FINITE TIME DEAD-CORE RATE FOR THE HEAT EQUATION WITH A STRONG ABSORPTION

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Abstract. We study the solution of the heat equation with a strong absorption. It is well-known that the solution develops a dead-core in finite time for a large class of initial data. It is also known that the exact dead-core rate is faster than the corresponding self-similar rate. By using the idea of matching, we formally derive the exact dead-core rates under a dynamical theory assumption. Moreover, we also construct some special solutions for the corresponding Cauchy problem satisfying this dynamical theory assumption. These solutions provide some examples with certain given polynomial rates.

1. Introduction

In this paper, we study the following initial boundary value problem (P) for the heat equation with a strong absorption:

$$u_t = u_{xx} - u^p, \quad 0 < x < 1,\ t > 0, \tag{1.1}$$

$$u_x(0,t) = 0,\ u(1,t) = k, \quad t > 0, \tag{1.2}$$

$$u(x,0) = u_0(x), \quad 0 \le x \le 1, \tag{1.3}$$

where $p \in (0,1)$, k is a positive constant, and u_0 is a smooth positive function defined on $[0,1]$ such that

$$u_0'(0) = 0,\ u_0(1) = k,\ u_0'(x) \ge 0 \quad \text{for } x \in [0,1]. \tag{1.4}$$

Problem (P) has been studied extensively for past years. It arises in the modeling of an isothermal reaction-diffusion process [2, 11] in which the solution u of (P) represents the concentration of the reactant which is injected with a fixed amount on the boundary $x = \pm 1$ (by a symmetric reflection), and p is the order of reaction. It also arises in the modeling of a description of thermal energy transport in plasma [8, 7]. For more references, we refer the reader to a recent work of Guo-Souplet [4] and the references cited therein.

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In the literature, the region where $u = 0$ is called the *dead-core* and the first time when u reaches zero is called the *dead-core time*.

It is shown in [2] that for a large class of initial data u_0 the solution of (P) develops a dead-core in a finite time, say, T. By the following transformation

$$u(x,t) = (T-t)^{\alpha} w(y,s), \quad \alpha := 1/(1-p), \tag{1.5}$$

$$y = x/\sqrt{T-t}, \quad s = -\ln(T-t), \tag{1.6}$$

we see that u is a solution of (P) if and only if w is a solution of the following initial and boundary value problem (Q):

$$w_s = w_{yy} - w^p + \alpha w - \frac{1}{2} y w_y, \quad 0 < y < R(s),\ s > s_0, \tag{1.7}$$

$$w_y(0,s) = 0,\ w(R(s),s) = k e^{\alpha s}, \quad s > s_0, \tag{1.8}$$

$$w(y,s_0) = w_0(y) := u_0(y\sqrt{T})/T^{\alpha}, \quad 0 \le y \le 1/\sqrt{T}, \tag{1.9}$$

where $R(s) := e^{s/2}$ and $s_0 := -\ln T$. Furthermore, it is shown in [4] that, as $s \to \infty$, $w(y,s) \to U(y)$ uniformly on compact subsets. Here $U(y) := k_p y^{2\alpha}$, $k_p := [2\alpha(2\alpha-1)]^{-\alpha}$. In particular, $w(0,s) \to 0$ as $s \to \infty$. Therefore, the exact convergence rate is still not determined.

The main purpose of this paper is to find the exact convergence rate of $w(0,s)$. For the same question to different problems, we refer to the reader to the recent works of Dold-Galaktionov-Lacey-Vazquez [3] and Souplet-Vazquez [10]. The main idea of these two works is to use a matching of the inner and outer expansions.

For the inner expansion, we enlarge the inner region near $y = 0$ by a re-scaling. Then the inner expansion is derived by studying a stabilization problem as the time goes to infinity.

For the outer expansion, we first study the linearized operator of the right-hand side of (1.7) around the singular steady state U. Then, from the dynamical point of view, we assume that there exist an integer $l \ge 1$ and positive constants ϵ, K, s_1 with ϵ sufficiently small such that

$$|w(y,s) - U(y) - \hat{c}_l e^{-(l-1/2)s} \phi_l(y)| \le \epsilon e^{-(l-1/2)s} y^{2\alpha-1} \tag{1.10}$$

for $y \in [Ke^{-(l-1/2)s}, 1]$ and $s \ge s_1$ for some nonzero constant $\hat{c}_l$, where ϕ_l is the l-th eigenfunction of the linearized operator.

Then, by a matching, the rate of convergence of $w(0,s)$ can be formally derived as

$$\{\ln[w(0,s)]/s\} = -2\alpha(l-1/2) + O(1/s) \ \text{ as } \ s \to \infty. \tag{1.11}$$

Note that the estimate (1.11) implies that

$$u(0,t) \sim (T-t)^{\alpha + 2\alpha(l-1/2)} \ \text{ as } \ t \uparrow T^-.$$

Hence, under the assumption (1.10), the dead-core rate is at most polynomially. But, it is faster than the so-called self-similar rate.

Although we are unable to verify the assumption (1.10) for general solutions, motivated by the works of Herrero-Velazquez [6] and Mizoguchi [9], we can construct some special solutions for the corresponding Cauchy problem such that the assumption (1.10) is satisfied by these solutions for any odd integer l.

We note that the dead-core rate of the solution of (P) should depend on the initial data u_0. We observe from the exact expression of ϕ_l (see Section 4 below) that there are exact l intersections of $w(y,s)$ (constructed in Theorem 6.1 below) and $U(y)$ in $(0,\infty)$ for any $s \geq s_1$. Notice that the number of intersections of $w(y,s)$ and $U(y)$ is the same as the number of intersections of $u(x,t)$ and $U(x)$ due to the scaling invariance of U under the scaling (1.6). Also, as $t \uparrow T^-$ (or $s \to \infty$), the y-domain of w tends to the whole real line. It is nature to expect that (1.10) is satisfied with an integer l which is related to the number of intersections of $u(x,t)$ and $U(x)$ in $(0,1)$. This gives a connection between the initial data and the assumption (1.10) on the solution.

The paper is organized as follows. In Section 2, we study the structure of steady states of (P). The inner expansion is given in Section 3. In Section 4, we first study the spectrum of the linearized operator around the singular steady state U. With this information on the spectrum, we then give a formal outer expansion. Then, in Section 5, we formally derive the exact convergence rate of $w(0,s)$ under the dynamical theory assumption (1.10). Finally, to illustrate the plausibility of the assumption (1.10), we construct some special solutions for the corresponding Cauchy problem with certain given rates in Section 6. These solutions satisfy the dynamical theory assumption (1.10). The proof of a key lemma in this construction is given in Section 7. This involves a quite heavy analysis.

2. Steady States

In this section, we shall study the steady states of (P). For any $\eta \geq 0$, let U_η be the solution of

$$u'' = u^p,\ u > 0 \text{ for any } y > 0; \quad u(0) = \eta,\ u'(0) = 0. \tag{2.1}$$

In particular, $U_0(y) = U(y) = k_p y^{2\alpha}$ for $y \geq 0$. Note that, by a re-scaling, we have

$$U_\eta(y) = \eta U_1(\eta^{(p-1)/2} y) \text{ for any } \eta > 0. \tag{2.2}$$

Also, by a simple comparison, we have $U_{\eta_1} > U_{\eta_2}$ if $\eta_1 > \eta_2 \geq 0$. Moreover, $U_\eta \to U_0$ as $\eta \to 0^+$.

Remark 1. For $\eta = 0$, there are non-negative solutions in the form

$$U_0^\varepsilon(y) := k_p (y - \varepsilon)_+^{2\alpha}$$

for any $\varepsilon > 0$. Also, these give all the possible non-negative non-trivial solutions of (2.1).

Concerning the asymptotic behavior of U_η as $\eta \to 0^+$, we have

Lemma 2.1. *As* $\eta \to 0^+$,

$$U_\eta(x) = U_0(x) + k(\eta)x^{2\alpha-1}(1 + o(1)) \tag{2.3}$$

for any $x > 0$, *where* $k(\eta) := a\eta^{(1-p)/2}$ *for some* $a > 0$.

Proof. First, we study the asymptotic behavior of $U_1(y)$ as $y \to \infty$. For this, we write $U_1 = U_0 + v$. Then v satisfies the equation

$$v'' = by^{-2}v + c_2y^{-2-2\alpha}v^2 + c_3y^{-2-4\alpha}v^3 + \cdots$$

for some constants c_i, $i \geq 2$, where $b := (2\alpha - 1)(2\alpha - 2)$. Assume that $v(y) \sim y^\gamma$ as $y \to \infty$ for some $\gamma > 0$. Then we must have

$$\gamma(\gamma - 1) = b.$$

By writing $\gamma = 2\alpha - \delta$, we obtain that either $\delta = 1$ or $\delta = 4\alpha - 2 > 2\alpha$, which is impossible. Hence we obtain that

$$U_1(y) = U_0(y) + ay^{2\alpha-1}(1 + o(1)) \ \text{ as } \ y \to \infty \tag{2.4}$$

for some constant a. The constant a is positive, since $U_1 > U_0$.

Now, for any $x > 0$, from (2.2) and (2.4) it follows that

$$U_\eta(x) = \eta U_1(\eta^{(p-1)/2}x) = U_0(x) + a\eta^{(1-p)/2}x^{2\alpha-1}(1 + o(1)) \ \text{ as } \ \eta \to 0^+.$$

The lemma is proved.

3. Inner Expansion

In the sequel, for convenience we denote $\sigma(s) := w(0, s)$. To derive the inner expansion, we make the following transformations

$$w(y, s) := \sigma(s)\theta(\xi, \tau), \quad \xi := \sigma(s)^{(p-1)/2}y, \quad \tau = \tau(s) := \int_{s_0}^{s} \sigma(r)^{p-1}dr. \tag{3.1}$$

Then it is easy to check that θ satisfies the equation

$$\theta_\tau = \theta_{\xi\xi} - \theta^p + [\sigma^{1-p}(s) + g(\tau)] \left(\alpha\theta - \frac{1}{2}\xi\theta_\xi\right), \tag{3.2}$$

where

$$g(\tau) := -(1 - p)\sigma'(s)\sigma(s)^{-p}. \tag{3.3}$$

Also, $\theta(0, \tau) = 1$ and $\theta_\xi(0, \tau) = 0$ for all $\tau > 0$. We shall study the stabilization of the solution θ of (3.2).

First, recall from (3.7) of [4] that

$$0 \leq u_x \leq Cu^{(p+1)/2} \ \text{ for any } x \in [0, 1],\ T/2 \leq t \leq T,$$

where C is a positive constant. Hence we have

$$0 \le w_y \le Cw^{(p+1)/2} \text{ for any } y \in [0, R(s)],\ s \ge -\ln(T/2) := s_1, \tag{3.4}$$

for some positive constant C. Consequently, by an integration, we deduce from (3.4) that

$$w(y,s) \le [\sigma(s)^{(1-p)/2} + cy]^{2\alpha} \text{ for any } y \in [0, R(s)],\ s \ge s_1, \tag{3.5}$$

for some positive constant c. Using (3.5), (3.1) and $w_y = \sigma^{(1+p)/2}\theta_\xi$, we obtain the following estimate

$$0 < \theta(\xi,\tau) \le (1 + c\xi)^{2\alpha}, \quad 0 \le \xi\theta_\xi(\xi,\tau) \le C(1+\xi)^{2\alpha} \tag{3.6}$$

for all $\xi \in [0, R(s)\sigma^{(p-1)/2}(s)]$, $\tau \ge \tau_1 := \tau(s_1)$, for some positive constants c and C.

On the other hand, recall from (1.7) of [4] that

$$u(x,t) \ge [u(0,t)^{1-p} + cx^2]^\alpha \text{ for any } x \in [0,1],\ T/2 \le t \le T, \tag{3.7}$$

for some positive constant c. Hence $w(y,s)$ grows at least as fast as $y^{2\alpha}$ for $y \gg 1$ and $s \gg 1$.

Next, it follows from Hopf's Lemma that $w_{yy}(0,s) > 0$ and so $w_s(0,s) > -w^p(0,s)$ by (1.7). Hence $g(\tau) < 1-p$ for all $\tau > 0$. In the sequel, we assume, in addition to (1.4), that u_0 satisfies the condition

$$u_0'' - u_0^p \le 0 \quad \text{in } [0,1]. \tag{3.8}$$

It follows from the maximum principle that $u_t \le 0$ in $[0,1] \times [0,T]$. From the relation

$$u_t(x,t) = (T-t)^{\alpha-1}[w_s(y,s) - \alpha w(y,s) + yw_y(y,s)/2],$$

it follows that $w_s(0,s) \le \alpha w(0,s)$ for all $s > s_0$. Hence $g(\tau) \ge -w^{1-p}(0,s)$ for all $s > s_0$. Therefore, g is bounded and $\liminf_{\tau\to\infty} g(\tau) \ge 0$.

Note that

$$\int_0^\infty g(\tau)d\tau = -(1-p)\int_{s_0}^\infty \sigma^{-1}(s)\sigma'(s)ds = \infty.$$

Hence either $\limsup_{\tau\to\infty} g(\tau) = 0$ or $\limsup_{\tau\to\infty} g(\tau) > 0$. Indeed, the first case holds as shown in the following useful lemma.

Lemma 3.1. *There holds* $\lim_{\tau\to\infty} g(\tau) = 0$.

Proof. Otherwise, there is a sequence $\{\tau_n\} \to \infty$ such that $g(\tau_n) \to \gamma$ as $n \to \infty$ for some constant $\gamma > 0$. By using (3.6) and the standard regularity theory, we can show that there is a subsequence, still denoted by $\{\tau_n\}$, such that

$$\theta(\xi, \tau + \tau_n) \to \tilde{\theta}(\xi,\tau) \text{ as } n \to \infty$$

uniformly on any compact subsets, where $\tilde{\theta}$ solves the equation

$$\tilde{\theta}_\tau = \tilde{\theta}_{\xi\xi} - \tilde{\theta}^p + \gamma(\alpha\tilde{\theta} - \frac{1}{2}\xi\tilde{\theta}_\xi), \quad \xi > 0,\ \tau > 0,$$

with $\tilde{\theta}(0,\tau) = 1$ and $\tilde{\theta}_\xi(0,\tau) = 0$. Moreover, it is easy to check that $\tilde{\theta}_\xi \geq 0$ and $\tilde{\theta}(\xi,\tau)$ grows at most polynomially as $\xi \to \infty$ for $\tau \gg 1$.

Now, it follows from the so-called energy argument (cf. the proof of Proposition 3.1 in [4]) that $\tilde{\theta}(\xi,\tau) \to V(\xi)$ as $\tau \to \infty$ for some V satisfying

$$V'' - V^p + \gamma(\alpha V - \frac{1}{2}\xi V') = 0, \quad \xi > 0,$$
$$V'(0) = 0,\ V(0) = 1.$$

Note that $V' \geq 0$ and V grows at most polynomially. Set

$$W(y) := \gamma^\alpha V(y/\sqrt{\gamma}).$$

Then W satisfies

$$W'' - W^p + \alpha W - \frac{1}{2}yW' = 0, \quad y > 0,$$
$$W'(0) = 0,\ W(0) = \gamma^\alpha.$$

Since $W > 0$, $W' \geq 0$ for $y > 0$, and taking into account of the polynomial boundedness of W, it follows from Proposition 3.3 of [4] that either $W = U$ or $W \equiv \alpha^{-\alpha}$. The first case is impossible, since $U(0) = 0$. The second case is also impossible, since θ is unbounded by (3.7). Hence the lemma follows. □

Again, by the standard limiting process with the estimate (3.6) and Lemma 3.1, for any given sequence $\{\tau_n\} \to \infty$ we can show that there is a limit $\tilde{\theta}$ satisfying

$$\tilde{\theta}_\tau = \tilde{\theta}_{\xi\xi} - \tilde{\theta}^p, \quad \xi > 0,\ \tau \in \mathbb{R},$$
$$\tilde{\theta}(0,\tau) = 1,\ \tilde{\theta}_\xi(0,\tau) = 0,$$

such that $\theta(\xi,\tau+\tau_n) \to \tilde{\theta}(\xi,\tau)$ as $n \to \infty$ uniformly on compact subsets of $(0,\infty) \times \mathbb{R}$. Since we also have

$$0 = U_1'' - U_1^p,\ \tilde{\theta}(0,\tau) = U_1(0),\ \tilde{\theta}_\xi(0,\tau) = (U_1)_\xi(0),$$

Hopf's Lemma and the real analyticity of $\tilde{\theta} - U_1$ imply that $\tilde{\theta} \equiv U_1$. Indeed, suppose on the contrary that $\tilde{\theta} \not\equiv U_1$. Taking any finite τ_0, it follows from the real analyticity of $v(\xi) := \tilde{\theta}(\xi,\tau_0) - U_1(\xi)$ in ξ that the zeros of v are isolated. Assume that the smallest positive zero of v is ξ_0 ($\xi_0 := \infty$, if there is no finite zero). Without loss of generality we may assume that $v < 0$ in $(0,\xi_0)$. In the connected component

$$Q := \{(\xi,\tau) \mid \xi > 0, \tau \geq \tau_0, \tilde{\theta} - U_1 < 0\}$$

containing $(0,\xi_0) \times \{\tau_0\}$, any point $(0,\tau)$ with $\tau > \tau_0$ is a maximum point. Then Hopf's Lemma implies that $(\tilde{\theta} - U_1)_\xi(0,\tau) < 0$, a contradiction. Therefore, we must have $\tilde{\theta} \equiv U_1$.

Since this limit is independent of the given sequence $\{\tau_n\}$, we see that $\theta(\xi,\tau) \to U_1(\xi)$ as $\tau \to \infty$ uniformly on any compact subsets. Returning to the original variables and using the relation (2.2), we thus have proved the following so-called inner expansion.

Theorem 3.2. *As $s \to \infty$, we have*

$$w(y,s) = U_{\sigma(s)}(y)(1+o(1))$$

uniformly in $\{0 \le \sigma^{(p-1)/2}(s)y \le C\}$ for any positive constant C.

4. Outer Expansion

In the matching process, we need to study the following linearized operator

$$\mathcal{L}v := -v'' + \frac{y}{2}v' + \frac{b}{y^2}v,\ b := (2\alpha-1)(2\alpha-2)$$

which comes from the linearization of (1.7) around the singular steady state U.

Consider the eigenvalue problem

$$\mathcal{L}\phi = \lambda\phi,\ y>0; \quad \phi'(0)=0,\ \phi(0)=0. \tag{4.1}$$

Set $\rho(y) := \exp(-y^2/4)$. We introduce the following weighted Hilbert spaces:

$$L^2_\rho := \left\{ \phi \text{ is measurable in } \mathbb{R} \;\middle|\; \phi(y)=\phi(|y|),\ \int_0^\infty \phi^2(y)\rho(y)dy < \infty \right\},$$

$$\mathbb{H} := \left\{ \phi \in L^2_\rho \;\middle|\; \int_0^\infty \phi_y^2(y)\rho(y)dy < \infty,\ \int_0^\infty \frac{\phi^2(y)}{y^2}\rho(y)dy < \infty \right\}.$$

Also, we set

$$J(\phi) := \int_0^\infty \phi_y^2(y)\rho(y)dy + b\int_0^\infty \frac{\phi^2(y)}{y^2}\rho(y)dy,$$

$$I(\phi) := \int_0^\infty \phi^2(y)\rho(y)dy.$$

Then the principal eigenvalue λ_0 of (4.1) can be characterized by

$$\lambda_0 := \inf\{J(\phi)/I(\phi) \mid \phi \in \mathbb{H},\ I(\phi)>0\}. \tag{4.2}$$

It is easy to see that $\lambda_0 > 0$. Also, by taking a minimization sequence, we can show that this λ_0 can be attained by a function $\phi_0 \in \mathbb{H}$ which is the eigenfunction of (4.1) such that

$$\phi_0 > 0 \text{ in } (0,\infty), \quad \int_0^\infty (\phi_0(y))^2\rho(y)dy = 1.$$

Moreover, from the standard theory on eigenvalue problem, there is a sequence of eigenpairs $\{(\lambda_n,\phi_n)\}_{n\ge 0}$ of (4.1) with $0<\lambda_0<\lambda_1\le\lambda_2\le\cdots$ and $\lambda_n\to\infty$ as $n\to\infty$. Since $\phi_n \not\equiv 0$, without loss of generality we may assume that $\phi_n>0$ for $y>0$ and small. Also, we take ϕ_n so that $\int_0^\infty(\phi_n(y))^2\rho(y)dy=1$. It is also easy to see that

$$\phi_n(y) = d_n y^{\gamma_+}(1+o(1)) \text{ as } y \to 0^+ \tag{4.3}$$

for some positive constant d_n, where $\gamma_+ := 2\alpha - 1$.

To compute the eigenvalues, we set

$$\phi(y) = |y|^{\gamma_+} H(\eta), \quad \eta = y^2/4.$$

Then ϕ satisfies (4.1) if and only if H satisfies

$$\eta H''(\eta) + (\hat{b} - \eta)H'(\eta) - \hat{a}H(\eta) = 0, \tag{4.4}$$

where $\hat{a} := \gamma_+/2 - \lambda$ and $\hat{b} := \gamma_+ + 1/2$. This is Kummer's Equation (cf. [1]) and its general solution is given by

$$c_1 M(\hat{a}, \hat{b}, \eta) + c_2 U(\hat{a}, \hat{b}, \eta)$$

for some constants c_1 and c_2, where

$$U(\hat{a}, \hat{b}, \eta) := \frac{\pi}{\sin \hat{b}\pi} \left[\frac{M(\hat{a}, \hat{b}, \eta)}{\Gamma(1 + \hat{a} - \hat{b})\Gamma(\hat{b})} - \eta^{1-\hat{b}} \frac{M(1 + \hat{a} - \hat{b}, 2 - \hat{b}, \eta)}{\Gamma(\hat{a})\Gamma(2 - \hat{b})} \right]$$

with M Kummer's function and Γ the Gamma function.

Since $H(0)$ is finite, $1 - \hat{b} = 3/2 - 2\alpha < 0$, and $M(\cdot, \cdot, 0) = 1$, it follows that the solution of (4.4) is given by

$$H(\eta) = cM(\hat{a}, \hat{b}, \eta)$$

for some constant c.

Since $\hat{b} > 0$, $H(\eta)$ is always well-defined. If $-\hat{a} \in \mathbb{N} \cup \{0\}$, then $M(\hat{a}, \hat{b}, \eta)$ is a polynomial of degree $-\hat{a}$ in η (cf. [1]). Indeed, $M \equiv 1$ if $\hat{a} = 0$. For $n \in \mathbb{N}$, we have

$$M(-n, \hat{b}, \eta) = 1 + \sum_{m=1}^{n} (-1)^m \frac{n!}{(n-m)!m!} \frac{1}{\hat{b} \cdots (\hat{b} + m - 1)} \eta^m.$$

Otherwise, if $-\hat{a} \notin \mathbb{N} \cup \{0\}$, then

$$M\left(\hat{a}, \hat{b}, \frac{y^2}{4}\right) \sim \frac{\Gamma(\hat{b})}{\Gamma(\hat{a})} \left(\frac{y^2}{4}\right)^{-q} \exp\left(\frac{y^2}{4}\right)$$

as $y \to \infty$, where $q := \gamma_+/2 + 1/2 + \lambda = \alpha + \lambda$. Since $\phi \in \mathbb{H}$, we conclude that the eigenvalues of $\mathcal{L}$ are given by $\lambda_0 = \alpha - 1/2$ and $\lambda_n = \gamma_+/2 + n = \alpha + n - 1/2$ for $n \in \mathbb{N}$.

Note that $\phi_0(y) = c_0|y|^{2\alpha-1}$ for some positive constant c_0. Since $\phi(y) = y^{\gamma_+} H(\eta)$, we have

$$\phi_n(y) = c_n y^{2\alpha-1}(1 + o(1)) \text{ as } y \to 0^+, \tag{4.5}$$

$$\phi_n(y) = \tilde{c}_n y^{2\lambda_n}(1 + o(1)) \text{ as } y \to \infty, \tag{4.6}$$

for some constants $c_n > 0$ and $(-1)^n \tilde{c}_n > 0$ for all $n \in \mathbb{N}$.

In this paper, we are unable to derive a good outer expansion rigorously. Since $w \to U$ as $s \to \infty$, from the dynamical theory point of view, we assume instead that there exist an integer $l \geq 1$ and positive constants ϵ, K, s_1 with ϵ sufficiently small such that

$$|w(y,s) - U(y) - \hat{c}_l e^{-(\lambda_l - \alpha)s}\phi_l(y)| \leq \epsilon e^{-(\lambda_l-\alpha)s} y^{2\alpha-1} \tag{4.7}$$

for $y \in [Ke^{-(\lambda_l-\alpha)s}, 1]$ and $s \geq s_1$ for some nonzero constant $\hat{c}_l$.

5. Rate of Convergence

In this section, we shall use the idea of matching to derive formally the exact convergence rate of $\sigma(s) := w(0,s)$ to zero as $s \to \infty$.

We recall from Lemma 2.1 and Theorem 3.2 that for any $y > 0$:

$$w(y,s) - U(y) = a\sigma^{(1-p)/2}(s) y^{2\alpha-1}(1 + o(1)) \ \text{ as } \ s \to \infty. \tag{5.1}$$

On the other hand, by (4.7) and (4.3), we have, for $y(s) := Ke^{-(\lambda_l-\alpha)s}$,

$$(\hat{c}_l d_l - \epsilon)e^{-(l-1/2)s} y(s)^{2\alpha-1} \leq w(y(s),s) - U(y(s)) \leq (\hat{c}_l d_l + \epsilon) e^{-(l-1/2)s} y(s)^{2\alpha-1} \tag{5.2}$$

for all $s \geq s_1$ for some $l \geq 1$. Consequently, (1.11) can be formally derived under the dynamical theory assumption (4.7).

6. Construction of Some Special Solutions

In this section, we shall construct some special solutions for the problem (CP):

$$w_s = w_{yy} - \frac{y}{2} w_y + \alpha w - w^p \quad \text{ for } \ y > 0, \ s > s_1, \tag{6.1}$$

$$w_y(0,s) = 0 \quad \text{ for } s > s_1, \tag{6.2}$$

$$w(y, s_1) = w_0(y) > 0 \quad \text{ for } y \in [0, \infty). \tag{6.3}$$

By setting $v := w - U$, (6.1) is equivalent to

$$v_s \ = \ -\mathcal{A}v + f(v), \tag{6.4}$$

where

$$\mathcal{A}v := -v_{yy} + \frac{y}{2}v_y + \frac{b}{y^2}v - \alpha v, \quad b := (2\alpha - 1)(2\alpha - 2),$$
$$f(v) := U^p - (v + U)^p + bv/y^2.$$

First, we fix some notation. For each $l \in \mathbb{N}$, let c_l be the constant in (4.5) with $n = l$. Let $k(\eta)$ be the constant in (2.3). We choose (for a fix l) two positive constants η_1 and η_2 so that $k(\eta_1) < c_l < k(\eta_2)$.

Theorem 6.1. *For each odd $l \in \mathbb{N}$, there exists a positive solution w of* (CP) *such that, for some positive constants s_1, K, σ, ϵ with $(\lambda_l - \alpha)/(2\lambda_l - 2\alpha + 1) < \sigma < 1/2$, $s_1 \gg 1$, $K \gg 1$, and $\epsilon \ll 1$,*

$$e^{-\gamma s} U_{\eta_1}(e^{\beta s} y) < w(y, s) < e^{-\gamma s} U_{\eta_2}(e^{\beta s} y) \tag{6.5}$$

for $y \in [0, Ke^{-\beta s}]$ and $s \geq s_1$, and

$$|w(y, s) - U(y) - e^{-\beta s}\phi_l(y)| \leq \epsilon e^{-\beta s}(y^{2\alpha-1} + y^{2\lambda_l}) \tag{6.6}$$

for $y \in [Ke^{-\beta s}, e^{\sigma s}]$ and $s \geq s_1$, where $\beta = \beta_l := \lambda_l - \alpha$ and $\gamma = \gamma_l := 2\alpha\beta$.

Hereafter l is a fixed odd positive integer so that $\beta = \lambda_l - \alpha$ and $\gamma = 2\alpha\beta$ are fixed.

For a given $d := (d_0, \cdots, d_{l-1}) \in \mathbb{R}^l$ with the property

(D) $\sum_{n=0}^{l-1} |d_n| < \epsilon e^{-\beta s_1}$,

we let $w(y, s; d)$ be the solution of (6.1)–(6.3) with the initial data $w_0(y) = v(y, s_1) + U(y)$, where

$$v(y, s_1) = v(y, s_1; d) := \sum_{n=0}^{l-1} d_n \phi_n(y) + e^{-\beta s_1}\tilde{\phi}_l(y) \tag{6.7}$$

satisfying, for some (fixed) constants $\tilde{\sigma}$ and $\tilde{K}$ with $\tilde{\sigma} \in ((\lambda_l - \alpha)/(2\lambda_l - 2\alpha + 1), 1/2)$ and $\tilde{K} \gg 1$,

($V1$) $\tilde{\phi}_l(y) := -e^{\beta s_1}\left\{ U(y) - e^{-\gamma s_1} U_*(e^{\beta s_1} y) + \sum_{n=0}^{l-1} d_n \phi_n(y) \right\}$, $y \in [0, \tilde{K}e^{-\beta s_1}]$,
where U_* is the solution of (2.1) so that $k(\eta) = c_l$ in (2.3);

($V2$) $\tilde{\phi}_l(y) := \phi_l(y)$, $y \in [\tilde{K}e^{-\beta s_1}, e^{\tilde{\sigma} s_1}]$;

($V3$) $\tilde{\phi}_l(y)$ is smooth for $y \geq \tilde{K}e^{-\beta s_1}$ and satisfies the bound
$\tilde{\phi}_l(y) \leq \min\{\phi_l(y), U(y)\}$ for $y \geq e^{\tilde{\sigma} s_1}$;

($V4$) $v(y, s_1) + U(y) > 0$ in $(0, \infty)$.

Fix a positive constant $\epsilon \ll 1$. Take $\sigma \in ((\lambda_l - \alpha)/(2\lambda_l - 2\alpha + 1), \tilde{\sigma})$, $s_1 \gg 1$, and $K \gg \tilde{K}$ such that $e^{\beta s_1/8} < K < e^{\beta s_1/2}$. For $s_2 \geq s_1 > 0$ and $\theta \in (0, 1]$, let $W^{\theta}_{s_1,s_2}$ be the set of (symmetric) continuous functions w on $[0, \infty)$ satisfying

$$|w(y, s) - U(y) - e^{-\beta s}\phi_l(y)| \leq \theta\epsilon e^{-\beta s}(y^{2\alpha-1} + y^{2\lambda_l}) \text{ if } y \in [Ke^{-\beta s}, e^{\sigma s}],\ s \in [s_1, s_2]. \tag{6.8}$$

Define U_{s_1,s_2} to be the the set of all $d \in \mathbb{R}^l$ with the property (D) such that $w(y, s; d) \in W^1_{s_1,s_2}$. Also, we define $\overline{U}_{s_1,s_2}$ to be the closure of U_{s_1,s_2}. Hereafter the constant σ is fixed.

Proposition 6.1. *Let $s_1 \gg 1$ and $K \gg \tilde{K}$. If $d \in \overline{U}_{s_1,s_2}$ with some $s_2 > s_1$, then there exists a positive constant $\delta_0 \in (0, 1)$ such that*

$$(1 + \delta_0)e^{-\gamma s} U_{\eta_1}(e^{\beta s} y) < w(y, s) < (1 - \delta_0)e^{-\gamma s} U_{\eta_2}(e^{\beta s} y) \tag{6.9}$$

for $y \in [0, Ke^{-\beta s}]$ *and* $s \in [s_1, s_2]$*, where* $w(y,s) := w(y,s;d)$*.*

Proof. First, we recall from the proof of Lemma 2.1 that, as $\eta^{(p-1)/2}e^{\beta s}y \to \infty$,

$$e^{-\gamma s}U_\eta(e^{\beta s}y) = U(y) + k(\eta)e^{-\beta s}y^{2\alpha-1}(1+o(1)). \tag{6.10}$$

By assumption, $w(y,s) \in W^1_{s_1,s_2}$, since $d \in \overline{U}_{s_1,s_2}$. By (6.8) and the choices of η_1, η_2, ϵ, there exists a positive constant $\delta_1 \in (0,1)$ such that

$$(1+\delta_1)e^{-\gamma s}U_{\eta_1}(e^{\beta s}y) < w(y,s) < (1-\delta_1)e^{-\gamma s}U_{\eta_2}(e^{\beta s}y) \tag{6.11}$$

in $D_1 := \{y = Ke^{-\beta s},\ s \in [s_1, s_2]\}$, if $K \gg 1$. From (D), (V_1) and (V_2), by choosing $\delta_2 \in (0, \delta_1)$ small enough, we have

$$(1+\delta_2)e^{-\gamma s}U_{\eta_1}(e^{\beta s_1}y) < w(y,s_1) < (1-\delta_2)e^{-\gamma s_1}U_{\eta_2}(e^{\beta s_1}y) \tag{6.12}$$

in $D_2 := \{y \in [0, Ke^{-\beta s_1}],\ s = s_1\}$, if $s_1 \gg 1$ and $K \gg \tilde{K}$. Set $\delta_0 := \delta_2$. From (6.11) and (6.12), it follows that (6.9) holds in $D_1 \cup D_2$, if $K \gg \tilde{K}$ and $s_1 \gg 1$.

Now, we define

$$\begin{aligned}\tilde{w}(y,s) &= (1-\delta_0)e^{-\gamma s}U_{\eta_2}(e^{\beta s}y),\\ \hat{w}(y,s) &= (1+\delta_0)e^{-\gamma s}U_{\eta_1}(e^{\beta s}y).\end{aligned}$$

Then from a direct calculation we get, for $y \in [0, Ke^{-\beta s}]$ and $s \in [s_1, s_2]$,

$$\begin{aligned}&\tilde{w}_s - \tilde{w}_{yy} + \frac{1}{2}y\tilde{w}_y - \alpha\tilde{w} + \tilde{w}^p\\ &\quad= (1-\delta_0)e^{(-\gamma+2\beta)s}\{[-1+(1-\delta_0)^{p-1}]U^p_{\eta_2} + e^{-2\beta s}B_2\} > 0 \ \text{ if } s_1 \gg 1,\\ &\hat{w}_s - \hat{w}_{yy} + \frac{1}{2}y\hat{w}_y - \alpha\hat{w} + \hat{w}^p\\ &\quad= (1+\delta_0)e^{(-\gamma+2\beta)s}\{[-1+(1+\delta_0)^{p-1}]U^p_{\eta_1} + e^{-2\beta s}B_1\} < 0 \ \text{ if } s_1 \gg 1,\end{aligned}$$

where $B_i := -(\gamma+\alpha)U_{\eta_i}(e^{\beta s}y) + (\beta+1/2)(e^{\beta s}y)U'_{\eta_i}(e^{\beta s}y)$, $i = 1,2$. Since $w_y(0,s) = 0 = U'_\eta(0)$, the proposition follows by a comparison principle. ☐

We next derive the estimate in the region $\{y \ge e^{\sigma s}\}$ as follows.

Proposition 6.2. *Let* $s_1 \gg 1$ *and* $K \gg \tilde{K}$*. If* $d \in \overline{U}_{s_1,s_2}$ *with some* $s_2 > s_1$*, then*

$$0 < w(y,s) \le U(y) \tag{6.13}$$

for $y \ge e^{\sigma s}$ *and* $s \in [s_1, s_2]$*, where* $w(y,s) := w(y,s;d)$*.*

Proof. Note that, by (V4), it follows from the maximum principle that $w > 0$. Since $d \in \overline{U}_{s_1,s_2}$, by (6.8) and (4.6), we have

$$w(y,s) \le U(y) + e^{-\beta s}(\tilde{c}_l - 2\epsilon)y^{2\lambda_l} < U(y) \ \text{ if } \epsilon \ll 1 \text{ and } s_1 \gg 1,$$

for $y = e^{\sigma s}$ and $s \in [s_1, s_2]$. Here l is assumed to be odd and so $\tilde{c}_l < 0$. Clearly, (6.13) holds for $y \geq e^{\sigma s}$ and $s = s_1$, if $s_1 \gg 1$, since $v(y, s_1) < 0$ for $y \geq e^{\sigma s_1}$ by using (D), (V2) and (V3). The proposition follows by a comparison principle. $\square$

Next, we define the operator $P(\cdot; s_1, s_2)$ from $\overline{U}_{s_1,s_2}$ to $\mathbb{R}^l$ by

$$P(d; s_1, s_2) = (p_0, \cdots, p_{l-1}), \quad p_n := \langle v(y, s_2; d), \phi_n(y) \rangle, \ n = 0, \cdots, l-1,$$

where $v(y, s; d) := w(y, s; d) - U(y)$ and $\langle f, g \rangle := \int_0^\infty f(y)g(y)\rho(y)dy$.

Proposition 6.3. *Let $s_1 \gg 1$ and $K \gg \tilde{K}$. If there is $d \in \overline{U}_{s_1,s_2}$ for some $s_2 > s_1$ such that $P(d; s_1, s_2) = 0$, then $w(y, s; d) \in W^\theta_{s_1,s_2}$ for some $\theta \in (0, 1)$.*

To prove this key proposition, we shall apply an idea of Herrero and Velazquez [6], which was modified by Mizoguchi [9]. Since it involves rather complicated computations, we shall postpone its proof at the end of this paper. We continue to prove Theorem 6.1.

Proposition 6.4. *Let $s_1 \gg 1$. If $\overline{U}_{s_1,s_2} \neq \emptyset$ for some $s_2 > s_1$, then there exists $d \in \overline{U}_{s_1,s_2}$ such that $P(d; s_1, s_2) = 0$.*

Proof. We shall apply the degree theory to prove that

$$\deg (P(d; s_1, s_2), 0, U_{s_1,s_2}) = 1,$$

where $\deg (P(d; s_1, s_2), 0, U_{s_1,s_2})$ denotes the degree of $P(\cdot; s_1, s_2)$ with respect to 0 in the set U_{s_1,s_2}. First, we claim that $\deg (P(d; s_1, s_1), 0, U_{s_1,s_1}) = 1$. Note that $p_n(d; s_1, s_1) = d_n + e^{-\beta s_1} \langle \tilde{\phi}_l, \phi_n \rangle$. Set

$$h^t(d) = (1 - t)I(d) + tP(d; s_1, s_1),$$

where $I(d)$ is the identity mapping in $\mathbb{R}^l$ and $t \in [0, 1]$. Through the choice of $\tilde{\phi}_l$, for $s_1 \gg 1$ we have $h^t(\partial U_{s_1,s_1}) \neq 0$ for $t \in [0, 1]$. By the standard degree theory, $\deg (h^t(d), 0, U_{s_1,s_1})$ is independent of $t \in [0, 1]$. Hence we obtain that

$$\deg (P(d; s_1, s_1), 0, U_{s_1,s_1}) = \deg (I(d), 0, U_{s_1,s_1}) = 1.$$

Moreover, by Proposition 6.3 and Lemma 7.3, we have $P(d; s_1, s_2) \neq 0$ for all $d \in \partial U_{s_1,s_2}$, if $s_1 \gg 1$. Therefore, it follows from the homotopy invariance of degree theory that $\deg (P(d; s_1, s_2), 0, U_{s_1,s_2}) = 1$. This proves the proposition.

Proposition 6.5. *Let $s_1 \gg 1$. Then $U_{s_1,s_2} \neq \emptyset$ for all $s_2 > s_1$.*

Proof. Set $s^* := \sup\{s \ge s_1 \mid U_{s_1,s} \ne \emptyset\}$. From the theory of continuous dependence on initial data, we see that $U_{s_1,s} \ne \emptyset$ for $s > s_1$ with $s - s_1 \ll 1$. Hence $s^* > s_1$. Claim that $s^* = \infty$. If not, then there exists a sequence $\{s_j\}$ with $s_j < s^* < \infty$ such that $s_j \to s^*$ as $j \to \infty$. By Proposition 6.4, there exists $d_j \in \overline{U}_{s_1,s_j}$ such that $P(d_j; s_1, s_j) = 0$ for each j. Since $\{d_j\}$ is bounded, without loss of generality, we may assume $d_j \to d^*$ as $j \to \infty$ for some $d^* \in \mathbb{R}^l$. Moreover, we have $d^* \in \overline{U}_{s_1,s^*}$ and $P(d^*; s_1, s^*) = 0$ from the continuity of P. From Proposition 6.3, there exists $\theta \in (0, 1)$ such that $w(y, s; d^*) \in W^\theta_{s_1,s^*}$. Hence we get $w(y, s; d^*) \in U_{s_1,s^*+\eta}$ for some $\eta > 0$. This is a contradiction to the definition of s^* and the proposition is proved.

Now, we are ready to prove Theorem 6.1.

Proof of Theorem 6.1: From Proposition 6.5 it follows that there exists a strictly increasing sequence $\{s_j\} \subset (s_1, \infty)$ with $s_j \to \infty$ as $j \to \infty$ such that $U_{s_1,s_j} \ne \emptyset$. By Proposition 6.4, there exists $d_j \in \overline{U}_{s_1,s_j}$ such that $P(d_j; s_1, s_j) = 0$ for each j. Hence, from Proposition 6.3, we have $w(y, s; d_j) \in W^1_{s_1,s_j}$ for each j. Since $\{d_j\}$ is bounded, without loss of generality we may assume that $d_j \to \bar{d}$ as $j \to \infty$ for some $\bar{d} \in \mathbb{R}^l$. Let $w(y, s; \bar{d})$ be the solution of (CP) with initial value $w(y, s_1; \bar{d}) = v(y, s_1; \bar{d}) + U(y)$. Since $d_k \in \overline{U}_{s_1,s_k} \subseteq \overline{U}_{s_1,s_j}$ for all $k > j$ for any given j, we have $\bar{d} \in \overline{U}_{s_1,s_j}$ for all j. Therefore, we conclude that $w(y, s; \bar{d}) \in W^1_{s_1,\infty}$. □

7. Proof of Proposition 6.3

In the sequel, C denotes a (universal) positive constant, which may be different from one line to another, depending on p and l.

Recall that $f(v) := U^p - (v + U)^p + bv/y^2$. Note that, by (V4), $w = v + U > 0$ and so f is well-defined. From

$$v_s = -\mathcal{A}v + f(v) \quad \text{in } \mathbb{R} \times (s_1, \infty),$$

it follows that (cf. [5])

$$v(y, s) = e^{-\mathcal{A}(s-s_1)}v(y, s_1) + \int_{s_1}^{s} e^{-\mathcal{A}(s-\tau)} f(v(y, \tau))d\tau \tag{7.1}$$

for $s \ge s_1$. Hereafter s_1 is a very large constant.

Lemma 7.1. *We have*

$$0 \le f(v) \le CU^{p-2}v^2 \tag{7.2}$$

for some $C > 0$.

Proof. First, since

$$f(v) = \frac{1}{2}p(1-p)(U+\hat{v})^{p-2}v^2$$

for some $\hat{v}$ between 0 and v, we see that $f(v) \geq 0$.

To prove the upper bound, we divide our proof into two cases.

Case 1: $-U/2 \leq v$. Applying the mean value theorem, we have

$$\begin{aligned} f(v) &= U^p - (v+U)^p + pU^{p-1}v \\ &= -p(U+\theta_1 v)^{p-1}v + pU^{p-1}v = -p[(U+\theta_1 v)^{p-1} - U^{p-1}]v \\ &= \theta_1 p(1-p)(U+\theta_1\theta_2 v)^{p-2}v^2 \\ &\leq \theta_1 p(1-p)\left(U - \frac{1}{2}\theta_1\theta_2 U\right)^{p-2} v^2 \\ &\leq \theta_1 p(1-p)\left(\frac{1}{2}U\right)^{p-2} v^2 \leq CU^{p-2}v^2, \end{aligned}$$

where the constants $\theta_i \in (0,1)$, $i = 1,2$.

Case 2: $-U \leq v \leq -U/2$. Since $-U \leq v \leq -U/2$, we have $-2 \leq U/v \leq -1$. Hence

$$\begin{aligned} f(v) &= U^p - (v+U)^p + pU^{p-1}v \\ &\leq U^p - (v+U)^p \\ &= U^{p-2}v^2\left[\left(\frac{U}{v}\right)^2 - \left(\frac{v}{U}+1\right)^p\left(\frac{U}{v}\right)^2\right] \\ &\leq 2U^{p-2}v^2\left(\frac{U}{v}\right)^2 \leq CU^{p-2}v^2. \end{aligned}$$

Combining Case 1 with Case 2, we have proved the lemma.

Lemma 7.2. *Let $\delta > 1$. If $d \in \overline{U}_{s_1,s_2}$ for some $s_2 > s_1$, then*

$$0 \leq f(v) \leq CK^{2\alpha}e^{-2\alpha\beta s}y^{-2} \quad \text{for } y \in [0, Ke^{-\beta s}], \tag{7.3}$$

$$0 \leq f(v) \leq Ce^{-2\beta s}y^{2\alpha-4} \quad \text{for } y \in [Ke^{-\beta s}, \delta], \tag{7.4}$$

$$0 \leq f(v) \leq Ce^{-2\beta s}y^{4\lambda_l-2\alpha-2} \quad \text{for } y \in [\delta, e^{\sigma s}], \tag{7.5}$$

$$0 \leq f(v) \leq Cy^{2\alpha-2} \quad \text{for } y \geq e^{\sigma s} \tag{7.6}$$

for each $s \in [s_1, s_2]$, where $v := v(y,s;d) = w(y,s;d) - U(y)$.

Proof. Given any $s \in [s_1, s_2]$. From (6.9), there exists a constant $C > 0$ such that

$$|v(y,s)| \leq CK^{2\alpha}e^{-2\alpha\beta s}$$

for $y \in [0, Ke^{-\beta s}]$. Then (7.3) follows from the definition of f. From (6.8), (4.5) and (4.6), we have

$$|v(y,s)| \leq Ce^{-\beta s}y^{2\alpha-1} \text{ if } y \in [Ke^{-\beta s}, \delta];$$
$$|v(y,s)| \leq Ce^{-\beta s}y^{2\lambda_l} \text{ if } y \in [\delta, e^{\sigma s}].$$

Moreover, from (6.13) it follows that $|v(y,s)| \le Cy^{2\alpha}$ for $y \ge e^{\sigma s}$. Therefore, (7.4) through (7.6) follow by using (7.2). ☐

Lemma 7.3. *Suppose that $d \in \overline{U}_{s_1,s_2}$ such that $P(d;s_1,s_2) = 0$. Then for any $\nu \in (0,1)$ there exists $s_1 \gg 1$ such that*

$$\sum_{n=0}^{l-1} |d_n| \le \nu e^{-\beta s_1}. \tag{7.7}$$

Proof. Since $P(d;s_1,s_2) = 0$, we have $\langle v(y,s_2),\phi_n\rangle = 0$ for $n = 0,\cdots,l-1$. Fix $n \in \{0,\cdots,l-1\}$. Then we get from (7.1) and (6.7) that $|d_n| \le I_{n,1} + I_{n,2}$, where

$$I_{n,1} := e^{-\beta s_1} |\langle \tilde{\phi}_l, \phi_n\rangle|,$$
$$I_{n,2} := e^{-(\lambda_n-\alpha)s_1} \int_{s_1}^{s_2} e^{(\lambda_n-\alpha)\tau} |\langle f, \phi_n\rangle| d\tau.$$

Since $\tilde{\phi}_l \to \phi_l$ in L^2_ρ as $s_1 \to \infty$, $|\langle \tilde{\phi}_l, \phi_n\rangle| \to 0$ as $s_1 \to \infty$. Thus for any $\nu \in (0,1)$ there exists $s_1 \gg 1$ such that

$$I_{n,1} \le \frac{\nu}{2l} e^{-\beta s_1}.$$

Now, we shall apply Lemma 7.2 to estimate $|\langle f,\phi_n\rangle|$ for each $\tau \in [s_1,s_2]$. Recall that $K < e^{\beta s_1/2}$. First, from (7.3) it follows that

$$\left| \int_0^{Ke^{-\beta\tau}} f(v(y,\tau))\phi_n(y)\rho(y)dy \right| \le CK^{2\alpha} e^{-2\alpha\beta\tau} \le Ce^{(1-\alpha)\beta\tau} e^{-\beta\tau}. \tag{7.8}$$

Next, by (7.4) and (7.5), we have

$$\left| \left\{ \int_{Ke^{-\beta\tau}}^{\delta} + \int_{\delta}^{e^{\sigma\tau}} \right\} f(v(y,\tau))\phi_n(y)\rho(y)dy \right| \le Ce^{-2\beta\tau}. \tag{7.9}$$

Finally, by (7.6), we have

$$\left| \int_{e^{\sigma\tau}}^{\infty} f(v(y,\tau))\phi_n(y)\rho(y)dy \right| \le Ce^{-e^{2\sigma\tau}/8} \int_{e^{\sigma\tau}}^{\infty} y^{2\lambda_l+2\alpha-2} e^{-y^2/8} dy \le Ce^{-e^{2\sigma\tau}/8}. \tag{7.10}$$

Combining (7.8), (7.9) and (7.10), we could find a constant $\kappa > 0$ such that

$$|\langle f(v(y,\tau)),\phi_n\rangle| \le Ce^{-(\beta+\kappa)\tau} \quad \text{if } s_1 \gg 1. \tag{7.11}$$

Hence

$$I_{n,2} \le \frac{\nu}{2l} e^{-\beta s_1} \quad \text{if } s_1 \gg 1.$$

This completes the proof of the lemma.

Throughout the following, we suppose that $d \in \overline{U}_{s_1,s_2}$ with $1 \ll s_1 < s_2$ such that $P(d;s_1,s_2) = 0$.

7.1. **Short Time:** $s_1 \le s \le s_1 + 1$.

Since $v = \sum_{n=0}^{\infty} \langle v, \phi_n \rangle \phi_n$, we may rewrite $v(y, s_1)$ as

$$v(y,s_1) = \sum_{n=0}^{l-1} (d_n + e^{-\beta s_1} \langle \tilde{\phi}_l, \phi_n \rangle) \phi_n + e^{-\beta s_1} \langle \tilde{\phi}_l, \phi_l \rangle \phi_l + \sum_{n=l+1}^{\infty} e^{-\beta s_1} \langle \tilde{\phi}_l, \phi_n \rangle \phi_n.$$

Thus from (7.1) we may write, for $s \ge s_1$,

$$v(y,s) = S_1(y,s) + S_2(y,s) + S_3(y,s),$$

where

$$S_1(y,s) := e^{-\beta s} \langle \tilde{\phi}_l, \phi_l \rangle \phi_l, \tag{7.12}$$

$$S_2(y,s) := \sum_{n \ne l} e^{-(\lambda_n - \alpha)(s-s_1)} \langle \tilde{\phi}_l, \phi_n \rangle e^{-\beta s_1} \phi_n + \sum_{n=0}^{l-1} d_n e^{-(\lambda_n - \alpha)(s-s_1)} \phi_n, \tag{7.13}$$

$$S_3(y,s) := \int_{s_1}^{s} e^{-\mathcal{A}(s-\tau)} f(v(y,\tau)) d\tau. \tag{7.14}$$

Lemma 7.4. *For any $\nu \in (0,1)$, there exist $s_1 \gg 1$ and $K \gg \tilde{K}$ such that*

$$|S_2(y,s)| \le \nu e^{-\beta s} (y^{2\alpha-1} + y^{2\lambda_l}),$$

if $K e^{-\beta s} \le y \le e^{\sigma s}$ and $s_1 \le s \le s_1 + 1$, where $\sigma \in (0, \tilde{\sigma})$.

Proof. Given any $\nu \in (0,1)$. Note that $S_2(y,s)$ satisfies

$$(S_2)_s = (S_2)_{yy} - \frac{1}{2} y (S_2)_y + \alpha S_2 - \frac{b}{y^2} S_2.$$

Set $S_2(y,s) = y^{2\alpha-1} V(y,s)$. We have

$$V_s = V_{yy} + \left(\frac{2(2\alpha-1)}{y} - \frac{y}{2} \right) V_y + \frac{1}{2} V.$$

Moreover, by setting $\overline{V} = e^{-s/2} V$, we obtain that

$$\overline{V}_s = \overline{V}_{yy} + \left(\frac{2(2\alpha-1)}{y} - \frac{y}{2} \right) \overline{V}_y.$$

From Proposition 6.1 of [1], it follows that

$$\begin{aligned} \overline{V}(y,s) &= c_\alpha \frac{\exp\left((4\alpha-3)(s-s_1)/4\right)}{1-e^{-(s-s_1)}} \int_0^\infty I_{(4\alpha-3)/2} \left(\frac{e^{-(s-s_1)/2} r y}{2(1-e^{-(s-s_1)})} \right) \\ &\quad \cdot r^{(4\alpha-1)/2} y^{-(4\alpha-3)/2} \exp\left(-\frac{y^2 e^{-(s-s_1)} + r^2}{4(1-e^{-(s-s_1)})} \right) \overline{V}(r, s_1) dr, \end{aligned}$$

where $I_\mu(z)$ is the modified Bessel function of order μ and c_α is a positive constant depending on α. Hence

$$S_2(y,s) = Cy^{2\alpha-1}\frac{\exp\left((4\alpha-1)(s-s_1)/4\right)}{1-e^{-(s-s_1)}}\int_0^\infty H(r,y;s-s_1)r^{1/2}v_0 dr, \tag{7.15}$$

where $v_0(y,s_1) := v(y,s_1) - e^{-\beta s_1}\langle\tilde{\phi}_l,\phi_l\rangle\phi_l(y)$ and

$$H(r,y;s-s_1) := I_{(4\alpha-3)/2}\left(\frac{e^{-(s-s_1)/2}ry}{2(1-e^{-(s-s_1)})}\right)\exp(-\frac{y^2e^{-(s-s_1)}+r^2}{4(1-e^{-(s-s_1)})})y^{-(4\alpha-3)/2}. \tag{7.16}$$

Since

$$|I_\mu(z)| \le \frac{Cz^\mu e^z}{(1+z)^{\mu+1/2}},$$

from (7.15) we have for $s_1 \le s \le s_1+1$

$$|S_2(y,s)| \le \frac{Cy^{2\alpha-1}}{1-e^{-(s-s_1)}}\int_0^\infty T(r,y;s-s_1)r^{1/2}|v_0|dr := S_{2,1}+S_{2,2},$$

where

$$\begin{aligned} S_{2,1} &:= \frac{Cy^{2\alpha-1}}{1-e^{-(s-s_1)}}\int_0^{\tilde{K}e^{-\beta s_1}} T(r,y;s-s_1)r^{1/2}|v_0|dr,\\ S_{2,2} &:= \frac{Cy^{2\alpha-1}}{1-e^{-(s-s_1)}}\int_{\tilde{K}e^{-\beta s_1}}^\infty T(r,y;s-s_1)r^{1/2}|v_0|dr,\\ T(r,y;s-s_1) &:= \frac{\left(e^{-(s-s_1)/2}ry/2(1-e^{-(s-s_1)})\right)^{(4\alpha-3)/2}}{\left(1+e^{-(s-s_1)/2}ry/2(1-e^{-(s-s_1)})\right)^{2\alpha-1}}\\ &\quad\cdot\exp\left(-\frac{|ye^{-(s-s_1)/2}-r|^2}{4(1-e^{-(s-s_1)})}\right)y^{-2\alpha+3/2}. \end{aligned}$$

Firstly, we consider $S_{2,1}$. Using $(V1)$ and (6.10), for $s_1 \gg 1$ we have

$$|v_0(y,s_1)| \le C\tilde{K}^{2\alpha}e^{-2\alpha\beta s_1} \quad \text{for } y\in[0,\tilde{K}e^{-\beta s_1}]. \tag{7.17}$$

From (7.17), we have

$$S_{2,1}(y,s) \le \frac{C\tilde{K}^{2\alpha}e^{-2\alpha\beta s_1}y^{2\alpha-1}}{1-e^{-(s-s_1)}}\int_0^{\tilde{K}e^{-\beta s_1}} T(r,y;s-s_1)r^{1/2}dr.$$

For $y \ge Ke^{-\beta s}$ and $r \le \tilde{K}e^{-\beta s_1}$, we have (using $s_1 \le s \le s_1+1$)

$$ye^{-(s-s_1)/2} - r \ge r \;\text{ if } K \ge 2\tilde{K}e^{\beta+1/2}.$$

Hence we obtain

$$S_{2,1}(y,s) \le \frac{C\tilde{K}^{2\alpha}e^{-2\alpha\beta s_1}}{(1-e^{-(s-s_1)})^{1/2}}\int_0^{\tilde{K}e^{-\beta s_1}} \exp\left(-\frac{r^2}{4(1-e^{-(s-s_1)})}\right)dr.$$

Set $t = r/(1-e^{-(s-s_1)})^{1/2}$. We then have

$$\begin{aligned} S_{2,1}(y,s) &\le C\tilde{K}^{2\alpha}e^{-2\alpha\beta s_1}\int_0^\infty \exp\left(-\frac{t^2}{4}\right)dt \\ &\le C\tilde{K}^{2\alpha}e^{-2\alpha\beta s_1}y^{2\alpha-1}(Ke^{-\beta s})^{-2\alpha+1} \\ &\le \nu e^{-\beta s}y^{2\alpha-1} \qquad \text{if } K \gg \tilde{K}. \end{aligned}$$

Secondly, we consider $S_{2,2}(y,s)$. From Lemma 7.3, (4.5) and (4.6), for the given $\nu \in (0,1)$ there exists $s_1 \gg 1$ such that

$$|v_0(y,s_1)| \le \nu e^{-\beta s_1}(y^{2\alpha-1}+y^{2\lambda_l}) \quad \text{for } \tilde{K}e^{-\beta s_1} \le y \le e^{\tilde{\sigma}s_1}. \tag{7.18}$$

Moreover, it also holds

$$|v_0(y,s_1)| \le Cy^{2\lambda_l} \quad \text{for } \ y \ge e^{\tilde{\sigma}s_1}. \tag{7.19}$$

From (7.18) and (7.19), for $s_1 \gg 1$ we can estimate $S_{2,2}$ by

$$S_{2,2} \le S_{2,2}^1 + S_{2,2}^2 + S_{2,2}^3,$$

where

$$\begin{aligned} S_{2,2}^1 &= \frac{Cy^{2\alpha-1}}{1-e^{-(s-s_1)}}\int_{e^{\tilde{\sigma}s_1}}^\infty T(r,y;s-s_1)r^{1/2+2\lambda_l}dr, \\ S_{2,2}^2 &= \frac{C\nu e^{-\beta s_1}y^{2\alpha-1}}{1-e^{-(s-s_1)}}\int_{2ye^{-(s-s_1)/2}}^\infty T(r,y;s-s_1)r^{2\alpha-1/2}(1+r^{2\lambda_l-2\alpha+1})dr, \\ S_{2,2}^3 &= \frac{C\nu e^{-\beta s_1}y^{2\alpha-1}}{1-e^{-(s-s_1)}}\int_0^{2ye^{-(s-s_1)/2}} T(r,y;s-s_1)r^{2\alpha-1/2}(1+r^{2\lambda_l-2\alpha+1})dr. \end{aligned}$$

Since

$$r - ye^{-(s-s_1)/2} \ge r(1-e^{-s_1(\tilde{\sigma}-\sigma)}) \ge r/2 \quad \text{if } r \ge e^{\tilde{\sigma}s_1},\ y \le e^{\sigma s}, \text{ and } s_1 \gg 1,$$

we have

$$\begin{aligned} S_{2,2}^1 &\le \frac{Cy^{2\alpha-1}}{1-e^{-(s-s_1)}}\int_{e^{\tilde{\sigma}s_1}}^\infty \left(\frac{r}{2(1-e^{-(s-s_1)})}\right)^{(4\alpha-3)/2}\exp\left(-\frac{r^2}{16(1-e^{-(s-s_1)})}\right)r^{1/2+2\lambda_l}dr \\ &\le \frac{Cy^{2\alpha-1}}{1-e^{-(s-s_1)}}e^{-e^{2\tilde{\sigma}s_1}/32}\int_{e^{\tilde{\sigma}s_1}}^\infty \left(\frac{r}{2(1-e^{-(s-s_1)})}\right)^{(4\alpha-3)/2} \\ &\qquad \cdot\exp\left(-\frac{r^2}{32(1-e^{-(s-s_1)})}\right)r^{1/2+2\lambda_l}dr. \end{aligned}$$

Set $t = r/(1-e^{-(s-s_1)})^{1/2}$. Then

$$\begin{aligned} S_{2,2}^1 &\le Cy^{2\alpha-1}e^{-Ce^{2\tilde{\sigma}s_1}}\int_0^\infty t^{2\alpha+2\lambda_l-1}e^{-t^2/32}dt \\ &\le \nu e^{-\beta s}y^{2\alpha-1} \qquad \text{if } s_1 \gg 1. \end{aligned}$$

For $S_{2,2}^2$, since

$$r - ye^{-(s-s_1)/2} \geq r/2 \quad \text{if } r \geq 2ye^{-(s-s_1)/2},$$

we have

$$\begin{aligned} S_{2,2}^2 \;\leq\; & \frac{C\nu e^{-\beta s_1} y^{2\alpha-1}}{1-e^{-(s-s_1)}} \int_{2ye^{-(s-s_1)/2}}^{\infty} \left(\frac{r}{2(1-e^{-(s-s_1)})}\right)^{(4\alpha-3)/2} \\ & \cdot \exp\left(\frac{-r^2}{16(1-e^{-(s-s_1)})}\right) r^{2\alpha-1/2}(1+r^{2\lambda_l-2\alpha+1})dr. \end{aligned}$$

Set $t = r/(1-e^{-(s-s_1)})^{1/2}$. Then

$$\begin{aligned} S_{2,2}^2 &\leq C\nu e^{-\beta s_1} y^{2\alpha-1} \int_0^\infty t^{4\alpha-2} e^{-t^2/16}(1+t^{2\lambda_l-2\alpha+1})dt \\ &\leq C\nu e^{-\beta s} y^{2\alpha-1} \qquad \text{if } s_1 \gg 1. \end{aligned}$$

Finally, since

$$e^{-(s-s_1)/2} ry \geq r^2/2 \quad \text{if } r \leq 2ye^{-(s-s_1)/2},$$

we have

$$\begin{aligned} S_{2,2}^3 \;\leq\; & \frac{C\nu e^{-\beta s_1} y^{2\alpha-1}}{1-e^{-(s-s_1)}} \int_0^{2ye^{-(s-s_1)/2}} \left(\frac{r}{2(1-e^{-(s-s_1)})}\right)^{(4\alpha-3)/2} \left(\frac{r^2}{4(1-e^{-(s-s_1)})}\right)^{1-2\alpha} \\ & \cdot \exp\left(-\frac{|ye^{-(s-s_1)/2}-r|^2}{4(1-e^{-(s-s_1)})}\right) r^{2\alpha-1/2}(1+r^{2\lambda_l-2\alpha+1})dr \\ \leq\; & \frac{C\nu e^{-\beta s_1} y^{2\alpha-1}}{(1-e^{-(s-s_1)})^{1/2}}(1+y^{2\lambda_l-2\alpha+1}) \int_0^\infty \exp\left(-\frac{|ye^{-(s-s_1)/2}-r|^2}{4(1-e^{-(s-s_1)})}\right) dr \\ \leq\; & C\nu e^{-\beta s_1}(y^{2\alpha-1}+y^{2\lambda_l}) \qquad \text{if } s_1 \gg 1. \end{aligned}$$

This completes the proof.

Lemma 7.5. *For any $\nu \in (0,1)$, there exist $s_1 \gg 1$ such that*

$$|S_3(y,s)| \leq \nu e^{-\beta s}(y^{2\alpha-1}+y^{2\lambda_l}),$$

if $Ke^{-\beta s} \leq y \leq e^{\sigma s}$ and $s_1 \leq s \leq s_1+1$, where $\sigma \in ((\lambda_l-\alpha)/(2\lambda_l-2\alpha+1), \tilde{\sigma})$.

Proof. Let $Z(y,s) = e^{-\mathcal{A}(s-\tau)} f(v(y,\tau))$, $s_1 < \tau < s$. Then

$$Z_s = -\mathcal{A}Z = Z_{yy} - \frac{y}{2}Z_y - \frac{b}{y^2}Z + \alpha Z.$$

Following the same reasoning as in the proof of Lemma 7.4, we have

$$Z(y,s) = Cy^{2\alpha-1}\frac{\exp\left((4\alpha-1)(s-\tau)/4\right)}{1-e^{-(s-\tau)}} \int_0^\infty H(r,y;s-\tau) r^{1/2} f(v(r,\tau))dr,$$

where H is defined in (7.16). Therefore, we obtain

$$S_3(y,s)=Cy^{2\alpha-1}\int_{s_1}^{s}\frac{\exp\left((4\alpha-1)(s-\tau)/4\right)}{1-e^{-(s-\tau)}}\int_0^\infty H(r,y;s-\tau)r^{1/2}f(v(r,\tau))drd\tau.$$

Thus

$$\begin{aligned}|S_3(y,s)| &\le Cy^{2\alpha-1}\int_{s_1}^{s}\frac{1}{(1-e^{-(s-\tau)})^{(4\alpha-1)/2}}\int_0^\infty\left(1+\frac{e^{-(s-\tau)/2}ry}{2(1-e^{-(s-\tau)})}\right)^{-2\alpha+1}\\ &\quad\cdot\exp\left(-\frac{|ye^{-(s-\tau)/2}-r|^2}{4(1-e^{-(s-\tau)})}\right)r^{2\alpha-1}|f(v(r,\tau))|drd\tau\\ &\le S_{3,1}+S_{3,2}+S_{3,3},\end{aligned}$$

where

$$\begin{aligned}S_{3,1}&:=Cy^{2\alpha-1}\int_{s_1}^{s}(s-\tau)^{-2\alpha+1/2}\int_0^{Le^{-\beta\tau}}\left(1+\frac{Cry}{s-\tau}\right)^{-2\alpha+1}\\ &\quad\cdot\exp\left(-C\frac{|ye^{-(s-\tau)/2}-r|^2}{s-\tau}\right)r^{2\alpha-1}|f(v(r,\tau))|drd\tau,\\ S_{3,2}&:=Cy^{2\alpha-1}\int_{s_1}^{s}(s-\tau)^{-2\alpha+1/2}\int_{Le^{-\beta\tau}}^{e^{\sigma\tau}}\left(1+\frac{Cry}{s-\tau}\right)^{-2\alpha+1}\\ &\quad\cdot\exp\left(-C\frac{|ye^{-(s-\tau)/2}-r|^2}{s-\tau}\right)r^{2\alpha-1}|f(v(r,\tau))|drd\tau,\\ S_{3,3}&:=Cy^{2\alpha-1}\int_{s_1}^{s}(s-\tau)^{-2\alpha+1/2}\int_{e^{\sigma\tau}}^{\infty}\left(1+\frac{Cry}{s-\tau}\right)^{-2\alpha+1}\\ &\quad\cdot\exp\left(-C\frac{|ye^{-(s-\tau)/2}-r|^2}{s-\tau}\right)r^{2\alpha-1}|f(v(r,\tau))|drd\tau,\end{aligned}$$

by using $(s-\tau)/e\le 1-e^{-(s-\tau)}\le(s-\tau)$ for $s-\tau\in[0,1]$. Here $1\ll L\ll K$.

I. Estimate of $S_{3,1}$. By the same way of estimating (7.3), we have

$$\begin{aligned}S_{3,1} &\le CL^{2\alpha}e^{-2\alpha\beta s_1}y^{2\alpha-1}\int_{s_1}^{s}(s-\tau)^{-2\alpha+1/2}\\ &\quad\cdot\int_0^{Le^{-\beta\tau}}\exp\left(-C\frac{|ye^{-(s-\tau)/2}-r|^2}{s-\tau}\right)r^{2\alpha-3}drd\tau.\end{aligned}$$

Since $y\ge Ke^{-\beta s}$ and $r\le Le^{-\beta\tau}$, we get

$$\exp\left(-C\frac{|ye^{-(s-\tau)/2}-r|^2}{s-\tau}\right)\le\exp\left(\frac{-C(y^2+r^2)}{s-\tau}\right).$$

Set $u=r/(s-\tau)^{1/2}$. Then we obtain

$$S_{3,1}\le CL^{2\alpha}e^{-2\alpha\beta s_1}y^{2\alpha-1}\int_{s_1}^{s}(s-\tau)^{-\alpha-1/2}e^{-Cy^2/(s-\tau)}\int_0^{Le^{-\beta\tau}/\sqrt{s-\tau}}e^{-Cu^2}u^{2\alpha-3}dud\tau.$$

Set $t = y^2/(s-\tau)$. Then

$$S_{3,1} \le CL^{2\alpha} e^{-2\alpha\beta s_1} y^{2\alpha-1} \int_{y^2/(s-s_1)}^{\infty} t^{\alpha-3/2} e^{-Ct} y^{-2\alpha+1} \int_0^{\infty} e^{-Cu^2} u^{2\alpha-3} du dt.$$

Since $y \ge Ke^{-\beta s}$, we obtain that

$$S_{3,1} \le CL^{2\alpha} e^{-2\alpha\beta s_1} y^{2\alpha-1} (Ke^{-\beta s})^{-2\alpha+1} \le \nu e^{-\beta s_1} y^{2\alpha-1} \qquad \text{if } K \gg L.$$

II. Estimate of $S_{3,2}$**.** From (7.4), (7.5) and Proposition 6.1, we have $S_{3,2} \le S_{3,2}^1 + S_{3,2}^2$, where

$$\begin{aligned} S_{3,2}^1 &:= Ce^{-2\beta s_1} y^{2\alpha-1} \int_{s_1}^{s} (s-\tau)^{-2\alpha+1/2} \int_{Le^{-\beta\tau}}^{e^{\sigma\tau}} \left(1 + \frac{Cry}{s-\tau}\right)^{-2\alpha+1} \\ &\quad \cdot \exp\left(-C\frac{|ye^{-(s-\tau)/2} - r|^2}{s-\tau}\right) r^{4\alpha-5} dr d\tau, \\ S_{3,2}^2 &:= Ce^{-2\beta s_1} y^{2\alpha-1} \int_{s_1}^{s} (s-\tau)^{-2\alpha+1/2} \int_{Le^{-\beta\tau}}^{e^{\sigma\tau}} \left(1 + \frac{Cry}{s-\tau}\right)^{-2\alpha+1} \\ &\quad \cdot \exp\left(-C\frac{|ye^{-(s-\tau)/2} - r|^2}{s-\tau}\right) r^{4\lambda_l-3} dr d\tau. \end{aligned}$$

Firstly, we consider $S_{3,2}^1$. Note that $S_{3,2}^1 \le S_{3,2}^{1,1} + S_{3,2}^{1,2}$, where

$$\begin{aligned} S_{3,2}^{1,1} &:= Ce^{-2\beta s_1} y^{2\alpha-1} \int_{s_1}^{s} (s-\tau)^{-2\alpha+1/2} \int_{Le^{-\beta\tau}}^{4y} \left(1 + \frac{Cry}{s-\tau}\right)^{-2\alpha+1} \\ &\quad \cdot \exp\left(-C\frac{|ye^{-(s-\tau)/2} - r|^2}{s-\tau}\right) r^{4\alpha-5} dr d\tau, \\ S_{3,2}^{1,2} &:= Ce^{-2\beta s_1} y^{2\alpha-1} \int_{s_1}^{s} (s-\tau)^{-2\alpha+1/2} \int_{4y}^{\infty} \left(1 + \frac{Cry}{s-\tau}\right)^{-2\alpha+1} \\ &\quad \cdot \exp\left(-C\frac{|ye^{-(s-\tau)/2} - r|^2}{s-\tau}\right) r^{4\alpha-5} dr d\tau. \end{aligned}$$

Consider $S_{3,2}^{1,1}$. Since $ry \ge r^2/4$, if $r \le 4y$, we have

$$\begin{aligned} S_{3,2}^{1,1} &\le Ce^{-2\beta s_1} y^{2\alpha-1} \int_{s_1}^{s} (s-\tau)^{-2\alpha+1/2} \int_{Le^{-\beta\tau}}^{4y} \left(1 + \frac{Cr^2}{s-\tau}\right)^{-2\alpha+1} \\ &\quad \cdot \exp\left(-C\frac{|ye^{-(s-\tau)/2} - r|^2}{s-\tau}\right) r^{4\alpha-5} dr d\tau. \end{aligned}$$

Set $u = r/(s-\tau)^{1/2}$. Then

$$\begin{aligned} S_{3,2}^{1,1} &\le Ce^{-2\beta s_1} y^{2\alpha-1} \int_{s_1}^{s} (s-\tau)^{-3/2} \int_{(Le^{-\beta\tau})/\sqrt{s-\tau}}^{4y/\sqrt{s-\tau}} (1 + Cu^2)^{-2\alpha+1} \\ &\quad \cdot \exp\left(-C\left|\frac{ye^{-(s-\tau)/2}}{\sqrt{s-\tau}} - u\right|^2\right) u^{4\alpha-5} du d\tau. \end{aligned}$$

To estimate $S^{1,1}_{3,2}$, we divide it into two cases and define

$$\begin{aligned} D_1 &:= \left\{ u \in \left[\frac{Le^{-\beta\tau}}{\sqrt{s-\tau}}, \frac{4y}{\sqrt{s-\tau}} \right];\ \left| \frac{ye^{-(s-\tau)/2}}{\sqrt{s-\tau}} - u \right| \geq \frac{ye^{-(s-\tau)/2}}{2\sqrt{s-\tau}} \right\}, \\ D_2 &:= \left\{ u \in \left[\frac{Le^{-\beta\tau}}{\sqrt{s-\tau}}, \frac{4y}{\sqrt{s-\tau}} \right];\ \left| \frac{ye^{-(s-\tau)/2}}{\sqrt{s-\tau}} - u \right| \leq \frac{ye^{-(s-\tau)/2}}{2\sqrt{s-\tau}} \right\}. \end{aligned}$$

Since

$$\begin{aligned} &\int_{D_1} (1+Cu^2)^{-2\alpha+1} \exp\left(-C \left| \frac{ye^{-(s-\tau)/2}}{\sqrt{s-\tau}} - u \right|^2 \right) u^{4\alpha-5} du \\ &\leq \int_0^{4y/\sqrt{s-\tau}} e^{-Cy^2/(s-\tau)} u^{4\alpha-5} du \leq Ce^{-Cy^2/(s-\tau)} (y^2/(s-\tau))^{2\alpha-2}, \end{aligned}$$

by setting $t = y^2/(s-\tau)$, we have

$$\begin{aligned} S^{1,1,1}_{3,2} := \quad & Ce^{-2\beta s_1} y^{2\alpha-1} \int_{s_1}^{s} (s-\tau)^{-3/2} \int_{D_1} (1+Cu^2)^{-2\alpha+1} \\ & \cdot \exp\left(-C \left| \frac{ye^{-(s-\tau)/2}}{\sqrt{s-\tau}} - u \right|^2 \right) u^{4\alpha-5} du d\tau \\ \leq \quad & Ce^{-2\beta s_1} y^{2\alpha-2} \int_0^{\infty} t^{2\alpha-5/2} e^{-Ct} dt \\ \leq \quad & Ce^{-2\beta s_1} y^{2\alpha-1} (Ke^{-\beta s})^{-1} \\ \leq \quad & \nu e^{-\beta s_1} y^{2\alpha-1} \qquad \text{if } K \gg 1. \end{aligned}$$

Moreover, for any $a \in (0, 1/8)$, we have

$$\begin{aligned} &\int_{D_2} (1+Cu^2)^{-2\alpha+1} \exp\left(-C \left| \frac{ye^{-(s-\tau)/2}}{\sqrt{s-\tau}} - u \right|^2 \right) u^{4\alpha-5} du \\ &\leq \int_{D_2} u^{4\alpha-3+a} (1+Cu^2)^{-2\alpha+3/2-a/2} (1+Cu^2)^{-1/2+a/2} u^{-2-a} du \\ &\leq \int_{D_2} u^{-2-a} du \leq Cy^{-1-a} (s-\tau)^{1/2+a/2}, \end{aligned}$$

since

$$\frac{ye^{-(s-\tau)/2}}{2\sqrt{s-\tau}} \leq u \leq \frac{3ye^{-(s-\tau)/2}}{2\sqrt{s-\tau}} \qquad \text{if } u \in D_2.$$

Hence, by noting that $K \in (e^{\beta s_1/8}, e^{\beta s_1/2})$,

$$\begin{aligned} S_{3,2}^{1,1,2} := \quad & Ce^{-2\beta s_1} y^{2\alpha-1} \int_{s_1}^{s} (s-\tau)^{-3/2} \int_{D_2} (1+Cu^2)^{-2\alpha+1} \\ & \cdot \exp\left(-C\left|\frac{ye^{-(s-\tau)/2}}{\sqrt{s-\tau}} - u\right|^2\right) u^{4\alpha-5} du d\tau \\ \leq \quad & Ce^{-2\beta s_1} y^{2\alpha-1} y^{-1-a} \int_{s_1}^{s} (s-\tau)^{-1+a/2} d\tau \\ \leq \quad & Ce^{-2\beta s_1} y^{2\alpha-1} (Ke^{-\beta s})^{-1-a} \\ \leq \quad & \nu e^{-\beta s_1} y^{2\alpha-1} \qquad \text{if } s_1 \gg 1. \end{aligned}$$

We conclude that

$$S_{3,2}^{1,1} \leq \nu e^{-\beta s_1} y^{2\alpha-1} \qquad \text{if } s_1 \gg 1.$$

Consider $S_{3,2}^{1,2}$. Since

$$\left|ye^{-(s-\tau)/2} - r\right|^2 \geq y^2 e^{-1} + r^2/2 \qquad \text{if } r \geq 4y,$$

we have

$$S_{3,2}^{1,2} \;\leq\; Ce^{-2\beta s_1} y^{2\alpha-1} \int_{s_1}^{s} (s-\tau)^{-2\alpha+1/2} \int_{4y}^{\infty} e^{-Cy^2/(s-\tau)} e^{-Cr^2/(s-\tau)} r^{4\alpha-5} dr d\tau.$$

Set $u = r/(s-\tau)^{1/2}$. Then

$$S_{3,2}^{1,2} \leq Ce^{-2\beta s_1} y^{2\alpha-1} \int_{s_1}^{s} (s-\tau)^{-3/2} \int_{4y/\sqrt{s-\tau}}^{\infty} e^{-Cy^2/(s-\tau)} e^{-Cu^2} u^{4\alpha-5} du d\tau.$$

Set $t = y^2/(s-\tau)$. Then we get

$$\begin{aligned} S_{3,2}^{1,2} \;\leq\; & Ce^{-2\beta s_1} y^{2\alpha-1} y^{-1} \int_{y^2/(s-s_1)}^{\infty} t^{-1/2} e^{-Ct} \int_{4\sqrt{t}}^{\infty} e^{-Cu^2} u^{4\alpha-5} du dt \\ \leq\; & Ce^{-2\beta s_1} y^{2\alpha-1} (Ke^{-\beta s})^{-1} \leq \nu e^{-\beta s_1} y^{2\alpha-1} \qquad \text{if } K \gg 1. \end{aligned}$$

Secondly, we consider $S_{3,2}^{2}$. Note that $S_{3,2}^{2} \leq S_{3,2}^{2,1} + S_{3,2}^{2,2}$, where

$$\begin{aligned} S_{3,2}^{2,1} := & Ce^{-2\beta s_1} y^{2\alpha-1} \int_{s_1}^{s} (s-\tau)^{-2\alpha+1/2} \int_{Le^{-\beta\tau}}^{4y} \left(1 + \frac{Cry}{s-\tau}\right)^{-2\alpha+1} \\ & \cdot \exp\left(-C\frac{|ye^{-(s-\tau)/2} - r|^2}{s-\tau}\right) r^{4\lambda_l - 3} dr d\tau, \\ S_{3,2}^{2,2} := & Ce^{-2\beta s_1} y^{2\alpha-1} \int_{s_1}^{s} (s-\tau)^{-2\alpha+1/2} \int_{4y}^{\infty} \left(1 + \frac{Cry}{s-\tau}\right)^{-2\alpha+1} \\ & \cdot \exp\left(-C\frac{|ye^{-(s-\tau)/2} - r|^2}{s-\tau}\right) r^{4\lambda_l - 3} dr d\tau. \end{aligned}$$

Consider $S_{3,2}^{2,1}$. Since $ry \geq r^2/4$, if $r \leq 4y$, we have

$$\begin{aligned} S_{3,2}^{2,1} &\leq Ce^{-2\beta s_1} y^{2\alpha-1} \int_{s_1}^{s} (s-\tau)^{-2\alpha+1/2} \int_{Le^{-\beta\tau}}^{4y} \left(1+\frac{Cr^2}{s-\tau}\right)^{-2\alpha+1} \\ &\qquad \cdot \exp\left(-C\frac{|ye^{-(s-\tau)/2}-r|^2}{s-\tau}\right) r^{4\lambda_l-3} dr d\tau. \end{aligned}$$

Set $u = r/(s-\tau)^{1/2}$. Then we have

$$\begin{aligned} S_{3,2}^{2,1} &\leq Ce^{-2\beta s_1} y^{2\alpha-1} \int_{s_1}^{s} (s-\tau)^{2\lambda_l-2\alpha-1/2} \\ &\qquad \cdot \int_{Le^{-\beta\tau}/\sqrt{s-\tau}}^{4y/\sqrt{s-\tau}} \exp\left(-C\left|\frac{ye^{-(s-\tau)/2}}{\sqrt{s-\tau}}-u\right|^2\right) u^{4\lambda_l-4\alpha-1} du d\tau \\ &\leq Ce^{-2\beta s_1} y^{2\alpha-1} \int_{s_1}^{s} (s-\tau)^{2\lambda_l-2\alpha-1/2} \left(\frac{4y}{\sqrt{s-\tau}}\right)^{4\lambda_l-4\alpha} d\tau \\ &\leq Ce^{-2\beta s_1} y^{4\lambda_l-2\alpha-1} \leq Ce^{-\beta s_1} y^{2\lambda_l} (e^{-\beta s_1} e^{\sigma(2\lambda_l-2\alpha-1)s}) \\ &\leq \nu e^{-\beta s_1} y^{2\lambda_l} \qquad \text{if } s_1 \gg 1. \end{aligned}$$

Consider $S_{3,2}^{2,2}$. Since

$$\left|ye^{-(s-\tau)/2}-r\right|^2 \geq y^2 e^{-1} + r^2/2 \qquad \text{if } r \geq 4y,$$

we have

$$S_{3,2}^{2,2} \leq Ce^{-2\beta s_1} y^{2\alpha-1} \int_{s_1}^{s} (s-\tau)^{-2\alpha+1/2} \int_{4y}^{\infty} e^{-Cy^2/(s-\tau)} e^{-Cr^2/(s-\tau)} r^{4\lambda_l-3} dr d\tau.$$

Set $u = r/\sqrt{s-\tau}$. Then

$$\begin{aligned} S_{3,2}^{2,2} &\leq Ce^{-2\beta s_1} y^{2\alpha-1} \int_{s_1}^{s} (s-\tau)^{2\lambda_l-2\alpha-1/2} e^{-Cy^2/(s-\tau)} \int_{4y/\sqrt{s-\tau}}^{\infty} e^{-Cu^2} u^{4\lambda_l-3} du d\tau \\ &\leq Ce^{-2\beta s_1} y^{2\alpha-1} \leq \nu e^{-\beta s_1} y^{2\alpha-1} \qquad \text{if } s_1 \gg 1. \end{aligned}$$

III. Estimate of $S_{3,3}$. From (7.6), we have

$$\begin{aligned} S_{3,3} &\leq Cy^{2\alpha-1} \int_{s_1}^{s} (s-\tau)^{-2\alpha+1/2} \int_{e^{\sigma\tau}}^{\infty} \left(1+\frac{Cry}{s-\tau}\right)^{-2\alpha+1} \\ &\qquad \cdot \exp\left(-C\frac{|ye^{-(s-\tau)/2}-r|^2}{s-\tau}\right) r^{4\alpha-3} dr d\tau. \end{aligned}$$

To estimate $S_{3,3}$, we divide it into two cases: $y \leq e^{\sigma s}/4$ and $y \geq e^{\sigma s}/4$.
Case 1: $y \leq e^{\sigma s}/4$. Since

$$\exp\left(-C\frac{|ye^{-(s-\tau)/2}-r|^2}{s-\tau}\right) \leq e^{-Ce^{2\sigma\tau}} e^{-Cr^2/(s-\tau)} \qquad \text{if } r \geq e^{\sigma\tau} \text{ and } y \leq e^{\sigma s}/4,$$

we have

$$S_{3,3} \leq Cy^{2\alpha-1}\int_{s_1}^{s}(s-\tau)^{-2\alpha+1/2}\int_{e^{\sigma\tau}}^{\infty} e^{-Ce^{2\sigma\tau}}e^{-Cr^2/(s-\tau)}r^{4\alpha-3}drd\tau.$$

Set $u = r/\sqrt{s-\tau}$. Then we get

$$\begin{aligned} S_{3,3} &\leq Cy^{2\alpha-1}\int_{s_1}^{s}(s-\tau)^{-1/2}\int_{e^{\sigma\tau}/\sqrt{s-\tau}}^{\infty} e^{-Ce^{2\sigma\tau}}e^{-Cu^2}u^{4\alpha-3}dud\tau \\ &\leq Ce^{-Ce^{2\sigma s_1}}y^{2\alpha-1} \leq \nu e^{-\beta s_1}y^{2\alpha-1} \qquad \text{if } s_1 \gg 1. \end{aligned}$$

Case 2: $y \geq e^{\sigma s}/4$. Note that $S_{3,3} = S_{3,3}^1 + S_{3,3}^2$, where

$$\begin{aligned} S_{3,3}^1 &:= Cy^{2\alpha-1}\int_{s_1}^{s}(s-\tau)^{-2\alpha+1/2}\int_{e^{\sigma\tau}}^{4y}\left(1+\frac{Cry}{s-\tau}\right)^{-2\alpha+1} \\ &\qquad \cdot \exp\left(-C\frac{|ye^{-(s-\tau)/2}-r|^2}{s-\tau}\right)r^{4\alpha-3}drd\tau, \\ S_{3,3}^2 &:= Cy^{2\alpha-1}\int_{s_1}^{s}(s-\tau)^{-2\alpha+1/2}\int_{4y}^{\infty}\left(1+\frac{Cry}{s-\tau}\right)^{-2\alpha+1} \\ &\qquad \cdot \exp\left(-C\frac{|ye^{-(s-\tau)/2}-r|^2}{s-\tau}\right)r^{4\alpha-3}drd\tau. \end{aligned}$$

Consider $S_{3,3}^1$. Since $ry \geq r^2/4$, if $r \leq 4y$, we have

$$S_{3,3}^1 \leq Cy^{2\alpha-1}\int_{s_1}^{s}(s-\tau)^{-2\alpha+1/2}\int_{e^{\sigma\tau}}^{4y}\left(1+\frac{Cr^2}{s-\tau}\right)^{-2\alpha+1}r^{4\alpha-3}drd\tau.$$

Set $u = r/\sqrt{s-\tau}$. Then we have

$$\begin{aligned} S_{3,3}^1 &\leq Cy^{2\alpha-1}\int_{s_1}^{s}(s-\tau)^{-1/2}\int_{e^{\sigma\tau}/\sqrt{s-\tau}}^{4y/\sqrt{s-\tau}} u^{-1}dud\tau \\ &\leq Cy^{2\alpha-1}\int_{s_1}^{s}(s-\tau)^{-1/2}\left(\frac{e^{\sigma\tau}}{\sqrt{s-\tau}}\right)^{-1}\left(\frac{4y}{\sqrt{s-\tau}}\right)d\tau \\ &\leq Cy^{2\alpha}e^{-\sigma s_1} \\ &\leq Ce^{-\beta s_1}y^{2\lambda_l}e^{\beta s_1+\sigma(-2\lambda_l+2\alpha-1)s_1} \\ &\leq \nu e^{-\beta s_1}y^{2\lambda_l} \qquad \text{if } s_1 \gg 1, \end{aligned}$$

where the assumption $\sigma > (\lambda_l - \alpha)/(2\lambda_l - 2\alpha + 1)$ is used.

Consider $S_{3,3}^2$. Since

$$\left|ye^{-(s-\tau)/2} - r\right|^2 \geq y^2e^{-1} + r^2/2 \qquad \text{if } r \geq 4y,$$

we have

$$S_{3,3}^2 \leq Cy^{2\alpha-1}\int_{s_1}^{s}(s-\tau)^{-2\alpha+1/2}\int_{4y}^{\infty} e^{-Cr^2/(s-\tau)}e^{-Cy^2/(s-\tau)}r^{4\alpha-3}drd\tau.$$

Set $u = r/\sqrt{s-\tau}$. Then we get

$$\begin{aligned} S_{3,3}^2 &\le Cy^{2\alpha-1}\int_{s_1}^{s}(s-\tau)^{-1/2}\int_{4y/\sqrt{s-\tau}}^{\infty} e^{-Cu^2}u^{4\alpha-3}dud\tau \\ &\le Cy^{2\alpha-1} \le Ce^{-\beta s_1}y^{2\lambda_l}e^{\beta s_1+\sigma(-2\lambda_l+2\alpha-1)s_1} \\ &\le \nu e^{-\beta s_1}y^{2\lambda_l} \qquad \text{if } s_1 \gg 1, \end{aligned}$$

where the assumption $\sigma > (\lambda_l - \alpha)/(2\lambda_l - 2\alpha + 1)$ is used.

Putting together all the above estimates, the lemma is proved.

7.2. **Long Time:** $s_2 > s_1 + 1$**.**

Since $P(d; s_1, s_2) = 0$, we have

$$d_n = -e^{-\beta s_1}\langle \tilde{\phi}_l, \phi_n\rangle - \int_{s_1}^{s_2} e^{-(\lambda_n-\alpha)(s_1-\tau)}\langle f(v(\tau)), \phi_n\rangle d\tau$$

for $n = 0, 1, \cdots, l-1$. From (7.1) we may write

$$v(y,s) = I_1(y,s) + I_2(y,s) + I_3(y,s) + I_4(y,s),$$

where

$$I_1 := e^{-\beta s}\langle \tilde{\phi}_l, \phi_l\rangle \phi_l, \tag{7.20}$$

$$I_2 := \sum_{n=l+1}^{\infty} e^{-(\lambda_n-\alpha)(s-s_1)}\langle \tilde{\phi}_l, \phi_n\rangle e^{-\beta s_1}\phi_n, \tag{7.21}$$

$$I_3 := \sum_{n=0}^{\infty}\int_{s_1}^{s} e^{-(\lambda_n-\alpha)(s-\tau)}\langle f(v(\tau)), \phi_n\rangle \phi_n d\tau, \tag{7.22}$$

$$I_4 := -\sum_{n=0}^{l-1}\int_{s_1}^{s_2} e^{-(\lambda_n-\alpha)(s-\tau)}\langle f(v(\tau)), \phi_n\rangle \phi_n d\tau. \tag{7.23}$$

Notice that $I_3 = S_3$ and $S_2 = I_2 + I_4$.

Lemma 7.6. *For any* $\nu \in (0,1)$*, there exists* $s_1 \gg 1$ *such that*

$$|I_4(y,s)| \le \nu e^{-\beta s}(y^{2\alpha-1} + y^{2\lambda_l})$$

for all $y \ge 0$ *and* $s_1 + 1 \le s \le s_2$*.*

Proof. From (4.5), (4.6) and (7.11), we get

$$\begin{aligned}
|I_4| &\leq C(y^{2\alpha-1}+y^{2\lambda_l})\int_{s_1}^{s_2} e^{-(\lambda_n-\alpha)(s-\tau)}e^{-(\beta+\kappa)\tau}d\tau \\
&= Ce^{-\beta s}(y^{2\alpha-1}+y^{2\lambda_l})\int_{s_1}^{s_2} e^{(\lambda_l-\lambda_n)(s-\tau)}e^{-\kappa\tau}d\tau \\
&\leq Ce^{-\beta s}(y^{2\alpha-1}+y^{2\lambda_l})e^{-\kappa s_1} \\
&\leq \nu e^{-\beta s}(y^{2\alpha-1}+y^{2\lambda_l}) \qquad \text{if } s_1 \gg 1,
\end{aligned}$$

for any $y \geq 0$ and $s \in [s_1+1, s_2]$. □

From now on, we shall fix $R > 1$.

Lemma 7.7. *For any $\nu \in (0,1)$, there exists $s_1 \gg 1$ such that*

$$|I_2(y,s)| \leq \nu e^{-\beta s} y^{2\alpha-1}$$

for $Ke^{-\beta s} \leq y \leq R$ and $s_1+1 \leq s \leq s_2$.

Proof. Fix $s \geq s_1+1$. Since $\tilde{\phi}_l \to \phi_l$ in L^2_ρ as $s_1 \to \infty$, $|\langle\tilde{\phi}_l, \phi_n\rangle| \to 0$ as $s_1 \to \infty$ uniformly on n for $n \geq l+1$. Moreover, we have $e^{-(\lambda_n-\alpha)(s-s_1)}e^{-\beta s_1} = e^{-\beta s}e^{-(\lambda_n-\lambda_l)(s-s_1)}$. Hence for a given $\nu \in (0,1)$ there exists $s_1 \gg 1$ such that

$$|I_2(y,s)| \leq \nu e^{-\beta s}\sum_{n=l+1}^{\infty} e^{-(n-l)}|\phi_n|.$$

Since $\int_0^\infty |\phi_n(y)|^2\rho(y)dy = 1$ and $\phi_n(y) = c_n y^{2\alpha-1}M(-n, 2\alpha-1/2; y^2/4)$, we have

$$c_n^2 \cdot 2^{4\alpha-2}\int_0^\infty \xi^{2\alpha-3/2}e^{-\xi}(M(-n,2\alpha-1/2;\xi))^2 d\xi = 1.$$

Making use of

$$\int_0^\infty x^{t-1}e^{-x}(M(-n,\hat{\alpha};\xi))^2 dx = \frac{[\Gamma(\hat{\alpha})]^2\Gamma(n+1)}{\Gamma(\hat{\alpha}+n)},$$

we have

$$\int_0^\infty \xi^{2\alpha-3/2}e^{-\xi}(M(-n,2\alpha-1/2;\xi))^2 d\xi = \frac{[\Gamma(2\alpha-1/2)]^2\Gamma(n+1)}{\Gamma(2\alpha+n-1/2)}.$$

Hence

$$c_n^2 = \frac{\Gamma(2\alpha+n-1/2)}{2^{4\alpha-2}[\Gamma(2\alpha-1/2)]^2\Gamma(n+1)}.$$

Since we know that $\Gamma(x) \sim \sqrt{2\pi}e^{-x}x^{x-1/2}$ as $x \to \infty$, we get

$$\left(\frac{\Gamma(2\alpha+n-1/2)}{\Gamma(n+1)}\right)^{1/2} \sim Cn^{\alpha-3/4} \qquad \text{for } n \gg 1.$$

Thus

$$c_n \sim Cn^{\alpha-3/4} \qquad \text{for } n \gg 1. \tag{7.24}$$

Moreover, since

$$\begin{aligned}&M(-n,2\alpha-1/2;\frac{y^2}{4})\\&=\pi^{-1/2}\Gamma(2\alpha-1/2)e^{y^2/4}[(\alpha+n-1/4)y^2/4]^{1/2-\alpha}\\&\cdot\cos\left(\frac{\sqrt{4\alpha+4n-1}}{2}y-(\alpha-1/2)\pi\right)(1+o(1))\quad\text{ as } n\to\infty\end{aligned}$$

uniformly for $y\in[0,R]$, there exits a positive constant $C=C(R)$ such that

$$|M(-n,2\alpha-1/2;y^2/4)|\le Cn^{1/2-\alpha}\ \text{ for } y\in[0,R] \text{ and } n\gg 1. \tag{7.25}$$

From (7.24) and (7.25), we obtain

$$|\phi_n|\le Cy^{2\alpha-1}n^{-1/4}\ \text{ for } y\in[0,R] \text{ and } n\gg 1. \tag{7.26}$$

Since

$$\sum_{n=l+1}^{\infty}e^{-n}n^{-1/4}<\infty,$$

it follows that there exists $s_1\gg 1$ such that

$$|I_2(y,s)|\le\nu e^{-\beta s}y^{2\alpha-1}.$$

This proves the lemma.

Lemma 7.8. *For any $\nu\in(0,1)$, there exists $s_1\gg 1$ such that*

$$|I_3(y,s)|\le\nu e^{-\beta s}(y^{2\alpha-1}+y^{2\lambda_l})$$

for $Ke^{-\beta s}\le y\le R$ and $s_1+1\le s\le s_2$.

Proof. Given any $\nu\in(0,1)$, note that $I_3=I_{3,1}+I_{3,2}$, where

$$I_{3,1}:=\sum_{n=1}^{l-1}\int_{s_1}^{s}e^{-(\lambda_n-\alpha)(s-\tau)}\langle f(v(\tau)),\phi_n\rangle\phi_n d\tau,$$

$$I_{3,2}:=\sum_{n=l}^{\infty}\int_{s_1}^{s}e^{-(\lambda_n-\alpha)(s-\tau)}\langle f(v(\tau)),\phi_n\rangle\phi_n d\tau.$$

By the same reasoning as the proof of Lemma 7.6, it follows immediately that

$$|I_{3,1}|\le\nu e^{-\beta s}(y^{2\alpha-1}+y^{2\lambda_l})\qquad\text{if } s_1\gg 1.$$

We write $I_{3,2}=I_{3,2}^1+I_{3,2}^2$, where

$$I_{3,2}^1:=\sum_{n=l}^{\infty}\int_{s-1}^{s}e^{-(\lambda_n-\alpha)(s-\tau)}\langle f(v(\tau)),\phi_n\rangle\phi_n d\tau,$$

$$I_{3,2}^2:=\sum_{n=l}^{\infty}\int_{s_1}^{s-1}e^{-(\lambda_n-\alpha)(s-\tau)}\langle f(v(\tau)),\phi_n\rangle\phi_n d\tau.$$

Since

$$I_{3,2}^1 = \int_{s-1}^{s} e^{-\mathcal{A}(s-\tau)} f(v(\tau))d\tau - \sum_{n=0}^{l-1} \int_{s-1}^{s} e^{-(\lambda_n-\alpha)(s-\tau)} \langle f(v(\tau)), \phi_n\rangle \phi_n d\tau,$$

by the same arguments as the proofs of Lemma 7.5 and the estimate of $I_{3,1}$, we get

$$I_{3,2}^1 \le \nu e^{-\beta s}(y^{2\alpha-1} + y^{2\lambda_l}) \quad \text{if } s_1 \gg 1.$$

For $I_{3,2}^2$, we have

$$I_{3,2}^2 = \int_{s_1}^{s-1} e^{-\beta(s-\tau)} \sum_{n=l}^{\infty} e^{-(\lambda_n-\lambda_l)(s-\tau)} \langle f(v(\tau)), \phi_n\rangle \phi_n d\tau.$$

Thus

$$|I_{3,2}^2| \le \int_{s_1}^{s-1} e^{-\beta(s-\tau)} \left(\sum_{n=l}^{\infty} e^{-2(\lambda_n-\lambda_l)(s-\tau)}|\phi_n|^2\lambda_n^2\right)^{1/2} \left(\sum_{n=l}^{\infty} \frac{|\langle f(v(\tau)), \phi_n\rangle|^2}{\lambda_n^2}\right)^{1/2} d\tau.$$

From (7.26), we have for $n \gg 1$ and $\tau \in [s_1, s-1]$

$$e^{-2(\lambda_n-\lambda_l)(s-\tau)}|\phi_n|^2\lambda_n^2 \le Ce^{-2n}n^{3/2}y^{4\alpha-2}.$$

Since $\sum_{n=l}^{\infty} e^{-2n}n^{3/2} < \infty$, we obtain

$$\left(\sum_{n=l}^{\infty} e^{-2(\lambda_n-\lambda_l)(s-\tau)}|\phi_n|^2\lambda_n^2\right)^{1/2} \le Cy^{2\alpha-1}.$$

Moreover, from (7.26)

$$\begin{aligned}
\left(\sum_{n=l}^{\infty} \frac{|\langle f(v(r,\tau)), \phi_n\rangle|^2}{\lambda_n^2}\right)^{1/2} &\le \sum_{n=l}^{\infty} \frac{|\langle f(v(r,\tau)), \phi_n\rangle|}{\lambda_n} \\
&\le \sum_{n=l}^{\infty} \frac{\int_0^\infty |f(v(r,\tau))||\phi_n|e^{-r^2/4}dr}{\lambda_n} \\
&\le C\sum_{n=l}^{\infty} \frac{n^{-1/4}}{\alpha - 1/2 + n} \int_0^\infty |f(v(r,\tau))|r^{2\alpha-1}e^{-r^2/4}dr \\
&\le C\int_0^\infty |f(v(r,\tau))|r^{2\alpha-1}e^{-r^2/4}dr,
\end{aligned}$$

since $\sum_{n=l}^{\infty} n^{-5/4} < \infty$. Hence we have

$$|I_{3,2}^2| \le Cy^{2\alpha-1} \int_{s_1}^{s-1} e^{-\beta(s-\tau)} \int_0^\infty |f(v(r,\tau))|r^{2\alpha-1}e^{-r^2/4}drd\tau. \tag{7.27}$$

From Lemma 7.2, we have the estimate

$$\int_0^\infty |f(v(r,\tau))|r^{2\alpha-1}e^{-r^2/4}dr \le J_1 + J_2 + J_3,$$

where

$$J_1 := CK^{2\alpha}e^{-2\alpha\beta\tau}\int_0^{Ke^{-\beta\tau}} r^{2\alpha-3}e^{-r^2/4}dr,$$

$$J_2 := Ce^{-2\beta\tau}\int_{Ke^{-\beta\tau}}^{e^{\sigma\tau}} (r^{2\alpha-4}+r^{4\lambda_l-2\alpha-2})r^{2\alpha-1}e^{-r^2/4}dr,$$

$$J_3 := Ce^{-2\beta\tau}\int_{e^{\sigma\tau}}^{\infty} r^{4\alpha-3}e^{-r^2/4}dr.$$

By a similar argument to the proof of Lemma 7.3, we have

$$\int_0^\infty |f(v(r,\tau))|r^{2\alpha-1}e^{-r^2/4}dr \ll e^{-\beta\tau} \qquad \text{if } s_1 \gg 1.$$

Plugging into (7.27), we obtain

$$|I^2_{3,2}| \;\le\; \nu e^{-\beta s}y^{2\alpha-1} \qquad \text{if } s_1 \gg 1.$$

This proves the lemma.

Lemma 7.9. *For any* $\nu \in (0,1)$, *there exists* $s_1 \gg 1$ *such that*

$$|I_2(y,s)| \le \nu e^{-\beta s}(y^{2\alpha-1}+y^{2\lambda_l})$$

for $y \in \{Ke^{-\beta s} \le y \le e^{\sigma s}\} \cap \{y \ge e^{(s-s_1)/2}\}$ *and* $s_1+1 \le s \le s_2$.

Proof. Since $I_2 = S_2 - I_4$, by Lemma 7.6, we only need to estimate S_2.

From (7.17), (7.18) and (7.19), we could easily find that for a given $\nu \in (0,1)$ there exists $s_1 \gg 1$ such that

$$|v_0(y,s_1)| \;\le\; \nu e^{-\beta s_1}(y^{2\alpha-1}+y^{2\lambda_l}) \qquad \text{if } Ke^{-\beta s_1} \le y \le e^{\sigma s_1}, \tag{7.28}$$

$$|v_0(y,s_1)| \;\le\; CK^{2\alpha}e^{-2\alpha\beta s_1} \qquad \text{if } y \le Ke^{-\beta s_1}, \tag{7.29}$$

$$|v_0(y,s_1)| \;\le\; Cy^{2\lambda_l} \qquad \text{if } y \ge e^{\sigma s_1}. \tag{7.30}$$

Moreover, from (7.15) and Lemma 7.2, we have $|S_2| \le S_2^1 + S_2^2 + S_2^3$, where

$$S_2^1 := \quad CK^{2\alpha}e^{-2\alpha\beta s_1}y^{1/2}e^{(4\alpha-1)(s-s_1)/4}\int_0^{Ke^{-\beta s_1}} r^{1/2}dr,$$

$$S_2^2 := \quad C\nu e^{-\beta s_1}y^{2\alpha-1}e^{(s-s_1)/2}\int_{Ke^{-\beta s_1}}^{e^{\sigma s}} (1+Ce^{-(s-s_1)/2}ry)^{-2\alpha+1} \cdot \exp(-C|ye^{-(s-s_1)/2}-r|^2)r^{2\alpha-1}(r^{2\alpha-1}+r^{2\lambda_l})dr,$$

$$S_2^3 := \quad Cy^{2\alpha-1}e^{(s-s_1)/2}\int_{e^{\sigma s}}^{\infty} r^{2\lambda_l+2\alpha-1}\exp(-C|ye^{-(s-s_1)/2}-r|^2)dr.$$

First, for S_2^1, since $y \ge e^{(s-s_1)/2}$ and $K \in (e^{\beta s_1/8}, e^{\beta s_1/2})$, we have

$$\begin{aligned} S_2^1 &\le Ce^{-\beta s}y^{2\lambda_l}\{K^{2\alpha}e^{-2\alpha\beta s_1}y^{-2\lambda_l+1/2}e^{\beta s}e^{(4\alpha-1)(s-s_1)/4}(Ke^{-\beta s_1})^{3/2}\},\\ &\le \nu e^{-\beta s}y^{2\lambda_l} \qquad \text{if } s_1 \gg 1. \end{aligned}$$

Next, we consider S_2^2. Define

$$D_1 := \{|ye^{-(s-s_1)/2} - r| \le r/2\}, \quad D_2 := \{|ye^{-(s-s_1)/2} - r| \ge r/2\},$$

and write $S_2^2 = S_2^{2,1} + S_2^{2,2}$, where

$$\begin{aligned} S_2^{2,1} &:= C\nu e^{-\beta s_1} y^{2\alpha-1} e^{(s-s_1)/2} \int_{D_1} (1 + Ce^{-(s-s_1)/2} ry)^{-2\alpha+1} \\ &\quad \cdot \exp(-C|ye^{-(s-s_1)/2} - r|^2) r^{2\alpha-1}(r^{2\alpha-1} + r^{2\lambda_l}) dr, \\ S_2^{2,2} &:= C\nu e^{-\beta s_1} y^{2\alpha-1} e^{(s-s_1)/2} \int_{D_2} (1 + Ce^{-(s-s_1)/2} ry)^{-2\alpha+1} \\ &\quad \cdot \exp(-C|ye^{-(s-s_1)/2} - r|^2) r^{2\alpha-1}(r^{2\alpha-1} + r^{2\lambda_l}) dr. \end{aligned}$$

Since $e^{-(s-s_1)/2} ry \ge r^2/2$, if $|ye^{-(s-s_1)/2} - r| \le r/2$. We get

$$\begin{aligned} S_2^{2,1} &\le C\nu e^{-\beta s_1} y^{2\alpha-1} e^{(s-s_1)/2} \int_{D_1} \exp(-C|ye^{-(s-s_1)/2} - r|^2)(1 + r^{2\lambda_l - 2\alpha+1}) dr \\ &\le C\nu e^{-\beta s_1} y^{2\alpha-1} e^{(s-s_1)/2} [1 + (ye^{-(s-s_1)/2})^{2\lambda_l - 2\alpha+1}] \\ &\le C\nu e^{-\beta s} y^{2\lambda_l} \{e^{(\beta + \frac{1}{2})(s-s_1)} [y^{-2\lambda_l + 2\alpha - 1} + e^{-(2\lambda_l - 2\alpha+1)(s-s_1)/2}]\} \\ &\le C\nu e^{-\beta s} y^{2\lambda_l}. \end{aligned}$$

Moreover, we get

$$\begin{aligned} S_2^{2,2} &\le C\nu e^{-\beta s_1} y^{2\alpha-1} e^{(s-s_1)/2} \int_{D_2} \exp(-Cr^2) r^{2\alpha-1}(r^{2\alpha-1} + r^{2\lambda_l}) dr \\ &\le C\nu e^{-\beta s} y^{2\lambda_l} [e^{(\beta+1/2)(s-s_1)} y^{-2\lambda_l + 2\alpha - 1}] \\ &\le C\nu e^{-\beta s} y^{2\lambda_l}. \end{aligned}$$

Finally, we consider S_2^3. Since $r - ye^{-(s-s_1)/2} \ge Cr$ if $r \ge e^{\sigma s_1}$ and $y \le e^{\sigma s}$, we have

$$\begin{aligned} S_2^3 &\le Cy^{2\alpha-1} e^{(s-s_1)/2} \int_{e^{\sigma s}}^{\infty} r^{2\lambda_l + 2\alpha - 1} \exp(-Cr^2) dr, \\ &\le Cy^{2\alpha-1} e^{(s-s_1)/2} e^{-Ce^{2\sigma s_1}}, \\ &\le \nu e^{-\beta s} y^{2\lambda_l} \qquad \text{if } s_1 \gg 1. \end{aligned}$$

This completes the proof of the lemma.

Lemma 7.10. *For any $\nu \in (0, 1)$, there exists $s_1 \gg 1$ such that*

$$|I_3(y, s)| \le \nu e^{-\beta s} y^{2\lambda_l}$$

for $y \in \{Ke^{-\beta s} \le y \le e^{\sigma s}\} \cap \{y \ge e^{(s-s_1)/2}\}$ and $s_1 + 1 \le s \le s_2$.

Proof. By Lemma 7.5, we only need to prove that

$$|B(y, s)| \le \nu e^{-\beta s} y^{2\lambda_l},$$

where

$$B(y,s) = \int_{s_1}^{s-1} e^{-\mathcal{A}(s-\tau)} f(v(y,\tau))d\tau.$$

Following the same argument as the proof of Lemma 7.5, we have

$$\begin{aligned} |B(y,s)| &\le Cy^{2\alpha-1}\int_{s_1}^{s-1} e^{(s-\tau)/2}\int_0^\infty (1+Ce^{-(s-\tau)/2}ry)^{-2\alpha+1} \\ &\quad \cdot \exp(-C|ye^{-(s-\tau)/2}-r|^2)r^{2\alpha-1}|f(v(r,\tau))|drd\tau \\ &= I_3^1 + I_3^2 + I_3^3, \end{aligned}$$

where

$$\begin{aligned} I_3^1 &:= Cy^{2\alpha-1}\int_{s_1}^{s-1} e^{(s-\tau)/2}\int_0^{Ke^{-\beta\tau}} (1+Ce^{-(s-\tau)/2}ry)^{-2\alpha+1} \\ &\quad \cdot \exp(-C|ye^{-(s-\tau)/2}-r|^2)r^{2\alpha-1}|f(v)|drd\tau, \\ I_3^2 &:= Cy^{2\alpha-1}\int_{s_1}^{s-1} e^{(s-\tau)/2}\int_{Ke^{-\beta\tau}}^{e^{\sigma\tau}} (1+Ce^{-(s-\tau)/2}ry)^{-2\alpha+1} \\ &\quad \cdot \exp(-C|ye^{-(s-\tau)/2}-r|^2)r^{2\alpha-1}|f(v)|drd\tau, \\ I_3^3 &:= Cy^{2\alpha-1}\int_{s_1}^{s-1} e^{(s-\tau)/2}\int_{e^{\sigma\tau}}^{\infty} (1+Ce^{-(s-\tau)/2}ry)^{-2\alpha+1} \\ &\quad \cdot \exp(-C|ye^{-(s-\tau)/2}-r|^2)r^{2\alpha-1}|f(v)|drd\tau. \end{aligned}$$

Consider I_3^1. From (7.3) we have

$$\begin{aligned} I_3^1 &\le CK^{2\alpha}e^{-2\alpha\beta s_1}y^{2\alpha-1}\int_{s_1}^{s-1} e^{(s-\tau)/2}\int_0^{Ke^{-\beta\tau}} r^{2\alpha-3}drd\tau \\ &= CK^{2\alpha}e^{-2\alpha\beta s_1}y^{2\alpha-1}\int_{s_1}^{s-1} e^{(s-\tau)/2}(Ke^{-\beta\tau})^{2\alpha-2}d\tau \\ &\le CK^{4\alpha-2}e^{-2\alpha\beta s_1}y^{2\alpha-1}e^{s/2}\int_{s_1}^{s-1} e^{-(1/2+\beta(2\alpha-2))\tau}d\tau \\ &\le Ce^{-\beta s}y^{2\lambda_l}\{K^{4\alpha-2}y^{-2\lambda_l+2\alpha-1}e^{(\beta+1/2)s}e^{-(2\alpha\beta+\beta(2\alpha-2)+1/2)s_1}\} \\ &\le \nu e^{-\beta s}y^{2\lambda_l} \quad \text{ if } s_1 \gg 1 \text{ and } y \ge e^{(s-s_1)/2}. \end{aligned}$$

Consider I_3^2. From (7.4) and (7.5) we get $I_3^2 \le I_3^{2,1} + I_3^{2,2}$, where

$$\begin{aligned} I_3^{2,1} &:= Cy^{2\alpha-1}\int_{s_1}^{s-1} e^{-2\beta\tau}e^{(s-\tau)/2}\int_{Ke^{-\beta\tau}}^{e^{\sigma\tau}} (1+Ce^{-(s-\tau)/2}ry)^{-2\alpha+1} \\ &\quad \cdot \exp(-C|ye^{-(s-\tau)/2}-r|^2)r^{4\alpha-5}drd\tau, \\ I_3^{2,2} &:= Cy^{2\alpha-1}\int_{s_1}^{s-1} e^{-2\beta\tau}e^{(s-\tau)/2}\int_{Ke^{-\beta\tau}}^{e^{\sigma\tau}} (1+Ce^{-(s-\tau)/2}ry)^{-2\alpha+1} \\ &\quad \cdot \exp(-C|ye^{-(s-\tau)/2}-r|^2)r^{4\lambda_l-3}drd\tau. \end{aligned}$$

Firstly, we consider $I_3^{2,1}$. Define

$$D_1 := \{|ye^{-(s-\tau)/2} - r| \le r/2\}, \quad D_2 := \{|ye^{-(s-\tau)/2} - r| \ge r/2\},$$

and write $I_3^{2,1} = I_3^{2,1,1} + I_3^{2,1,2}$, where

$$\begin{aligned} I_3^{2,1,1} &:= Cy^{2\alpha-1} \int_{s_1}^{s-1} e^{-2\beta\tau} e^{(s-\tau)/2} \int_{D_1} (1 + Ce^{-(s-\tau)/2} ry)^{-2\alpha+1} \\ &\quad \cdot \exp(-C|ye^{-(s-\tau)/2} - r|^2) r^{4\alpha-5} dr d\tau, \\ I_3^{2,1,2} &:= Cy^{2\alpha-1} \int_{s_1}^{s-1} e^{-2\beta\tau} e^{(s-\tau)/2} \int_{D_2} (1 + Ce^{-(s-\tau)/2} ry)^{-2\alpha+1} \\ &\quad \cdot \exp(-C|ye^{-(s-\tau)/2} - r|^2) r^{4\alpha-5} dr d\tau. \end{aligned}$$

Consider $I_3^{2,1,1}$. If $r \in D_1$, we have

$$r/2 \le ye^{-(s-\tau)/2} \le 3r/2.$$

Hence

$$\begin{aligned} I_3^{2,1,1} &\le Cy^{2\alpha-1} \int_{s_1}^{s-1} e^{-2\beta\tau} e^{(s-\tau)/2} \int_{D_1} r^{-3} dr d\tau \\ &\le Cy^{2\alpha-1} \int_{s_1}^{s-1} e^{-2\beta\tau} e^{(s-\tau)/2} (ye^{-(s-\tau)/2})^{-2} d\tau \\ &\le Cy^{2\alpha-3} \int_{s_1}^{s-1} e^{-2\beta\tau} e^{3(s-\tau)/2} d\tau \\ &\le Ce^{-\beta s} y^{2\lambda_l} [y^{-2\lambda_l+2\alpha-3} e^{(\beta+3/2)s} e^{-(2\beta+3/2)s_1}] \\ &\le \nu e^{-\beta s} y^{2\lambda_l} \qquad \text{if } s_1 \gg 1 \text{ and } y \ge e^{(s-s_1)/2}. \end{aligned}$$

For $I_3^{2,1,2}$, we have

$$\begin{aligned} I_3^{2,1,2} &\le Cy^{2\alpha-1} \int_{s_1}^{s-1} e^{-2\beta\tau} e^{(s-\tau)/2} \int_{D_2} e^{-Cr^2} r^{4\alpha-5} dr d\tau \\ &\le Cy^{2\alpha-1} \int_{s_1}^{s-1} e^{-2\beta\tau} e^{(s-\tau)/2} d\tau \\ &\le Ce^{-\beta s} y^{2\lambda_l} (y^{-2\lambda_l+2\alpha-1} e^{(\beta+1/2)s} e^{(-2\beta+1/2)s_1}) \\ &\le \nu e^{-\beta s} y^{2\lambda_l} \qquad \text{if } s_1 \gg 1 \text{ and } y \ge e^{(s-s_1)/2}. \end{aligned}$$

Secondly, we consider $I_3^{2,2}$ and as before we write $I_3^{2,2} = I_3^{2,2,1} + I_3^{2,2,2}$, where

$$\begin{aligned} I_3^{2,2,1} &:= Cy^{2\alpha-1} \int_{s_1}^{s-1} e^{-2\beta\tau} e^{(s-\tau)/2} \int_{D_1} (1 + Ce^{-(s-\tau)/2} ry)^{-2\alpha+1} \\ &\quad \cdot \exp(-C|ye^{-(s-\tau)/2} - r|^2) r^{4\lambda_l-3} dr d\tau, \\ I_3^{2,2,2} &:= Cy^{2\alpha-1} \int_{s_1}^{s-1} e^{-2\beta\tau} e^{(s-\tau)/2} \int_{D_2} (1 + Ce^{-(s-\tau)/2} ry)^{-2\alpha+1} \\ &\quad \cdot \exp(-C|ye^{-(s-\tau)/2} - r|^2) r^{4\lambda-3} dr d\tau. \end{aligned}$$

For $I_3^{2,2,1}$, by the same reasoning as the case of $I_3^{2,2,1}$, we obtain

$$\begin{aligned} I_3^{2,2,1} &\le Cy^{4\lambda_l-2\alpha-1}\int_{s_1}^{s-1} e^{-2\beta\tau}e^{-(2\lambda_l-2\alpha-1/2)(s-\tau)}d\tau \\ &\le Ce^{-\beta s}y^{2\lambda_l}(y^{2\lambda_l-2\alpha-1}e^{(-\lambda_l+\alpha+1/2)s}e^{-s_1/2}) \\ &\le \nu e^{-\beta s}y^{2\lambda_l} \qquad \text{if } s_1 \gg 1 \text{ and } y \le e^{\sigma s}. \end{aligned}$$

For $I_3^{2,2,2}$, by the same reasoning as the case of $I_3^{2,1,2}$, we also get

$$I_3^{2,2,2} \le \nu e^{-\beta s}y^{2\lambda_l} \qquad \text{if } s_1 \gg 1 \text{ and } y \ge e^{(s-s_1)/2}.$$

Now, we consider I_3^3. Since $r - ye^{-(s-\tau)/2} \ge Cr$ if $r \ge e^{\sigma\tau}$, $y \le e^{\sigma s}$ and $\tau \le s-1$, it follows from (7.6) that

$$\begin{aligned} I_3^3 &\le Cy^{2\alpha-1}\int_{s_1}^{s-1} e^{(s-\tau)/2}\int_{e^{\sigma\tau}}^{\infty} e^{-Cr^2}r^{4\alpha-3}drd\tau \\ &\le Ce^{-Ce^{2\sigma s_1}}y^{2\alpha-1}\int_{s_1}^{s-1} e^{(s-\tau)/2}d\tau \\ &\le Ce^{-\beta s}y^{2\lambda_l}e^{-Ce^{2\sigma s_1}}(y^{-2\lambda_l+2\alpha-1}e^{(\beta+1/2)s}e^{-s_1/2}) \\ &\le \nu e^{-\beta s}y^{2\lambda_l} \qquad \text{if } s_1 \gg 1 \text{ and } y \ge e^{(s-s_1)/2}. \end{aligned}$$

This proves the lemma.

Proof of Proposition 6.3: By Lemmas 7.4 and 7.5, it suffices to consider the case when $s_2 > s_1 + 1$. Fix $\tilde{s}_1$ such that $s_1 + 1 < \tilde{s}_1 < s_2$. Let $R = e^{(\tilde{s}_1 - s_1)/2}$. Set

$$v_0(y,s) = v(y,s) - e^{-\beta s}\langle\tilde{\phi}_l, \phi_l\rangle\phi_l.$$

Note that $v_0(y,s) = S_2(y,s) + S_3(y,s) = I_2(y,s) + I_3(y,s) + I_4(y,s)$. We easily see that (7.29) and (7.30) are true if s_1 is replaced by s for any $s \in [s_1, s_2]$. Moreover, from Lemmas 7.6, 7.7 and 7.8, we have

$$|v_0(y,s)| \le \nu e^{-\beta s}(y^{2\alpha-1} + y^{2\lambda_l}) \tag{7.31}$$

for $Ke^{-\beta s} \le y \le R$, $s \in [s_1+1, s_2]$; and from Lemmas 7.6, 7.9 and 7.10, we have

$$|v_0(y,s)| \le \nu e^{-\beta s}(y^{2\alpha-1} + y^{2\lambda_l}) \tag{7.32}$$

for $R \le y \le e^{\sigma s}$, $s \in [s_1+1, \tilde{s}_1]$. Hence (7.28) is true if s_1 is replaced by s for any $s \in [s_1, \tilde{s}_1]$.

Let $s \in [\tilde{s}_1, \tilde{s}_1 + 2\log R]$. If $R \le y \le e^{\sigma s}$, then we have $y \ge e^{(s-\tilde{s}_1)/2}$. Hence, by the same arguments as Lemmas 7.6, 7.9 and 7.10, we have

$$|I_2(y,s)| + |I_3(y,s)| + |I_4(y,s)| \le \nu e^{-\beta s}(y^{2\alpha-1} + y^{2\lambda_l}) \tag{7.33}$$

for $R \le y \le e^{\sigma s}$ and $\tilde{s}_1 + 1 \le s \le \tilde{s}_1 + 2\log R$. Moreover, by the same arguments as those of Lemmas 7.4 and 7.5, we obtain

$$|S_2(y,s)| + |S_3(y,s)| \le \nu e^{-\beta s}(y^{2\alpha-1} + y^{2\lambda_l}) \tag{7.34}$$

for $Ke^{-\beta s} \le y \le e^{\sigma s}$ and $\tilde{s}_1 \le s \le \tilde{s}_1 + 1$. From (7.33) and (7.34), we have

$$|I_2(y,s)| + |I_3(y,s)| + |I_4(y,s)| \le \nu e^{-\beta s}(y^{2\alpha-1} + y^{2\lambda_l}) \tag{7.35}$$

for $R \le y \le e^{\sigma s}$ and $\tilde{s}_1 \le s \le \tilde{s}_1 + 2\log R$. Continuing to set $\tilde{s}_2 = \tilde{s}_1 + 2\log R$, $\tilde{s}_3 = \tilde{s}_2 + 2\log R$, $\cdots$, and following the same argument as above, we have

$$|I_2(y,s)| + |I_3(y,s)| + |I_4(y,s)| \le \nu e^{-\beta s}(y^{2\alpha-1} + y^{2\lambda_l}) \tag{7.36}$$

for $R \le y \le e^{\sigma s}$ and $\tilde{s}_1 \le s \le s_2$.

Combining (7.31), (7.32) and (7.36), with the help of Lemmas 7.4 and 7.5, we have

$$|v(y,s) - e^{-\beta s}\langle \tilde{\phi}_l, \phi_l \rangle \phi_l| \le \nu e^{-\beta s}(y^{2\alpha-1} + y^{2\lambda_l}) \tag{7.37}$$

for $Ke^{-\beta s} \le y \le e^{\sigma s}$ and $s_1 \le s \le s_2$. Since $\langle \tilde{\phi}_l, \phi_l \rangle \to 1$ as $s_1 \to \infty$, we get

$$|v(y,s) - e^{-\beta s}\phi_l| \le 2\nu e^{-\beta s}(y^{2\alpha-1} + y^{2\lambda_l})$$

for $Ke^{-\beta s} \le y \le e^{\sigma s}$ and $s_1 \le s \le s_2$. Hence we have proved Proposition 6.3.

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