QUENCHING BEHAVIOR FOR A SINGULAR PREDATOR-PREY MODEL

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Abstract. In this paper, we study the quenching behavior for a system of two reaction-diffusion equations arising in the modelling of the spatio-temporal interaction of prey and predator populations in fragile environment. We first provide some sufficient conditions on the initial data to have finite time quenching. Then we classify the initial data to distinguish type I quenching and type II quenching, by introducing a delicate energy functional along with the help of some a priori estimates. Finally, we present some results on the quenching set. It can be a singleton, the whole domain, or a compact subset of the domain.

1. Introduction

We consider in this work the following reaction-diffusion system

$$\begin{cases} u_t - u_{xx} = -v \\ v_t - v_{xx} = -r\frac{v^2}{u} \end{cases} \tag{1.1}$$

posed for $x \in (0,\infty)$ and $t \in (0,T)$ and supplemented together with homogeneous Neumann boundary condition and initial condition

$$\begin{cases} u_x(t,0) = v_x(t,0) = 0, \quad t \in (0,T) \\ (u(0,x), v(0,x)) = (u_0(x), v_0(x)), \quad x \geq 0, \end{cases} \tag{1.2}$$

where u_0 and v_0 are two positive and smooth functions on $[0,\infty)$.

System (1.1) is a simplification close to quenching of a more complex reaction-diffusion system proposed by Gaucel and Langlais in [7]. The complete system they proposed reads with normalized parameters as

$$\begin{cases} u_t - d\Delta u = r_u u(1-u) - v \\ v_t - \Delta v = rv\left(1 - \frac{v}{u}\right). \end{cases} \tag{1.3}$$

The above system of equations models the spatio-temporal interaction of prey and predator populations in fragile environment. More specifically, u (respectively v) denotes the spatial density of prey (respectively predator). In the absence of predator

Date: April 5, 2012. Corresponding author: J.-S. Guo.

2000 Mathematics Subject Classification. 35K45, 35K57, 35K55.

This work was supported in part by the Orchid program between France and Taiwan with grants NSC 99-2911-I-032-004 and NSC 100-2911-I-032-001. The second author is also supported in part by the National Science Council of the Republic of China under the grant NSC 99-2115-M-032-006-MY3. We would like to thank Michel Langlais and Jean-Baptiste Burie for some valuable discussions. Thanks are also to the anonymous referees for some valuable comments.

(namely $v \equiv 0$), the prey density follows the usual logistic growth with parameter $r_u \geq 0$ and a normalized carrying capacity. On the other hand, the equation for the density of predators follows a logistic dynamics with a varying carrying capacity proportional to the density of prey. From a modelling point of view, this specific form allows the density of predator to reach high density when the food is unlimited ($u \to \infty$). It also increases the competition between predators when the density of prey is low ($u \to 0$). In that case, the carrying capacity of predators goes to zero and leads to the extinction of the population of predators. We refer to Courchamp and Sugihara in [2] for more details on the derivation of the model and applicability for cats and birds dynamics in insular environment.

The main property of the above system is its ability to exhibit a finite time and simultaneous extinction of both species. The dynamical property of kinetic system, namely the underlying ordinary differential equation, has been well studied. Firstly introduced by Courchamp and Sugihara in [2] and Courchamp et al in [3], it has been further investigated by Gaucel and Langlais [7]. Extensions to the reaction-diffusion system (1.3) posed on a bounded domain and supplemented together with the zero flux boundary condition have been provided in [7] in the equi-diffusional case, namely $d = 1$. One can also notice that when the quenching occurs for (1.3) then the ratio v/u blows up in finite time. This remark allows us to introduce (1.1) as a formal simplification of (1.3) close to quenching. Indeed, close to quenching, u becomes negligible with respect to v leading to (1.1).

The aim of this paper is to consider (1.1) and to give some information on the quenching behavior. Since the parameter $r > 0$ plays a crucial role on the quenching behavior of (1.3) but also (1.1), throughout this work, we always assume that the parameter r satisfies $0 < r < 1$. Let us however mention that parameter r_u in (1.3) also plays an important role. The logistic dynamics for prey acts in favor of an increase of the prey density and therefore acts against the quenching phenomenon. This balance may induce temporal and spatio-temporal oscillations that become difficult to control and to analyze. This is the reason why we will restrict our analysis to the simplified system (1.1).

In order to fulfil this analysis, we allow the initial data to be unbounded when $x \to \infty$. More precisely we will assume that

Assumption 1.1. *Function* $(u_0, v_0) \in C^2\left([0,\infty)\right)^2$ *is assumed to satisfy*

(i) $u_0'(0) = v_0'(0) = 0$.
(ii) *There exists* $\varepsilon > 0$ *and* $k > 0$ *such that*

$$\varepsilon \leq u_0(x) \leq k(1 + x^2), \quad \forall x \in [0, \infty).$$

(iii) *For each* $x \geq 0$, $v_0(x) \geq 0$ *and there exists* $M > 0$ *such that for all* $x \geq 0$:

$$P_0(x) := \frac{v_0(x)}{u_0(x)} \leq M.$$

Here we allow the initial data to be unbounded. This property will be used in the sequel to derive the existence of type II quenching (see Corollary 1.9 and 1.10). See also Remark 1.11 below.

Recall that from usual existence result for reaction-diffusion systems, under Assumption 1.1, system (1.1)-(1.2) has a unique classical maximal solution $(u,v) \equiv (u,v)(t,x)$ on a maximal time interval $[0,T)$ with $T>0$ such that

$$(u,v) \in C^{1,2}\left([0,T)\times[0,\infty)\right)^2, \quad u>0, \quad v \geq 0,$$

and if $T<\infty$ then

$$\liminf_{t\nearrow T} \inf_{x\geq 0} u(t,x) = 0. \tag{1.4}$$

The property (1.4) with $T<\infty$ is referred to as finite time quenching and one can introduce the so-called quenching set $\mathcal{Q} = \mathcal{Q}(u_0,v_0)$ defined by

$$\mathcal{Q} := \{x \geq 0 | \; \exists \{(t_n,x_n)\}_{n\geq 0} \subset [0,T)\times[0,\infty) \text{ such that } \lim_{n\to\infty}(t_n,x_n) = (T,x) \text{ and } \lim_{n\to\infty} u(t_n,x_n) = 0\}.$$

Definition 1.2. *Suppose that finite time quenching occurs at time T. Then it is called type I quenching if*

$$\liminf_{t\nearrow T}(T-t)^{-\frac{1}{1-r}} \inf_{x\geq 0} u(t,x) > 0.$$

Otherwise, it is referred to as type II quenching.

Then the main results of this work are the following:

Theorem 1.3. *Let Assumption 1.1 be satisfied. Assume moreover that there exists $\lambda>0$ such that*

$$\lambda u_0(x) = v_0(x), \quad \forall x \geq 0.$$

Then system (1.1)-(1.2) exhibits a finite time quenching. It is of type I and the quenching set $\mathcal{Q}$ satisfies $\mathcal{Q} = [0,\infty)$.

For more general initial data, let us introduce the following assumption:

Assumption 1.4. *We assume that $(u_0,v_0) \in C^2\left([0,\infty)\right)^2$ such that $\inf_{x\geq 0} u_0(x) > 0$, $v_0 \not\equiv 0$, $v_0(x) \geq 0$ for all $x \geq 0$ and together with the compatibility condition at $x=0$, that is $u_0'(0) = v_0'(0) = 0$. Finally we assume that*

$$u_0(x) = O(x^2) \text{ when } x \to \infty \text{ and } P_0 := v_0/u_0 \text{ is bounded.}$$

Then the following result holds true:

Theorem 1.5. *Let Assumption 1.4 be satisfied. We assume furthermore that*

$$u_0'(x)^2 \leq 2u_0(x)v_0(x) \text{ and } P_0'(x) \leq 0 \;\; \forall x \in [0,\infty). \tag{1.5}$$

Then system (1.1)-(1.2) exhibits a finite time quenching.

Next, in order to derive the behavior of type I quenching, let us introduce the following set of assumption:

Assumption 1.6. *Let (u_0, v_0) be two given functions satisfying Assumption 1.1. Furthermore assume that*

(i) *For all $x \geq 0$, $u_0'(x) \geq 0$, $v_0'(x) \geq 0$ and $P_0'(x) \leq 0$.*
(ii) *There exists two constants $k_1 > 0$ and $k_2 > 0$ such that for all $x \geq 0$:*

$$v_0(x) \leq k_1 \left(u_0(x)\right)^r, \quad \left(u_0'(x)\right)^2 \leq k_2 u_0(x)^{1+r}. \tag{1.6}$$

Then we have the following information on the quenching behavior.

Theorem 1.7. *Let Assumption 1.6 be satisfied. Assume furthermore that the solution has a finite time quenching at time T. If the quenching is of type I, then the following behavior holds true:*

$$\lim_{t \uparrow T} [P(t,x)(T-t)] = 1/(1-r),$$
$$\lim_{t \uparrow T} [u_x(t,x)(T-t)^{-1/(1-r)+1/2}] = 0,$$
$$\lim_{t \uparrow T} [v_x(t,x)(T-t)^{-r/(1-r)+1/2}] = 0,$$

uniformly on $\{(t,x) \mid 0 \leq x \leq L\sqrt{T-t}\}$ for any $L > 0$, wherein we have set $P(t,x) := v(t,x)/u(t,x)$.

As a consequence of Theorem 1.7, we are able to provide a class of initial data leading to type II quenching. To do so let us consider the following assumption.

Assumption 1.8. *Let (u_0, v_0) be two given functions satisfying Assumption 1.6. We assume furthermore that*

$$x^2 v_0(x) \leq \frac{2}{1-r} u_0(x), \quad \forall x \geq 0.$$

Then the following results hold true.

Corollary 1.9. *Let Assumption 1.8 be satisfied. If the solution of system* (1.1)-(1.2) *exhibits finite time quenching at time T, then the quenching occurs at the single point $x = 0$, namely $\mathcal{Q} = \{0\}$, and it is of type II quenching.*

Corollary 1.10. *Let us consider the following initial data*

$$u_0(x) = \alpha + \beta x^2, \quad v_0(x) \equiv \kappa,$$

for some constants $\alpha > 0$, $\beta > 0$ and $\kappa > 0$. If $\kappa \in [2\beta, 2\beta/(1-r)]$, then the corresponding solution (u, v) of system (1.1)-(1.2) *exhibits a finite time quenching at time T, the quenching set satisfies $\mathcal{Q} = \{0\}$ and the quenching is of type II.*

Remark 1.11. *For the technical reason, the aforementioned results require the initial data (at least for u) to be unbounded. From an applicative point of view, this assumption can be understood by looking at the ratio $P_0 := \frac{v_0}{u_0}$. Indeed, the inequality in Assumption 1.8 can be re-written as $P_0(x) \leq \frac{2}{1-r}\frac{1}{x^2}$. This means that, close to infinity, the prey is abundant while the density of predator is very low. For another technical reason, we require u_0 and v_0 to be increasing to enforce u_0 being* *unbounded. We expect to extend the above results to the case when u_0 is bounded and v_0 is compactly supported. This situation corresponds to the localized introduction of predators into the environment.*

Before going further, let us comment some other recent works concerning systems (1.1) and (1.3). From a biological point of view, the quenching (especially one point quenching) does not mean the global extinction of the species. It is therefore relevant to deal with the continuation of the solution beyond quenching. In this direction, let us mention the recent work of Ducrot and Langlais [5] who deal with the existence of solution for the following system of equations posed on some bounded domain together with Neumann boundary conditions:

$$\begin{cases} u_t - \Delta u = r_u u(1-u) - v1_{\{u>0\}} \\ v_t - \Delta v = rv\left(1 - \frac{v}{u}1_{\{u>0\}}\right). \end{cases}$$

The existence of globally defined classical solution (with the right hand side in some suitable L^p space with $p > 1$) for the above problem is proved for a large class of initial data (possibly singular), but uniqueness or non-uniqueness of solutions is still an open problem. We refer to [5] for more details. Some numerical simulations of this problem (see [1]) suggest that the formation as well as the dynamics of the dead-core can exhibit a very complex dynamics including pattern formations and regularization effect after quenching. This complex dynamics seems to be essentially due to parameter r_u that increases the prey population. In absence of vital dynamics for the prey population, namely $r_u = 0$, a first study of the travelling solutions for the above problem, recently provided by Ducrot and Langlais in [4], allows us to expect a very simple dynamics for the dead-core in that case. More precisely, we expect that under some suitable assumptions, the dead-core arising during the predator invasion process would asymptotically expand with a constant speed and without any regularization effect after quenching.

Coming back to (1.1), note that most of the above mentioned results will be obtained by using some nice properties of the function $P := v/u$ that satisfies the following equations

$$\begin{cases} P_t(t,x) = P_{xx}(t,x) + 2\frac{u_x(t,x)}{u(t,x)}P_x(t,x) + (1-r)P(t,x)^2, \\ P_x(t,0) = 0, \\ P(0,x) = P_0(x) := \frac{v_0(x)}{u_0(x)}, \quad \forall x \geq 0. \end{cases} \tag{1.7}$$

The study of singularity formation (e.g., blow-up, quenching, dead-core, etc.) in parabolic problems has attracted a lot of attention during the past years. Before 1994, it was only found that the singularity is always of self-similar type (or, type I singularity). Here the self-similarity means solutions are invariant under certain scaling of independent and dependent variables. The works by Herrero and Velazquez [13, 14] provide the first example of non-self-similar type singularity for a blow-up problem. We call such a singularity as type II. This result was later extended by Mizoguchi [17, 18, 19, 20]. Moreover, in [21], Mizoguchi and Senba studied a system of parabolic-elliptic equations which exhibits a type II singularity.

For the dead-core problem, it was shown in [12] that the dead-core rate is of type II. Unlike in blow-up problem where we need to impose the higher spatial dimensions, here the spatial domain is only required to be 1D. Later a fast diffusion equation was also studied to exhibit the type II singularity (cf. [9]). Another example of type II singularity is found for the gradient blow-up. We refer the reader to the works [8, 15, 16] and the references cited therein. To the authors' knowledge, this work provides the first example of type II quenching.

It is interesting to remark that the singularity temporal asymptotic rates are not unique for the type II singularity. Its rate depends on the initial datum. On the contrary, there is a unique rate for the self-similar type I singularity. It is a very interesting question to determine the exact rates for any given initial data when type II singularity occurs. For this topic, we refer the reader to [14, 20, 10]. But, the exact quenching rates for our problem are left for open.

This work is organized as follows: Section 2 is devoted to providing sufficient conditions on the initial data to have finite time quenching and also deriving some preliminary estimates of the solutions. In Section 3, we first investigate the quenching behavior of the solution under the type I quenching assumption. This corresponds to the proof of Theorem 1.7. Then two results on type II quenching are proved, namely, Corollaries 1.9 and 1.10. Finally, Section 4 gives some conditions for finite time quenching to occur with a compact quenching set.

2. Finite time quenching and preliminary estimates

In this section, we will give sufficient conditions on the initial data u_0 and v_0 so that system (1.1)-(1.2) exhibits finite time quenching. More specifically, we aim to prove Theorems 1.3 and 1.5.

First, we prove Theorem 1.3.

Proof of Theorem 1.3. The proof of this result relies on (1.7). Indeed, if $P_0 \equiv \lambda$, then $P(t,x) \equiv P(t)$ and therefore the blow-up time T and P can be explicitly written as

$$T = \frac{1}{\lambda(1-r)}, \quad P(t) = \frac{\lambda T}{T-t}.$$

Thus the u-equation becomes

$$u_t - u_{xx} = -\frac{\lambda T}{T-t}u, \quad u_x(t,0) = 0, \quad u(0,x) = u_0(x).$$

Due to Assumption 1.1, there exists $\eta > 0$ such that

$$\eta T^{\frac{1}{1-r}} \leq u_0(x), \quad \forall x \geq 0,$$

and using the comparison principle, one obtains that

$$\eta (T-t)^{\frac{1}{1-r}} \leq u(t,x), \quad \forall x \geq 0, \ t \in [0,T).$$

The above inequality implies type I quenching.

It remains to prove that the quenching set $\mathcal{Q} = [0, \infty)$. To prove this, let us notice that $u(t,x)P(t) = v(t,x)$ and, by (ii) of Assumption 1.1, we have

$$v_0(x) = \lambda u_0(x) \leq \lambda k(1 + x^2), \ \forall x \geq 0.$$

It follows that there exists $K > 0$ such that

$$v(t,x) \leq \frac{K}{\sqrt{t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} (1 + y^2) dy, \quad \forall t \in (0,T), \ x \geq 0. \tag{2.1}$$

Therefore, we have for each $t \in (0,T)$ and $x \geq 0$:

$$u(t,x) \leq (T-t) \frac{\widetilde{K}}{\sqrt{t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} (1 + y^2) dy$$

for some constant $\widetilde{K} > 0$. This implies that $\mathcal{Q} = [0, \infty)$ and thereby completes the proof of Theorem 1.3. $\square$

Next, we rewrite system (1.1)-(1.2) in terms of the variables $Q := u_x/u$ and P. Then system (1.1)-(1.2) becomes

$$Q_t - Q_{xx} = -P_x + 2QQ_x, \tag{2.2}$$

$$P_t - P_{xx} = 2QP_x + (1-r)P^2, \tag{2.3}$$

posed for $x \geq 0$ and $t \in (0,T)$ together with

$$Q(t,0) = 0, \quad P_x(t,0) = 0,$$

$$Q(0,x) = Q_0(x) := \frac{u_0'(x)}{u_0(x)}, \quad P(0,x) = P_0(x).$$

Then the following lemma holds true.

Lemma 2.1. *The set* $\Sigma := \{(Q,P) \in \mathbb{R} \times [0,\infty) : G(Q,P) := Q^2 - 2P \leq 0\}$ *is positively invariant under* (2.2)-(2.3).

Proof. The proof of this lemma directly follows from the results on invariant region in the book of Smoller [23] (see Theorems 14.7 and 14.11).

To do so, let us introduce for each $Z = \begin{pmatrix} Q \\ P \end{pmatrix} \in \mathbb{R}^2$ the following functions

$$M(Z) = \begin{pmatrix} 2Q & -1 \\ 0 & 2Q \end{pmatrix}, \quad f(Z) = \begin{pmatrix} 0 \\ (1-r)P^2 \end{pmatrix},$$

so that system (2.2)-(2.3) is re-written in a vectorial form as

$$Z_t - Z_{xx} = M(Z)Z_x + f(Z).$$

Then for each $Z_0 = (Q_0, P_0)^T$ such that $G(Q_0, P_0) = 0$, one has

$$dG(Z_0) = 2\left(Q_0, -1\right),$$

and we have

$$dG(Z_0)M(Z_0) = 2Q_0 dG(Z_0) \text{ and } dG(Z_0)f(Z_0) \leq 0.$$

Hence the result follows. $\square$

The following lemma is concerned with the monotonicity of solutions.

Lemma 2.2. *If $P_0'(x) \leq 0$ for all $x \in [0, \infty)$ then*

$$P_x(t,x) \leq 0, \ \ \forall t \in (0,T), \ x \geq 0.$$

If furthermore $u_0'(x) \geq 0$ (respectively $v_0'(x) \geq 0$) then

$$u_x(t,x) \geq 0 \ \textit{(respectively } v_x(t,x) \geq 0\textit{)}, \ \ \forall t \in (0,T), \ x \geq 0.$$

Remark 2.3. *This lemma will be crucial in the sequel. Let us notice that such a result seems not to be true when dealing with system* (1.3) *with $r_u > 0$.*

Proof. From (1.7), the map $w = P_x$ satisfies

$$\begin{cases} w_t - w_{xx} = 2Qw_x + 2Q_x w + 2(1-r)Pw, \\ w(t,0) = 0, \\ w(0,x) = P_0'(x). \end{cases}$$

Then the comparison principle can be applied to obtain the first result.

If we furthermore assume that $u_0' \geq 0$, then due to the u-equation in (1.1), we obtain that $z = u_x$ satisfies

$$\begin{cases} z_t - z_{xx} = -P_x u - Pz \geq -Pz, \\ z(t,0) = 0, \\ z(0,x) = u_0'(x), \end{cases}$$

and the assertion $u_x \geq 0$ follows by using the comparison principle. The case for v is similar and the lemma is proved. $\square$

We are now ready to prove Theorem 1.5.

Proof of Theorem 1.5. As a consequence of (1.5), Lemma 2.1 and Lemma 2.2, we have $P_x \leq 0$ and $Q \leq \sqrt{2}P^{1/2}$. Hence

$$P_t - P_{xx} = 2QP_x + (1-r)P^2 \geq 2^{3/2}P^{1/2}P_x + (1-r)P^2.$$

This implies that for some constant $a > 0$

$$P_t - P_{xx} - a\left(P^{3/2}\right)_x - (1-r)P^2 \geq 0$$

We are now able to apply the comparison principle together with Theorem 37.4 in [22] to derive that P exhibits a finite time blow-up at some time T. Moreover since $P_x \leq 0$, one obtains that

$$\liminf_{t \nearrow T} P(t,0) = \infty.$$

Recalling that $u = v/P$. Then it follows from (2.1) that

$$\liminf_{t \nearrow T} u(t,0) = 0.$$

The theorem is proved. $\square$

As a direct consequence of Theorem 1.5, one concludes the finite time quenching for system (1.1)-(1.2) in the two following situations:

$$u_0 \equiv \alpha > 0 \text{ (for some constant } \alpha) \text{ and } v_0 \text{ is decreasing;}$$

and

$$u_0(x) = \alpha + \beta x^2, \quad v_0(x) \equiv \kappa > 0,$$

wherein $\alpha > 0$, $\beta > 0$ and $\kappa > 0$ are constants such that $2\beta \le \kappa$.

We continue this section by deriving some basic estimates when quenching occurs. More precisely, we prove the following upper bound for the quenching rate.

Lemma 2.4 (Upper bound for the quenching rate)**.** *Let u_0, v_0 be a couple of initial data satisfying Assumption 1.4 such that $u_0' \ge 0$, $P_0' \le 0$ and*

$$\frac{v_0}{u_0^r} \in L^\infty(0,\infty). \tag{2.4}$$

Assume that the corresponding solution (u, v) of system (1.1)-(1.2) *exhibits the finite time quenching at time $T > 0$. Then there exists some constant $K > 0$ such that*

$$\inf_{x\ge 0} u(t,x) \le K(T-t)^{\frac{1}{1-r}}, \tag{2.5}$$

$$\inf_{x\ge 0} v(t,x) \le K(T-t)^{\frac{r}{1-r}}, \tag{2.6}$$

$$\sup_{x\ge 0} P(t,x) \ge K^{-1}(T-t)^{-1} \tag{2.7}$$

for $t \in [0,T)$.

The proof of this lemma relies on the following estimate.

Lemma 2.5. *Let u_0, v_0 be a couple of initial data satisfying Assumption 1.4 such that $u_0' \ge 0$. $P_0' \le 0$ and* (2.4) *holds. Let (u, v) be the corresponding solution of system* (1.1)-(1.2) *on $(0,T)$. Then there exists some constant $M > 0$ such that*

$$v(t,x) \le M u^r(t,x), \quad \forall x \ge 0, \ t \in [0,T).$$

Proof. Set $J(t,x) = v(t,x) - Mu(t,x)^r$ wherein $M > 0$ denotes some constant such that $v_0(x) \le Mu_0^r(x)$ for all $x \ge 0$.. Then J satisfies

$$J_t = J_{xx} - r\frac{v}{u}\left(v - Mu^r\right) + Mr(r-1)u^{r-2}u_x^2.$$

Since $r < 1$, we have

$$J_t \le J_{xx} - r\frac{v}{u}J \ \text{ in } (0,T)\times(0,\infty)\ , \quad J_x = 0 \text{ at } x = 0,$$

and from the definition of M we have $J(0,x) \le 0$. Thus one gets that $J \le 0$ and the result follows. □

Proof of Lemma 2.4. First, from Lemma 2.2, one obtain that for each $t \in (0,T)$

$$\sup_{x\ge 0} P(t,x) = P(t,0), \quad \inf_{x\ge 0} u(t,x) = u(t,0).$$

As a consequence, we get that $P_{xx}(t,0) \leq 0$ and $u_{xx}(t,0) \geq 0$ so that

$$\begin{aligned} &u_t(t,0) \geq -v(t,0),\\ &P_t(t,0) \leq (1-r)P^2(t,0). \end{aligned}$$

From the second inequality, one gets that for some constant

$$\frac{1}{(1-r)(T-t)} \leq P(t,0).$$

Thus (2.7) is derived.

Next, from Lemma 2.5, there exists some constant $M > 0$ such that $v \leq Mu^r$. Thus

$$u_t(t,0) \geq -v(t,0) \geq -Mu(t,0)^r.$$

Integrating this inequality from t to T leads us to

$$u(t,0) \leq K(T-t)^{\frac{1}{1-r}}$$

for some positive constant K. This gives (2.5). Finally, since $v \leq Mu^r$, the result (2.6) follows and the lemma is proved. □

3. Quenching behavior

In this section, we shall study the quenching behavior of the solution (u,v) of problem (1.1)-(1.2) and give a proof of Theorem 1.7. Let T be the quenching time of (u,v). Then T is also the blow-up time of P. For notational convenience, we let $Q_T := (0,T) \times (0,\infty)$. Also, throughout this section, let Assumption 1.6 be satisfied.

Recall that u is of type I quenching, if

$$\liminf_{t \nearrow T} (T-t)^{-\frac{1}{1-r}} u(t,0) > 0.$$

Before studying the behavior of the solution, let us first prove the following estimate.

Lemma 3.1. *Let Assumption 1.6 be satisfied. Then there exists some constant $K > 0$ such that*

$$u_x(t,x) \leq K u(t,x)^{\frac{1+r}{2}},\ (t,x) \in [0,T) \times [0,\infty), \tag{3.1}$$

$$u(t,x) \leq \left(u(t,0)^{\frac{1-r}{2}} + \frac{1-r}{2} Kx \right)^{\frac{2}{1-r}}, \tag{3.2}$$

$$v(t,x) \leq k_1 \left(u(t,0)^{\frac{1-r}{2}} + \frac{1-r}{2} Kx \right)^{\frac{2r}{1-r}}, \tag{3.3}$$

where $k_1 > 0$ is the constant defined in Assumption 1.6.

Proof. Following [6], we consider the map $J = \frac{1}{2}u_x^2 - Ku^{1+r}$ wherein $K > \frac{k_2}{2}$ is some constant that will be specified later on. Then we have

$$\begin{aligned}
J_t &= u_{tx}u_x - K(1+r)u^r u_t,\\
J_x &= u_{xx}u_x - K(1+r)u^r u_x,\\
J_{xx} &= u_{xxx}u_x + u_{xx}^2 - K(1+r)ru^{r-1}u_x^2 - K(1+r)u^r u_{xx}.
\end{aligned}$$

Thus we obtain that

$$\begin{aligned}
J_t - J_{xx} &= (u_t - u_{xx})_x u_x - K(1+r)u^r(u_t - u_{xx}) - u_{xx}^2\\
&\qquad +Kr(1+r)u^{r-1}u_x^2\\
&= -v_x u_x + K(1+r)u^r v - \left(\frac{J_x}{u_x} + K(1+r)u^r\right)^2\\
&\qquad +Kr(1+r)u^{r-1}\left(2J + 2Ku^{1+r}\right)\\
&\le K(1+r)\delta u^{2r} - \frac{2K(1+r)u^r}{u_x}J_x + 2Kr(1+r)u^{r-1}J\\
&\qquad +2K^2r(1+r)u^{2r} - K^2(1+r)^2u^{2r}\\
&= -\frac{2K(1+r)u^r}{u_x}J_x + 2Kr(1+r)u^{r-1}J\\
&\qquad +Ku^{2r}[(1+r)\delta + K(1+r)\,(r-1)].
\end{aligned}$$

Recalling that $r \in (0,1)$, let K be sufficiently large so that

$$(1+r)\delta + K(1+r)\,(r-1) < 0,$$

and since $K > \frac{k_2}{2}$, recalling Assumption 1.6, one has $\frac{1}{2}u_{0x}^2 \le Ku_0^{1+r}$ on $[0,\infty)$. Then together with such a choice of K, the function J satisfies

$$J_t - J_{xx} + \frac{2K(1+r)u^r}{u_x}J_x - 2Kr(1+r)u^{r-1}J \le 0.$$

Since $J(t,0) \le 0$, the comparison principle implies that $J \le 0$. Hence (3.1) follows. Integrating (3.1) and using Lemma 2.5, we obtain (3.2) and (3.3). The lemma is proved. $\square$

To study the quenching behavior, we introduce the following self similar change of variables

$$s = -\log(T-t),\ \ y = x(T-t)^{-1/2},$$

and the change of unknown functions

$$u(t,x) = (T-t)^{\sigma+1}U(s,y),\ v(t,x) = (T-t)^{\sigma}V(s,y),\ P(t,x) = W(s,y)/(T-t),$$

wherein we have set $\sigma = r/(1-r)$. This transformation leads us to the following system of equations

$$U_s = U_{yy} - \frac{y}{2}U_y - V + (\sigma+1)U, \tag{3.4}$$

$$V_s = V_{yy} - \frac{y}{2}V_y - r\frac{V^2}{U} + \sigma V, \tag{3.5}$$

$$W_s = W_{yy} - \frac{y}{2}W_y + 2\frac{U_y}{U}W_y + (1-r)W^2 - W \tag{3.6}$$

defined on the domain

$$D = \{(s,y);\ \ s_0 := -\log T < s < \infty,\ y > 0\} = (s_0,\infty)\times(0,\infty)$$

together with the boundary conditions

$$(U_y, V_y, W_y)(s,0) = (0,0,0), \tag{3.7}$$

for all $s > s_0$.

Recall from Lemmas 2.4 and 3.1 that

$$u(t,x) \le K\left((T-t)^{\frac{1}{2}} + x\right)^{\frac{2}{1-r}},\ \ v(t,x) \le K\left((T-t)^{\frac{1}{2}} + x\right)^{\frac{2r}{1-r}},\ \ (t,x)\in Q_T,$$

for some constant $K > 0$ large enough. Then we have

$$U(s,y) \le K(1+y)^{\frac{2}{1-r}},\ \ V(s,y) \le K(1+y)^{\frac{2r}{1-r}} \quad \text{in } \ D. \tag{3.8}$$

Then we have the following result on the asymptotic behavior.

Proposition 3.2. *Assume that u is of type I quenching. Then*

$$\lim_{s\to\infty} W(s,y) = \frac{1}{1-r}, \quad \lim_{s\to\infty} U_y(s,y) = \lim_{s\to\infty} V_y(s,y) = 0,$$

locally uniform with respect to $y \in [0,\infty)$.

Proof. Under the type I quenching assumption for u, one obtains that there exists $m > 0$ such that

$$m \le U(s,0) \le U(s,y),\ \ \forall (s,y)\in\overline{D}. \tag{3.9}$$

Then due to (3.8), $W = V/U$, Lemma 2.2 and (2.7), there exist $0 < k_1 < k_2 < \infty$ such that

$$W(s,y) \le W(s,0) \le \frac{K}{m} := k_2,\ \forall (s,y)\in\overline{D},\ \text{and } k_1 \le W(s,0),\ \forall s > s_0. \tag{3.10}$$

Recall that $r \in (0,1)$. Take $\alpha \in (1/(1-r),\infty)$, $\beta < 0$ and consider the map

$$\mathcal{V}[U,V](s) = \int_0^\infty \rho(y)V(s,y)^\alpha U(s,y)^\beta dy, \quad \rho(y) := \exp(-y^2/4).$$

Note that such a functional is well defined due to (3.8) and (3.9). Then for a solution (U, V) of (3.4)-(3.5) we compute

$$\begin{aligned} &\frac{d\mathcal{V}[U,V](s)}{ds} \\ = &-\int_0^\infty \rho(y)\{\alpha(\alpha-1)V^{\alpha-2}U^\beta V_y^2 + \beta(\beta-1)V^\alpha U^{\beta-2}U_y^2 + 2\alpha\beta V^{\alpha-1}U^{\beta-1}U_y V_y\}dy \\ &\quad +(\alpha r+\beta)\int_0^\infty \rho(y)\left(\frac{V^\alpha U^\beta}{1-r} - V^{\alpha+1}U^{\beta-1}\right)dy. \end{aligned}$$

By choosing $\beta = -r\alpha$, we obtain that

$$\begin{aligned} &\frac{d\mathcal{V}[U,V](s)}{ds} \\ = &-\int_0^\infty \rho(y)\{\alpha(\alpha-1)V^{\alpha-2}U^\beta V_y^2 + \beta(\beta-1)V^\alpha U^{\beta-2}U_y^2 + 2\alpha\beta V^{\alpha-1}U^{\beta-1}U_y V_y\}dy. \end{aligned}$$

Moreover, since $\alpha + \beta = \alpha(1-r) > 1$, we conclude that

$$\frac{d\mathcal{V}[U,V](s)}{ds} \le 0 \text{ with } \frac{d\mathcal{V}[U,V](s)}{ds} = 0 \text{ when } U_y(s,\cdot) \equiv V_y(s,\cdot) \equiv 0.$$

Next, we take any sequence $s_n \to \infty$ and consider the sequence of maps

$$U_n(s,y) := U(s+s_n,y), \;\; V_n(s,y) := V(s+s_n,y).$$

Note that the equation (3.5) can be rewritten by

$$V_s = V_{yy} - \frac{y}{2}V_y - rWV + \sigma V.$$

Due to parabolic estimates, using (3.8) and (3.10), up to a subsequence, one may assume that U_n and V_n converges in the topology of $C^{1,2}_{loc}(\mathbb{R}\times[0,\infty))$ towards some positive functions, denoted by $(\hat{U},\hat{V}) \in C^{1,2}(\mathbb{R}\times[0,\infty))$, a complete orbit of the problem

$$\begin{aligned} U_s &= U_{yy} - \frac{y}{2}U_y - V + (\sigma+1)U, \;\; (s,y)\in\mathbb{R}\times(0,\infty), \\ V_s &= V_{yy} - \frac{y}{2}V_y - r\frac{V^2}{U} + \sigma V, \;\; (s,y)\in\mathbb{R}\times(0,\infty), \end{aligned}$$

with $\hat{U}_y(s,0) = \hat{V}_y(s,0) = 0$ for all $s\in\mathbb{R}$ and such that

$$\frac{d\mathcal{V}[\hat{U},\hat{V}](s)}{ds} = 0, \;\; \forall s\in\mathbb{R}.$$

This implies that

$$\hat{U}_y(s,\cdot) \equiv \hat{V}_y(s,\cdot) \equiv 0.$$

Thus $(\hat{U},\hat{V}) \equiv (\hat{U},\hat{V})(s)$ is a bounded complete orbit of

$$\left\{ \begin{array}{l} U_s = -V + U/(1-r), \;\; s\in\mathbb{R}, \\ V_s = -r\frac{V^2}{U} + \frac{rV}{1-r}, \;\; s\in\mathbb{R}. \end{array} \right. \tag{3.11}$$

By differentiating the ratio $\hat{W} := \hat{V}/\hat{U}$ with respect to s and using (3.11), we can deduce that $\hat{W}$ satisfies

$$W_s = (1-r)W^2 - W, \quad s \in \mathbb{R}.$$

It follows from the uniqueness theory of ODE that $\hat{W}$ is strictly monotone, if $\hat{W}$ is not a constant function. Then it is easy to check that the existence of a complete orbit is possible only if

$$0 < k_1 \le \hat{W}(s) \le \frac{1}{1-r}, \quad \forall s \in \mathbb{R}. \tag{3.12}$$

Now, if there exists $s_0 \in \mathbb{R}$ such that $\hat{W}(s_0) \in \left(0, \frac{1}{1-r}\right)$ then it is easy to see that

$$\lim_{s\to\infty} \hat{W}(s) = 0. \tag{3.13}$$

Indeed, by comparison argument, one obtains that $\hat{W}(s) \le \overline{W}(s)$ for all $s \ge s_0$ wherein $\overline{W}$ is defined by

$$\overline{W}(s) = \frac{1}{1-r+e^{s-s_0}\frac{1-(1-r)w(s_0)}{w(s_0)}}.$$

Hence (3.13) holds which violates (3.12). Thus we have that $\hat{W}(s) \equiv 1/(1-r)$, which is independent of the choice of $\{s_n\}$. This completes the proof of the proposition. □

Returning to the original variables, we obtain Theorem 1.7.

As a direct corollary of Theorem 1.7, one will prove Corollaries 1.9 and 1.10. To do so let us first prove the following result:

Lemma 3.3. *Let Assumption 1.6 be satisfied. If we furthermore assume that*

$$P_0(x) \le \frac{2}{1-r}\frac{1}{x^2}, \quad \forall x > 0,$$

then

$$P(t,x) \le \frac{2}{1-r}\frac{1}{x^2}, \quad \forall t \in (0,T), \ \forall x > 0.$$

Proof. Consider the map $H(t,x) = u(t,x) - kx^2 v(t,x)$ with $k = \frac{1-r}{2}$. Then one has

$$\begin{aligned} H_t &= u_t - kx^2 v_t, \\ H_x &= u_x - 2kxv - kx^2 v_x, \\ H_{xx} &= u_{xx} - 2kv - 4kxv_x - kx^2 v_{xx}, \end{aligned}$$

so that we get

$$\begin{aligned}H_t - H_{xx} &= (u_t - u_{xx}) - kx^2(v_t - v_{xx}) + 2kv + 4kxv_x \\ &= -v + rkx^2\frac{v^2}{u} + 2kv + 4kxv_x \\ &= -r\frac{v}{u}H + (r-1+2k)v + 4kxv_x.\end{aligned}$$

Since $v_x \geq 0$ and $r - 1 + 2k = 0$, one obtains that

$$H_t - H_{xx} + r\frac{v}{u}H \geq 0.$$

Since $H(t,0) \geq 0$ and, by assumption,

$$H(0,x) = u_0(x) - kx^2 v_0(x) \geq 0,$$

the result follows from the comparison principle. □

We are now able to complete the proof of Corollaries 1.9 and 1.10. Using Lemma 3.3, one obtains that $\mathcal{Q} = \{0\}$, if finite time quenching occurs. Furthermore, using the self-similar transformation, one obtains from Lemma 3.3 that

$$W(s,y) \leq \frac{C}{y^2}, \quad \forall s > s_0, \ y \geq 0,$$

for some constant $C > 0$. The above inequality prevents $W(s,y)$ from locally uniformly tending to $\frac{1}{1-r}$ as $s \to \infty$. This completes the proof of Corollary 1.9.

Finally, Corollary 1.10 directly follows from combining Theorem 1.5 together with Corollary 1.9.

4. Quenching set

In this section, we shall study the quenching set under the following assumptions:

(i) The limits $N_1 := \lim_{x\to\infty} u_0(x)$ and $N_2 := \lim_{x\to\infty} v_0(x)$ exist such that $0 < N_1 < \infty$ and $0 < N_2 < \infty$.
(ii) There hold $u_0'(x) \geq 0$ and $P_0'(x) \leq 0$ for all $x \geq 0$.

We will prove the following result:

Proposition 4.1. *Under the above assumptions, system* (1.1)-(1.2) *has a finite time quenching at time T such that*

$$0 < T \leq T_0(N_1, N_2) := \frac{1}{1-r}\frac{N_1}{N_2}.$$

Furthermore, the quenching set $\mathcal{Q}$ is compact if and only if $T < T_0(N_1, N_2)$.

Proof. Using the above monotonic assumptions (ii) and Lemma 2.2, one obtains that for each $t \in [0,T)$, where T is the maximal time of existence of the solution, (u,v) the solution satisfies

$$\sup_{x \geq 0} u(t,x) = \lim_{x \to \infty} u(t,x), \quad t \in (0,T). \tag{4.1}$$

Consider now the associated kinetic system

$$U_t = -V, \quad V_t = -r\frac{V^2}{U}. \tag{4.2}$$

Note that $(V/U^r)_t = 0$. Then it is easy to see that the solution of (4.2) with the initial condition $(U,V)(0) = (N_1, N_2)$ is given by

$$U(t) = \left[N_1^{1-r} - (1-r)\frac{N_2 t}{N_1^r}\right]^{1/(1-r)}, \quad V(t) = \frac{N_2 U^r(t)}{N_1^r}, \quad t \in [0, T_0),$$

where

$$T_0 := T_0(N_1, N_2) = \frac{1}{1-r}\frac{N_1}{N_2}.$$

In particular, $U(t)$ and $V(t)$ quench at the finite time T_0.

To proceed further, we apply [11, Theorem 4.1] to system (1.1)-(1.2) with $a_n = 4n$ and $r_n = n$ to obtain that

$$\lim_{x \to \infty} u(t,x) = U(t), \quad \lim_{x \to \infty} v(t,x) = V(t)$$

for all $t \in [0, \min\{T, T_0\})$. It is clear that $T \leq T_0$. Moreover, due to (4.1), when $T = T_0$, we have the total quenching, i.e., $\mathcal{Q} = [0, \infty)$.

On the other hand, the quenching set $\mathcal{Q}$ is compact, if $T < T_0$. Indeed, recalling that v_0 is bounded, one obtains that $v \in L^\infty((0,T) \times (0,\infty))$. Notice that $U(t)$ is decreasing in $t \in (0,T)$. As a consequence of parabolic estimates one obtains that

$$\lim_{x \to \infty} u(t,x) = U(t) \text{ uniformly with respect to } t \in [T/2, T).$$

Since $T < T_0$, $U(T) > 0$ and so there exists $M > 0$ large enough such that

$$u(t,x) \geq U(T)/2, \ \ \forall t \in [T/2, T), \ x \geq M.$$

Thus the result follows. □

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