{\tau}\int_{-R}^{R} e^{-\beta_2 l}K(y,l)V(x_0-y,t_0-l)dydl \ge \frac{\delta}{4}.$$

Choose now $\eta > 0$ small enough such that

$$ab(b-r_2)\left[\int_0^{\eta} e^{-\beta_1 l}dl + \int_0^{\eta}\int_{-R}^{R} e^{-\beta_2 l}K(y,l)dydl\right] \le \frac{\delta}{8},$$

so that, using the same argument as above, we get

$$ab\int_{\eta}^{\tau} e^{-\beta_1 l}V(x_0,t_0-l)dl + ab\int_{\eta}^{\tau}\int_{-R}^{R} e^{-\beta_2 l}K(y,l)V(x_0-y,t_0-l)dydl \ge \frac{\delta}{8}.$$

Hence, setting $\theta > 0$ defined by

$$\frac{\delta}{8} = \theta\left[ab\int_{\eta}^{\tau} e^{-\beta_1 l}dl + ab\int_{\eta}^{\tau}\int_{-R}^{R} e^{-\beta_2 l}K(y,l)dydl\right],$$

there exists $l_0 \in [t_0-\tau, t_0-\eta]$ and $y_0 \in [x_0-R, x_0+R]$ such that $V(y_0, l_0) \geq \theta$. Moreover, since the function V is uniformly continuous on $\mathbb{R}\times[0,\infty)$, there exists $\varrho > 0$ independent of (y_0, l_0) such that

$$V(y, l_0) \geq \frac{\theta}{2},\ \forall y \in [y_0-\varrho, y_0+\varrho].$$

Finally, we consider a uniformly continuous function $Z_0 \leq \theta/2$ in $\mathbb{R}$ such that

$$Z_0(x) = \frac{\theta}{2},\ \forall x \in \left[-\frac{\varrho}{2}, \frac{\varrho}{2}\right], \quad Z_0(x) = 0,\ \forall |x| \geq \varrho.$$

Observe that the function V satisfies

$$\frac{\partial V}{\partial t}(x,t) \geq d_2\mathcal{N}_2[V(\cdot,t)](x) - r_2\left(1+b-r_2\right)V(x,t),\ x \in \mathbb{R},\ t>0.$$

Since $Z_0(x) \leq V(y_0+x, l_0)$, it follows from the comparison principle that

$$Z(x,t) \leq V(y_0+x, l_0+t),\ \forall x \in \mathbb{R},\ t \geq 0,$$

wherein we have set $Z(\cdot,t) = e^{r_2(1+b-r_2)t}\exp\left(d_2t\mathcal{N}_2\right)[Z_0](\cdot)$. Recalling that $Z(x,t) > 0$ for all $x \in \mathbb{R}$, $t>0$. Thus one obtains that

$$V(x_0,t_0) \geq Z\left(x_0-y_0, t_0-l_0\right) \geq \gamma := \min_{x\in[-R,R],\, t\in[\eta,\tau]} Z(x,t) > 0.$$

Then, using (2.11), (2.10), (2.9) and $V \leq b-r_2$, we get

$$b\overline{u}(x_0,t_0) \leq \frac{\delta}{2} + 3ab\tau(b-r_2) \leq \delta + MV(x_0,t_0),$$

by setting $M = 3\gamma^{-1}ab\tau(b-r_2) > 0$ (which is independent of (x_0,t_0)). This completes the proof of the claim. □

3. Traveling waves and minimal wave speed

In this section, we investigate the existence of traveling wave solutions for system (1.1) and our purpose is to show that the constant c^* defined by (1.5) corresponds to the minimal wave speed of traveling wave solutions connecting the predator-free state to a nontrivial state.

To perform our analysis, we set

$$u(x,t) = 1-U(x,t),\ \ v(x,t) = V(x,t),$$
$$\bar{a} = \frac{a}{r_1} \text{ and } \bar{b} = \frac{b}{r_2}.$$

With these notations, problem (1.1) is re-written as

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = d_1\mathcal{N}_1[u(\cdot,t)](x) + r_1[1-u(x,t)]\left[\bar{a}v(x,t)-u(x,t)\right], \\ \dfrac{\partial v(x,t)}{\partial t} = d_2\mathcal{N}_2[v(\cdot,t)](x) + r_2v(x,t)[\bar{b}-1-v(x,t)-\bar{b}u(x,t)]. \end{cases} \tag{3.1}$$

Note also that condition (1.3) becomes

$$\bar{b} > 1,\ \ \bar{a}(\bar{b}-1) < 1, \tag{3.2}$$

and the condition $ab < r_1r_2$ becomes $\bar{a}\bar{b} < 1$.

As already mentioned above, we are interested in traveling wave solution for (1.1) connecting the predator-free equilibrium at $\xi = -\infty$ and a nontrivial state at $\xi = \infty$ in the sense of (1.6). This is equivalent to finding a solution for system (3.1) in the form

$$u(x,t) = \phi(\xi), \quad v(x,t) = \psi(\xi), \quad \text{where } \xi := x + ct,$$

such that $(\phi, \psi)(-\infty) = (0,0)$ and

$$\limsup_{\xi\to\infty} \phi(\xi) < 1, \quad \liminf_{\xi\to\infty} \psi(\xi) > 0. \tag{3.3}$$

Observe that looking for such special solutions yields the following nonlocal system of equation for the profile function (ϕ, ψ)

$$\begin{cases} c\phi'(\xi) = d_1\mathcal{N}_1[\phi](\xi) + r_1[1 - \phi(\xi)][\bar{a}\psi(\xi) - \phi(\xi)], \ \xi \in \mathbb{R}, \\ c\psi'(\xi) = d_2\mathcal{N}_2[\psi](\xi) + r_2\psi(\xi)[\bar{b} - 1 - \psi(\xi) - \bar{b}\phi(\xi)], \ \xi \in \mathbb{R}, \end{cases} \tag{3.4}$$

where, as before, the linear operators $\mathcal{N}_i$ are defined by

$$\mathcal{N}_i[\varphi](\xi) := \int_{\mathbb{R}} J_i(y)\varphi(\xi - y)dy - \varphi(\xi), \ i = 1, 2.$$

To study the existence of solution to (3.4), we shall use the standard partial ordering in $\mathbb{R}^2$ defined by

$$u \le v \ \Leftrightarrow \ u_i \le v_i, \ i = 1, 2$$

for any $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $\mathbb{R}^2$. Hereafter we set $X_{\bar{b}}^2 := X_1 \times X_{\bar{\alpha}}$, where $\bar{\alpha} := \bar{b} - 1$. Next we introduce the following definition.

Definition 3.1. *A pair of functions $(\overline{\phi}, \overline{\psi}), (\underline{\phi}, \underline{\psi}) \in X_{\bar{b}}^2$ is called a pair of upper and lower solutions of* (3.4) *if $(\underline{\phi}(\xi), \underline{\psi}(\xi)) \le (\overline{\phi}(\xi), \overline{\psi}(\xi))$ for all $\xi \in \mathbb{R}$ and the following inequalities*

$$c\overline{\phi}'(\xi) \ge d_1\mathcal{N}_1[\overline{\phi}](\xi) + r_1[1 - \overline{\phi}(\xi)][\bar{a}\overline{\psi}(\xi) - \overline{\phi}(\xi)], \tag{3.5}$$

$$c\overline{\psi}'(\xi) \ge d_2\mathcal{N}_2[\overline{\psi}](\xi) + r_2\overline{\psi}(\xi)[\bar{b} - 1 - \overline{\psi}(\xi) - \bar{b}\underline{\phi}(\xi)], \tag{3.6}$$

$$c\underline{\phi}'(\xi) \le d_1\mathcal{N}_1[\underline{\phi}](\xi) + r_1[1 - \underline{\phi}(\xi)][\bar{a}\underline{\psi}(\xi) - \underline{\phi}(\xi)], \tag{3.7}$$

$$c\underline{\psi}'(\xi) \le d_2\mathcal{N}_2[\underline{\psi}](\xi) + r_2\underline{\psi}(\xi)[\bar{b} - 1 - \underline{\psi}(\xi) - \bar{b}\overline{\phi}(\xi)] \tag{3.8}$$

hold for all $\xi \in \mathbb{R} \setminus E$, where E denotes some finite set $E \subset \mathbb{R}$.

The following lemma shall be proved by applying Schauder's fixed point theorem. Since it is a rather standard methodology, we only outline the proof and refer the reader to [15, 23, 34, 33, 29, 5] and references cited therein.

Lemma 3.2. *Let $c > 0$ be given. Let $(\overline{\phi}, \overline{\psi}), (\underline{\phi}, \underline{\psi})$ be a pair of upper and lower solutions of* (3.4)*. Then system* (3.4) *admits a solution (ϕ, ψ) such that*

$$(\underline{\phi}(\xi), \underline{\psi}(\xi)) \le (\phi(\xi), \psi(\xi)) \le (\overline{\phi}(\xi), \overline{\psi}(\xi)), \ \xi \in \mathbb{R}.$$

Proof. First, we introduce the nonlinear operators F_1 and F_2 defined on $X_{\bar{b}}^2$ by

$$\begin{aligned} F_1(\phi, \psi)(\xi) &:= \beta\phi(\xi) + d_1\mathcal{N}_1[\phi](\xi) + r_1[1 - \phi(\xi)][\bar{a}\psi(\xi) - \phi(\xi)], \\ F_2(\phi, \psi)(\xi) &:= \beta\psi(\xi) + d_2\mathcal{N}_2[\psi](\xi) + r_2\psi(\xi)[\bar{b} - 1 - \psi(\xi) - \bar{b}\phi(\xi)] \end{aligned}$$

for each $(\phi,\psi)\in X_b^2$. Here β denotes some large positive constant to be determined later. We also define the operator $P=(P_1,P_2):X_b^2\to X^2$ by

$$\begin{cases} P_1(\phi,\psi)(\xi) := \frac{1}{c}\int_{-\infty}^{\xi} e^{-\frac{\beta(\xi-s)}{c}}F_1(\phi,\psi)(s)ds, \\ P_2(\phi,\psi)(\xi) := \frac{1}{c}\int_{-\infty}^{\xi} e^{-\frac{\beta(\xi-s)}{c}}F_2(\phi,\psi)(s)ds, \end{cases} \quad \forall(\phi,\psi)\in X_b^2.$$

Then a fixed point of P is a solution to (3.4). Hence to prove Lemma 3.2, it is sufficient to investigate the existence of fixed points for the nonlinear operator P.

To that aim let $\mu>0$ be a small constant. Denote

$$B_\mu(\mathbb{R},\mathbb{R}^2) := \left\{(\phi,\psi)\in X^2 : |(\phi,\psi)|_\mu := \sup_{\xi\in\mathbb{R}}\max\left(|\phi(\xi)|,|\psi(\xi)|\right)e^{-\mu|\xi|}<\infty\right\}.$$

Observe that $\left(B_\mu(\mathbb{R},\mathbb{R}^2),|\cdot|_\mu\right)$ is a Banach space. Define a set $\Gamma\subset B_\mu(\mathbb{R},\mathbb{R}^2)$ by

$$\Gamma{:=}\{(\phi,\psi)\in X_b^2 : \left(\underline{\phi},\underline{\psi}\right)\le(\phi,\psi)\le\left(\overline{\phi},\overline{\psi}\right)\}.$$

We also observe that Γ is a nonempty convex, closed and bounded subset of $B_\mu(\mathbb{R},\mathbb{R}^2)$.

By the definition of upper and lower solutions, one can check that

$$\left(\underline{\phi}(\xi),\underline{\psi}(\xi)\right)\le\left(P_1(\phi,\psi)(\xi),P_2(\phi,\psi)(\xi)\right)\le\left(\overline{\phi}(\xi),\overline{\psi}(\xi)\right),\ \forall\,\xi\in\mathbb{R},$$

for any $(\phi,\psi)\in X_b^2$, when β is chosen sufficiently large. Indeed, using $(\phi,\psi)\ge(\underline{\phi},\underline{\psi})$ and $(\phi,\psi)\in X_b^2$, one has

$$\begin{aligned} F_1(\phi,\psi)(\xi) &\ge \beta\phi(\xi)+d_1(J_1*\phi-\phi)(\xi)+r_1[(1-\phi)(\bar{a}\underline{\psi}-\phi)](\xi) \\ &= (\beta-d_1-r_1-r_1\bar{a}\underline{\psi}(\xi))\phi(\xi)+r_1\bar{a}\underline{\psi}(\xi)+r_1\phi^2(\xi)+d_1(J_1*\phi)(\xi) \\ &\ge (\beta-d_1-r_1-r_1\bar{a}\underline{\psi}(\xi))\underline{\phi}(\xi)+r_1\bar{a}\underline{\psi}(\xi)+r_1\underline{\phi}^2(\xi)+d_1(J_1*\underline{\phi})(\xi), \end{aligned}$$

as long as we choose β such that $\beta\ge d_1+r_1+r_1\bar{a}(b-1)$. Then $F_1(\phi,\psi)(\xi)\ge F_1(\underline{\phi},\underline{\psi})(\xi)$ for all $\xi\in\mathbb{R}$ and so

$$\begin{aligned} P_1(\phi,\psi)(\xi) &\ge \frac{1}{c}\int_{-\infty}^{\xi} e^{-\beta(\xi-s)/c}F_1(\underline{\phi},\underline{\psi})(s)ds \\ &\ge \int_{-\infty}^{\xi} e^{-\beta(\xi-s)/c}\left[\underline{\phi}'(s)+\frac{\beta}{c}\underline{\phi}(s)\right]ds=\underline{\phi}(\xi),\ \forall\,\xi\in\mathbb{R}. \end{aligned}$$

The other cases can be treated similarly, by choosing a suitable β sufficiently large. We safely omit it. Hence we have obtained that $P(\Gamma)\subset\Gamma$. Moreover, with a suitable choice of μ (see [5]), the mapping $P:\Gamma\to\Gamma$ is completely continuous with respect to the norm $|\cdot|_\mu$. As a consequence, Schauder's fixed point theorem applies and ensures the existence of a fixed point in Γ for the map P. This completes the proof of the lemma. □

In order to go further and construct traveling wave solution, we consider the function

$$\Delta(\lambda,c) := d_2\left[\int_{\mathbb{R}} J_2(y)e^{\lambda y}dy-1\right]-c\lambda+r_2(\bar{b}-1).$$

Then it is easy to see that $\lambda\mapsto\Delta(\lambda,c)$ is strictly convex with respect to λ for each given c. Moreover, due to $\bar{b}>1$ and the definition of c^* in (1.5), it enjoys the following properties

(1) For any given $c>c^*$, the equation $\Delta(\lambda,c)=0$ admits two positive roots $\lambda_1(c)<\lambda_2(c)<\bar{\lambda}_2$ such that $\Delta(\lambda,c)<0$ if and only if $\lambda\in(\lambda_1(c),\lambda_2(c))$.

(2) There exists $\lambda^* \in \big[0, \bar{\lambda}_2\big)$ such that $\Delta(\lambda^*, c^*) = 0$, $\Delta(\lambda, c^*) > 0$ for all $\lambda \in [0, \bar{\lambda}_2) \setminus \{\lambda^*\}$ while

$$\frac{\partial \Delta(\lambda, c)}{\partial \lambda}\bigg|_{(\lambda,c)=(\lambda^*,c^*)} = d_2 \int_{\mathbb{R}} J_2(y) y e^{\lambda^* y} dy - c^* = 0. \tag{3.9}$$

(3) For any given $c \in (0, c^*)$, one has $\Delta(\lambda, c) > 0$ for all $\lambda \in [0, \bar{\lambda}_2)$.

We now derive the existence of solution to system (3.4) for $c > c^*$ as follows.

Theorem 3.3. *Let $c > c^*$ be given and fixed. Then* (3.4) *admits a nonnegative solution (ϕ, ψ) such that*

$$\lim_{\xi \to -\infty} (\phi(\xi), \psi(\xi)) = (0, 0).$$

Proof. Let $c > c^*$ be given and fixed. Due to Lemma 3.2, to prove the above theorem it is sufficient to construct a suitable pair of upper and lower solutions for system (3.4) with $(\overline{\phi}, \overline{\psi})(-\infty) = (\underline{\phi}, \underline{\psi})(-\infty) = (0, 0)$. To that aim set

$$A(\lambda) := d_1 \left[\int_{\mathbb{R}} J_1(y) e^{\lambda y} dy - 1 \right] - c\lambda.$$

Since $A(0) = 0$ and

$$A'(0) = \lim_{\lambda \to 0} \left\{ d_1 \int_{\mathbb{R}} J_1(y) y e^{\lambda y} dy - c \right\} = -c < 0,$$

there exists $\lambda \in (0, \min\{\bar{\lambda}_1, \lambda_1(c)\})$ such that

$$A(\lambda) = d_1 \left[\int_{\mathbb{R}} J_1(y) e^{\lambda y} dy - 1 \right] - c\lambda < 0. \tag{3.10}$$

Also, for notational simplicity, since c is fixed along this proof, we simply denote $\lambda_i = \lambda_i(c)$, $i = 1, 2$.

Now, we define the following continuous functions:

$$\begin{aligned} &\overline{\phi}(\xi) = \min\{1, K e^{\lambda \xi}\}, \quad \underline{\phi}(\xi) = 0, \\ &\overline{\psi}(\xi) = \min\{\bar{b} - 1, e^{\lambda_1 \xi}\}, \quad \underline{\psi}(\xi) = \max\{0, e^{\lambda_1 \xi} - q e^{\eta \lambda_1 \xi}\}, \end{aligned}$$

where the constants K, η, q shall be chosen in order below.

First, define ξ_1 by $e^{\lambda_1 \xi_1} = \bar{b} - 1$ and we choose a constant $\xi_0 < \min\{0, \xi_1\}$ such that $\bar{a} e^{(\lambda_1 - \lambda)\xi_0} < 1$. Next set $K = e^{-\lambda \xi_0}$ and observe that $K > 1$ and

$$\bar{a} e^{\lambda_1 \xi} < e^{\lambda \xi}, \ \forall \, \xi < \xi_0. \tag{3.11}$$

Now, choose $\eta \in (1, 2)$ such that

$$\eta \lambda_1 < \min\{\lambda_2, \lambda_1 + \lambda\}. \tag{3.12}$$

Finally, for $q > 1$ define $\xi_2 = \xi_2(q) < 0$ by $e^{(\eta - 1)\lambda_1 \xi_2} = 1/q$. Since $\xi_2(q) \to -\infty$ as $q \to \infty$, one can choose a constant $q > 1$ large enough such that $\xi_2 \le \xi_1$ and

$$q > \frac{r_2 + r_2 b K}{-\Delta(\eta \lambda_1, c)} + 1. \tag{3.13}$$

It is clear that $\underline{\phi}(\xi) \leq \overline{\phi}(\xi)$ for any $\xi \in \mathbb{R}$. Since $\xi_2 \leq \xi_1$, one also has $\underline{\psi}(\xi) \leq \overline{\psi}(\xi)$ for all $\xi \in \mathbb{R}$. Note also that one has

$$\overline{\phi}(\xi) = \begin{cases} Ke^{\lambda\xi}, & \xi \leq \xi_0, \\ 1, & \xi \geq \xi_0, \end{cases} \quad \overline{\psi}(\xi) = \begin{cases} e^{\lambda_1\xi}, & \xi \leq \xi_1, \\ \bar{b} - 1, & \xi \geq \xi_1, \end{cases}$$

$$\underline{\psi}(\xi) = \begin{cases} e^{\lambda_1\xi} - qe^{\eta\lambda_1\xi}, & \xi \leq \xi_2, \\ 0, & \xi \geq \xi_2. \end{cases}$$

We now check that the functions $(\overline{\phi}, \overline{\psi})$ and $(\underline{\phi}, \underline{\psi})$ are a pair of upper and lower solutions of (3.4). To that aim, first note that for $\xi > \xi_0$, we have $\overline{\phi}(\xi) = 1$, so that

$$r_1[1 - \overline{\phi}(\xi)][\bar{a}\overline{\psi}(\xi) - \overline{\phi}(\xi)] = 0, \quad \overline{\phi}'(\xi) = 0.$$

Since

$$d_1[J_1 * \overline{\phi}](\xi) \leq d_1 \int_{\mathbb{R}} J_1(y)dy = d_1 = d_1\overline{\phi}(\xi),$$

condition (3.5) holds for $\xi > \xi_0$.

For $\xi < \xi_0$, one has $\overline{\phi}(\xi) = Ke^{\lambda\xi} < 1$ and it follows from (3.11) and the equality $\overline{\psi}(\xi) = e^{\lambda_1\xi}$ for all $\xi < \xi_0 \leq \xi_1$, that

$$r_1[1 - \overline{\phi}(\xi)][\bar{a}\overline{\psi}(\xi) - \overline{\phi}(\xi)] \leq 0.$$

Since $\overline{\phi}(\xi) \leq Ke^{\lambda\xi}$ for all $\xi \in \mathbb{R}$, then one obtains

$$\begin{aligned} & d_1\mathcal{N}_1[\overline{\phi}](\xi) - c\overline{\phi}'(\xi) \\ = \ & d_1 \left[\int_{\mathbb{R}} J_1(y)\overline{\phi}(\xi - y)dy - Ke^{\lambda\xi} \right] - c\lambda Ke^{\lambda\xi} \\ \leq \ & d_1 \left[\int_{\mathbb{R}} J_1(y)e^{\lambda(\xi-y)}dy - Ke^{\lambda\xi} \right] - c\lambda Ke^{\lambda\xi} \\ = \ & Ke^{\lambda\xi} \left\{ d_1 \left[\int_{\mathbb{R}} J_1(y)e^{\lambda y}dy - 1 \right] - c\lambda \right\} \\ \leq \ & 0 \end{aligned}$$

for $\xi < \xi_0$. Here we have used (3.10) and the symmetry of the function J_1. Hence condition (3.5) also holds for $\xi < \xi_0$ and thus for any $\xi \neq \xi_0$.

We now turn to (3.6). For $\xi > \xi_1$, we have $\overline{\psi}(\xi) = \bar{b} - 1$. Since $\overline{\psi}(\xi) \leq \bar{b} - 1$ for all $\xi \in \mathbb{R}$, we have

$$\mathcal{N}_2[\overline{\psi}](\xi) \leq \int_{\mathbb{R}} (\bar{b} - 1)J_2(y)dy - (\bar{b} - 1) = 0, \ \forall\, \xi > \xi_1.$$

Hence (3.6) holds for $\xi > \xi_1$.

For $\xi < \xi_1$, we have $\overline{\psi}(\xi) = e^{\lambda_1 \xi}$. Using $\overline{\psi}(\xi) \leq e^{\lambda_1 \xi}$ for all $\xi \in \mathbb{R}$, we compute

$$\begin{aligned}
& d_2\mathcal{N}_2[\overline{\psi}](\xi) - c\overline{\psi}'(\xi) + r_2\overline{\psi}(\xi)[\bar{b} - 1 - \overline{\psi}(\xi) - \bar{b}\underline{\phi}(\xi)] \\
\leq\ & d_2\mathcal{N}_2[\overline{\psi}](\xi) - c\overline{\psi}'(\xi) + r_2(\bar{b} - 1)\overline{\psi}(\xi) \\
=\ & d_2\left[\int_{\mathbb{R}} \overline{\psi}(\xi - y)J_2(y)dy - e^{\lambda_1\xi}\right] - c\lambda_1 e^{\lambda_1\xi} + r_2(\bar{b} - 1)e^{\lambda_1\xi} \\
\leq\ & e^{\lambda_1\xi}\Delta(\lambda_1, c) \\
=\ & 0
\end{aligned}$$

for all $\xi < \xi_1$. Hence (3.6) also holds for $\xi < \xi_1$ and thus for all $\xi \in \mathbb{R} \setminus \{\xi_1\}$.

Note now that (3.7) is trivial since $\underline{\phi}(\xi) \equiv 0$. It remains to check that (3.8) holds true. To do so, observe that for $\xi > \xi_2$, we have $\underline{\psi}(\xi) = 0$ and (3.8) holds true for $\xi > \xi_2$.

For $\xi < \xi_2$, we have $\underline{\psi}(\xi) = e^{\lambda_1\xi} - qe^{\eta\lambda_1\xi}$ and, recalling that for any $\xi \in \mathbb{R}$ one has $\underline{\psi}(\xi) \leq e^{\lambda_1\xi}$ and $\overline{\phi}(\xi) \leq Ke^{\lambda\xi}$, we compute

$$\begin{aligned}
& r_2\underline{\psi}(\xi)[\bar{b} - 1 - \underline{\psi}(\xi) - \bar{b}\overline{\phi}(\xi)] \\
=\ & r_2(\bar{b} - 1)\underline{\psi}(\xi) - r_2\underline{\psi}^2(\xi) - r_2\bar{b}\overline{\phi}(\xi)\underline{\psi}(\xi) \\
\geq\ & r_2(\bar{b} - 1)(e^{\lambda_1\xi} - qe^{\eta\lambda_1\xi}) - r_2e^{2\lambda_1\xi} - r_2\bar{b}Ke^{(\lambda+\lambda_1)\xi}.
\end{aligned}$$

Also, using $\underline{\psi}(\xi) \geq e^{\lambda_1\xi} - qe^{\eta\lambda_1\xi}$ for all $\xi \in \mathbb{R}$, one gets

$$\begin{aligned}
& d_2\mathcal{N}_2[\underline{\psi}](\xi) - c\underline{\psi}'(\xi) \\
=\ & d_2\left[\int_{\mathbb{R}} \underline{\psi}(\xi - y)J_2(y)dy - (e^{\lambda_1\xi} - qe^{\eta\lambda_1\xi})\right] \\
& -c(\lambda_1 e^{\lambda_1\xi} - q\eta\lambda_1 e^{\eta\lambda_1\xi}) \\
\geq\ & d_2\left[\int_{\mathbb{R}} (e^{\lambda_1(\xi-y)} - qe^{\eta\lambda_1(\xi-y)})J_2(y)dy - (e^{\lambda_1\xi} - qe^{\eta\lambda_1\xi})\right] \\
& -c(\lambda_1 e^{\lambda_1\xi} - q\eta\lambda_1 e^{\eta\lambda_1\xi}).
\end{aligned}$$

Hence, coupling the above estimates, one obtains that for each $\xi < \xi_2$

$$\begin{aligned}
& d_2\mathcal{N}_2[\underline{\psi}](\xi) - c\underline{\psi}'(\xi) + r_2\underline{\psi}(\xi)[\bar{b} - 1 - \underline{\psi}(\xi) - \bar{b}\overline{\phi}(\xi)] \\
\geq\ & e^{\lambda_1\xi}\Delta(\lambda_1, c) + e^{\eta\lambda_1\xi}\left\{-q\Delta(\eta\lambda_1, c) - r_2e^{(2-\eta)\lambda_1\xi} - r_2\bar{b}Ke^{(\lambda+\lambda_1-\eta\lambda_1)\xi}\right\} \\
\geq\ & e^{\eta\lambda_1\xi}[-q\Delta(\eta\lambda_1, c) - r_2 - r_2\bar{b}K].
\end{aligned}$$

Here we have used $\Delta(\lambda_1, c) = 0$, (3.12) and $\xi < \xi_2 < 0$. Finally, recalling the choice of q in (3.13), it follows that (3.8) also holds for $\xi < \xi_2$ and thus for any $\xi \in \mathbb{R} \setminus \{\xi_2\}$.

We conclude that $(\overline{\phi}, \overline{\psi})$ and $(\underline{\phi}, \underline{\psi})$ are a pair of upper and lower solutions of (3.4). And finally, it holds $(\phi, \psi)(-\infty) = (0, 0)$ since $(\underline{\phi}, \underline{\psi})(-\infty) = (\overline{\phi}, \overline{\psi})(-\infty) = (0, 0)$. The proof is complete. □

The proof of the existence of traveling waves for $c = c^*$ is more involved. To obtain our existence result we require an additional hypothesis on the compactness of the support of J_2. Our precise result reads as follows.

Theorem 3.4. *Assume that the function J_2 is compactly supported. Then system* (3.4) *with $c = c^*$ admits a nonnegative solution (ϕ, ψ) such that* $\lim_{\xi\to-\infty}(\phi(\xi), \psi(\xi)) = (0, 0)$.

Proof. Similarly, as for the proof of Theorem 3.3, we construct a suitable pair of upper and lower solutions satisfying the limit condition $(\overline{\phi}, \overline{\psi})(-\infty) = (\underline{\phi}, \underline{\psi})(-\infty) = (0, 0)$.

We introduce the following functions:

$$\begin{aligned}\overline{\phi}(\xi) &= \min\{1, Ke^{\lambda\xi}\}, \quad \underline{\phi}(\xi) = 0,\\ \overline{\psi}(\xi) &= \min\{-L\xi e^{\lambda^*\xi}, \bar{b}-1\}, \quad \underline{\psi}(\xi) = \max\{(-L\xi - q\sqrt{-\xi})e^{\lambda^*\xi}, 0\},\end{aligned}$$

where L, K, q are constants to be determined (in order) as follows while the constant $\lambda \in (0, \min\{\bar{\lambda}_1, \lambda^*/2\})$ is defined as before so that (3.10) holds with $c = c^*$.

First, since J_2 has a compact support, there exists a positive constant S such that

$$J_2(y) = 0, |y| > S.$$

For any large enough constant L, let $\xi_1 < \xi_2$ be two negative roots of the equation

$$-L\xi e^{\lambda^*\xi} = \bar{b} - 1.$$

Note that

$$\xi_1 < -\frac{1}{\lambda^*} < \xi_2 < 0 \text{ and } -L\xi e^{\lambda^*\xi} > \bar{b} - 1 \text{ for all } \xi \in (\xi_1, \xi_2).$$

We first choose $L > 0$ large enough so that

$$\xi_2 - \xi_1 > S. \tag{3.14}$$

Next fix $\lambda_* \in (\lambda^* - \lambda, \lambda^*)$ and observe that $\lambda_* > \lambda$ by noting $\lambda < \lambda^*/2$. Since $\lambda_* - \lambda^* - \lambda < 0$, we can choose $\xi_0 < 0$ such that

$$\xi_0 < \min\{-1/(\lambda^* - \lambda_*), \xi_1\}, \; e^{(\lambda_*-\lambda^*-\lambda)\xi_0}/(-\xi_0) > \bar{a}L.$$

For such a fixed ξ_0, we set $K := e^{-\lambda\xi_0}$. Since the function $\xi \mapsto (-\xi)e^{(\lambda^*-\lambda_*)\xi}$ is increasing on $(-\infty, -1/(\lambda^* - \lambda_*))$, we can easily verify that

$$-\bar{a}L\xi e^{\lambda^*\xi} < Ke^{\lambda_*\xi} < Ke^{\lambda\xi} \quad \text{for all } \xi \le \xi_0. \tag{3.15}$$

For any $q > 0$, the function $z \mapsto g(z) := [-Lz - q(-z)^{1/2}]e^{\lambda^* z}$ is positive and has a unique critical (maximal) point in $(-\infty, \xi_3)$, where $\xi_3 := -(q/L)^2$ (see [5]). Then we take q large enough such that

$$\xi_3 < \xi_0, \tag{3.16}$$

$$q \ge \frac{\max_{\xi<0}\left\{8(-\xi+S)^{3/2}\left[r_2\frac{K^2}{\bar{a}^2}e^{(2\lambda_*-\lambda^*)\xi} + r_2\bar{b}\frac{K^2}{\bar{a}}e^{(\lambda_*+\lambda-\lambda^*)\xi}\right]\right\}}{d_2\int_{\mathbb{R}} J_2(y)y^2e^{-\lambda^* y}dy}. \tag{3.17}$$

Note that the right-hand side of (3.17) is a well-defined finite number, since $\xi_3 < 0$, $2\lambda_* - \lambda^* > 2(\lambda^* - \lambda) - \lambda^* = \lambda^* - 2\lambda > 0$, and $\lambda_* + \lambda - \lambda^* > 0$.

Now observe that we have

$$\overline{\phi}(\xi) = \begin{cases} Ke^{\lambda\xi}, & \xi \le \xi_0,\\ 1, & \xi \ge \xi_0,\end{cases} \quad \overline{\psi}(\xi) = \begin{cases} -L\xi e^{\lambda^*\xi}, & \xi \le \xi_1,\\ \bar{b}-1, & \xi \ge \xi_1,\end{cases}$$

$$\underline{\psi}(\xi) = \begin{cases} (-L\xi - q\sqrt{-\xi})e^{\lambda^*\xi}, & \xi \le \xi_3,\\ 0, & \xi \ge \xi_3.\end{cases}$$

It is clear that $\underline{\phi} \le \overline{\phi}$ on $\mathbb{R}$. Also, by (3.16), $\xi_3 < \xi_0 < \xi_1$ so that we have $\underline{\psi} < \overline{\psi}$ on $\mathbb{R}$.

We now verify that (3.5) holds for any $\xi \in \mathbb{R}$ with $1 \neq Ke^{\lambda\xi}$, namely $\xi \neq \xi_0$. It is trivial for $\xi > \xi_0$. For $\xi < \xi_0$, we have $\overline{\phi}(\xi) = Ke^{\lambda\xi} < 1$. Then, by (3.15) and using $\xi_0 < \xi_1$, we have

$$r_1[1 - \overline{\phi}(\xi)][\bar{a}\overline{\psi}(\xi) - \overline{\phi}(\xi)] \le 0.$$

On the other hand, using (3.10) and noting that $\overline{\phi}(\xi) \le Ke^{\lambda\xi}$ for all $\xi \in \mathbb{R}$, we compute

$$\begin{aligned} d_1\mathcal{N}_1[\overline{\phi}](\xi) - c^*\overline{\phi}'(\xi) &= d_1\left[\int_{\mathbb{R}} J_1(y)\overline{\phi}(\xi - y)dy - Ke^{\lambda\xi}\right] - c^*\lambda Ke^{\lambda\xi} \\ &\le Ke^{\lambda\xi}\left\{d_1\left[\int_{\mathbb{R}} J_1(y)e^{\lambda y}dy - 1\right] - c^*\lambda\right\} < 0 \end{aligned}$$

for all $\xi < \xi_0$. Hence (3.5) holds for all $\xi \neq \xi_0$.

For $\xi > \xi_1$, we have $\overline{\psi}(\xi) = \bar{b} - 1$. Since $\overline{\psi} \le \bar{b} - 1$, (3.6) is trivial for $\xi > \xi_1$.

For $\xi < \xi_1$, with the assumption that J_2 has a compact support in $[-S, S]$ and (3.14), we have $\xi - y \le \xi_1 + S < \xi_2$ for all $y \in [-S, S]$ so that we get

$$\begin{aligned} \int_{\mathbb{R}} \overline{\psi}(\xi - y)J_2(y)dy &= \int_{-S}^{S} \overline{\psi}(\xi - y)J_2(y)dy \\ &\le \int_{\mathbb{R}} \left\{-L(\xi - y)e^{\lambda^*(\xi - y)}\right\} J_2(y)dy. \end{aligned}$$

Hence we obtain

$$\begin{aligned} & d_2\mathcal{N}_2[\overline{\psi}](\xi) - c^*\overline{\psi}'(\xi) + r_2\overline{\psi}(\xi)[\bar{b} - 1 - \overline{\psi}(\xi) - \bar{b}\underline{\phi}(\xi)] \\ \le\ & d_2\mathcal{N}_2[\overline{\psi}](\xi) - c^*\overline{\psi}'(\xi) + r_2(\bar{b} - 1)\overline{\psi}(\xi) \\ =\ & d_2\left[\int_{\mathbb{R}} \overline{\psi}(\xi - y)J_2(y)dy + L\xi e^{\lambda^*\xi}\right] + c^*L(\lambda^*\xi + 1)e^{\lambda^*\xi} - r_2L(\bar{b} - 1)\xi e^{\lambda^*\xi} \\ \le\ & d_2\left[\int_{\mathbb{R}} \left\{-L(\xi - y)e^{\lambda^*(\xi - y)}\right\} J_2(y)dy + L\xi e^{\lambda^*\xi}\right] + c^*L(\lambda^*\xi + 1)e^{\lambda^*\xi} - r_2L(\bar{b} - 1)\xi e^{\lambda^*\xi} \\ =\ & -L\xi e^{\lambda^*\xi}\left\{d_2\left[\int_{\mathbb{R}} e^{\lambda^* y}J_2(y)dy - 1\right] - c^*\lambda^* + r_2(\bar{b} - 1)\right\} \\ & + Le^{\lambda^*\xi}\left[d_2\int_{\mathbb{R}} ye^{-\lambda^* y}J_2(y)dy + c^*\right] = 0 \end{aligned}$$

for $\xi < \xi_1$, where (3.9) is used with a change of variable $z = -y$ yielding

$$d_2\int_{-\infty}^{\infty} ye^{-\lambda^* y}J_2(y)dy = -d_2\int_{-\infty}^{\infty} ze^{\lambda^* z}J_2(z)dz = -c^*. \tag{3.18}$$

Therefore, (3.6) holds true for all $\xi \neq \xi_1$.

Next, since $\overline{\phi}(\xi) \equiv 0$, (3.7) is clear and it remains to verify (3.8). On the one hand, one has $\underline{\psi}(\xi) = 0$ for $\xi > \xi_3$ and (3.8) holds true for $\xi > \xi_3$.

On the other hand for $\xi < \xi_3$, we have $\underline{\psi}(\xi) = (-L\xi - q\sqrt{-\xi})e^{\lambda^*\xi}$. Using $\underline{\psi}(z) \geq (-Lz - q\sqrt{-z})e^{\lambda^* z}$ for all $z \in \mathbb{R}$, we compute

$$
\begin{aligned}
&d_2\mathcal{N}_2[\underline{\psi}](\xi) - c^*\underline{\psi}'(\xi) = d_2\left[\int_{\mathbb{R}} J_2(y)\underline{\psi}(\xi-y)dy - \underline{\psi}(\xi)\right] - c^*\underline{\psi}'(\xi)\\
\geq\ & d_2\int_{\mathbb{R}} J_2(y)\left\{\left[-L(\xi-y) - q\sqrt{-(\xi-y)}\right]e^{\lambda^*(\xi-y)}\right\}dy - d_2(-L\xi - q\sqrt{-\xi})e^{\lambda^*\xi}\\
&+c^*L\left(1+\lambda^*\xi\right)e^{\lambda^*\xi} + c^*q\left(\lambda^*\sqrt{-\xi} - \frac{1}{2\sqrt{-\xi}}\right)e^{\lambda^*\xi} := I_1(\xi) + I_2(\xi),
\end{aligned}
$$

wherein we have set

$$
\begin{aligned}
&I_1(\xi) := d_2e^{\lambda^*\xi}\left[\int_{\mathbb{R}} J_2(y)[-L(\xi-y)]e^{-\lambda^* y}dy + L\xi\right] + c^*L\left(1+\lambda^*\xi\right)e^{\lambda^*\xi},\\
&I_2(\xi) := -qd_2e^{\lambda^*\xi}\left[\int_{\mathbb{R}} J_2(y)\sqrt{-(\xi-y)}e^{-\lambda^* y}dy - \sqrt{-\xi}\right] + c^*q\left(\lambda^*\sqrt{-\xi} - \frac{1}{2\sqrt{-\xi}}\right)e^{\lambda^*\xi}.
\end{aligned}
$$

Using (3.15) and (3.16), we also compute

$$
\begin{aligned}
&r_2\underline{\psi}(\xi)[\bar{b} - 1 - \underline{\psi}(\xi) - \bar{b}\overline{\phi}(\xi)]\\
\geq\ & r_2\underline{\psi}(\xi)(\bar{b}-1) - r_2\overline{\psi}^2(\xi) - r_2\bar{b}\overline{\phi}(\xi)\overline{\psi}(\xi)\\
\geq\ & r_2(\bar{b}-1)\left[(-L\xi - q\sqrt{-\xi})e^{\lambda^*\xi}\right] - r_2\frac{K^2}{\bar{a}^2}e^{2\lambda_*\xi} - r_2\bar{b}\frac{K^2}{\bar{a}}e^{(\lambda_*+\lambda)\xi}.
\end{aligned}
$$

Combining the above two estimates and using

$$
d_2\left[\int_{\mathbb{R}} J_2(y)e^{-\lambda^* y}dy - 1\right] - c^*\lambda^* + r_2(\bar{b}-1) = 0,\ \int_{\mathbb{R}} J_2(y)ye^{-\lambda^* y}dy + c^* = 0,
$$

we infer that

$$
\begin{aligned}
&d_2\mathcal{N}_2[\underline{\psi}](\xi) - c^*\underline{\psi}'(\xi) + r_2\underline{\psi}(\xi)[\bar{b} - 1 - \underline{\psi}(\xi) - \bar{b}\overline{\phi}(\xi)]\\
\geq\ & I_1(\xi) + I_2(\xi) + r_2(\bar{b}-1)\left[(-L\xi - q\sqrt{-\xi})e^{\lambda^*\xi}\right] - r_2\frac{K^2}{\bar{a}^2}e^{2\lambda_*\xi} - r_2\bar{b}\frac{K^2}{\bar{a}}e^{(\lambda_*+\lambda)\xi}\\
=\ & -qd_2e^{\lambda^*\xi}\left[\int_{\mathbb{R}} J_2(y)\sqrt{-(\xi-y)}e^{-\lambda^* y}dy - \sqrt{-\xi}\right] + c^*q\left(\lambda^*\sqrt{-\xi} - \frac{1}{2\sqrt{-\xi}}\right)e^{\lambda^*\xi}\\
&-r_2(\bar{b}-1)q\sqrt{-\xi}e^{\lambda^*\xi} - r_2\frac{K^2}{\bar{a}^2}e^{2\lambda_*\xi} - r_2\bar{b}\frac{K^2}{\bar{a}}e^{(\lambda_*+\lambda)\xi}\\
:=\ & e^{\lambda^*\xi}[qI_3(\xi) - I_4(\xi)],
\end{aligned}
$$

with

$$
\begin{aligned}
I_3(\xi)\ &:=\ -d_2\left[\int_{\mathbb{R}} J_2(y)\sqrt{-(\xi-y)}e^{-\lambda^* y}dy - \sqrt{-\xi}\right]\\
&\qquad +c^*\left(\lambda^*\sqrt{-\xi} - \frac{1}{2\sqrt{-\xi}}\right) - r_2(\bar{b}-1)\sqrt{-\xi},\\
I_4(\xi)\ &:=\ r_2\frac{K^2}{\bar{a}^2}e^{(2\lambda_*-\lambda^*)\xi} + r_2\bar{b}\frac{K^2}{\bar{a}}e^{(\lambda_*+\lambda-\lambda^*)\xi}.
\end{aligned}
$$

To conclude, we need to derive a lower bound estimate of $I_3(\xi)$ for $\xi < \xi_3$. For this, first note, using $\Delta(\lambda^*, c^*) = 0$, that one has

$$\begin{aligned} I_3(\xi) &= -d_2 \left\{ \int_{\mathbb{R}} J_2(y) \left[\sqrt{-\xi} + \sqrt{-(\xi - y)} - \sqrt{-\xi} \right] e^{-\lambda^* y} dy - \sqrt{-\xi} \right\} \\ &\quad + c^* \left(\lambda^* \sqrt{-\xi} - \frac{1}{2\sqrt{-\xi}} \right) - r_2(\bar{b} - 1)\sqrt{-\xi} \\ &= -d_2 \left\{ \int_{\mathbb{R}} J_2(y) \left[\sqrt{-(\xi - y)} - \sqrt{-\xi} \right] e^{-\lambda^* y} dy \right\} - \frac{c^*}{2\sqrt{-\xi}}, \end{aligned}$$

Next, it follows from (3.18) that

$$\begin{aligned} I_3(\xi) &= d_2 \left\{ \int_{\mathbb{R}} J_2(y) \left[\sqrt{-\xi} - \sqrt{-(\xi - y)} \right] e^{-\lambda^* y} dy \right\} + \frac{d_2}{2\sqrt{-\xi}} \int_{\mathbb{R}} J_2(y) y e^{-\lambda^* y} dy \\ &:= d_2 \int_{\mathbb{R}} J_2(y) Q(\xi, y) e^{-\lambda^* y} dy, \end{aligned}$$

where $Q = Q(\xi, y)$ is defined by

$$Q(\xi, y) := \frac{y}{2\sqrt{-\xi}} + \sqrt{-\xi} - \sqrt{-(\xi - y)}.$$

Now observe that

$$\begin{aligned} Q(\xi, y) &= \frac{y}{2\sqrt{-\xi}} - \frac{y}{\sqrt{-\xi} + \sqrt{-(\xi - y)}} = \frac{y[\sqrt{-(\xi - y)} - \sqrt{-\xi}]}{2\sqrt{-\xi}[\sqrt{-\xi} + \sqrt{-(\xi - y)}]} \\ &= \frac{y^2}{2\sqrt{-\xi}[\sqrt{-\xi} + \sqrt{-(\xi - y)}]^2}. \end{aligned}$$

Since $J_2 = 0$ outside $[-S, S]$ and

$$2\sqrt{-\xi}[\sqrt{-\xi} + \sqrt{-(\xi - y)}]^2 \le 8(-\xi + S)^{3/2} \text{ for } |y| < S,$$

we end up with the following lower estimate

$$I_3(\xi) \ge \frac{d_2}{8(-\xi + S)^{3/2}} \int_{\mathbb{R}} J_2(y) y^2 e^{-\lambda^* y} dy \quad \text{for } \xi < \xi_3.$$

Finally, we infer from the above computations and using the definition q in (3.17) that

$$d_2 \mathcal{N}_2[\underline{\psi}](\xi) - c^* \underline{\psi}'(\xi) + r_2 \underline{\psi}(\xi)[\bar{b} - 1 - \underline{\psi}(\xi) - \bar{b}\overline{\phi}(\xi)] \ge 0,$$

for all $\xi < \xi_3$. Thereby the proof is complete. □

The following theorem completes the proof of the existence part of Theorem 1.3.

Theorem 3.5. *Let $c \ge c^*$ and let (ϕ, ψ) be any solution provided by Theorem 3.3 or Theorem 3.4. Then it holds that $\limsup_{\xi \to \infty} \phi(\xi) < 1$. Moreover, the second properties in (3.3) also holds true, if we further assume that $\bar{a}\bar{b} < 1$.*

Proof. First, recall the parameter condition $\bar{a}(\bar{b} - 1) < 1$. Set

$$(U, V)(x, t) := (1 - \phi, \psi)(x + ct).$$

Then, since $V \le \bar{b} - 1$ in $\mathbb{R} \times \mathbb{R}$, we infer from (1.1) that $U = U(x,t)$ satisfies

$$\frac{\partial U(x,t)}{\partial t} \ge d_1 \mathcal{N}_1[U](x,t) + r_1 U(x,t)\{[1 - \bar{a}(\bar{b}-1)] - U(x,t)\},\ x \in \mathbb{R},\ t > 0,$$

such that $U(x,0) = 1 - \phi(x) \ge 0$ and $U \not\equiv 0$. Indeed, $U(x,0) \to 1$ as $x \to -\infty$. It then follows from Proposition 2.1 that $U \ge z$, where z is the solution to

$$\begin{cases} \dfrac{\partial z(x,t)}{\partial t} = d_1 \mathcal{N}_1[z(\cdot,t)](x) + r_1\{z[1 - \bar{a}(\bar{b}-1) - z]\}(x,t),\ x \in \mathbb{R},\ t > 0, \\ z(x,0) = U(x,0),\ x \in \mathbb{R}. \end{cases}$$

Now, Proposition 2.2 applies and ensures that $z(x,t) \to 1 - a(b-1)$ as $t \to \infty$ for all $x \in \mathbb{R}$. Hence, one obtains in particular that

$$\liminf_{t\to\infty} U(0,t) \ge \liminf_{t\to\infty} z(0,t),$$

that re-writes as

$$\limsup_{t\to\infty} \phi(ct) = 1 - \liminf_{t\to\infty} U(0,t) \le 1 - \liminf_{t\to\infty} z(0,t) = \bar{a}(\bar{b}-1),$$

and, recalling that $c > 0$, this proves the first statement of Theorem 3.5, namely

$$\limsup_{\xi\to\infty} \phi(\xi) \le \bar{a}(\bar{b}-1) < 1. \tag{3.19}$$

Next, we claim that

$$B := \sup_{\xi\in\mathbb{R}} \phi(\xi) \le \bar{a}(\bar{b}-1). \tag{3.20}$$

To prove this claim, we argue by contradiction by assuming that $B > \bar{a}(\bar{b}-1)$. Then, due to (3.19) and recalling that $\phi(\xi) \to 0$ as $\xi \to -\infty$, there exists $\xi_0 \in \mathbb{R}$ such that $\phi(\xi_0) = B$. Hence $\phi(\xi_0) = B \ge \phi(\xi)$ for all $\xi \in \mathbb{R}$. On the other hand, using $\phi(-\infty) = 0$, $\phi \le 1$ and $\psi \le \bar{b} - 1$, we get

$$\begin{aligned} &(J_1 * \phi)(\xi_0) = \int_{\mathbb{R}} J_1(y)\phi(\xi_0 - y)dy < \phi(\xi_0)\int_{\mathbb{R}} J_1(y)dy = \phi(\xi_0), \\ &r_1[1 - \phi(\xi_0)][\bar{a}\psi(\xi_0) - \phi(\xi_0)] \le 0, \end{aligned}$$

and $\phi'(\xi_0) = 0$. This contradicts the first equation of (3.4). Hence (3.20) holds.

Now, using this upper bound we shall complete the proof of Theorem 3.5 by proving the second limit condition in (3.3). Here recall that we assume the parameter condition $\bar{a}\bar{b} < 1$. Using (3.20), the function $V(x,t) = \psi(x+ct)$ satisfies

$$\begin{cases} \dfrac{\partial V(x,t)}{\partial t} \ge d_2 \mathcal{N}_2[V(\cdot,t)](x) + r_2 v(x,t)[(1-\bar{a}\bar{b})(\bar{b}-1) - V(x,t)],\ x \in \mathbb{R},\ t > 0, \\ V(x,0) = \psi(x) > 0. \end{cases}$$

Then a similar argument as above we deduce that

$$\liminf_{\xi\to\infty} \psi(\xi) \ge (1-\bar{a}\bar{b})(\bar{b}-1) > 0$$

and this completes the proof of the theorem. □

We now give a proof of the nonexistence part of Theorem 1.3 as follows.

Theorem 3.6. *If $c \in (0, c^*)$, then* (3.4) *does not admit any non-trivial nonnegative solution (ϕ, ψ) satisfying* (3.3) *and $(\phi, \psi)(-\infty) = (0, 0)$.*

Proof. To prove this result we argue by contradiction by assuming that (3.4) admits a non-trivial nonnegative solution (ϕ, ψ) satisfying $(\phi, \psi)(-\infty) = (0, 0)$ and (3.3) for some wave speed $c_1 \in (0, c^*)$. Here recall that $\phi > 0$ and $\psi > 0$ on $\mathbb{R}$ (see the remark after Proposition 2.2).

Let $\epsilon \in (0, (\bar{b} - 1)/2)$ be given such that

$$c_2 := \inf_{\lambda > 0} \frac{d_2 \left[\int_{\mathbb{R}} J_2(y) e^{\lambda y} dy - 1 \right] + r_2(\bar{b} - 1 - \epsilon)}{\lambda} > c_1.$$

Since $\phi(-\infty) = 0$, there exists a $\xi_0 \in \mathbb{R}$ such that $\bar{b}\phi(\xi) < \epsilon$ for all $\xi \leq \xi_0$. When $\xi > \xi_0$, the strict positivity of ψ and (3.3) imply that there exists $\delta > 0$ such that

$$\inf_{\xi > \xi_0} \psi(\xi) \geq \delta.$$

Define $M > 1$ by $\delta(M - 1) = \bar{b}$. Then, since $\phi \leq 1$, we have

$$\bar{b}\phi(\xi) \leq (M - 1)\psi(\xi) \text{ for all } \xi \geq \xi_0.$$

Hence

$$c_1 \psi'(\xi) \geq d_2 \mathcal{N}_2[\psi](\xi) + r_2 \psi(\xi)[\bar{b} - 1 - \epsilon - M\psi(\xi)], \ \xi \in \mathbb{R},$$

and the function $V(x, t) := \psi(x + c_1 t)$ satisfies

$$\begin{cases} \dfrac{\partial V(x,t)}{\partial t} \geq d_2 \mathcal{N}_2[V(\cdot, t)](x) + r_2 v(x,t)[\bar{b} - 1 - \epsilon - MV(x,t)], \ x \in \mathbb{R}, \ t > 0, \\ V(x, 0) = \psi(x) > 0. \end{cases} \tag{3.21}$$

Now, since $(c_1 + c_2)/2 < c_2$, Propositions 2.1 and 2.2 lead us to

$$\liminf_{t \to \infty} V\left(-\frac{c_2 + c_1}{2} t, t \right) \geq \frac{\bar{b} - 1 - 2\epsilon}{M} > 0. \tag{3.22}$$

However,

$$-(c_2 + c_1)t/2 + c_1 t = (c_1 - c_2)t/2 \to -\infty \text{ as } t \to \infty,$$

and (3.22) yields

$$\liminf_{\xi \to -\infty} \psi(\xi) = \liminf_{t \to \infty} V(-(c_2 + c_1)t/2, t) \geq \frac{\bar{b} - 1 - 2\epsilon}{M} > 0,$$

that contradicts the limit condition $\psi(-\infty) = 0$. Thus the theorem is proved. $\square$

Finally, we prove Corollary 1.4.

Proof of Corollary 1.4. To prove the corollary, we shall show that, under the parameter condition $d_2 < b - r_2$ (or, $d_2 < r_2 \left(\bar{b} - 1 \right)$), any nonnegative solution (ϕ, ψ) of system (3.4) together with $(\phi, \psi)(-\infty)(0, 0)$ and (3.3) satisfies $c > 0$.

To that aim, we argue by contradiction by assuming that problem (3.4) admits a nonnegative solution (ϕ, ψ) with speed $c \leq 0$ satisfying both (3.3) and $(\phi, \psi)(-\infty) = (0, 0)$.

First, since $(\phi, \psi)(-\infty) = (0, 0)$, there is a sufficiently large constant $K > 0$ such that

$$r_2[\bar{b} - 1 - \psi(\xi) - \bar{b}\phi(\xi)] > \alpha := \frac{r_2(\bar{b} - 1)}{2} + \frac{d_2}{2}, \ \forall \xi \leq -K.$$

Next, since $J_2 \geq 0$ and $\psi > 0$, one has $(J_2 * \psi)(\xi) \geq 0$ for all $\xi \in \mathbb{R}$. It then follows from the second equation in (3.4) that

$$c\psi'(\xi) \geq \gamma\psi(\xi),\ \forall\, \xi \leq -K,$$

where $\gamma := (\alpha - d_2) > 0$, due to the assumption $d_2 < r_2(\bar{b} - 1)$. Integrating the above inequality from $-\infty$ to $y \leq -K$ yields

$$c\psi(y) \geq \gamma \int_{-\infty}^{y} \psi(\xi)d\xi,\ \forall\, y \leq -K.$$

This contradicts $c \leq 0$ and proves that $c > 0$. The proof of the corollary is complete. □

The following proposition shows that the nontrivial state at $\xi = \infty$ is actually the co-existence state, if the wave tail does converge at $\xi = \infty$.

Proposition 3.7. *Let (U, V) be a traveling wave of (1.1) connecting $(1, 0)$ and a nontrivial state. Suppose that the limit*

$$(\bar{u}, \bar{v}) := \lim_{\xi \to \infty} (U, V)(\xi)$$

exists. Then $(\bar{u}, \bar{v}) = (u^*, v^*)$, where (u^*, v^*) is defined in (1.7).

Proof. First, by assumption, $(U_t, V_t)(\xi) \to (0, 0)$ as $\xi \to \infty$. Note that $\bar{u} > 0$ and $\bar{v} > 0$, by (1.6). Moreover, by the a priori estimates, $\bar{u} \leq 1$ and $\bar{v} \leq \bar{b} - 1$. Then the lemma follows if we can show that

$$(J_1 * U)(\xi) \to \bar{u}, \quad (J_1 * V)(\xi) \to \bar{v} \quad \text{as } \xi \to \infty, \tag{3.23}$$

since the only nontrivial solution to

$$r_1 U(1 - U) - aUV = 0, \quad -r_2 V(1 + V) + bUV = 0,$$

is (u^*, v^*).

Next, since the proof of the second limit in (3.23) is similar to the first one, we only give a proof of the first one. Given $\epsilon \in (0, \bar{u})$. Since $U(+\infty) = \bar{u}$ and $\int_{\mathbb{R}} J_1(y)dy = 1$, there is $M > 0$ sufficiently large such that

$$\int_{M}^{\infty} J_1(y)dy < \frac{\epsilon}{2} \quad \text{and} \quad \bar{u} - \frac{\epsilon}{2} < U(\xi) < \bar{u} + \frac{\epsilon}{2}, \quad \forall\, \xi \geq M.$$

We write

$$\int_{\mathbb{R}} J_1(y)U(\xi - y)dy = \int_{-\infty}^{M} J_1(y)U(\xi - y)dy + \int_{M}^{\infty} J_1(y)U(\xi - y)dy := I_1(\xi) + I_2(\xi).$$

Then we have

$$I_1(\xi) + I_2(\xi) \leq \left(\bar{u} + \frac{\epsilon}{2}\right) + \frac{\epsilon}{2} = \bar{u} + \epsilon \tag{3.24}$$

for all $\xi \geq 2M$, by using $\xi - y \geq M$ for $y \leq M$ and $\xi \geq 2M$, $\int_{\mathbb{R}} J_1(y)dy = 1$ and $U \leq 1$. Also, we have

$$I_1(\xi) + I_2(\xi) \geq I_1(\xi) \geq \left(\bar{u} - \frac{\epsilon}{2}\right)\left(1 - \frac{\epsilon}{2}\right) \geq \bar{u} - \epsilon \tag{3.25}$$

for all $\xi \geq 2M$, by using $\bar{u} \leq 1$. Combining (3.24) and (3.25), we have shown that $(J_1 * U)(\xi) \to \bar{u}$ as $\xi \to \infty$ and so (3.23) follows. Therefore, the proposition is proved. □

However, whether the traveling wave solutions converge to the co-existence state as $\xi \to \infty$ is still left open. One of the difficulties is due to the nonlocal dispersal term.

Finally, comparing equations with or without nonlocal dispersal, we have the following remark. First, the spreading speed here is actually the asymptotically spreading speed of the *predator*. At the leading edge, our solution behaves like the predator-free state. Hence it is expected that the predator propagates as what Theorem 1.2 described. Since we are dealing with the nonlocal dispersal, the formula for the spreading speed involves the infimum over a rational function with an integral term. However, if the kernel is the classical Gaussian kernel, it is expected that the spreading speed of the nonlocal dispersal problem tends to the spreading speed of the classical Fisher-KPP equation, by taking a suitable scaling with a small parameter tending to zero.

We provide some detailed justifications of the above result as follows. For the classical Fisher-KPP equation with Laplace operator, namely,

$$u_t = du_{xx} + f(u), \quad x \in \mathbb{R},\ t > 0, \quad f(u) := ru(s-u),$$

the spreading speed c^* is given by

$$c^* := \inf_{\lambda>0} \frac{d\lambda^2 + f'(0)}{\lambda} = 2\sqrt{f'(0)}.$$

In fact, c^* is exactly the limit of the following spreading speeds

$$c^*_\delta := \inf_{\lambda>0} \frac{d[\int_{\mathbb{R}} J_\delta(y)e^{\lambda y} - 1]/\delta^2 + f'(0)}{\lambda}$$

when we take $\delta \downarrow 0$, where

$$J_\delta(y) := \frac{1}{\delta} J\left(\frac{y}{\delta}\right), \quad J(y) := \frac{1}{2\sqrt{\pi}} e^{-y^2/4} \text{ (the Gaussian kernel)}.$$

The corresponding equation with nonlocal dispersal is given by

$$w_t = \frac{1}{\delta^2} d[J_\delta * w - w] + f(w)$$

and the solution $w = w_\delta$ converges to the solution u of the classical Fisher-KPP equation.

For our predator-prey model, since we are concerned with the spreading of the predator, the predator equation plays the key role. At the leading edge, the solution behaves as the predator-free state. Therefore, the spreading speed is given by

$$\inf_{0<\lambda<\bar{\lambda}_2} \frac{d_2[\int_{\mathbb{R}} J_2(y)e^{\lambda y}dy - 1] + G_V(1,0)}{\lambda}, \quad G_V(U,V) := \frac{\partial G}{\partial V}(U,V) = bU - r_2(1+2V),$$

which is exactly the quantity c^* defined in (1.5).

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Normandie Univ, UNIHAVRE, LMAH, FR-CNRS-3335, ISCN, 76600 Le Havre, France
E-mail address: arnaud.ducrot@univ-lehavre.fr

Department of Mathematics, Tamkang University, 151, Yingzhuan Road, Tamsui, New Taipei City 25137, Taiwan
E-mail address: jsguo@mail.tku.edu.tw

School of Mathematics and Statistics, Lanzhou University, Lanzhou, Gansu 730000, People's Republic of China
E-mail address: ling@lzu.edu.cn

School of Science, Lanzhou University of Technology, Lanzhou, Gansu 730050, People's Republic of China
E-mail address: shxpan@yeah.net