# SELF-SIMILAR SOLUTIONS OF A 2-D MULTIPLE PHASE CURVATURE FLOW

XINFU CHEN AND JONG-SHENQ GUO

Abstract. This article studies self-similar shrinking, stationary, and expanding solutions of a 2-dimensional motion by curvature equation modelling evolution of grain boundaries in polycrystals. Here the interfacial energy densities are assumed to depend only on the grains and the Herring condition is used for triple junctions (the intersections of three grain boundaries). In particular, in the isotropic case, a total of six configurations are classified as the only self-similar shrinking solutions.

*Keywords:* Self-similar; Curvature flow; Triple junction; Grain boundary; Herring condition

## 1. Introduction

A typical pure solid crystal has a fundamental property that its constituent atoms or molecules are located on a periodic network which we call a **lattice**. Consider the two space dimensional case and denote by $\theta \in [0, 2\pi]$ the angle between the orientation of a lattice and the horizontal axis, and call $e^{\mathbf{i}\theta}$ ($\mathbf{i} = \sqrt{-1}$) the **orientation** or **phase** of the lattice. Depending on the manufacture process (e.g., solidified from a liquid with multiple seeds of different orientation), quite often one obtains polycrystals—a crystal consists of multiple pure crystals, say, occupying regions $\Omega_1, \cdots, \Omega_n$ with their respective orientations $e^{\mathbf{i}\theta_1}, \cdots, e^{\mathbf{i}\theta_n}$. We call each $\Omega_i$, $i = 1, \cdots, n$, a **grain** or a **phase** $e^{\mathbf{i}\theta_i}$ **region**. We call $\cup_i \partial\Omega_i = \cup_{i,j}\gamma_{ij}$ with $\gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$ the **grain boundaries**.

Consider the grain boundary $\gamma_{ij}$ between phase domains $\Omega_i$ and $\Omega_j$. Denote by $e^{\mathbf{i}\theta_{ij}(x)}$ the unit normal of $\gamma_{ij}$ at $x \in \gamma_{ij}$. When $\theta_i = \theta_j$ (or more generally $\theta_i = \theta_j \text{mod } (\frac{2\pi}{k})$ where $k$ is a positive integer depending on the rotational symmetry of the lattice), molecular distances can generally be adjusted so that the two phase regions join and become one pure phase region and there is no grain boundary between them (i.e., $\gamma_{ij}$ is indeed empty). On the other hand, when the two orientations $\theta_i$ and $\theta_j$ are different, molecular distance adjustment (and probably some tiny rotations of $\Omega_i$ and $\Omega_j$) cannot eliminate the grain boundary. As a result, an excess **interfacial energy** with **density** $\sigma(\theta_j - \theta_i, \theta_{ij}(x) - \theta_i)$ (Joule/cm) are stored along $\gamma_{ij}$, allowing atoms or molecules near the interface a certain degree of freedom of movement. Here $\sigma(\cdot, \cdot)$ is a non-negative function with certain periods and symmetries. Thus, compare to a pure crystal, the polycrystal occupy region $\Omega$ has an excess energy

$$\mathbf{E}(\Omega) = \sum_{i,j} \int_{\gamma_{ij}\cap\Omega} \sigma(\theta_i - \theta_j, \theta_{ij}(x) - \theta_i) d\ell$$

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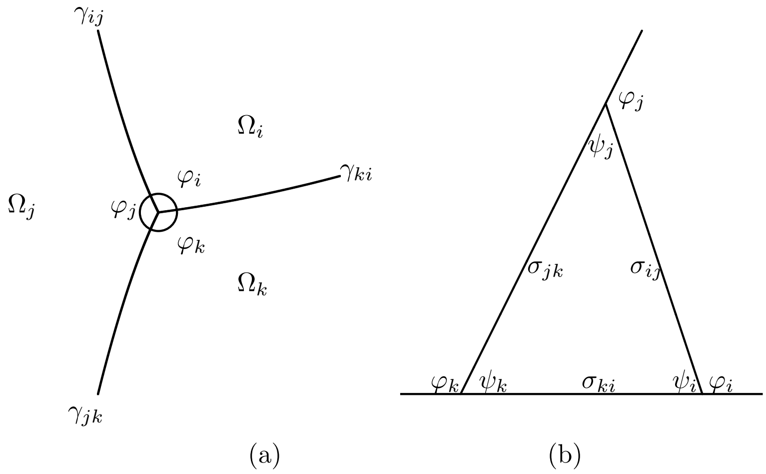


FIGURE 1

where $d\ell$ is the arc length element.

In the literature, a model is called **isotropic** if $\sigma$ is a constant, otherwise it is called **anisotropic**. We shall consider a model where $\sigma$ does not depend on the normal of an interface. In such a case, $\sigma_{ij} = \sigma(\theta_j - \theta_i, \theta_{ij}(x) - \theta_i)$ depends only on the phase $\theta_i$ and $\theta_j$ and is a constant along $\gamma_{ij}$, so that

$$\mathbf{E}(\Omega) = \sum_{i,j} \sigma_{ij}\, |\gamma_{ij} \cap \Omega|$$

where $|\gamma_{ij} \cap \Omega|$ is the arclength of $\gamma_{ij} \cap \Omega$.

There will be points, called **junctions**, at which at least three phase regions or grain boundaries meet. Typically what can be seen are **triple junctions**, the intersections of three phase regions and three interfaces. The three intersection angles of interfaces at a triple junction are not arbitrary; they obey certain rules. Here in this paper we use the following **Herring Condition** [24, 25]: Referring to Figure 1 (a), at a triple junction $p_{ijk}$ of three interfaces $\gamma_{ij}, \gamma_{jk}, \gamma_{ki}$ with their respective interfacial energy densities $\sigma_{ij}, \sigma_{jk}$ and $\sigma_{ki}$, their intersection angles $\varphi_i, \varphi_j, \varphi_k$ satisfy the Herring condition

$$\frac{\sin\varphi_i}{\sigma_{jk}} = \frac{\sin\varphi_j}{\sigma_{ki}} = \frac{\sin\varphi_k}{\sigma_{ij}}, \qquad \varphi_i, \varphi_j, \varphi_k \in (0, \pi), \qquad \varphi_i + \varphi_j + \varphi_k = 2\pi.$$

Recalling the sine rule of trigonometry, the three angles $\varphi_i, \varphi_j, \varphi_k$ are the three exterior angles of the triangle with sides of lengths $\sigma_{ij}, \sigma_{jk}, \sigma_{ki}$; see Figure 1 (b). In terms of the interior angles $\psi_i = \pi - \varphi_i, \psi_j = \pi - \varphi_j$ and $\psi_k = \pi - \varphi_k$, we have

$$\frac{\sin\psi_i}{\sigma_{jk}} = \frac{\sin\psi_j}{\sigma_{ki}} = \frac{\sin\psi_k}{\sigma_{ij}}, \qquad \psi_i, \psi_j, \psi_k \in (0, \pi), \qquad \psi_i + \psi_j + \psi_k = \pi.$$

Hence, given positive interfacial energy densities $\sigma_{ij}, \sigma_{jk}, \sigma_{ki}$ satisfying

$$0 < \sigma_{ik} < \sigma_{ij} + \sigma_{jk}, \quad 0 < \sigma_{jk} < \sigma_{ki} + \sigma_{ik}, \quad 0 < \sigma_{ki} < \sigma_{ij} + \sigma_{jk},$$

there are unique $\varphi_i, \varphi_j, \varphi_k$ satisfying the Herring condition. The physical interpretation of these inequalities means that a direct connection between phases $\theta_i$ and $\theta_j$ regions has smaller interfacial energy density than that by adding an intermediate phase $\theta_k$ region.

For more technical details, see Angenent and Gurtin [4, 20], Herring [24, 25], Mullins [33, 34, 35], Sutton and Baluffi [37], and Woodruff [42].

The excess (free) energy $\mathbf{E}$ attributed from the mismatch of lattices at the grain boundary provides certain degree of freedom of movement of atoms or molecules near the boundary, allowing them to first detach from one lattice and then attach to another lattice, resulting new grain boundary of shorter length so the total free energy is decreased in time.

In this paper, we use a free boundary model obtained from a gradient flow of the total excess energy to describe the evolution of grain boundary. A detailed derivation such as that in [30] demonstrates that, in an appropriate units of time and length, the grain boundary evolves according to the mean curvature flow

$$V = \kappa,$$

where $V$ is the normal velocity and $\kappa$ is the curvature of the grain boundary. Indeed, as different grain boundary carries different energy density, the resulting motion should be a weighted mean curvature flow. Here for simplicity, we assume that all the weights are the same.

In the two dimensional case, the motion by mean curvature equation $V = \kappa$ can be described as follows. Using polar coordinates, we can express a generic grain boundary (a smooth curve) as

$$\mathbf{x} = r(\theta, t)e^{\mathbf{i}\theta}.$$

Then a tangent of the curve is given by

$$\mathbf{x}_\theta = [r_\theta + \mathbf{i}r]e^{\mathbf{i}\theta} = \sqrt{r_\theta^2 + r^2}\, e^{\mathbf{i}(\theta+\psi)} = \frac{r}{\sin\psi} e^{\mathbf{i}(\theta+\psi)}, \qquad \psi := \operatorname{arccot}\frac{r_\theta}{r} \in (0, \pi).$$

The unit tangent $\mathbf{t}$, unit normal $\mathbf{n}$, normal velocity $V$ and curvature $\kappa$ can be written as

$$\mathbf{t} = e^{\mathbf{i}(\theta+\psi)}, \quad \mathbf{n} = -\mathbf{i}\mathbf{t}, \quad V = \mathbf{x}_t \cdot \mathbf{n} = r_t \sin\psi, \quad \kappa = \frac{\mathbf{n}\cdot\mathbf{t}_\theta}{|\mathbf{x}_\theta|} = -\frac{1+\psi_\theta}{r}\, \sin\psi.$$

Thus, the motion by curvature equation $V = \kappa$ can be written as a system

$$r r_t = -[1 + \psi_\theta], \qquad r_\theta = r\cot\psi.$$

The main purpose of this paper is to find self-similar solutions of the form:

(1) Self-similar expanding: $r(\theta, t) = R(\theta)\sqrt{2t}, \quad t > 0$;
(2) Self-similar shrinking: $r(\theta, t) = R(\theta)\sqrt{-2t}, \quad t < 0$;
(3) Stationary: $r(\theta, t) = R(\theta), \quad t \in \mathbb{R}$.

These solutions provide characteristic understanding in topological changes of grains in polycrystals; in particular, they provide asymptotic behavior in nucleation and diminishing of grains.

Substituting the expression of $r(\theta, t)$ into the system we obtain an ode system

$$\frac{d\psi(\theta)}{d\theta} = \mu R^2(\theta) - 1, \qquad \frac{dR(\theta)}{d\theta} = R\cot\psi(\theta), \tag{1.1}$$

where $\mu = 1, 0, -1$ are for self-similar shrinking, stationary, and self-similar expanding respectively.

Note that in the stationary case, $\kappa = V = 0$, so grain boundaries are line segments; similarly, for self-similar expanding, $\kappa = V > 0$, so the grain boundaries are "concave"; for self-similar shrinking, $\kappa = V < 0$, so the grain boundary is "convex"; see Figure 2 for the basic configurations.

We consider the self-similar evolution of a phase domain, denoted by $\Omega_0$, surrounded by phase domains $\Omega_1, \cdots, \Omega_n$, in a counterclockwise order. We use notation $\Omega_{n+1} = \Omega_1$. As we consider only self-similar solutions, we shall assume that for each $i = 1, \cdots, n$, the interface $\gamma_{i,i+1}$ between $\Omega_i$ and $\Omega_{i+1}$ is a straight ray (half-line).

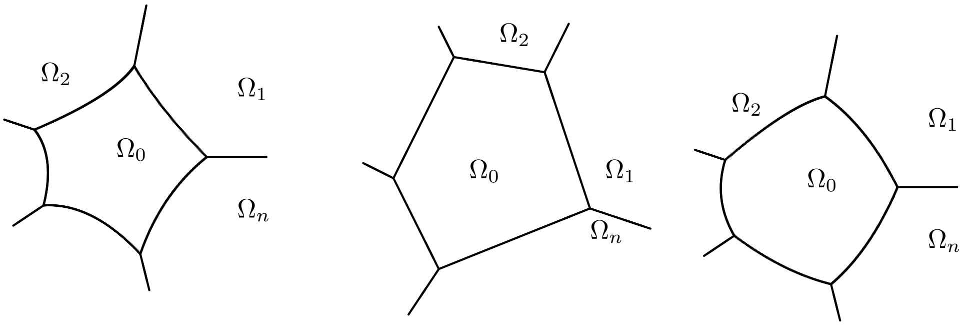


FIGURE 2. Left: self-similar expanding; middle: stationary; right: self-similar shrinking.

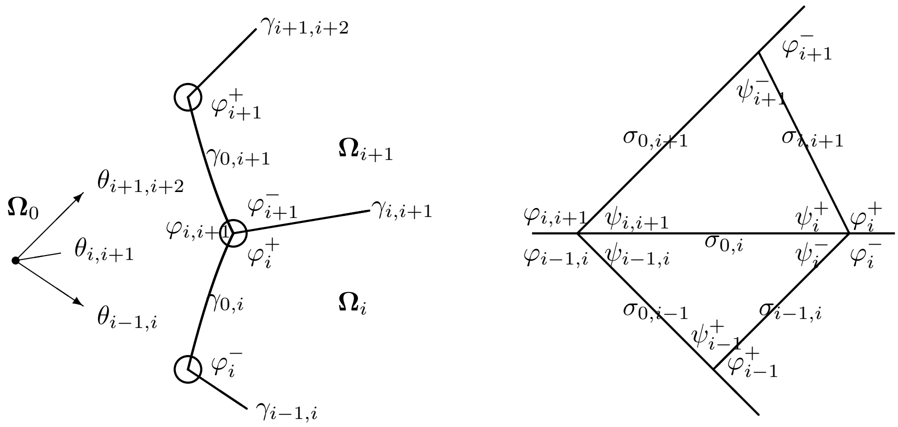


FIGURE 3

In the stationary case, we shall show that there is a special configuration where the linear extensions of $\gamma_{i,i+1}$, $i = 1, \cdots, n$, intersect at a common point, which we name as the origin. In the self-similar expanding and self-similar shrinking case, we shall assume that the linear extensions of $\gamma_{i,i+1}$, $i = 1, \cdots, n$, intersect at a single point, denoted as the origin. In general this may not be true since $\gamma_{i,i+1}$ are not straight. However, in the process of self-shrinking (i.e., $\Omega_0(t) := \Omega_0\sqrt{-2t}$ goes to a single point as $t \uparrow 0^-$), one knows that the speed goes faster and faster as the size of the domain gets smaller and smaller. In this process, one can roughly assume that the boundaries $\gamma_{i,i+1}$, $i = 1, \cdots, n$, are flat near $\Omega_0(t)$. Hence we assume that $\gamma_{i,i+1}$, $i = 1, \cdots, n$, are straight lines and whose directions do not change during the shrinking process. Thus, for a self-shrinking solution, we may assume that all the lines $\gamma_{i,i+1}$, $i = 1, \cdots, n$, intersect at a common point.

Referring to Figure 3, consider, for each fixed $i = 1, \cdots, n$, the triple junction $P_{i,i+1} = R_{i,i+1}e^{\mathbf{i}\theta_{i,i+1}}$ of phase domains $\Omega_0, \Omega_i$, and $\Omega_{i+1}$. Denote by $\sigma_{0,i}, \sigma_{0,i+1}$, and $\sigma_{i,i+1}$ the interfacial energy densities of the corresponding interfaces $\gamma_{0,i}$, $\gamma_{0,i+1}$, and $\gamma_{i,i+1}$. Also denote by $\varphi_{i,i+1}, \varphi_{i+1}^-, \varphi_i^+$ the open angles of $\Omega_0$, $\Omega_{i+1}$, and $\Omega_i$ at $P_{i,i+1}$. Since the unit tangent of grain boundary at $Re^{\mathbf{i}(\theta)}$ is $e^{\mathbf{i}(\theta+\psi)}$, we have

$$\psi(\theta_{i-1,i}+0) = \varphi_i^-, \quad \psi(\theta_{i,i+1}-0) = \pi - \varphi_i^+ \qquad \forall\, i = 1, \cdots, n. \tag{1.2}$$

Thus, finding a self-similar solution is to find a $2\pi$ periodic solution $(\psi, R)$ to (1.1) in the set $\cup_{i=1}^{n}(\theta_{i-1,i}, \theta_{i,i+1})$ subject to the continuity of $R$ and the boundary conditions (1.2), where $\{\theta_{i-1,i}\}_{i=1}^{n}$ is a set of unknowns. By rotation, we may assume that

$$0 = \theta_{0,1} < \theta_{1,2} < \cdots < \theta_{n-1,n} < \theta_{n,n+1} = 2\pi.$$

Motion by mean curvature has been well-studied; see [1, 5, 8, 9, 10, 11, 12, 14, 15, 17, 21, 22, 23, 26, 27, 28, 29, 36] for the pure motion by mean curvature modelling evolution of two phases (i.e. no triple junctions). For the mean curvature flow for multiple phases (i.e. with triple junctions), most of the research is in the two space dimensional case; see [2, 3, 6, 30, 31, 32, 38, 39, 40, 41].

We shall study stationary, self-similar expanding and self-similar shrinking solutions in §2, §3, and §4 respectively. In particular, in §2 we show that *there is a stationary solution if and only if*

$$\sum_{i=1}^{n} \varphi_{i,i+1} = (n-2)\pi, \qquad \varphi_i^+ + \varphi_i^- > \pi \quad \forall\, i = 1, \cdots, n.$$

*In this case, there exists a special stationary solution in which the linear extensions of all* $\gamma_{i,i+1}, i = 1, \cdots, n$*, intersect at a common point, say* $O$*. Moreover, the energy of this special stationary solution in a bigger enough ball centered at* $O$ *is independent of the size of* $\Omega_0$*; more precisely, the excess energy in the ball* $B(O,R)$ $(R \gg 1)$ *is*

$$\mathbf{E}(B(O,R)) = R \sum_{i=1}^{n} \sigma_{i,i+1}.$$

In §3, we show that *there is a self-similar expanding solution if and only if*

$$\sum_{i=1}^{n} \varphi_{i,i+1} < (n-2)\pi, \qquad \varphi_i^+ + \varphi_i^- > \pi \quad \forall\, i = 1, \cdots, n.$$

*The solution is unique up to a shift of the origin and a rotation.*

Finally in §4 we study self-similar shrinking solutions. A necessary condition for the existence of a solution is

$$\sum_{i=1}^{n} \varphi_{i,i+1} > (n-2)\pi.$$

Since in general it is very complicated to classify all solutions, we provide certain partial existence and uniqueness result. Nevertheless, for the isotropic case, i.e., in the case

$$\varphi_{i,i+1} = \varphi_i^+ = \varphi_{i+1}^- = \frac{2\pi}{3} \quad \forall\, i,$$

we can classify all the solutions, depicted in Figure 9.

The 2-D curvature flow in a fixed sector has been studied by Guo and Kohsaka [19], Chang, Guo and Kohsaka [7], Guo and Hu [18], where inside the sector, of open angle $\Delta\theta$, the curve moves by its curvature, with fixed contact angles $\psi^{\pm} \in (0, \pi)$ at the intersections of the curve and the boundary of the sector. It is shown that the area of the region enclosed by the curve and the boundary of the sector is expanding if $\psi^+ + \psi^- + \Delta\theta < \pi$, preserving if $\psi^+ + \psi^- + \Delta\theta = \pi$, and shrinking if $\psi^+ + \psi^- + \Delta\theta > \pi$. Furthermore, in the non-shrinking case, the solution is global and tends to the unique self-similar solution as $t \to \infty$; in the shrinking case, the curve shrinks to the origin in finite time $T$ and the solution tends to a self-similar solution as $t \nearrow T$, under a technical condition $\Delta\theta + \psi^{\pm} < \pi$. For self-similar solution for crystalline flow in a sector, we refer the reader to the work of Giga, Giga and Hontani [16].

## 2. Stationary Solutions

In this section we consider a stationary configuration consists of $n+1$ phase domains: $\Omega_i$, $i = 0, 1, \cdots, n$, where $\Omega_0$ is bounded and is surrounded by unbounded phase domains $\Omega_i$, $i = 1, \cdots, n$, in the counterclockwise order. Then all phase boundaries are line segments or half lines. Hence, $\Omega_0$ is a $n$-polygon with sides $\gamma_{0,1}, \cdots, \gamma_{0,n}$; each phase domain $\Omega_i$ $(i = 1, \cdots, n)$ is bounded by line segment $\gamma_{0,i}$ and half lines $\gamma_{i-1,i}$ and $\gamma_{i,i+1}$.

**Theorem 1.** *Let $n \geqslant 3$, $\Omega_0, \Omega_1, \cdots, \Omega_n, \Omega_{n+1} = \Omega_1$, be phase regions, $\sigma_{i,j} > 0$ be the interfacial energy density of the grain boundary between phase regions $\Omega_i$ and $\Omega_j$. Assume that for each $i = 1, \cdots, n$, the three line segments of respective length $\sigma_{i,i+1}, \sigma_{0,i}, \sigma_{0,i+1}$ forms a non-degenerate triangle. Then the following two statements are equivalent:*

(a) *There exists a stationary configuration such that $\Omega_0$ is a polygon surrounded by $\Omega_1, \cdots, \Omega_n$ in the counterclockwise order with $\varphi_i^+ + \varphi_i^- > \pi$ for all $i = 1, \cdots, n$, and $\gamma_{i,i+1} = \partial\Omega_i \cap \partial\Omega_{i+1}$, $i = 1, \cdots, n$, being disjoint half lines; see Figure* 2 (b)*;*

(b) *There exists a strictly convex $n$-polygon with vertices $p_1, \cdots, p_n$, side length $|\overline{p_i p_{i+1}}| = \sigma_{i,i+1}$, and spoke radius $|\overline{op_i}| = \sigma_{0,i}$ for $i = 1, \cdots, n$; see Figure* 4.

*Suppose* (b) *holds. Then there exists a* **self-similar stationary configuration** *in the sense that all the linear extensions of $\gamma_{i,i+1}$, $i = 1, \cdots, n$, intersect at a single point in $\Omega_0$; see Figure* 5(a). *In addition, all self-similar stationary configurations are similar, i.e., subject to a shift of the origin, a rotation, and a scaling, any two self-similar stationary configurations are identical. Furthermore, restrict to any large ball, any two self-similar stationary configurations have the same total excess energy.*

*Proof.* **(a) $\Rightarrow$ (b).** Suppose there exists a stationary configuration as stated. Here stationary means that the curvature of each grain boundary is zero, so each grain boundary is a line segment or a half line. Denote by $\varphi_{i,i+1}$ the angle between $\gamma_{0,i}$ and $\gamma_{0,i+1}$, by $\varphi_i^+$ the angle between $\gamma_{0,i}$ and $\gamma_{i,i+1}$, and by $\varphi_i^-$ the angle between $\gamma_{0,i}$ and $\gamma_{i-1,i}$. Since $\gamma_{0,1}, \cdots, \gamma_{0,n}$ form a polygon

$$\sum_{i=1}^{n} \varphi_{i,i+1} = (n-2)\pi \qquad \text{i.e.} \qquad \sum_{i=1}^{n} (\pi - \varphi_{i,i+1}) = 2\pi.$$

Now consider a triangle with side lengths $\sigma_{0,i}, \sigma_{0,i+1}$ and $\sigma_{i,i+1}$; see Figure 3 (b). By the Herring condition, the angle opposite to the side of length $\sigma_{i,i+1}$ is $\psi_{i,i+1} := \pi - \varphi_{i,i+1}$, the angle opposite to side of length $\sigma_{0,i+1}$ is $\psi_i^+ := \pi - \varphi_i^+$ and the angle opposite to the side of length $\sigma_{0,i}$ is $\psi_{i+1}^- := \pi - \varphi_{i+1}^-$. Since $\sum_{i=1}^n \psi_{i,i+1} = \sum_{i=1}^n (\pi - \varphi_{i,i+1}) = 2\pi$, if we join all these triangles counterclockwise with the same vertex $o$, we have a polygon with vertices $p_1, p_2, \cdots, p_n$, sides lengths $|\overline{p_i p_{i+1}}| = \sigma_{i,i+1}$ $(p_{n+1} := p_1)$ and spoke radius $|\overline{op_i}| = \sigma_{0,i}$ for all $i = 1, \cdots, n$.

Finally, since $\gamma_{i-1,i}$ and $\gamma_{i,i+1}$ do not intersect and we have $\varphi_i^+ + \varphi_i^- > \pi$, the interior angle of the polygon at $p_i$ is given by $\psi_i^+ + \psi_i^- = 2\pi - (\varphi_i^+ + \varphi_i^-) < \pi$. Hence, the polygon with vertices $p_1, \cdots, p_n$ is strictly convex. Thus (b) holds.

**(b) $\Rightarrow$ (a).** Note that for each $i$, $0 < \varphi_{i,i+1} < \pi$. That $p_1, \cdots, p_n$ form a polygon implies $\sum_{i=1}^n \varphi_{i,i+1} = (n-2)\pi$. It follows from a simple geometric fact that there are numerous convex

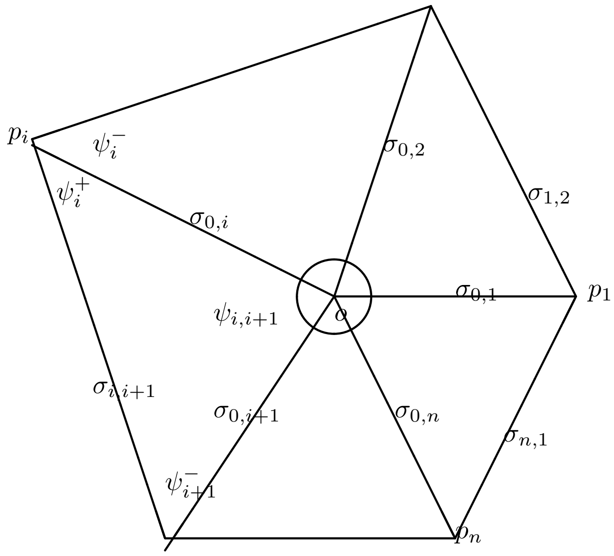


FIGURE 4

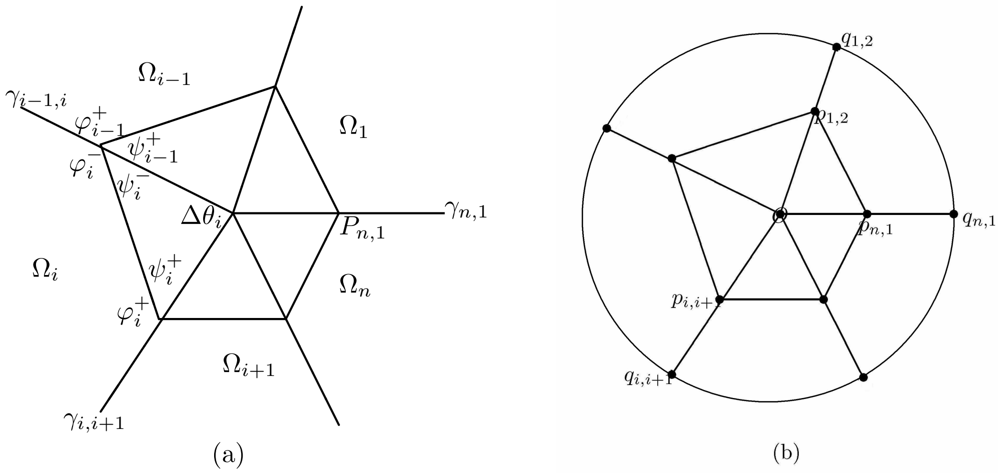


FIGURE 5

$n$-polygons $\Omega_0$ whose interior angles are $\varphi_{i,i+1}, i = 1, \cdots, n$. Pick arbitrarily such a polygon. On each vertex, separate the exterior sector of opening $2\pi - \varphi_{i,i+1} = \varphi_i^+ + \varphi_{i+1}^-$ by a half-line $\gamma_{i,i+1}$ into two sectors of open angle $\varphi_i^+$ and $\varphi_{i+1}^-$ respectively. Since the polygon with vertices $p_1, \cdots, p_n$ is strictly convex, $\psi_i^+ + \psi_i^- < \pi$ for each $i$. Hence $\varphi_i^+ + \varphi_i^- = 2\pi - (\psi_i^+ + \psi_i^-) > \pi$ and so $\gamma_{i-1,i}$ does not intersect $\gamma_{i,i+1}$. Thus, there exists a stationary configuration and (a) holds.

Note that any $n$-polygon with interior angle $\varphi_{i,i+1}$, $i = 1, \cdots, n$, can be used as $\Omega_0$ to construct a stationary configuration, there are infinitely many stationary configurations. In the sequel, we shall construct a special stationary configuration which we call self-similar.

Now we assume that (b) holds and that the polygon with vertices $p_1, \cdots, p_n$ is strictly convex, i.e., $\psi_i^+ + \psi_i^- < \pi$ for each $i$.

Referring to Figure 5 (a), denote

$$\Delta\theta_i := \pi - \psi_i^+ - \psi_i^- = \varphi_i^+ + \varphi_i^- - \pi, \quad \theta_{0,1} = 0,\ \theta_{i,i+1} = \sum_{k=0}^{i} \Delta\theta_k,\ i = 1, \cdots, n.$$

Then $\Delta\theta_i > 0$ and

$$\sum_{i=1}^{n} \Delta\theta_i = \sum_{i=1}^{n}(\varphi_i^+ + \varphi_{i+1}^-) - n\pi = \sum_{i=1}^{n}(2\pi - \varphi_{i,i+1}) - n\pi = n\pi - \sum_{i=1}^{n} \varphi_{i,i+1} = 2\pi.$$

Define, for $i = 1, \cdots, n$,

$$R_{i,i+1} := \frac{\sigma_{i,i+1}}{\sigma_{0,i}\sigma_{0,i+1}\sin\varphi_{i,i+1}}, \qquad P_{i,i+1} := R_{i,i+1}e^{\mathbf{i}\theta_{i,i+1}},$$
$$\gamma_{i,i+1} := \{re^{\mathbf{i}\theta_{i,i+1}} \mid r \geq R_{i,i+1}\}, \qquad \gamma_{0,i} := \overline{P_{i-1,i}P_{i,i+1}}.$$

Using the Herring condition, we have

$$R_{i,i+1} = \frac{\sigma_{i,i+1}}{\sigma_{0,i}\sigma_{0,i+1}\sin\varphi_{i,i+1}} = \frac{1}{\sigma_{0,i}\sin\psi_i^+} = \frac{1}{\sigma_{0,i+1}\sin\psi_{i+1}^-}.$$

Let $\Omega_0$ be the polygon with vertices $P_{1,2}, \cdots, P_{n-1,n}, P_{n,n+1} = P_{n,1}$. Let $\Omega_i$ be the domain bounded by $\gamma_{0,i}$, $\gamma_{i-1,i}$ and $\gamma_{i,i+1}$.

Now consider the triangle with vertices $O, P_{i-1,i}$ and $P_{i,i+1}$. We want to show that the three interior angles at $O, P_{i-1,i}$ and $P_{i,i+1}$ are $\Delta\theta_i, \psi_i^-$, and $\psi_i^+$ respectively.

Clearly, by the definition of $P_{i-1,i}$ and $P_{i,i+1}$, $\angle P_{i-1,i}OP_{i,i+1} = \Delta\theta_i$. Also, we see that

$$\frac{|\overline{OP_{i-1,i}}|}{|\overline{OP_{i,i+1}}|} = \frac{R_{i-1,i}}{R_{i,i+1}} = \frac{\sigma_{0,i}\sin\psi_i^+}{\sigma_{0,i}\sin\psi_i^-} = \frac{\sin\psi_i^+}{\sin\psi_i^-}.$$

It then follows that $\triangle P_{i-1,i}Op_{i,i+1}$ is similar to the triangle with three interior angles $\psi_i^-, \Delta\theta_i, \psi_i^+$. Thus, for every $i = 1, \cdots, n$,

$$\angle P_{i-1,i}Op_{i,i+1} = \Delta\theta_i, \quad \angle OP_{i-1,i}P_{i,i+1} = \psi_i^-, \qquad \angle P_{i-1,i}P_{i,i+1}O = \psi_i^+.$$

Hence, the three angles at the triple junction $P_{i,i+1}$ of $\gamma_{0,i}, \gamma_{0,i+1}, \gamma_{i,i+1}$ are $\pi - \psi_i^+ = \varphi_i^+$, $\pi - \psi_{i+1}^- = \varphi_{i+1}^-$, and $\psi_i^+ + \psi_{i+1}^- = \pi - \psi_{i,i+1} = \varphi_{i,i+1}$. That is, the so constructed phase domains constitute a self-similar stationary configuration.

Now we show that all self-similar stationary configurations are similar.

Let $\tilde\Omega_0, \cdots, \tilde\Omega_n$ be a self-similar stationary configuration. Denote by $O$ the common intersection point of the linear extensions of $\gamma_{i,i+1}$, $i = 1, \cdots, n$. Denote by $p_{i,i+1}, i = 1, \cdots, n$, the vertices of $\tilde\Omega_0$.

Consider the triangle $Op_{1,2}p_{2,3}$ and triangle $OP_{1,2}P_{2,3}$. These two triangles are similar since they have the same set of interior angles. Hence, we have

$$\frac{|\overline{Op_{1,2}}|}{R_{1,2}} = \frac{|\overline{Op_{2,3}}|}{R_{2,3}} = \frac{|\overline{p_{1,2}p_{2,3}}|}{|\overline{P_{1,2}P_{2,3}}|}.$$

By induction, one sees that $\tilde\Omega_0$ is similar to $\Omega_0$. Thus, all self-similar stationary configurations are similar.

Finally, we consider the "local" excess energy of any self-similar pattern.

Suppose we have a self-similar stationary configuration. Let the origin $O$ be the common intersection point of the linear extensions of $\gamma_{i,i+1}, i = 1, \cdots, n$. Pick a sufficiently large $R$. Denote by $q_{i,i+1}$ the intersection of $\gamma_{i,i+1}$ with the ball (circle) $B(O,R)$ of radius $R$ centered at $O$; see Figure **??** (b).

The total excess energy of the stationary configuration in the ball $B(O,R)$ is

$$\mathbf{E}(B(O,R)) := \sum_{i=1}^{n} \sigma_{i,i+1}|\overline{p_{i,i+1}q_{i,i+1}}| + \sum_{i=1}^{n} \sigma_{0,i}|\overline{p_{i-1,i}p_{i,i+1}}|.$$

We now show that $\mathbf{E}(B(O,R))$ is independent of the size of the stationary configuration. In particular, we show that

$$\mathbf{E}(B(O,R)) = R\sum_{i=1}^{n} \sigma_{i,i+1}.$$

For this, if suffices to show the following:

$$\sum_{i=1}^{n} \sigma_{0,i}|\overline{p_{i-1,i}p_{i,i+1}}| = \sum_{i=1}^{n} \sigma_{i,i+1}|\overline{Op_{i,i+1}}|.$$

For this purpose, denote $\rho = |\overline{Op_{1,2}}|/R_{1,2}$. Then, for every $i = 1, \cdots, n$,

$$|\overline{p_{i-1,i+1}p_{i,i+1}}| = \rho|\overline{P_{i-1,i}P_{i,i+1}}|, \qquad |\overline{Op_{i,i+1}}| = \rho R_{i,i+1} = \rho|\overline{OP_{i,i+1}}|.$$

Consider the triangle $P_{i-1,i}OP_{i,i+1}$. We have

$$|\overline{OP_{i-1,i}}| = R_{i-1,i} = \tfrac{1}{\sigma_{0,i}\sin\psi_i^-}, \qquad |\overline{OP_{i,i+1}}| = R_{i,i+1} = \tfrac{1}{\sigma_{0,i}\sin\psi_i^+},$$
$$\angle P_{i-1,i}OP_{i,i+1} = \Delta\theta_i = \pi - (\psi_i^+ + \psi_{i+1}^-).$$

It follows that

$$\begin{aligned}
\sigma_{0,i}|\overline{P_{i-1,i}P_{i,i+1}}| &= \sigma_{0,i}\sqrt{R_i^2 + R_{i+1}^2 - 2R_iR_{i+1}\cos\Delta\theta_i} \\
&= \sqrt{\frac{1}{\sin^2\psi_i^+} + \frac{1}{\sin^2\psi_i^-} + 2\frac{\cos\psi_i^+\cos\psi_i^- - \sin\psi_i^+\sin\psi_i^-}{\sin\psi_i^+\sin\psi_i^-}} \\
&= \sqrt{\cot^2\psi_i^+ + \cot^2\psi_i^- + 2\cot\psi_i^+\cot\psi_i^-} \\
&= \cot\psi_i^+ + \cot\psi_i^-.
\end{aligned}$$

Here we used the fact that $\psi_i^+ + \psi_i^- \in (0,\pi)$ so that $\cot\psi_i^+ + \cot\psi_i^- > 0$. It follows that

$$\begin{aligned}
\sum_{i=1}^{n} \sigma_{0,i}|\overline{P_{i-1,i}P_{i,i+1}}| &= \sum_{i=1}^{n}(\cot\psi_i^+ + \cot\psi_i^-) = \sum_{i=1}^{n}(\cot\psi_i^+ + \cot\psi_{i+1}^-) \\
&= \sum_{i=1}^{n}\frac{\sin(\psi_i^+ + \psi_{i+1}^-)}{\sin\psi_i^+\sin\psi_{i+1}^-} = \sum\frac{\sin\varphi_{i,i+1}}{\sin\varphi_i^+\sin\varphi_{i+1}^-} \\
&= \sum_{i=1}^{n}\frac{\sigma_{i,i+1}^2}{\sigma_{0,i}\sigma_{0,i+1}\sin\varphi_{i,i+1}} = \sum_{i=1}^{n}\sigma_{i,i+1}|\overline{OP_{i,i+1}}|
\end{aligned}$$

by the Herring condition and definition of $R_{i,i+1}$. Thus, $\mathbf{E}(B(O,R)) = R\sum_{i=1}^{n}\sigma_{i,i+1}$ is independent of the self-similar stationary configuration. The assertion of the Theorem thus follows. $\square$

## 3. Self-Similar Expanding Solutions

We are looking for $2\pi$ periodic continuous and positive function $R(\cdot)$ and angles $\theta_{0,1} := 0 < \theta_{1,2} < \cdots < \theta_{n-1,n} < \theta_{n,n+1} := 2\pi$ such that the phase domains $\Omega_{0t}, \cdots, \Omega_{nt}$ at time $t$ are given by $(\Omega_{0t}, \cdots, \Omega_{nt}) = \sqrt{2t}\,(\Omega_0, \cdots, \Omega_n)$ for $t > 0$, where

$$\begin{aligned} \Omega_0 &:= \{re^{\mathbf{i}\theta} \mid 0 \leqslant \theta \leqslant 2\pi, 0 \leqslant r < R(\theta)\}, \\ \Omega_i &:= \{re^{\mathbf{i}\theta} \mid \theta_{i-1,i} < \theta < \theta_{i,i+1}, r > R(\theta)\},\ i = 1, \cdots, n. \end{aligned}$$

For the evolution to be the motion by curvature, we introduce $\psi$ such that the unit tangent of boundary $\Omega_0$ at $R(\theta)e^{\mathbf{i}\theta}$ is $e^{\mathbf{i}(\theta+\psi)}$. Then we are seeking a $2\pi$ periodic function $(\psi, R)$ and "free boundaries" $\{\theta_{i,i+1}\}_{i=0}^n$ such that $R$ is continuous and

$$\begin{cases} 0 = \theta_{0,1} < \theta_{1,2} < \cdots < \theta_{n-1,n} < \theta_{n,n+1} = 2\pi, \\ \psi_\theta = -(1 + R^2), \quad R_\theta = R\cot\psi \quad \text{in} \ \ (\theta_{i-1,i}, \theta_{i,i+1}), \quad i = 1, \cdots, n, \\ \psi(\theta_{i-1,i} + 0) = \pi - \psi_i^-, \quad \psi(\theta_{i,i+1} - 0) = \psi_i^+, \quad i = 1, \cdots, n. \end{cases} \tag{3.1}$$

See Figure 3 for boundary conditions. Note that $\psi_i^\pm = \pi - \varphi_i^\pm$. Here the angles $\psi_i^\pm$ are uniquely determined by the Herring condition (cf. Figure 3)

$$\begin{cases} \psi_i^+, \psi_{i+1}^-, \psi_{i,i+1} \in (0, \pi), \quad \psi_i^+ + \psi_{i+1}^- + \psi_{i,i+1} = \pi, \\ \dfrac{\sin\psi_i^+}{\sigma_{0,i+1}} = \dfrac{\sin\psi_{i+1}^-}{\sigma_{0,i}} = \dfrac{\sin\psi_{i,i+1}}{\sigma_{i,i+1}}, \quad i = 1, \cdots, n, \\ \psi_1^- = \psi_{n+1}^-, \qquad \psi_{0,1} = \psi_{n,n+1}. \end{cases} \tag{3.2}$$

**Problem (PE):** Given a set $\{\psi_i^\pm\}_{i=1}^n$ of angles from the Herring condition (3.2), find a $2\pi$ periodic function $(\psi, R)$ and angles $\{\theta_{i,i+1}\}_{i=0}^n$ such that $R$ is continuous and (3.1) holds.

Now we solve the problem.

The system $\psi_\theta = -(1 + R^2)$, $R_\theta = R\cot\psi$ can be written as

$$\frac{dR}{R\cot\psi} = -\frac{d\psi}{1 + R^2} = d\theta.$$

There is a first integral given by

$$H(R) = c\ \sin\psi, \qquad H(R) := \frac{1}{Re^{R^2/2}},$$

where $c$ is a positive constant. Note that

$$H' < 0 \quad \text{in} \quad (0, \infty), \quad H(\infty) = 0, \quad H(0+) = \infty.$$

Therefore, there exist positive constants $c_1, \cdots, c_n$ such that

$$H(R) = c_i\ \sin\psi \qquad \text{in} \quad (\theta_{i-1,i}, \theta_{i,i+1}).$$

The continuity of $R$ at $\theta_{i,i+1}$ and boundary conditions $\psi(\theta_{i,i+1}+0) = \pi - \psi_{i+1}^-$ and $\psi(\theta_{i,i+1}-0) = \psi_i^+$ are equivalent to

$$H(R(\theta_{i,i+1})) = c_i \sin\psi_i^+ = c_{i+1}\sin\psi_{i+1}^-.$$

By the Herring condition (3.2), we see that

$$\frac{c_i}{\sigma_{0,i}} = \frac{c_{i+1}}{\sigma_{0,i+1}} \qquad \forall\, i.$$

Hence there exists $a > 0$ such that

$$c_i = a\,\sigma_{0,i} \qquad \forall\, i = 1, \cdots, n.$$

Next, we determine $a$.

Denote by $G$ the inverse function of $H$:

$$s = H(r) = \frac{1}{re^{r^2/2}} \qquad \Longleftrightarrow \qquad r = G(s).$$

Then on $(\theta_{i-1,i}, \theta_{i,i+1})$ we have

$$H(R) = c_i \,\sin\psi = a\,\sigma_{0,i}\,\sin\psi \qquad \text{or} \qquad R = G(a\,\sigma_{0,i}\sin\psi).$$

Now, using

$$d\theta = -\frac{d\psi}{R^2+1},$$

we see that

$$\Delta\theta_i := \theta_{i,i+1} - \theta_{i-1,i} = \int_{\psi_i^+}^{\pi-\psi_i^-} \frac{d\psi}{G^2(a\,\sigma_{0,i}\sin\psi)+1}.$$

Note that

$$\Delta\theta_i > 0 \quad \Longleftrightarrow \quad \psi_i^+ + \psi_i^- < \pi, \tag{3.3}$$

$$\sum_{i=1}^n \Delta\theta_i = 2\pi \quad \Longleftrightarrow \quad \sum_{i=1}^n \int_{\psi_i^+}^{\pi-\psi_i^-} \frac{d\psi}{G^2(a\,\sigma_{0,i}\sin\psi)+1} = 2\pi. \tag{3.4}$$

Thus, $a$ is determined by the relation $Q(a) = 2\pi$, where

$$Q(x) := \sum_{i=1}^n \int_{\psi_i^+}^{\pi-\psi_i^-} \frac{d\psi}{G^2(x\,\sigma_{0,i}\sin\psi)+1}, \qquad x \in (0,\infty).$$

Assume (3.3). Then $Q(0) = 0$ and $Q'(x) > 0$ for all $x \in (0,\infty)$, since $G' < 0$ on $(0,\infty)$ and $G(0) = \infty$. Hence there exists at most one solution to $Q(a) = 2\pi$. Therefore, a necessary and sufficient condition for the existence of a solution to $Q(a) = 2\pi$ is $Q(\infty) > 2\pi$, i.e.,

$$2\pi < \sum_{i=1}^n \{\pi - \psi_i^+ - \psi_i^-\} = \sum_{i=1}^n \{\pi - \psi_i^+ - \psi_{i+1}^-\} = \sum_{i=1}^n \psi_{i,i+1}.$$

Once $a$ is found to satisfy $Q(a) = 2\pi$, it is easy to verify that we have a solution to **(PE)**.

Hence, we have proved the following theorem.

**Theorem 2.** *Given a set $\{\psi_i^\pm\}_{i=1}^n$ of angles from the Herring condition* (3.2)*, there exists a solution to* **(PE)** *(that corresponds to a self-similar expanding solution) if and only if*

$$\psi_i^+ + \psi_i^- < \pi \quad \forall\, i = 1, \cdots, n, \qquad \sum_{i=1}^n \psi_{i,i+1} > 2\pi.$$

*In addition, under the above conditions, solutions to* **(PE)** *are unique.*

## 4. Self-Similar Shrinking Solutions

We are looking for $2\pi$ periodic continuous and positive function $R(\cdot)$ and angles $\theta_{0,1} := 0 < \theta_{1,2} < \cdots < \theta_{n-1,n} < \theta_{n,n+1} := 2\pi$ such that the phase domains $\Omega_{0t}, \cdots, \Omega_{nt}$ at time $t$ are given by $(\Omega_{0t}, \cdots, \Omega_{nt}) = \sqrt{-2t}\,(\Omega_0, \cdots, \Omega_n)$ for $t < 0$, where

$$\begin{aligned} \Omega_0 &:= \{re^{\mathbf{i}\theta} \mid 0 \leqslant \theta \leqslant 2\pi, 0 \leqslant r < R(\theta)\}, \\ \Omega_i &:= \{re^{\mathbf{i}\theta} \mid \theta_{i-1,i} < \theta < \theta_{i,i+1}, r > R(\theta)\}, \quad i = 1, \cdots, n. \end{aligned}$$

As derived in the earlier sections, we are seeking a $2\pi$ periodic function $(\psi, R)$ and "free boundaries" $\{\theta_{i,i+1}\}_{i=0}^n$ such that $R$ is continuous and

$$\begin{cases} 0 = \theta_{0,1} < \theta_{1,2} < \cdots < \theta_{n-1,n} < \theta_{n,n+1} = 2\pi, \\ \psi_\theta = R^2 - 1, \quad R_\theta = R\cot\psi \quad \text{in } (\theta_{i-1,i}, \theta_{i,i+1}), \quad i = 1, \cdots, n, \\ \psi(\theta_{i-1,i} + 0) = \pi - \psi_i^-, \quad \psi(\theta_{i,i+1} - 0) = \psi_i^+, \quad i = 1, \cdots, n. \end{cases} \tag{4.1}$$

**Problem (PS):** Given a set $\{\psi_i^\pm\}_{i=1}^n$ of angles from the Herring condition (3.2), find a $2\pi$ periodic function $(\psi, R)$ and angles $\{\theta_{i,i+1}\}_{i=0}^n$ such that $R$ is continuous and (4.1) holds.

To solve the problem, we observe the following:

**(1)** If there is a solution, we must have

$$\int_0^{2\pi} (1 - R^2(\theta))d\theta = -\sum_{i=1}^n \int_{\theta_{i-1,i}}^{\theta_{i,i+1}} \psi_\theta d\theta = \sum_{i=1}^n (\pi - \psi_i^- - \psi_i^+) = \sum_{i=1}^n \psi_{i,i+1}.$$

Hence, a necessary condition for the existence of a solution to **(PS)** is

$$\sum_{i=1}^n \psi_{i,i+1} < 2\pi. \tag{4.2}$$

**(2)** The system $\psi_\theta = R^2 - 1,\ R_\theta = R\cot\psi$ can be written as

$$\frac{dR}{R\cot\psi} = \frac{d\psi}{R^2 - 1} = d\theta.$$

There is a first integral given by

$$K(R) = c\ \sin\psi, \qquad K(R) := \frac{e^{(R^2-1)/2}}{R},$$

where $c$ is a positive constant. The function $K$ has the property that

$$\begin{aligned} K' < 0 \text{ in } (0, 1), &\qquad K' > 0 \text{ in } (1, \infty), \\ K(0) = K(\infty) = \infty, &\qquad K(1) = \min_{R\in(0,\infty)} K(R) = 1. \end{aligned}$$

In the sequel, we denote by $r = r_1(s)$ and $r = r_2(s)$ the two inverses of $s = K(r)$:

$$\begin{aligned} s = r^{-1}e^{(r^2-1)/2}, \quad 0 < r < 1 &\qquad \Longleftrightarrow \qquad r = r_1(s), \\ s = r^{-1}e^{(r^2-1)/2}, \quad r > 1 &\qquad \Longleftrightarrow \qquad r = r_2(s). \end{aligned}$$

On the $(R, \psi)$ phase plane, we denote by $\gamma(c)$ the trajectory $K(R) = c\sin\psi$; that is,

$$\gamma(c) := \{(R, \psi) \in (0, \infty) \times (0, \pi) \mid K(R) = c\sin\psi\} \qquad \forall\, c \geqslant 1.$$

Figure 6 is a few samples of the trajectory $\gamma(c)$. Since $d\psi = (R^2 - 1)d\theta$, positive $\theta$ direction corresponds to counterclockwise rotation.

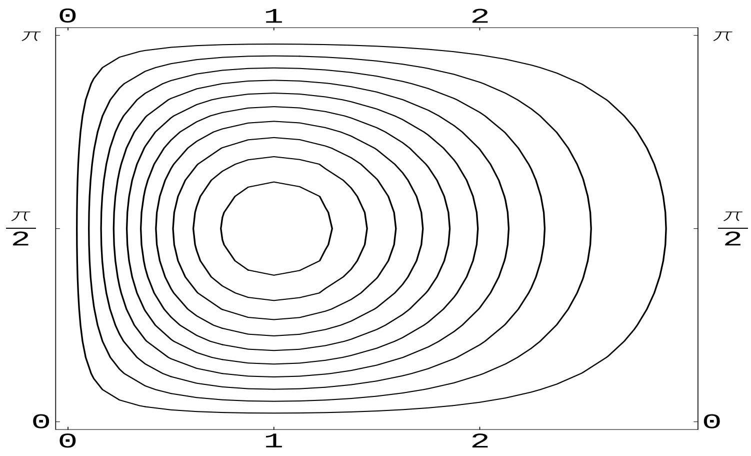


FIGURE 6. Trajectory $\gamma(c)$ on the $R$-$\psi$ phase plane; $\gamma(1) = \{(1, \pi/2)\}$.

**(3)** If $(R, \psi, \{\theta_{i,i+1}\})$ is a solution to **(PS)**, then there exist constants $c_1, \cdots, c_n$ with $c_i \geqslant 1$ for each $i$ such that

$$K(R(\theta)) = c_i \sin\psi(\theta) \qquad \forall\, \theta \in (\theta_{i-1,i}, \theta_{i,i+1}), \quad i = 1, \cdots, n.$$

The continuity of $R$ and the boundary condition $\psi(\theta_{i,i+1} + 0) = \pi - \psi_{i+1}^-, \psi(\theta_{i,i+1} - 0) = \psi_i^+$ at $\theta_{i,i+1}$ require

$$K(R(\theta_{i,i+1})) = c_i \sin\psi_i^+ = c_{i+1} \sin\psi_{i+1}^-.$$

By the Herring condition, there exists $a > 0$ such that

$$c_i = a\,\sigma_{0,i} \quad \forall i.$$

As $\min_{R>0} K(R) = 1$, we need $a\sigma_{0,i} \sin\psi_i^\pm \geq 1$ for all $i$. Hence, a necessary condition is

$$a \geqslant a_* := \max_{1 \leqslant i \leqslant n} \left\{ \frac{1}{\sigma_{0,i} \sin\psi_i^+} \right\} = \max_{1 \leqslant i \leqslant n} \left\{ \frac{1}{\sigma_{0,i} \sin\psi_i^-} \right\}.$$

Since the trajectory can make a number of loops, a complete analysis is quite complicated. Here we discuss a few special cases.

4.1. **The Case** $n = 1$**.** We consider the case when $\Omega_0$ is surrounded by $\Omega_1$, so there is no triple junctions. This renders to find smooth solutions to

$$R_\theta = R\cot\psi, \quad \psi_\theta = R^2 - 1 \quad \text{on } \mathbb{R}, \quad R(0) = R(2\pi) > 0, \quad \psi(0) = \psi(2\pi). \tag{4.3}$$

An obvious solution is given by

$$R \equiv 1, \qquad \psi \equiv \frac{\pi}{2}.$$

This solution corresponds to the unit circle. This should be the only solution, since in the 2-dimensional case, if $\Gamma_t$ is the boundary of a simply connected domain $\Omega_t$ and $\Gamma_t$ shrinks to a single point as $t \nearrow 0$ according to the motion by curvature, then $\Gamma_t$ is, asymptotically, a circle of radius $\sqrt{-2t}$; see Gage [13], Gage and Hamilton [14], and Grayson [17].

Without using the general result in [13, 14, 17], here we study directly (4.3). On the phase plane, the solution trajectory is given by $K(R) = c\ \sin\psi$ for some constant $c \geqslant 1$. Denote by $R_* = \min R$ and $R^* = \max R$. Then

$$c = K(R_*) = K(R^*) \qquad \text{or} \qquad R_* = r_1(c), \qquad R^* = r_2(c).$$

Also denote by $T(c)$ the period of the trajectory $\gamma(c)$:

$$T(c) = \oint_{\gamma(c)} \frac{dR}{R_\theta} = \oint_{\gamma(c)} \frac{d\psi}{\psi_\theta}.$$

Using

$$R_\theta = R\cot\psi = R\frac{\cos\psi}{\sin\psi} = \pm R\sqrt{c^2R^2e^{1-R^2}-1}$$

we obtain, by symmetry,

$$T(c) \quad = \quad 2\int_{r_1(c)}^{r_2(c)} f(c,R)\,dR, \qquad f(c,R) := \frac{1}{R\sqrt{c^2R^2e^{1-R^2}-1}}. \tag{4.4}$$

The parametric curve $\{(r_1(c), T(c)/\pi)\}_{c>1}$ is plotted in Figure 7.

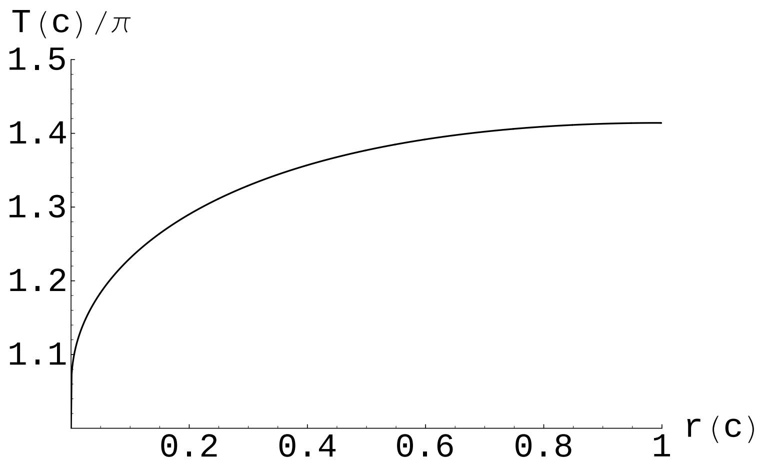


FIGURE 7. Parametric curve $(r(c) = r_1(c), T(c)/\pi)$).

Using the change of variable $\rho = (R/r_1(c))^2$ we have

$$T(c) = \int_1^{(r_2(c)/r_1(c))^2} \frac{d\rho}{\rho\sqrt{\rho e^{r_1^2(c)(1-\rho)}-1}}.$$

From this it is not very difficult to show that $\lim_{c\to\infty} T(c) = \pi$ and $\lim_{c\searrow 1} T(c) = \sqrt{2}\pi$.

From Figure 7, one may believe that $T'(c) < 0$ for $c > 1$; a rigorous proof for $T' < 0$ can be derived from the paper of Abresch and Langer [1]. Hence,

$$T'(c) < 0 \quad \text{in} \quad (1,\infty), \qquad T(\infty) = \pi, \qquad T(1) = \sqrt{2}\pi.$$

Consequently, when $c > 1$, $2\pi$ is not an integer multiple of $T(c)$, so $\gamma(c)$ is not a trajectory of (4.3). Hence, the only solution to (4.3) is $R \equiv 1, \psi \equiv \pi/2$.

4.2. **A classification of solutions.** In the sequel, we assume that $n \geq 2$. We classify solutions to **(PS)** as follows.

(1) **Type I solution**: $0 < R \leqslant 1$ for all $\theta$.
(2) **Type II solution**: None of $\{(R(\theta), \psi(\theta))\}_{\theta_{i-1,i}<\theta<\theta_{i,i+1}}$, $i = 1, \cdots, n$, contains a whole loop $\gamma(c)$ for some $c > 1$ and there exists an integer $k \in \{1, \cdots, n\}$ such that
$$R(\theta_{k-1,k}) > 1 \geqslant R(\theta_{i-1,i}) \quad \forall\, i \in \{1, \cdots, n\} \setminus \{k\}.$$
(3) **Type III solution**: None of $\{(R(\theta), \psi(\theta))\}_{\theta_{i-1,i}<\theta<\theta_{i,i+1}}$, $i = 1, \cdots, n$, contains a whole loop $\gamma(c)$ for some $c > 1$ and there exist two different integers $k, j \in \{1, \cdots, n\}$ such that
$$R(\theta_{k-1,k}) > 1, \quad R(\theta_{j-1,j}) > 1, \quad R(\theta_{i-1,i}) \leqslant 1 \quad \forall\, i \in \{1, \cdots, n\} \setminus \{k, j\}.$$
(4) **Type IV solution**: None of the above.

We remark that when $n \geq 2$, there is no solution satisfying $R(\cdot) \geqslant 1$ in $[0, 2\pi]$, since
$$\int_0^{2\pi} (1 - R^2)\, d\theta = \sum_{i=1}^{n} \psi_{i,i+1} > 0.$$

4.3. **Type I solutions.** For a type I solution, the trajectory $\{(R(\theta), \psi(\theta))\}_{\theta_{i-1,i}<\theta<\theta_{i,i+1}}$ is a part of $\gamma(a\sigma_{0,i})$ in the region $\{(R, \psi) \mid 0 < R \leqslant 1, 0 < \psi < \pi\}$. It follows from the equation $d\psi = (R^2 - 1)d\theta$ that
$$\Delta\theta_i := \theta_{i,i+1} - \theta_{i-1,i} = \int_{\psi_i^+}^{\pi-\psi_i^-} \frac{d\psi}{1 - r_1^2(a\sigma_{0,i} \sin\psi)}.$$
As $\Delta\theta_i > 0$, we need
$$\psi_i^+ < \pi - \psi_i^-, \qquad \text{i.e.,} \qquad \psi_i^+ + \psi_i^- < \pi \qquad \forall\, i = 1, \cdots, n. \tag{4.5}$$

Denote
$$Q_I(x) := \sum_{i=1}^{n} \int_{\psi_i^+}^{\pi-\psi_i^-} \frac{d\psi}{1 - r_1^2(x\, \sigma_{0,i} \sin\psi)}, \qquad x \geqslant a_*.$$
Then, under (4.5), a type I solution exists if and only if $Q_I(a) = 2\pi$ for some $a \geqslant a_*$.

Note that $r_1(\infty) = 0$ and $r_1'(c) < 0$ for all $c \in (1, \infty)$. It implies that $Q_I'(x) < 0$ for all $x > a_*$ and
$$Q_I(\infty) = \sum_{i=1}^{n} (\pi - \psi_i^+ - \psi_i^-) = \sum_{i=1}^{n} \psi_{i,i+1}.$$
Thus, we have the following result.

**Proposition 4.1.** *There exists a unique type I solution to* **(PS)** *if and only if*
$$\psi_i^+ + \psi_i^- < \pi \quad \forall\, i, \qquad \sum_{i=1}^{n} \psi_{i,i+1} < 2\pi, \quad Q_I(a_*) \geqslant 2\pi.$$

4.4. **Type II solutions.** Assume (4.2), (4.5), and $Q_I(a_*) < 2\pi$. Let $i_*$ be an index such that

$$\sigma_{0,i_*}\sin\psi_{i_*}^- = \frac{1}{a_*} := \min_{1\leqslant i\leqslant n}\{\sigma_{0,i}\sin\psi_i^-\} = \min_{1\leqslant i\leqslant n}\{\sigma_{0,i}\sin\psi_i^+\}. \tag{4.6}$$

Assume without loss of generality that $i^* = 1$. Then

$$\sigma_{0,1}\sin\psi_1^- = \sigma_{0,n}\sin\psi_n^+ = \frac{1}{a_*}\ .$$

This equation, together with (4.5), imply $\pi - \psi_1^- > \pi/2$ and $\psi_n^+ < \pi/2$. Indeed, if $\pi - \psi_1^- \leqslant \pi/2$, then (4.5) implies that $\psi_1^+ < \pi - \psi_1^- \leqslant \pi/2$ so that $\sigma_{0,1}\sin\psi_1^+ < \sigma_{0,1}\sin\psi_1^-$ contradicting the assumption that $i_* = 1$. The proof for $\psi_n^+ < \pi/2$ is similar.

We seek a type II solution such that

$$R(0) > 1 \geqslant R(\theta_{i,i+1}) \quad \forall\, i = 1,\cdots,n-1.$$

If such a solution exists, then there exists $a \geqslant a_*$ such that for each $i = 2,\cdots,n-1$, the trajectory $\{(R(\theta),\psi(\theta))\}_{\theta_{i-1,i}<\theta<\theta_{i,i+1}}$ is the part of $\gamma(a\sigma_{0,i})$ in $\{(R,\psi) \mid R \leqslant 1,\ \psi_i^+ < \psi < \pi - \psi_i^-\}$. For $i = 1$ and $i = n$, the trajectory $\{(R(\theta),\psi(\theta))\}_{\theta_{i-1,i}<\theta<\theta_{i,i+1}}$ is given by

$$\{(R,\psi) \mid \psi_1^+ \leqslant \psi \leqslant \pi,\ 0 < R < r_2(a\sigma_{0,1}\sin\psi_1^-), \quad K(R) = a\sigma_{0,1}\sin\psi\},$$
$$\{(R,\psi) \mid 0 < \psi \leqslant \pi - \psi_n^-,\ 0 < R < r_2(a\sigma_{0,n}\sin\psi_n^-), \quad K(R) = a\sigma_{0,n}\sin\psi\},$$

respectively. Then the corresponding solution satisfies, using $d\theta = f(c,R)|dR|$,

$$\sum_{i=1}^n \Delta\theta_i \quad = \quad Q_{II}^{i_*}(a),$$

where

$$\begin{aligned} Q_{II}^i(x) &:= Q_I(x) + \Delta_i(x), \\ \Delta_i(x) &:= \int_{r_1(x\sigma_{0,i}\sin\psi_i^-)}^{r_2(x\sigma_{0,i}\sin\psi_i^-)} f(x\sigma_{0,i},R)dR + \int_{r_1(x\sigma_{0,i-1}\sin\psi_{i-1}^+)}^{r_2(x\sigma_{0,i-1}\sin\psi_{i-1}^+)} f(x\sigma_{0,i-1},R)dR \end{aligned}$$

with $f(c,R)$ being defined as in (4.4). Thus, under (4.5), there exists a type II solution if $Q_{II}^{i_*}(a) = 2\pi$ for some $a > a_*$.

It is easy to see that

$$\Delta_{i_*}(a_*) = 0, \qquad \Delta_i(\infty) = \psi_i^- + \psi_{i-1}^+ \quad \forall\, i = 1,\cdots,n.$$

It then follows that

$$Q_{II}^{i_*}(\infty) = \psi_{i_*}^- + \psi_{i_*-1}^+ + \sum_{i=1}^n \psi_{i,i+1} = \pi + \sum_{i\neq i_*}\psi_{i-1,i}.$$

Thus, if $Q_{II}^{i_*}(\infty) > 2\pi$, there exists at least one $a > a_*$ such that $Q_{II}^{i_*}(a) = 2\pi$, and we have a solution to (4.1). In conclusion we have the following result.

**Proposition 4.2.** *Assume* (4.2), (4.5), *and* $Q_I(a_*) < 2\pi$. *Also assume that*

$$\sum_{i\neq i_*}\psi_{i-1,i} > \pi,$$

*where* $i_*$ *is as in* (4.6). *Then* (4.1) *admits a type II solution.*

4.5. **Type III solutions.** Assume (4.2) and (4.5). We consider a type III solution such that for a particular $j \neq i^*$, there holds $\min\{R_{i_*-1,i_*}, R_{j-1,j}\} > 1 \geqslant R_{i-1,i}$ for $i \neq i^*, i \neq j$. Assume that $\psi_j^- \leqslant \pi/2 \leqslant \pi - \psi_{j-1}^+$. Then the period of the solution is

$$Q_{III}(a) := Q_I(a) + \Delta_j(a) + \Delta_{i_*}(a).$$

Note that

$$\begin{aligned} Q_{III}(a^*) &= Q_{II}^j(a^*), \\ Q_{III}(\infty) &= Q_I(\infty) + [\psi_{i_*-1}^+ + \psi_{i_*}^-] + [\psi_{j-1}^+ + \psi_j^+] = 2\pi + \sum_{i \neq i_*, j} \psi_{i-1,i}. \end{aligned}$$

Hence, if $Q_{II}^j(a^*) < 2\pi$ and $n \geqslant 3$, there exists an $a > a_*$ such that $Q_{III}(a) = 2\pi$ and we have a type III solution. We summarize the result as follows.

**Proposition 4.3.** *Assume* (4.2) *and* (4.5)*. Let $i_*$ be as in* (4.6)*. Let $j \in \{1, \cdots, n\} \setminus \{i_*\}$ be an integer such that $\psi_j^- \leqslant \pi/2 \leqslant \pi - \psi_{j-1}^+$. Assume that $Q_{II}^j(a_*) < 2\pi$ and $n \geqslant 3$. Then there is a type III solution.*

4.6. **Isotropic case.** Finally we consider a special case where all interfacial energy densities are identical; that is,

$$\psi_i^{\pm} = \psi_{i,i+1} = \frac{\pi}{3} \quad \forall i. \tag{4.7}$$

Then

$$\sum_{i=1}^{n} \psi_{i,i+1} = \frac{n\pi}{3}.$$

Hence, a necessary condition for the existence of a solution to **(PS)** is $n \leqslant 5$.

As derived earlier, if we have a solution, then on the phase plane, the trajectory is a subset of $\gamma(c)$ for some $c \geqslant 2/\sqrt{3}$. In addition,

$$K(R(\theta_{i,i+1})) = c\ \sin\frac{\pi}{3} = \frac{\sqrt{3}c}{2}.$$

For each $x \geqslant 2/\sqrt{3}$, we denote

$$\begin{aligned} \gamma_1(x) &= \{(R,\psi) \mid K(R) = x\ \sin\psi, \quad \frac{\pi}{3} < \psi < \frac{2\pi}{3}, R < 1\} \\ \gamma_2^+(x) &= \{(R,\psi) \mid K(R) = x\ \sin\psi, \quad \frac{2\pi}{3} < \psi < \pi\}, \\ \gamma_2^-(x) &= \{(R,\psi) \mid K(R) = x\ \sin\psi, \quad 0 < \psi < \frac{\pi}{3}\}, \\ \gamma_3(x) &= \{(R,\psi) \mid K(R) = x\ \sin\psi, \quad \frac{\pi}{3} < \psi < \frac{2\pi}{3}, R > 1\}. \end{aligned}$$

Note that $\gamma(x) = \gamma_1(x) + \overline{\gamma_2^+(x)} + \overline{\gamma_2^-(x)} + \gamma_3(x)$. Also, as a trajectory, the angle $\Delta\theta$ spent on $\gamma_1(x), \gamma_2^\pm(x)$ and $\gamma_3(x)$ are respectively given by

$$\begin{aligned} h_1(x) &:= \int_{\gamma_1(x)} \frac{d\psi}{\psi_\theta} = \int_{\pi/3}^{2\pi/3} \frac{d\psi}{1-[r_1(x\sin\psi)]^2}, \quad x \geq c_* := \frac{2}{\sqrt{3}}, \\ h_2(x) &:= \int_{\gamma_2^\pm(x)} \frac{dR}{R_\theta} = \int_{r_1(\sqrt{3}x/2)}^{r_2(\sqrt{3}x/2)} f(x,R)dR, \quad x \geq c_*, \\ h_3(x) &:= \int_{\gamma_1(x)} \frac{d\psi}{\psi_\theta} = \int_{\pi/3}^{2\pi/3} \frac{d\psi}{[r_2(x\sin\psi)]^2-1}, \quad x \geq c_*. \end{aligned}$$

The period of $\gamma(x)$ is

$$T(x) := h_1(x) + 2h_2(x) + h_3(x).$$

The functions $h_1, h_2, h_3$, and $T$ are displayed in Figure 8. We point out that

$$\lim_{c\to\infty} T(c) = \pi, \qquad \lim_{c\to\infty} h_1(c) = \frac{\pi}{3}, \qquad \lim_{c\to\infty} h_2(c) = \frac{\pi}{3}, \qquad \lim_{c\to\infty} h_3(c) = 0.$$

This implies that $h_2$ is not monotonic.

On phase plane, each $\gamma_i := \{(R(\theta), \psi(\theta))\}_{\theta_{i-1,i}<\theta<\theta_{i,i+1}}$ is one of the following:

(1) $\gamma_i = \gamma_1(c)$. In this case $R_{i-1,i} = R_{i,i+1} < 1$ and $\Delta\theta_i = h_1(c)$.
(2) $\gamma_i = \gamma_1(c) + \gamma_2^\pm(c)$. In this case, $\Delta\theta_i = h_1(c) + h_2(c)$ and either $R_{i-1,i} > 1 > R_{i,i+1}$ or $R_{i-1,i} < 1 < R_{i,i+1}$.
(3) $\gamma_i = \gamma_1(c) + \gamma_2^+(c) + \gamma_2^-(c)$. Then $R_{i-1,i} = R_{i,i+1} > 1$ and $\Delta\theta_i = h_1(c) + 2h_2(c)$.
(4) For some integer $m \geq 1$, $\gamma_i = m\gamma(c) + \gamma_1(c)$, or $\gamma_i = m\gamma(c) + \gamma_1(c) + \gamma_2^\pm(c)$, or $\gamma_i = (m+1)\gamma(c) - \gamma_3(c)$. In this case, $\Delta\theta_i = mT(c) + \hat{\Delta}\theta_i$, where $m$ is the number of loops and $\hat{\Delta}\theta_i$ is computed as in (1), (2), or (3). Since $T(c) > \pi$, we see that $m = 1$.

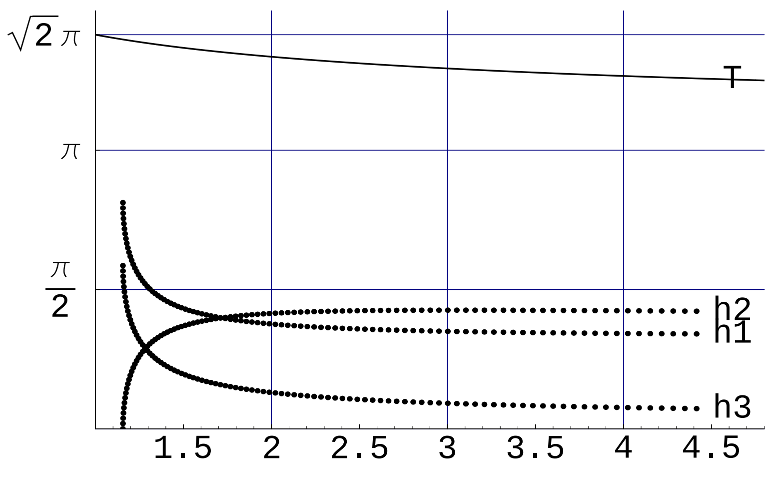


FIGURE 8. The functions $h_1(c), h_2(c), h_3(c)$ and $T(c)$.

(1) **Type I solution.** In this case, every $\{(R,\psi)\}_{\theta_{i,i-1}<\theta<\theta_{i,i+1}}$ is exactly $\gamma_1(c)$ for some $c \geqslant c_* = 2/\sqrt{3}$. As the total period is $nh_1(c)$, looking for a type I solution is equivalent to finding a constant $c$ from the algebraic equation

$$h_1(c) = \frac{2\pi}{n}.$$

Since $r_1' < 0$ in $[1,\infty)$ and $r_1(\infty) = 0$, we have

$$h_1'(x) < 0 \quad \forall x \in (c_*, \infty), \qquad h_1(\infty) = \frac{\pi}{3}.$$

Numerical evaluation gives

$$0.811\pi < h_1(c_*) < 0.812\pi.$$

Hence, when $n = 2$ or $n \geqslant 6$, there is no solution to $h_1(c) = 2\pi/n$. When $n = 3, 4, 5$, there exists a unique $c > c_*$ such that $h_1(c) = 2\pi/n$; indeed, numerical evaluation gives the following:

$$n = 3: \ c \approx 1.177; \qquad n = 4: \ c \approx 1.311; \qquad n = 5: \ c \approx 1.699.$$

The corresponding type I solutions to **(PS)** are depicted in Figure 9, (I3), (I4), (I5).

(2) **Type II solutions.** This renders to solve the equation, for $c \geqslant c_*$,

$$2\pi = nh_1(c) + 2h_2(c).$$

Numerical evaluation shows that there exists a solution if and only if $n = 3$ and $c \approx 1.659$; see Figure 9 (II3) for the corresponding type II solution to **(PS)**.

(3) **Type III solutions.** This renders to solve the equation

$$2\pi = nh_1(c) + 4h_2(c).$$

There exists a solution if and only if $n = 2$ and $c \approx 1.196$; see Figure 3 (III2) for the corresponding type III solution to **(PS)**.

(4) **Types IV solutions.**

Suppose $n \geqslant 3$ and there are at least three different $i$ such that $R(\theta_{i-1,i}) > 1$. Then the period is at least

$$3h_1(c) + 6h_2(c).$$

Since $\min_{c \geqslant c_*}\{3h_1(c) + 6h_2(c)\} > 2\pi$, there is no such kind of solutions. Similarly, one can also show that when $n \geqslant 3$, there is no type IV solutions.

Suppose $n = 2$. We seek a type IV solution.

If there is a type IV solution such that $R(0) > 1 > R(\theta_{1,2})$, then $\gamma_1 = m_1\gamma(c) + \gamma_1(c) + \gamma_2^+(c)$ and $\gamma_2 = m_2\gamma(c) + \gamma_1(c) + \gamma_2^-(c)$ for some non-negative integers $m_1$ and $m_2$ satisfying $m_1 + m_2 \geqslant 1$. We have such a solution if and only if

$$2\pi = (m_1 + m_2)T(c) + 2h_1(c) + 2h_2(c).$$

Since $\min_{c \geqslant c_*}\{T(c) + 2h_1(c) + 2h_2(c)\} > 2\pi$, there is no solution to this algebraic equation. Similarly, one can show that there is no type IV solution satisfying $\min\{R(0), R(\theta_{1,2})\} > 1$.

The remaining case is both $R(0)$ and $R(\theta_{1,2})$ are less than one. Then $\gamma_2 = m_1\gamma(c) + \gamma_1(c)$ and $\gamma_1 = m_2\gamma(c)+\gamma_1(c)$ where $m_1$ and $m_2$ are non-negative integers with $m_1+m_2 \geqslant 1$. This renders to solve

$$2\pi = (m_1 + m_2)T(c) + 2h_1(c).$$

This equation has a solution if and only if $m_1 + m_2 = 1$ and $c \approx 2.88$; see Figure 9 (IV2) for the corresponding type (IV) solutions to **(PS)**.

In conclusion, we have the following result.

**Theorem 3.** *Assume that $\psi_i^\pm = \psi_{i,i+1} = \pi/3$ for all $i$. Then there exists a self-similar shrinking solution if and only if $n \leqslant 5$. In addition, the following holds:*

(1) *When $n = 4$ and $n = 5$, solution to* **(PS)** *is unique and the solution satisfies $R(\theta) \in (0, 1)$, $\psi(\theta) \in [\frac{\pi}{3}, \frac{2\pi}{3}]$ for all $\theta \in [0, 2\pi]$.*
(2) *When $n = 3$,* **(PS)** *has exactly two solutions. One is of type I and the other is of type II.*
(3) *When $n = 2$,* **(PS)** *has exactly two solutions. One is of type III and the other is of type IV.*

All solutions to **(PS)** are depicted in Figure 9.

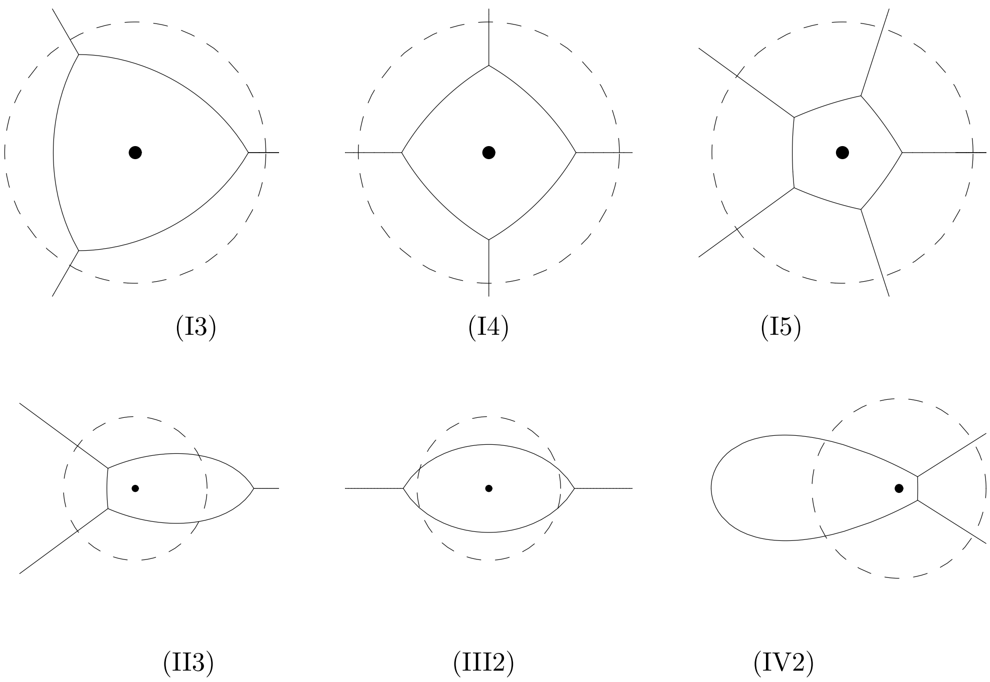


FIGURE 9. A classification of all self-similar shrinking solutions in the isotropic case. Dash-curve is the unit circle; (I3), (I4), (I5) are the type I solutions with $n = 3, 4, 5$ respectively; (II3) is the type II solution with $n = 3$; (III2) is the type III solution with $n = 2$; (IV2) is the type IV solution with $n = 2$ (the figure has been rotated so $\theta_{0,1} = 2\pi - \theta_{1,2} > 0$).

## References

[1] U. Abresch, J. Langer, *The normalized curve shortening flow and homothetic solutions*, J. Diff. Geometry **23** (1986), 175–196.

[2] B.L. Adams, S. Ta'asan, D. Kinderlehrer, I. Livshits, D.E. Mason, C.-T. Wu, W.W. Mullins, G.S. Rother, A.D.Rollett & D.M. Saylor, *Extracting grain boundary and surface energy measurement of triple junction geometry,* Interface Science **7** (1999), 321–338.

[3] B.L. Adams, D. Kinderlehrer, W.W. Mullins, A.D.Rollett & S. Ta'asan, *Extracting the relative grain boundary free energy and mobility function from the geometry of microstructures,* Scripta Materialia **38** (1998), 531–536.

[4] S. Angenent & M.E.Gurtin, *Multiphase thermomechanics with interfacial structure. 2. Evolution of an isothermal interface,* Arch. Rat. Mech. Anal. **108** (1989), 323–391.

[5] K.A. Brakke, The Motion of A Surface by Its Mean Curvature, Princeton Univ. Press, Princeton, 1978.

[6] L. Bronsard & F. Reitich, *On three-phase boundary motion and the singular limit of a vector valued Ginzburg-Landau equation,* Arch. Rat. Mech. Anal. **124** (1993), 355–379.

[7] Y.-L. Chang, J.-S. Guo & Y. Kohsaka, *On a two-point free boundary problem for a quasilinear parabolic equation*, Asymptotic Anal. **34** (2003), 333-358.

[8] Y.G. Chen, Y. Giga & S. Goto, *Uniqueness and existence of viscosity solution of generalized mean curvature flow equations,* J. Diff. Geom. **33** (1991), 749–786.

[9] L.C. Evans & J. Spruck, *Motion of level sets by mean curvature I,* J. Diff. Geom., **33** (1991), 635–681.

[10] L.C. Evans & J. Spruck, *Motion of level sets by mean curvature II,* Trans. Amer. Math. Soc. **330** (1992), 321–332.

[11] L.C. Evans & J. Spruck, *Motion of level sets by mean curvature III,* J. Geom. Anal. **2** (1992), 121–150.

[12] L.C. Evans & J. Spruck, *Motion of level sets by mean curvature IV,* J. Geom. Anal. **5** (1995), 77–114.

[13] M. Gage, *Curve shortening makes convex curves circular*, Invent. Math. **76** (1984), 357–364.

[14] M. Gage & R.S. Hamilton, *The shrinking of convex plane curves by the heat equation,* J. Diff. Geom. **26** (1987), 69–96.

[15] Y. Giga, Surface evolution equations-A level set approach, Birkhäuser, (2006).

[16] M.-H. Giga, Y. Giga & H. Hontani, *Self-similar expanding solutions in a sector for a crystalline flow*, SIAM J. Math. Anal. **37** (2005), 1207–1226.

[17] M.A. Grayson, *The heat equation shrinks embedded plane curves to round points,* J. Diff. Geom. **26** (1987), 285–314.

[18] J.-S. Guo & B. Hu, *On a two-point free boundary problem*, Quarterly Appl. Math. **64** (2006), 413–431.

[19] J.-S. Guo & Y. Kohsaka, *Self-similar solutions of two-point free boundary problem for heat equation*, RIMS Kokyuroku 1258, RIMS,Kyoto University, April, 2002, pp.94-107.

[20] M. Gurtin, Thermomechanics of evolving phase boundaries in the plane, Oxford Science Publication, London, 1993.

[21] R.S. Hamilton, *Four manifolds with positive ricci curvature*, J. Diff. Geom. **24** (1986), 153–179.

[22] R.S. Hamilton, *Harnack estimate for the mean curvature flow,* J. Diff. Geom. **41** (1995), 215–226.

[23] R.S. Hamilton, *Isoparametric estimates for the curve shrinking flow in the plane,* Modern Methods in Complex Analysis, Princeton University Press, NJ, 1995, 201–222.

[24] C. Herring, *Surface tension as motivation for sintering*, The Physica of Powder Metallury, (W. Kingston ed.), McGraw-Hill, New York, 1951.

[25] C. Herring, *The use of classical macroscopic concepts in surface-energy problems,* Structure and Properties of Solid Surfaces, (R. Gomer & S. Smith eds), U. Chicago Press, Chicago, 1952.

[26] G. Huisken, *Flow by mean curvature of convex surfaces into spheres,* J. Diff. Geom. **20** (1984), 237–266.

[27] G. Huisken, *Asymptotic behavior of singularities of the mean curvature flow,* J. Diff. Geom. **31** (1990), 285–299.

[28] G. Huisken, *A distance comparison principle for evolving curves,* Asian J. Math. **2** (1998), 127–133.

[29] T. Ilmanen, Elliptic Regularization and Partial Regularity for Motion by Mean Curvature, Mem. Amer. Math. Soc. vol **108** (520), AMS, 1994.

[30] D. Kinderlehrer & C. Liu, *Evolution of grain boundaries*, Math. Models Methods Appl. Sci. **11** (2001), 713–729.

[31] D. Kinderlehrer, I. Livshits, F. Manolache, A.D. Rollett & S. Ta'asan, *An approach to the mesoscale simulation of grain growth,* in Influences of interface and dislocation behavior on microstructure evolution (Aindow, M. et al., eds), Mat. Res. Soc. Symp. Proc. **652**, 2001.

[32] C. Mantegazza, M. Novaga & V.M. Tortorelli, *Motion by curvature of planar networks,* Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **3** (2004), 235–324.

[33] W. Mullins, *Solid surface morphologies governed by capillarity*, Metal Surface: Structure, Energetics and Kinetics, Am Soc. Metals, 17–66, 1963.

[34] W. Mullins, *Two dimensional motion of idealized grain boundaries,* J. Appl. Phys. **27** (1956), 900-904.

[35] W. Mullins, *On idealized two-dimensional grain growth,* Scripta Metall **22** (1988), 1441-1444.

[36] H.M. Soner, *Motion of a set by the mean curvature of its boundary,* J. Diff. Eqns. **101** (1993), 313-372

[37] A. Sutton & R. Baluffi, Interface in Crystalline Materials, Oxford Science Publication, London, 1995.

[38] J.E. Taylor, *The motion of multiple-phase junctions under prescribed phase-boundary velocities*, J. Diff. Eqns. **119** (1995), 109–136.

[39] J.E. Taylor, *Mean curvature and weighted mean curvature,* Acta Metall. Mater. **40** (1992), 1475–1485.

[40] J.E. Taylor & J.W. Cahn, *Linking anisotropic sharp and diffuse surface motion laws via gradient flow,* J. Stat. Phys. **77** (1994), 183–197.

[41] J.E. Taylor, J.W. Cahn & C.A. Handwerker, *Geometric models of crystal growth,* Acta Metall. Mater. **40** (1992) 1443–1474.

[42] D. Woodruff, The Solid-Liquid Interface, Cambridge University Press, 1973.

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