TRAVELING WAVE SOLUTION FOR A LATTICE DYNAMICAL SYSTEM WITH CONVOLUTION TYPE NONLINEARITY

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ABSTRACT. We study traveling wave solutions for a lattice dynamical system with convolution type nonlinearity. We consider the monostable case and discuss the asymptotic behaviors, monotonicity and uniqueness of traveling wave. First, we characterize the asymptotic behavior of wave profile at both wave tails. Next, we prove that any wave profile is strictly decreasing. Finally, we prove the uniqueness (up to translation) of wave profile for each given admissible wave speed.

1. INTRODUCTION

In this paper, we study the following lattice dynamical system (LDS) of convolution type:

$$u_j' = D(u_{j+1} + u_{j-1} - 2u_j) - du_j + \sum_{i\in\mathbb{Z}} J(i)b(u_{j-i}), \quad j \in \mathbb{Z}, \tag{1.1}$$

where $u_j = u_j(t)$, $D, d > 0$,

$$J(i) = J(-i) \geq 0, \quad \sum_{i\in\mathbb{Z}} J(i) = 1.$$

Hereafter the prime denotes the derivative with respect to the independent variable. In this paper, we shall always assume that b is an increasing Lipschitz continuous function on $[0, 1]$ such that

$$b(u) > du \quad \text{if } 0 < u < 1, \quad b(0) = b(1) - d = 0. \tag{1.2}$$

The system (1.1) can be thought as the spatial discrete version of the following nonlocal partial differential equation:

$$u_t = u_{xx} - du + J \star b(u), \quad [J \star b(u)](x,t) := \int_{-\infty}^{\infty} J(y)b(u(x-y,t))dy. \tag{1.3}$$

In ecology, u represents the population density, d is the death rate and the nonlinear function b is the birth function of population density which is interacting with neighbors by the

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nonnegative weighted function J. When J is the Dirac function, the equation (1.3) is reduced to the standard PDE:

$$u_t = u_{xx} + f(u), \quad f(u) := -du + b(u). \tag{1.4}$$

On the other hand, when the habitat is divided into discrete regions and the population density is measured at one point (e.g., center) in each region, then (1.3) is reduced to the system (1.1) in which the index j stands for the jth site in spatial domain.

We are interested in the traveling wave solutions of (1.1). We say that $\{u_j\}$ is a traveling wave solution of (1.1) with speed c if $u_j(t) = U(j - ct)$ for $j \in \mathbb{Z}$ and $t \in \mathbb{R}$ for some function U (called *wave profile*). Then (c, U) satisfies the following equation

$$cU'(x) + \mathcal{D}_2[U](x) - dU(x) + \sum_{i\in\mathbb{Z}} J(i)b(U(x-i)) = 0, \quad x \in \mathbb{R},$$

where

$$\mathcal{D}_2[U](x) := D[U(x+1) + U(x-1) - 2U(x)].$$

The spatial discrete version of the equation (1.4) has been studied very extensively for more general function f (cf. e.g., [2, 5, 6, 7, 11, 12, 19, 20, 21]). A related equation of convolution type is the following equation:

$$v_t = J \star v - v + f(v) \tag{1.5}$$

with J compactly supported and satisfying

$$J(-x) = J(x) \geq 0, \quad \int_{\mathbb{R}} J(y)dy = 1,$$

and f monostable. Schumacher [17] has derived the existence of traveling wave solution for the equation (1.5). Here a solution v is a traveling wave solution with speed c if $v(x,t) = V(x - ct)$ for some (wave profile) V. Carr and Chmaj [3] have obtained the uniqueness of traveling wave solution for (1.5). Moreover, the asymptotic behavior of traveling fronts and entire solutions of (1.5) are studied by Lv [13]. See also the works by Coville and Dupaigne [8, 9, 10]. For the same equation with bistable nonlinearity f, we refer the reader to [1]. We also refer to [2, 14, 15] for the corresponding discrete lattice case.

In this paper, we shall only study (1.1) with short range interaction so that $J(i) = 0$ for all $|i| \geq p$ with $p = 3$. Therefore, we shall study the following problem (P):

$$cU'(x) + \mathcal{D}_2[U](x) - dU(x) + \sum_{i=-2}^{2} J(i)b(U(x-i)) = 0, \quad x \in \mathbb{R}, \tag{1.6}$$

$$U(-\infty) = 1, \quad U(+\infty) = 0, \quad 0 \leq U(\cdot) \leq 1 \text{ on } \mathbb{R}, \tag{1.7}$$

where $J(i) = J(-i) \geq 0$ for $i \in \{0, 1, 2\}$ and $\sum_{i=-2}^{2} J(i) = 1$.

Let (c, U) be a solution of (1.6)-(1.7). By integrating (1.6) from $-a$ to a with $a \in (0, \infty)$ and letting $a \to \infty$, we obtain that

$$c = \int_{-\infty}^{\infty} [b(U(x)) - dU(x)]dx > 0.$$

Moreover, we have $0 < U(x) < 1$ for all $x \in \mathbb{R}$. Indeed, if $U(x_0) = 0$ for some $x_0 \in \mathbb{R}$, then using (1.6) and $U \geq 0$, by induction, we have $U(x_0 + n) = 0$ for all $n \in \mathbb{Z}$. This contradicts the boundary condition $U(-\infty) = 1$. Hence $U > 0$ in $\mathbb{R}$. Similarly, we have $U < 1$ in $\mathbb{R}$.

For the existence of traveling wave of (1.1), it is already well-studied in [16, 18] for more general settings, including with time delay. In particular, under the assumptions

(**H1**) b is differentiable at 0 and 1 such that $b'(1) < d < b'(0)$.

(**H2**) b is differentiable at 0 and there exist constants $M > 0, \alpha \in (0, 1]$ such that

$$b'(0)u - Mu^{1+\alpha} \leq b(u) \leq b'(0)u + Mu^{1+\alpha}, \quad \text{if} \quad u \in [0, 1],$$

by the result of [16] (see also [18]), we have

(i) There exists a positive constant $c_{\min}$ such that (P) admits a strictly decreasing solution if and only if $c \geq c_{\min}$. Moreover, if we assume the extra condition that

$$b(u) \leq b'(0)u \quad \text{for all } u \in [0, 1],$$

then we have $c_{\min} = c_*$, where

$$c_* := \inf_{r>0} \frac{1}{r} \left[D(e^r + e^{-r} - 2) - d + b'(0) \sum_{i=-2}^{2} J(i)e^{ir} \right].$$

(ii) For each $c > c_{\min}$, the traveling wave is unique (up to a translation) under the additional condition

$$\limsup_{x \to \infty} [U(x)e^{-\Lambda(c)x}] < \infty, \tag{1.8}$$

where $\Lambda(c)$ be the larger (negative) root of the following characteristic equation

$$\Phi(\lambda; c) := c\lambda + D(e^{\lambda} + e^{-\lambda} - 2) + b'(0) \sum_{i=-2}^{2} J(i)e^{i\lambda} - d = 0. \tag{1.9}$$

Indeed, the method of [16] is by investigating the asymptotic speed of propagation, and it works also for the case of infinite range interaction. Note that we can follow a (direct) method developed in [7] to derive the existence of solutions of (P) for any finite range interaction.

The main purpose of this paper is to investigate the asymptotic behavior of wave tails, the monotonicity of wave profiles and the uniqueness without the assumption (1.8).

Now, we list the main theorems of this paper as follows.

First, in order to study the asymptotic behavior of wave tails, we need to study the following equation

$$ar(x) + \sum_{i=-2}^{2} a_{|i|} \exp\left[\int_x^{x+i} r(y)dy\right] = 0, \tag{1.10}$$

where $a \neq 0, a_2 > 0, a_1 > 0, a_0 \in \mathbb{R}$. By using a method of [7], we have

Theorem 1. *Let $a \neq 0, a_2 > 0, a_1 > 0, a_0 \in \mathbb{R}$, and*

$$P(a, a_2, a_1, a_0, \lambda) := \lambda a + a_2 e^{2\lambda} + a_1 e^{\lambda} + a_0 + a_1 e^{-\lambda} + a_2 e^{-2\lambda}.$$

Then

(i) *If $P(a, a_2, a_1, a_0, \lambda) = 0$ has no roots, then* (1.10) *has no solutions.*

(ii) *If $P(a, a_2, a_1, a_0, \lambda) = 0$ has only one root Λ^*, then* (1.10) *has only the trivial solution $r(\cdot) \equiv \Lambda^*$.*

(iii) *If $P(a, a_2, a_1, a_0, \lambda) = 0$ has two real roots $\{\Lambda_1, \Lambda_2\}$ with $\Lambda_1 < \Lambda_2$, then all solutions of* (1.10) *are formed as*

$$r(x) = \frac{\theta \Lambda_1 e^{\Lambda_1 x} + (1-\theta)\Lambda_2 e^{\Lambda_2 x}}{\theta e^{\Lambda_1 x} + (1-\theta) e^{\Lambda_2 x}}, \quad \theta \in [0, 1].$$

Next, the asymptotic behaviors of wave profiles near both tails are given as follows.

Theorem 2. *Assume* (**H1**) *and let (c, U) be an arbitrary solution of* (P). *Then there exist constants $\Lambda = \Lambda(c), \sigma = \sigma(c)$ with $\Lambda(c) < 0 < \sigma(c)$ such that*

$$\lim_{x\to\infty} [U'(x)/U(x)] = \Lambda(c), \quad \lim_{x\to-\infty} [U'(x)/(U(x) - 1)] = \sigma(c),$$

where $\Lambda(c)$ is a root of the characteristic equation $\Phi(\lambda; c) = 0$ defined in (1.9) *and $\sigma(c)$ be the unique positive root of the following characteristic equation*

$$\Psi(\sigma; c) := c\sigma + D(e^{\sigma} + e^{-\sigma} - 2) + b'(1) \sum_{i=-2}^{2} J(i) e^{i\sigma} - d = 0. \tag{1.11}$$

Note that Theorem 2 also implies that $c \geq c_*$ for any solution (c, U) of (P). In particular, we always have $c_{\min} \geq c_*$. With this asymptotic behavior, we can derive the monotonicity of wave profile as follows.

Theorem 3. *Assume* (**H1**) *and let (c, U) be an arbitrary solution of* (P). *Then $U'(x) < 0$ for $x \in \mathbb{R}$.*

Moreover, combining this monotonicity property with an idea from [7], we can determine the tail behavior at $x = \infty$ more precisely as follows.

Theorem 4. *The limit $\Lambda(c)$ in Theorem 2 is the larger (negative) root of the characteristic equation $\Phi(\lambda; c) = 0$ for $c > c_{\min}$.*

Note that when $c = c_*$ there is the unique double root of $\Phi(\lambda; c) = 0$.

Finally, we prove the following theorem by using an idea from [5].

Theorem 5. *Assume* (**H1**) *and* (**H2**)*. Let* (c, U) *be an arbitrary solution of* (P) *with* $c > c_{\min}$*. Then the limit* $\lim_{x\to\infty}[U(x)e^{-\Lambda(c)x}]$ *exists and is (finite) positive, where* $\Lambda(c)$ *is the bigger root of the characteristic equation* $\Phi(\lambda; c) = 0$*.*

Combining this theorem with the (partial) uniqueness result of [16], we conclude that the wave profile is unique (up to a translation) for each given admissible wave speed. Notice that (1.8) is not needed in our uniqueness result. It is automatic satisfied for each wave profile.

We organize this paper as follows. First, some preliminaries and the proof of Theorem 1 are given in section 2. Then we study the asymptotic behavior of the tails of wave profile in section 3. In section 4, we prove Theorem 3 (the monotonicity of wave profiles) and Theorem 4. Finally, we derive the uniqueness of wave profiles in section 5. The main idea and method of proofs of this paper are from [5, 7]. For the reader's convenience, we provide some details of proofs for completeness. But, due to the convolution term some difficulties are presented. In particular, the proof of Proposition 2.3 is highly nontrivial comparing with the case treated in [7]. We remark that our results can be extended to any finite range interaction if the key proposition (Proposition 2.3 below) can be extended to general positive integer p. We left it as an open question.

2. Preliminaries

In this section, we shall give some preliminaries for the asymptotic behavior of wave profiles near both wave tails. Also, we shall give the proof of Theorem 1.

Lemma 2.1. *Let* (c, U) *be a solution of* (P)*. Then there exists a positive constant* K *such that*

$$\sup_{x\in\mathbb{R},|s|\le 1}\frac{U(x+s)}{U(x)}+\sup_{x\in\mathbb{R}}\frac{|\ U'(x)\ |}{U(x)}\le K.$$

Proof. Given a solution (c, U) of (P). Since

$$cU'(x) - (2D + d)U(x) \le 0,$$

we obtain that $U'(x) \le \mu U(x)$ for $\mu \ge (2D + d)/c$. This implies that $e^{-\mu x}U(x)$ is a non-increasing function. Therefore, we have

$$U(x+s) \le U(x)e^{\mu s} \le U(x)e^{\mu}, \quad \text{if} \quad x \in \mathbb{R}, 0 \le s \le 1. \tag{2.1}$$

So we obtain that

$$\sup_{x\in\mathbb{R},0\le s\le 1}\frac{U(x+s)}{U(x)} \le e^{\mu}. \tag{2.2}$$

Next, we focus on the case that $x \in \mathbb{R}$ and $-1 \le s \le 0$. By integrating (1.6) over $[x, n]$, we get

$$\begin{aligned} & c[U(n) - U(x)] + D \int_x^n [U(y+1) + U(y-1) - 2U(y)]dy \\ + \quad & d \sum_{i=-2}^{2} J(i) \int_x^n [U(y-i) - U(y)]dy + \sum_{i=-2}^{2} J(i) \int_x^n [b(U(y-i)) - dU(y-i)]dy = 0. \end{aligned}$$

Since $J(i) = J(-i)$ and $b(s) \ge ds$ if $s \in [0,1]$, by taking $n \to \infty$, we have

$$-cU(x) - D \int_x^{x+1} [U(y) - U(y-1)]dy - d \sum_{i=1}^{2} J(i) \int_x^{x+i} [U(y) - U(y-i)]dy \le 0.$$

Since $U \ge 0$, we have

$$cU(x) \ge [D + dJ(1) + dJ(2)] \int_x^{x+1/2} U(y-1)dy - [D + dJ(1) + dJ(2)] \int_x^{x+2} U(y)dy.$$

By (2.1), we have

$$\begin{aligned} & \int_x^{x+1/2} U(y-1)dy \ge \int_{x-1}^{x-1/2} U(x-1/2)e^{-\mu/2}dy = \frac{1}{2}e^{-\mu/2}U(x-1/2), \\ & \int_x^{x+2} U(y)dy \le \int_x^{x+2} U(x)e^{2\mu}dy = 2e^{2\mu}U(x). \end{aligned}$$

Hence

$$cU(x) \ge \frac{1}{2}e^{-\mu/2}[D + dJ(1) + dJ(2)]U(x-1/2) - 2e^{2\mu}[D + dJ(1) + dJ(2)]U(x).$$

This implies that $[U(x-1/2)/U(x)]$ is bounded uniformly for $x \in \mathbb{R}$. Combining with (2.2), we conclude that

$$\sup_{x \in \mathbb{R}, |s| \le 1} \frac{U(x+s)}{U(x)} \le M_1$$

for some positive constant $M_1 \in \mathbb{R}$.

Moreover, dividing (1.6) by $U(x)$ and using the Lipschitz continuity of b, we obtain that $\sup_{x \in \mathbb{R}} |U'(x)/U(x)| \le M_2$ for some constant M_2. Therefore, the lemma is proved. □

Lemma 2.2. *Let* (c, U) *be a solution of* (P). *Then there exists a positive constant* K *such that*

$$\sup_{x \in \mathbb{R}, |s| \le 1} \frac{1 - U(x+s)}{1 - U(x)} + \sup_{x \in \mathbb{R}} \frac{|U'(x)|}{1 - U(x)} \le K.$$

Proof. First, we define $V(x) = 1 - U(x)$. Then (1.6) can be re-written as

$$cV'(x) + \mathcal{D}_2[V](x) - dV(x) + \sum_{i=-2}^{2} J(i)[b(1) - b(1 - V(x-i))] = 0. \tag{2.3}$$

Following the same method as that of Lemma 2.1, we can get

$$\sup_{x\in\mathbb{R},0\le s\le 1}\frac{V(x+s)}{V(x)}\le e^{\mu}$$

for some positive constant μ,

Secondly, since $V(-\infty)=1-U(-\infty)=0$ and $0<V(\cdot)<1$ on $\mathbb{R}$, the quantity

$$K(x):=\max_{s\ge 0}\frac{V(x-s)}{V(x)}$$

is well-defined for each $x\in\mathbb{R}$. We claim that $K(x)<e^{2\mu}$ for all $x\in\mathbb{R}$. Suppose not, then we can find $x_1\in\mathbb{R}$ and $s_0>0$ such that $V(x_1-s_0)\ge V(x_1)e^{2\mu}$. Let $y\in(-\infty,x_1)$ be the smallest value attained $\max_{(-\infty,x_1)}V(\cdot)$. So we have

$$V'(y)=0,\ \max\{V(y-1),V(y-2)\}<V(y).$$

On the other hand, since

$$\max_{[x_1,x_1+2]}V(\cdot)\le V(x_1)e^{2\mu}\le V(x_1-s_0)\le V(y),$$

we have $V(y+i)\le V(y)$ for $i=1,2$. Hence, by (1.2),

$$\begin{aligned}
&cV'(y)+D[V(y+1)+V(y-1)-2V(y)]-dV(y)+\sum_{i=-2}^{2}J(i)[b(1)-b(1-V(y-i))]\\
\le\ & D[V(y+1)+V(y-1)]-(2D+d)V(y)+d-d(1-V(y))\\
\le\ & D[V(y-1)-V(y)]<0.
\end{aligned}$$

This contradicts (2.3). So we have proved that $K(x)<e^{2\mu}$ for all $x\in\mathbb{R}$. This implies that

$$\sup_{x\in\mathbb{R},|s|\le 1}\frac{V(x+s)}{V(x)}\le e^{2\mu}.$$

Finally, dividing (2.3) by $V(x)$, it is easy to see that

$$\sup_{x\in\mathbb{R}}\frac{|V'(x)|}{V(x)}\le K$$

for some positive constant K. Hence the lemma follows. □

We shall follow the method of [7] to prove Theorem 1. In the course of proof, we need to analyze the following recurrence equation

$$a_{n+3}a_{n+2}+l_1a_{n+2}+\frac{l_1}{a_{n+1}}+\frac{1}{a_{n+1}a_n}=l,\quad n\in\mathbb{Z}, \tag{2.4}$$

where l_1,l are positive constants. Recall that, for the case treated in [7], the recurrence equation is given by

$$a_{n+1}+\frac{1}{a_n}=l,\quad n\in\mathbb{Z},$$

for some positive constant l. It is easy to deduce the monotonicity of a_n, and we can easily obtain the convergence of a_n. But, the convergence of the sequence $\{a_n\}$ defined by (2.4) is not trivial.

By taking two consecutive equations from (2.4), we have

$$\begin{aligned} a_{n+2}(a_{n+3}-a_{n+1}) + l_1(a_{n+2}-a_{n+1}) \\ + \frac{l_1}{a_{n+1}a_n}(a_n - a_{n+1}) + \frac{1}{a_{n+1}a_n a_{n-1}}(a_{n-1}-a_{n+1}) = 0, \quad n \in \mathbb{Z}. \end{aligned} \tag{2.5}$$

Moreover, we have

(i) If $\{a_n\}_{n=-\infty}^{\infty}$ is a positive sequence satisfying (2.4), then $\{a_n\}_{n=-\infty}^{\infty}$ is a bounded sequence and (2.5) holds.
(ii) If $\{a_n\}_{n=-\infty}^{\infty}$ is a positive sequence satisfying (2.4) and any consecutive four terms are equal, then $\{a_n\}_{n=-\infty}^{\infty}$ is a constant sequence.

In the sequel, we shall prove the following very technical proposition.

Proposition 2.3. *If $\{a_n\}_{n=-\infty}^{\infty}$ is a positive sequence satisfying* (2.4)*, then both* $\lim_{n\to\infty} a_n$ *and* $\lim_{n\to-\infty} a_n$ *exist.*

To prove this proposition, we define

$$M_n := \max\{a_{n-2}, a_{n-1}, a_n, a_{n+1}, a_{n+2}\}, \ m_n := \min\{a_{n-2}, a_{n-1}, a_n, a_{n+1}, a_{n+2}\}.$$

Lemma 2.4. *Let $\{a_n\}_{n=-\infty}^{\infty}$ be a non-constant positive sequence satisfying* (2.4)*. Then*

$$m_n < a_n < M_n, \quad \forall\, n \in \mathbb{Z}.$$

Proof. By the definitions of M_n and m_n, we have $m_n \le a_n \le M_n$ for all $n \in \mathbb{Z}$. Suppose that there exists $k \in \mathbb{Z}$ such that $a_k = M_k$. This implies that

$$a_k \ge a_{k+2}, a_k \ge a_{k+1}, a_k \ge a_{k-1}, a_k \ge a_{k-2}.$$

By (2.5) with $n = k-1$, we have

$$a_{k-2} = a_{k-1} = a_k = a_{k+1} = a_{k+2}.$$

Therefore, $\{a_n\}_{n=-\infty}^{\infty}$ is a constant sequence. We have reached a contradiction.

The case that there exists $k \in \mathbb{Z}$ such that $a_k = m_k$ can be treated similarly. Therefore, we have proved the lemma. □

By Lemma 2.4, we have

$$M_n = \max\{a_{n-2}, a_{n-1}, a_{n+1}, a_{n+2}\}, \quad m_n = \min\{a_{n-2}, a_{n-1}, a_{n+1}, a_{n+2}\}, \ \forall n \in \mathbb{Z}.$$

Indeed, in the next lemma, we have more precise information.

Lemma 2.5. *Let $\{a_n\}_{n=-\infty}^{\infty}$ be a non-constant positive sequence satisfying* (2.4)*. Then, for each integer $k \in \mathbb{Z}$, we have*

(1) $\min\{a_{k-2}, a_{k-1}\} = m_k < \min\{a_{k+1}, a_{k+2}\}$, *if* $M_k = \max\{a_{k+1}, a_{k+2}\}$;

(2) $\min\{a_{k+2}, a_{k+1}\} = m_k < \min\{a_{k-1}, a_{k-2}\}$, *if* $M_k = \max\{a_{k-1}, a_{k-2}\}$.

Moreover, either $M_k = \max\{a_{k+1}, a_{k+2}\}$ *or* $M_k = \max\{a_{k-1}, a_{k-2}\}$, *and they are exclusive.*

Proof. Given any integer k. Suppose that $M_k = \max\{a_{k+1}, a_{k+2}\}$. We claim that $m_k < \min\{a_{k+1}, a_{k+2}\}$. Suppose not. Then we have $m_k = \min\{a_{k+1}, a_{k+2}\}$. Since a_n is a non-constant sequence, we have $M_k > m_k$. This implies $a_{k+1} \neq a_{k+2}$. Without loss of generality, we may assume $a_{k+1} > a_{k+2}$. So $M_k = a_{k+1}$ and $m_k = a_{k+2}$. By Lemma 2.4, we get

$$a_{k+3} = M_{k+1} > a_{k+1},\ a_{k+4} = m_{k+2} < a_{k+2}.$$

Continuing in this way, we obtain that

$$\cdots < a_{k+6} < a_{k+4} < a_{k+2} < a_{k+1} < a_{k+3} < a_{k+5} < \cdots$$

Using (2.5) with $n = k + 2t$ for any positive integer t, we have

$$\begin{aligned}
& a_{n+2}(a_{n+3} - a_{n+1}) > l_1(a_{n+1} - a_{n+2}) \\
\Rightarrow\ & a_{n+2}(a_{n+3} - a_{n+2}) + a_{n+2}(a_{n+2} - a_{n+1}) > l_1(a_{n+1} - a_{n+2}) \\
\Rightarrow\ & a_{n+2}(a_{n+3} - a_{n+2}) > (a_{n+1} - a_{n+2})(l_1 + a_{n+2}) \\
\Rightarrow\ & a_{n+3} - a_{n+2} > (1 + \frac{l_1}{a_{n+2}})(a_{n+1} - a_{n+2}) > (1 + \frac{l_1}{a_{k+2}})(a_{n+1} - a_n)
\end{aligned}$$

Repeating the above process, we get that

$$a_{k+2t+1} - a_{k+2t} \geq (1 + \frac{l_1}{a_{k+2}})^{t-1}(a_{k+3} - a_{k+2})$$

for any positive integer t. This contradicts the boundedness of $\{a_n\}$. Therefore,

$$m_k < \min\{a_{k+1}, a_{k+2}\}.$$

By Lemma 2.4, we must have

$$m_k = \min\{a_{k-2}, a_{k-1}\}.$$

The case when $M_k = \max\{a_{k-1}, a_{k-2}\}$ is similar. Finally, it is clear that only one of the options can hold for each k. The proof is completed. □

Lemma 2.6. *Let* $\{a_n\}_{n=-\infty}^{\infty}$ *be a non-constant positive sequence satisfying* (2.4). *Then either* $M_n = \max\{a_{n+1}, a_{n+2}\}$, $m_n = \min\{a_{n-2}, a_{n-1}\}$ *for all* $n \in \mathbb{Z}$ *and both* $\{M_n\}$ *and* $\{m_n\}$ *are non-decreasing; or,* $M_n = \max\{a_{n-1}, a_{n-2}\}$, $m_n = \min\{a_{n+2}, a_{n+1}\}$ *for all* $n \in \mathbb{Z}$ *and both* $\{M_n\}$ *and* $\{m_n\}$ *are non-increasing.*

Proof. Suppose that there is an integer $k \in \mathbb{Z}$ such that $M_k = \max\{a_{k+1}, a_{k+2}\}$. By

$$\max\{a_{k+1}, a_{k+2}\} = M_k \geq \max\{a_{k-1}, a_k, a_{k+1}, a_{k+2}\}$$

and the definition of M_{k+1}, we have

$$M_{k+1} \geq \max\{a_{k+1}, a_{k+2}, a_{k+3}\} \geq \max\{a_{k-1}, a_k, a_{k+1}, a_{k+2}, a_{k+3}\} = M_{k+1}.$$

This and Lemma 2.4 imply that

$$M_{k+1} = \max\{a_{k+1}, a_{k+2}, a_{k+3}\} = \max\{a_{k+2}, a_{k+3}\}.$$

Next, we claim that $M_{k-1} = \max\{a_k, a_{k+1}\}$. Suppose not. Then from Lemma 2.5 we have $M_{k-1} = \max\{a_{k-3}, a_{k-2}\}$. Moreover, we have $m_{k-1} = \min\{a_k, a_{k+1}\}$. Since, by assumption, $m_k = \min\{a_{k-2}, a_{k-1}\}$, we have

$$m_k = \min\{a_{k-2}, a_{k-1}\} \geq m_{k-1} = \min\{a_k, a_{k+1}\} \geq m_k.$$

From this we deduce that $m_k = \min\{a_k, a_{k+1}\} = a_{k+1}$, by Lemma 2.4. This contradicts Lemma 2.5. Therefore, we conclude that $M_{k-1} = \max\{a_k, a_{k+1}\}$.

By induction and Lemma 2.5, we get

$$M_n = \max\{a_{n+1}, a_{n+2}\}, m_n = \min\{a_{n-2}, a_{n-1}\}, \ \forall n \in \mathbb{Z}.$$

Since

$$M_{n+1} \geq \max\{a_{n+1}, a_{n+2}\} = M_n, \quad m_{n+1} = \min\{a_{n-1}, a_n\} \geq m_n,$$

both $\{M_n\}$ and $\{m_n\}$ are non-decreasing sequences.

The other case is similar. Therefore, the lemma is proved by combining Lemma 2.5. □

Proof of Proposition 2.3. By Lemma 2.6, without loss of generality, we may assume that

$$M_n = \max\{a_{n+1}, a_{n+2}\} \text{ and } \ m_n = \min\{a_{n-2}, a_{n-1}\}, \ \forall n \in \mathbb{Z}.$$

So $\{M_n\}$ and $\{m_n\}$ are bounded and non-decreasing sequences. Therefore, $\lim_{n\to\pm\infty} M_n$ and $\lim_{n\to\pm\infty} m_n$ must exist.

First, we consider $\lim_{n\to\infty} a_n$. We define $M := \lim_{n\to\infty} M_n$ and $m := \lim_{n\to\infty} m_n$. If $M = m$, then $\lim_{n\to\infty} a_n$ exists (since $m_n < a_n < M_n$).

Suppose that $M > m$. By hypothesis, we have

$$M_n = \max\{a_{n+1}, a_{n+2}\}, \quad m_{n+3} = \min\{a_{n+1}, a_{n+2}\}.$$

This implies

$$M_n + m_{n+3} = a_{n+1} + a_{n+2}.$$

Since $\{M_n\}$ and $\{m_n\}$ are non-decreasing sequences,

$$a_{n+1} + a_{n+2} \leq a_{n+2} + a_{n+3}, \quad \forall n \in \mathbb{Z}.$$

Hence $\{a_{2n}\}$ and $\{a_{2n+1}\}$ are non-decreasing sequences. So the limits

$$p := \lim_{n\to\infty} a_{2n}, \quad q := \lim_{n\to\infty} a_{2n+1}$$

are well-defined. Note that, by (2.4),

$$a_{n+2} < l/l_1, \quad a_{n+1} > l_1/l$$

for all n. Hence $p, q \in (0, \infty)$. By (2.5) with n even and taking $n \to \infty$, we obtain

$$l_1(p-q) + \frac{l_1}{pq}(p-q) = 0.$$

This implies that $p = q$. Therefore, $\lim_{n\to\infty} a_n$ exists.

Similarly, we can prove that $\lim_{n\to-\infty} a_n$ exists. This completes the proof. □

Now, we turn to the study of (1.10). First, we have

Lemma 2.7. *If $r(\cdot) \in L^1_{loc}(\mathbb{R})$ solves (1.10) in $\mathbb{R}$, then $r(\cdot) \in L^\infty(\mathbb{R}) \cap C^\infty(\mathbb{R})$.*

Proof. First, we may assume $a > 0$. (If $a < 0$, then we may define $\widetilde{r}(x) = -r(-x)$). We define

$$v(x) := \exp\left[\mu x + \int_0^x r(y)dy\right]$$

with $\mu := a_0/a$. Since

$$-ar(x) = \sum_{i=-2}^{2} a_{|i|} e^{-i\mu} \frac{v(x+i)}{v(x)}, \tag{2.6}$$

we get

$$-av'(x) = v(x)[-ar(x) - a_0] = v(x) \sum_{i\in\{\pm2,\pm1\}} a_{|i|} \exp\left[\int_x^{x+i} r(y)dy\right] > 0. \tag{2.7}$$

This implies that $v'(x) < 0$ for all $x \in \mathbb{R}$ and so $v(\infty)$ exists and $v(\infty) \geq 0$. By integrating (2.7) over $[x, M]$, we get

$$av(x) - av(M) = \sum_{i\in\{\pm1,\pm2\}} a_{|i|} \int_x^M v(t) \exp\left[\int_t^{t+i} r(y)dy\right] dt = \sum_{i\in\{\pm1,\pm2\}} a_{|i|} e^{-i\mu} \int_x^M v(t+i)dt.$$

Sending $M \to \infty$, we get

$$av(x) - av(\infty) > a_1 \int_x^{x+1/2} e^{\mu} v(t-1)dt \geq \frac{a_1}{2} e^{\mu} v(x - 1/2).$$

Therefore, we obtain that

$$v(x - 1/2) \leq \frac{2a}{a_1} e^{-\mu} v(x), \ \forall x \in \mathbb{R}.$$

It follows from (2.6) and the fact that v is non-increasing, we conclude that $r(\cdot) \in L^\infty(\mathbb{R})$. Furthermore, $r(\cdot) \in C^\infty(\mathbb{R})$ by using (1.10). This proves the lemma. □

Lemma 2.8. *A locally integrable solution of (1.10) that attains its global maximum or minimum must be a constant function.*

Proof. Let r be a locally integrable solution of (1.10). By differentiating (1.10), we get

$$ar'(x) + \sum_{i=-2}^{2} a_{|i|} \exp\left[\int_x^{x+i} r(y)dy\right] \cdot [r(x+i) - r(x)] = 0.$$

If we define

$$A(x) := \exp\left[\int_x^{x+1} r(y)dy\right],$$

then the above equality can be re-written as

$$\begin{aligned} ar'(x) &+ a_2 A(x+1)A(x)[r(x+2) - r(x)] + a_1 A(x)[r(x+1) - r(x)] \\ &+ a_1[A(x-1)]^{-1}[r(x-1) - r(x)] + a_2[A(x-1)A(x-2)]^{-1}[r(x-2) - r(x)] = 0. \end{aligned} \tag{2.8}$$

Suppose that r attains its global maximum. Without loss of generality and by a translation, we may assume that $r(\cdot)$ attains its global maximum r^* at $x = 0$. Hence $r'(0) = 0$ and $r(\pm 2) = r(\pm 1) = r^*$. By induction, we easily deduce that $r(j) = r^*$ and $r'(j) = 0$ for all $j \in \mathbb{Z}$. By (1.10), we get

$$a_2 A(j+1)A(j) + a_1 A(j) + a_1/A(j-1) + a_2/[A(j-1)A(j-2)] = -ar^* - a_0$$

for all $j \in \mathbb{Z}$. Proposition 2.3 implies that $A^{\pm} := \lim_{j\to\pm\infty} A(j)$ exists such that

$$a_2(A^{\pm})^2 + a_1(A^{\pm}) + a_1/(A^{\pm}) + a_2/[(A^{\pm})^2] = -ar^* - a_0. \tag{2.9}$$

Now, let $\{x_i\}$ be a sequence with

$$\lim_{i\to\infty} r(x_i) = r_* := \inf_{x\in\mathbb{R}}\{r(x)\}.$$

We claim that $r^* = r_*$. By Lemma 2.7, $\{r(x_i + \cdot)\}_{i=1}^{\infty}$ is a uniformly bounded and equicontinuous sequence. According to Ascoli-Arzelà Theorem, we can extract a subsequence (still called $\{r(x_i + \cdot)\}_{i=1}^{\infty}$) such that $\lim_{i\to\infty} r(x_i + \cdot) = \hat{r}(\cdot)$ uniformly in any compact subset of $\mathbb{R}$ for some $\hat{r}(\cdot) \in C(\mathbb{R})$. It is easy to check that $\hat{r}(\cdot)$ is also a solution of (1.10) and $\hat{r}(0) = r_*$ is a global minimum of $\hat{r}(\cdot)$. A similar argument as above, we can get

$$-ar_* - a_0 = a_2\hat{A}(j+1)\hat{A}(j) + a_1\hat{A}(j) + \frac{a_1}{\hat{A}(j-1)} + \frac{a_2}{\hat{A}(j-1)\hat{A}(j-2)}, \quad \forall j \in \mathbb{Z}.$$

Moreover, we also have

$$-ar_* - a_0 = a_2(\hat{A}^{\pm})^2 + a_1(\hat{A}^{\pm}) + \frac{a_1}{\hat{A}^{\pm}} + \frac{a_2}{(\hat{A}^{\pm})^2}, \tag{2.10}$$

where $\hat{A}(x) := \exp[\int_x^{x+1} \hat{r}(y)dy]$ and $\hat{A}^{\pm} := \lim_{j\to\pm\infty} \hat{A}(j)$.

Finally, without loss of generality (by taking a subsequence if necessary), we may assume that $x_i \geq -M$ for some positive constant M. Since

$$\int_j^{j+1} r(y)dy = \ln[A(j)] \to \ln[A^+] \text{ as } j \to \infty,$$

we have

$$\frac{1}{n}\int_x^{x+n} r(y)dy \to \ln[A^+] \text{ as } n\to\infty \text{ uniformly in } x\in[-M,\infty).$$

Therefore

$$\begin{aligned}\ln[A^+] &= \lim_{i\to\infty}\lim_{n\to\infty}\frac{1}{n}\int_{x_i}^{x_i+n} r(y)dy = \lim_{n\to\infty}\lim_{i\to\infty}\frac{1}{n}\int_{x_i}^{x_i+n} r(y)dy\\ &= \lim_{n\to\infty}\frac{1}{n}\lim_{i\to\infty}\int_0^n r(x_i+y)dy = \lim_{n\to\infty}\frac{1}{n}\int_0^n \hat{r}(y)dy = \ln[\hat{A}^+].\end{aligned}$$

This implies that $A^+ = \hat{A}^+$. So by (2.9) and (2.10), we have $r^* = r_*$. Therefore $r(\cdot)$ is a constant function. The case when $r(\cdot)$ attains its global minimum can be also treated similarly. The lemma is proved. □

Lemma 2.9. *If $r(\cdot)\in L^1_{loc}(\mathbb{R})$ solves (1.10), then $r(\pm\infty) := \lim_{x\to\pm\infty} r(x)$ exists and*

$$P(a,a_2,a_1,a_0,r(\pm\infty)) = 0.$$

Proof. First, we consider $\{x_i\}$ such that

$$\lim_{i\to\infty} x_i = \infty, \quad \lim_{i\to\infty} r(x_i) = r^* := \limsup_{x\to\infty} r(x).$$

We claim that

$$\lim_{i\to\infty}\max_{[x_i-2,x_i+2]}\{| r(\cdot)-r^* |\} = 0. \tag{2.11}$$

Since the family $\{r(x_i+\cdot)\}_{i=1}^{\infty}$ is uniformly bounded and equi-continuous, we can choose a subsequence (still denoted by $\{r(x_i+\cdot)\}$) such that $\lim_{i\to\infty} r(x_i+\cdot) = \hat{r}(\cdot)$ uniformly in any compact subset of $\mathbb{R}$ for some $\hat{r}(\cdot)\in C(\mathbb{R})$. This implies that $\hat{r}(\cdot)$ is also a solution of (1.10). By $\hat{r}(0) = \lim_{i\to\infty} r(x_i) = r^*$ and $\hat{r}(y) = \lim_{i\to\infty} r(x_i+y) \le r^*$ for all $y\in\mathbb{R}$, we have $\hat{r}(0) = \max_{x\in\mathbb{R}}\{\hat{r}(x)\}$. By Lemma 2.8, we get $\hat{r}(\cdot)\equiv r^*$. Then (2.11) follows form the uniform convergence of $\{r(x_i+\cdot)\}$.

Next, we claim that $r(\infty)$ exists. Otherwise, we have $r_* := \liminf_{x\to\infty} r(x) < r^*$. Since

$$\lim_{i\to\infty}\max_{[x_i-2,x_i+2]}\{| r(\cdot)-r^* |\} = 0,\ \liminf_{x\to\infty} r(x) = r_* < r^*,$$

we can find $j\in\mathbb{N}$ such that

$$r(\cdot) > \frac{r^*+r_*}{2} \text{ in } [x_j, x_j+2]\cup[x_{j+1}-2, x_{j+1}], \quad \min_{[x_j,x_{j+1}]}\{r(\cdot)\} < \frac{r^*+r_*}{2}.$$

Since $r(\cdot)$ is continuous, we can choose $\hat{x}$ be the left-most point in $[x_j, x_{j+1}]$ such that $r(\hat{x}) = \min_{[x_j,x_{j+1}]}\{r(\cdot)\}$. From the above fact, we have $\hat{x}\in[x_j+2, x_{j+1}-2]$ such that

$$r'(\hat{x}) = 0,\ r(\hat{x}) \le \min\{r(\hat{x}+2), r(\hat{x}+1)\},\ r(\hat{x}) < \min\{r(\hat{x}-1), r(\hat{x}-2)\}.$$

This leads a contradiction with (2.8). Therefore, $r(\infty)$ exists.

Similarly, $r(-\infty) := \lim_{x\to-\infty} r(x)$ exists. Finally, by sending $x \to \pm\infty$ in (1.10), we get

$$P(a, a_2, a_1, a_0, r(\pm\infty)) = 0.$$

Then the lemma follows. □

Lemma 2.10. *If $r(\cdot) \in L^1_{loc}(\mathbb{R})$ is a non-constant solution of (1.10), then $r(-\infty) < r(x) < r(\infty)$ for all $x \in \mathbb{R}$ and*

$$\int_0^\infty [r(\infty) - r(y)]dy + \int_{-\infty}^0 [r(y) - r(-\infty)]dy < \infty.$$

Proof. According to Lemma 2.8, $r(x)$ cannot attain its global maximum or minimum. This implies that $r(\infty) \neq r(-\infty)$ and

$$\Lambda_1 := \min\{r(-\infty), r(\infty)\} < r(x) < \Lambda_2 := \max\{r(-\infty), r(\infty)\},\ \forall x \in \mathbb{R}.$$

By Lemma 2.9, we have

$$P(a, a_2, a_1, a_0, \Lambda_1) = P(a, a_2, a_1, a_0, \Lambda_2) = 0.$$

By the graph of $P(a, a_2, a_1, a_0, \lambda)$, we have

$$\frac{\partial}{\partial\lambda} P(a, a_2, a_1, a_0, \Lambda_1) < 0 < \frac{\partial}{\partial\lambda} P(a, a_2, a_1, a_0, \Lambda_2).$$

So we can find $\epsilon > 0$ such that

$$a + 2a_2 e^{2(\Lambda_1+\epsilon)} + a_1 e^{(\Lambda_1+\epsilon)} - a_1 e^{-(\Lambda_1+\epsilon)} - 2a_2 e^{-2(\Lambda_1+\epsilon)} < 0, \tag{2.12}$$

$$a + 2a_2 e^{2(\Lambda_2-\epsilon)} + a_1 e^{(\Lambda_2-\epsilon)} - a_1 e^{-(\Lambda_2-\epsilon)} - 2a_2 e^{-2(\Lambda_2-\epsilon)} > 0. \tag{2.13}$$

Suppose that $r(\infty) = \Lambda_1$ and $r(-\infty) = \Lambda_2$. By translation, we may assume that

$$r(x) < \Lambda_1 + \epsilon,\ \text{if } x \geq -4.$$

If we define $l := \min_{x\in[-4,4]} r(x)$, then we have $l \in (\Lambda_1, \Lambda_1 + \epsilon)$. Compare the line $h = l - kx$ with the curve $h = r(x)$ for $x > 0$. If $k \geq \epsilon$ and $0 < x \leq 4$, then $r(x) \geq l > l - kx$. If $k \geq \epsilon$ and $x > 4$, then $r(x) > \Lambda_1 > l - \epsilon > l - kx$. Combining these two cases, we get that

$$r(x) > l - kx,\ \forall\ x > 0,\ k \geq \epsilon.$$

Therefore, the quantity

$$\delta := \inf\{k > 0 \mid r(x) \geq l - kx,\ \forall x > 0\}$$

is well-defined and we can easily see that $\delta \in (0, \epsilon)$. Moreover, there is a number $x_0 \in (4, \infty)$ such that

$$r(x_0) - (l - \delta x_0) = 0 \leq r(x) - (l - \delta x),\ \forall x > 0.$$

This implies that

$$r'(x_0) = -\delta, r(x_0 \pm i) - r(x_0) \geq \mp i\delta \text{ for } i = 1, 2.$$

Recall $A(x) = \exp[\int_x^{x+1} r(y)dy]$ and using $\Lambda_1 < r(\cdot) < \Lambda_1 + \epsilon$ in $[0, \infty)$, we have

$$\begin{aligned}
& a\delta \\
= \ & a_2 A(x_0+1)A(x_0)[r(x_0+2) - r(x_0)] + a_1 A(x_0)[r(x_0+1) - r(x_0)] \\
& + a_1 A(x_0-1)^{-1}[r(x_0-1) - r(x_0)] + a_2 A(x_0-1)^{-1}A(x_0-2)^{-1}[r(x_0-2) - r(x_0)] \\
\geq \ & \delta[-2a_2 e^{2(\Lambda_1+\epsilon)} - a_1 e^{(\Lambda_1+\epsilon)} + a_1 e^{-(\Lambda_1+\epsilon)} + 2a_2 e^{-2(\Lambda_1+\epsilon)}].
\end{aligned}$$

This leads to a contradiction with (2.12). Therefore,

$$r(\infty) = \Lambda_2 > r(-\infty) = \Lambda_1.$$

Finally, we claim that

$$\int_{-\infty}^{0} [r(y) - \Lambda_1]dy < \infty.$$

Let $R(x) := r(x) - \Lambda_1$. For the ϵ above, there exists $x_\epsilon \in \mathbb{R}$ such that

$$r(\cdot) \leq \Lambda_1 + \epsilon \text{ on } (-\infty, x_\epsilon + 4].$$

By a direct computation and the Mean Value Theorem, we have

$$aR(x) + \sum_{i\in\{\pm1,\pm2\}} a_{|i|} \exp[sgn\{i\}\theta_i(x)] \int_x^{x+i} R(y)dy = 0, \ x \in \mathbb{R}, \tag{2.14}$$

where

$$\theta_{\pm2}(x) \in [2\Lambda_1, 2 \max_{[x-2,x+2]}\{r(\cdot)\}], \quad \theta_{\pm1}(x) \in [\Lambda_1, \max_{[x-1,x+1]}\{r(\cdot)\}].$$

Integrating (2.14) over $[-M, x_\epsilon]$, we get

$$\begin{aligned}
0 &= a\int_{-M}^{x_\epsilon} R(x)dx + \sum_{i\in\{\pm1,\pm2\}} a_{|i|} \int_{-M}^{x_\epsilon}\int_x^{x+i} \exp[sgn\{i\}\theta_i(x)]R(y)dydx \\
&= a\int_{-M}^{x_\epsilon} R(y)dy + \sum_{i\in\{\pm1,\pm2\}} a_{|i|} \int_{-M}^{x_\epsilon}\int_{y-i}^{y} \exp[sgn\{i\}\theta_i(x)]R(y)dxdy + O(1) \\
&= \int_{-M}^{x_\epsilon} \{a + \sum_{i\in\{\pm1,\pm2\}} a_{|i|} \int_{y-i}^{y} \exp[sgn\{i\}\theta_i(x)]dx\}R(y)dy + O(1),
\end{aligned}$$

where $O(1)$ is uniformly bounded (independent of M). According to the range of $\theta_i(x)$ and $r(\cdot) \leq \Lambda_1 + \epsilon$ on $(-\infty, x_\epsilon + 4]$, we have

$$\begin{aligned}
& a + \sum_{i\in\{\pm1,\pm2\}} a_{|i|} \int_{y-i}^{y} \exp[sgn\{i\}\theta_i(x)]dx \\
< \ & a + 2a_2 e^{2(\Lambda_1+\epsilon)} + a_1 e^{(\Lambda_1+\epsilon)} - a_1 e^{-(\Lambda_1+\epsilon)} - 2a_2 e^{-2(\Lambda_1+\epsilon)} < 0.
\end{aligned}$$

By taking $M \to \infty$, we have

$$[2a_2e^{-2(\Lambda_1+\epsilon)} + a_1e^{-(\Lambda_1+\epsilon)} - a_1e^{(\Lambda_1+\epsilon)} - 2a_2e^{2(\Lambda_1+\epsilon)} - a] \int_{-\infty}^{x_\epsilon} R(y)dy$$

$$\leq \int_{-\infty}^{x_\epsilon} \left\{ -a - \sum_{i\in\{\pm1,\pm2\}} a_{|i|} \int_{y-i}^{y} \exp[sgn\{i\}\theta_i(x)]dx \right\} R(y)dy = O(1).$$

Therefore, $\int_{-\infty}^{x_\epsilon} R(y)dy < \infty$. We conclude that

$$\int_{-\infty}^{0} [r(y) - \Lambda_1]dy < \infty.$$

Similarly, using (2.13) we can deduce that

$$\int_{0}^{\infty} [\Lambda_2 - r(y)]dy < \infty.$$

The proof is now completed. □

Proof of Theorem 1. First, parts (i) and (ii) follows from Lemma 2.9 and Lemma 2.10.

Next, we focus on the case that $P(a, a_2, a_1, a_0, \cdot) = 0$ has two real roots $\{\Lambda_1, \Lambda_2\}$ with $\Lambda_1 < \Lambda_2$. Suppose that $r(x)$ is an arbitrary non-constant solution of (1.10). By Lemma 2.9 and Lemma 2.10, we have $r(\infty) = \Lambda_2 > \Lambda_1 = r(-\infty)$. We define

$$u(x) = \exp\left[\int_0^x r(y)dy\right],$$
$$u_1(x) = \theta e^{\Lambda_1 x}, \quad \theta = \exp\left[\int_0^{-\infty} (r(y) - \Lambda_1)dy\right],$$
$$u_2(x) = u(x) - u_1(x).$$

By a direct computation, we have

$$u'(x) = u(x)r(x),\ u_2(0) = 1 - \theta,\ 0 < \theta < 1.$$

Since

$$u_2(x)e^{-\Lambda_1 x} = \exp\left[\int_0^x (r(y) - \Lambda_1)dy\right] - \exp\left[\int_0^{-\infty} (r(y) - \Lambda_1)dy\right],$$

we obtain that $u_2(x)e^{-\Lambda_1 x} > 0$ for all $x \in \mathbb{R}$ and $u_2(x)e^{-\Lambda_1 x} \to 0$ as $x \to -\infty$.

Now we define

$$\hat{r}(x) := \frac{u_2'(x)}{u_2(x)}.$$

It is easy to see that $\hat{r}(x)$ is a solution of (1.10). We claim that $\hat{r}(x)$ is a constant function. Suppose not. Then it follows from Lemma 2.10 that

$$\int_{-\infty}^{0} [\hat{r}(y) - \Lambda_1]dy < \infty.$$

But,

$$\ln[u_2(x)e^{-\Lambda_1 x}] - \ln[u_2(0)] = \int_0^x \frac{(u_2(y)e^{-\Lambda_1 y})'}{(u_2(y)e^{-\Lambda_1 y})} dy = \int_0^x [\hat{r}(y) - \Lambda_1] dy.$$

By taking $x \to -\infty$, we have

$$\lim_{x \to -\infty} \ln[u_2(x)e^{-\Lambda_1 x}] < \infty,$$

a contradiction. Therefore, $\hat{r}(x)$ must be a constant solution of (1.10). This also implies that either $\hat{r}(x) \equiv \Lambda_1$ or $\hat{r}(x) \equiv \Lambda_2$.

If $\hat{r}(x) \equiv \Lambda_1$, then $u(x) = ce^{\Lambda_1 x}$. This implies that $r(\cdot)$ is a constant function, a contradiction. Hence we have $\hat{r}(x) \equiv \Lambda_2$ and so $u_2(x) = u_2(0)e^{\Lambda_2 x} = (1-\theta)e^{\Lambda_2 x}$. Therefore, by adding two constant solutions $r(x) \equiv \Lambda_1$ and $r(x) \equiv \Lambda_2$ together, all solutions of (1.10) are given by

$$r(x) = \frac{\theta \Lambda_1 e^{\Lambda_1 x} + (1-\theta)\Lambda_2 e^{\Lambda_2 x}}{\theta e^{\Lambda_1 x} + (1-\theta)e^{\Lambda_2 x}}$$

for $\theta \in [0,1]$. This proves the theorem. □

3. Asymptotic behavior

Let (c, U) be a solution of (P). This section is devoted to the proof of Theorem 2, the asymptotic behavior of U near $x = \pm\infty$.

Proof of Theorem 2. To study the behavior at $x = \infty$, we define $\rho(x) := U'(x)/U(x)$. First, we claim that $\lim_{x\to\infty} \rho(x)$ exists. Suppose not. Then we have that

$$\overline{\lambda} := \limsup_{x \to \infty} \rho(x) > \underline{\lambda} := \liminf_{x \to \infty} \rho(x).$$

By the definition of $\rho(x)$, (1.6) can be re-written as

$$\begin{aligned}
c\rho(x) &+ \left[J(2)\frac{b(U(x+2))}{U(x+2)}\right] \exp\left[\int_x^{x+2} \rho(s)ds\right] \\
&+ \left[D + J(1)\frac{b(U(x+1))}{U(x+1)}\right] \exp\left[\int_x^{x+1} \rho(s)ds\right] \\
&+ \left[D + J(1)\frac{b(U(x-1))}{U(x-1)}\right] \exp\left[\int_x^{x-1} \rho(s)ds\right] \\
&+ \left[J(2)\frac{b(U(x-2))}{U(x-2)}\right] \exp\left[\int_x^{x-2} \rho(s)ds\right] \\
&+ \left[-2D - d + J(0)\frac{b(U(x))}{U(x)}\right] = 0.
\end{aligned} \tag{3.1}$$

Choose λ_1, λ_2 such that $\underline{\lambda} < \lambda_1 < \lambda_2 < \overline{\lambda}$ and choose $\lambda \in (\lambda_1, \lambda_2)$ such that

$$P(a, a_2, a_1, a_0, \lambda) \neq 0,$$

where

$$a = c,\ a_2 = a_{-2} = J(2)b'(0),\ a_1 = a_{-1} = D + J(1)b'(0),\ a_0 = -2D - d + J(0)b'(0).$$

Let $\{\eta_i^1\} \to \infty$ and $\{\eta_i^2\} \to \infty$ as $i \to \infty$ such that $\rho(\eta_i^1) = \lambda_1$, $\rho(\eta_i^2) = \lambda_2$ and $\eta_i^2 < \eta_i^1$ for all $i \in \mathbb{N}$. Since $\rho(\cdot)$ is a uniformly continuous function, we can choose that ξ_i to be the right-most point in (η_i^2, η_i^1) such that $\rho(\xi_i) = \lambda$. Since $\rho(\xi_i) - \rho(\eta_i^1) = \lambda - \lambda_1 > 0$, we get

$$\rho(\cdot) \le \lambda \text{ on } [\xi_i, \eta_i^1] \text{ and } \delta := \inf_{i\in\mathbb{N}}\{\eta_i^1 - \xi_i\} > 0.$$

By Lemma 2.1, $\{\rho(\xi_i + \cdot)\}_{i=1}^{\infty}$ is uniformly bounded and equi-continuous. By Ascoli-Arzelà Theorem, we can extract a subsequence (still denote $\rho(\xi_i + \cdot)$) such that $\rho(\xi_i + \cdot) \to r(\cdot)$ uniformly in any compact subset of $\mathbb{R}$ for some function $r \in C(\mathbb{R})$. Therefore,

$$r(0) = \lim_{i\to\infty} \rho(\xi_i) = \lambda, r(\cdot) \le \lambda \text{ on } [0, \delta]$$

and $r(x)$ is a solution of the equation (1.10). But, this contradicts Theorem 1, since all non-constant solutions of (1.10) are strictly increasing. Therefore, the limit $\Lambda := \lim_{x\to\infty} \rho(x)$ exists. By taking $x \to \infty$ in (3.1), we see that Λ is a root of (1.9). Since $b'(0) > d$, we have $\Lambda < 0$.

On the other hand, by the same argument as above and using Lemma 2.2, we can also prove that the limit

$$\sigma := \lim_{x\to-\infty} \frac{U'(x)}{U(x) - 1}$$

exists and satisfies (1.11). Note that $\sigma \neq 0$, since $b'(1) < d$. Indeed, (1.11) has a unique positive root and a unique negative root.

Finally, it remains to determine the sign of σ. If $\sigma < 0$, then there exists $M > 0$ such that $U'(x) > 0$ if $x \le -M$. This contradicts $U(-\infty) = 1$. Thus $\sigma > 0$ which is the unique positive root of the characteristic equation (1.11). This completes the proof of the theorem. □

4. Monotonicity and the proof of Theorem 4

In this section, we shall prove Theorems 3 and 4. Let (c, U) be a solution of (P). For the notational convenience, we define

$$W[U](y) := c\mu U(y) + \mathcal{D}_2[U](y) - dU(y) + \sum_{i=-2}^{2} J(i)b(U(y-i)),$$

$$T[U](x) := \frac{e^{\mu x}}{c}\int_x^{\infty} e^{-\mu y} W[U](y)dy.$$

where μ is a constant satisfying $\mu \ge (d + 2D)/c$. Note that $T[U](x) = U(x)$ for all $x \in \mathbb{R}$.

We now prove the following strong comparison principle.

Lemma 4.1. *If $(c, U_1), (c, U_2)$ are solutions of* (P) *and satisfy $U_1 \leq U_2$ on $\mathbb{R}$, then either $U_1 \equiv U_2$ or $U_1 < U_2$ on $\mathbb{R}$.*

Proof. Suppose that $U_1 < U_2$ on $\mathbb{R}$ does not hold. Then we can find $x_0 \in \mathbb{R}$ such that $U_1(x_0) = U_2(x_0)$. Since $U_1(\cdot), U_2(\cdot)$ are solution of (P), we have $T[U_1](x_0) = T[U_2](x_0)$. By the definition of $T[U](x)$, we can get

$$\int_{x_0}^{\infty} e^{-\mu y}\{W[U_1] - W[U_2]\}(y)dy = 0.$$

Since $W[U_1] \leq W[U_2]$ on $\mathbb{R}$, we have $W[U_1] \equiv W[U_2]$ on $[x_0, \infty)$. By the definition of $W[U]$ and the monotonicity of b, we get that $U_1(y-1) = U_2(y-1)$ and $U_1(y-2) = U_2(y-2)$ for $y \in [x_0, \infty)$. Hence $U_1 \equiv U_2$ on $[x_0 - 2, \infty)$. Continuing in this way, we conclude that $U_1 \equiv U_2$ on $\mathbb{R}$. The lemma is proved. □

Next, we use the sliding method to prove the following lemma.

Lemma 4.2. *If (c, U) is a solution of* (P) *such that $U'(x) \leq 0$ for all $|x| \gg 1$, then $U'(x) < 0$ on $\mathbb{R}$.*

Proof . By hypothesis, we can find $M > 0$ such that $U'(x) \leq 0$ on $\mathbb{R} \setminus (-M, M)$. Since $U(\infty) = 0$, $U(-\infty) = 1$ and $U(\cdot)$ is continuous, the set

$$\mathbb{A} := \{\xi > 0 \mid U(x + \eta) \leq U(x), \forall x \in \mathbb{R},\ \forall \eta \geq \xi\}$$

is nonempty. Hence $\xi^* := \inf \mathbb{A}$ is well-defined. Note that $U(x + \xi^*) \leq U(x)$ for all $x \in \mathbb{R}$. We claim that $\xi^* = 0$. Suppose $\xi^* > 0$. Since $U(\infty) \neq U(-\infty)$, by Lemma 4.1, we have

$$U(x + \xi^*) < U(x),\ \forall x \in \mathbb{R}.$$

Since $U(x)$ is a continuous function, we can choose $\epsilon \in (0, \xi^*)$ such that

$$U(x + \eta) < U(x),\ \text{if } x \in [-M - 2\xi^*, M + 2\xi^*] \text{ and } \eta \in [\xi^* - \epsilon, \xi^*].$$

For $x \in \mathbb{R} \setminus (-M - \xi^*, M + \xi^*)$ and $\eta \in (0, \xi^*]$, by the Mean Value Theorem, we obtain

$$U(x + \eta) - U(x) = \eta U'(f(x, \eta)),$$

where $x < f(x, \eta) < x + \eta$. Hence $U(x + \eta) \leq U(x)$ for all $x \in \mathbb{R} \setminus (-M - \xi^*, M + \xi^*)$ for all $\eta \in (0, \xi^*]$. We conclude that

$$U(x + \eta) \leq U(x),\ \forall x \in \mathbb{R},\ \forall \eta \in [\xi^* - \epsilon, \xi^*].$$

But this contradicts the definition of ξ^*. Therefore, $\xi^* = 0$ and we have $U'(x) \leq 0$ for all $x \in \mathbb{R}$. Finally, differentiating $U = T[U]$, we get that $U'(\cdot) < 0$ on $\mathbb{R}$. The proof is completed. □

Proof of Theorem 3. Let (c,U) be a solution of (P). By Theorem 2, we have

$$\lim_{x\to\infty}\frac{U'(x)}{U(x)}=\Lambda(<0)\quad\text{and}\quad\lim_{x\to-\infty}\frac{U'(x)}{U(x)-1}=\sigma(>0).$$

This implies that $U'(x)<0$, if $|x|\gg 1$. By Lemma 4.2, we have $U'(x)<0$ for any $x\in\mathbb{R}$. The theorem is proved. □

Now, we turn to the proof of Theorem 4.

Proof of Theorem 4. Let $c>c_{\min}$. Then the characteristic equation (1.9) always has two negative roots, denoted by $\lambda(c)<\Lambda(c)<0$. We claim that

$$\lim_{x\to\infty}\frac{U'(x)}{U(x)}=\Lambda(c).$$

Suppose on the contrary that

$$\lim_{x\to\infty}\frac{U'(x)}{U(x)}=\lambda(c).$$

Choose $\hat c\in(c_{\min},c)$ and $(\hat c,\hat U(x))$ a solution of (P). Then, by Theorem 2, we have

$$\lim_{x\to\infty}\frac{\hat U'(x)}{\hat U(x)}\ge\lambda(\hat c).$$

So we obtain

$$\lim_{x\to\infty}[\ln(\hat U(x)/U(x))]'=\lim_{x\to\infty}\{[\hat U'(x)/\hat U(x)]-[U'(x)/U(x)]\}\ge\lambda(\hat c)-\lambda(c)>0.$$

This implies that $\lim_{x\to\infty}\ln[\hat U(x)/U(x)]=+\infty$. Therefore, there exists a positive number M such that

$$\hat U(x)>U(x)\quad\forall\, x\ge M. \tag{4.1}$$

On the other hand, using $b(1)=d$ and

$$\lim_{x\to-\infty}[U'(x)/(U(x)-1)]=\sigma(c)>0,$$

it follows that

$$\begin{aligned}
&\lim_{x\to-\infty}\left[\int_{U(x)}^{\hat U(x)}\frac{1}{b(s)-ds}ds\right]'\\
=&\lim_{x\to-\infty}\left[\frac{\hat U'(x)}{\hat U(x)-1}\cdot\frac{\hat U(x)-1}{b(\hat U(x))-d\hat U(x)}-\frac{U'(x)}{U(x)-1}\cdot\frac{U(x)-1}{b(U(x))-dU(x)}\right]\\
=&\lim_{x\to-\infty}\left\{\frac{\hat U'(x)}{\hat U(x)-1}\cdot\frac{\hat U(x)-1}{[b(\hat U(x))-b(1)]-d[\hat U(x)-1]}\right.\\
&\qquad\left.-\frac{U'(x)}{U(x)-1}\cdot\frac{U(x)-1}{[b(U(x))-b(1)]-d[U(x)-1]}\right\}\\
=&\frac{\sigma(\hat c)-\sigma(c)}{b'(1)-d}<0,
\end{aligned}$$

since $b'(1) < d$ and, by (1.11), $\sigma(c)$ is strictly decreasing in c. Hence there exists $M_1 > 0$ such that

$$\hat{U}(x) > U(x) \quad \forall\, x \le -M_1. \tag{4.2}$$

By (4.1), (4.2) and Theorem 3, we obtain that

$$\hat{U}(x - M_1) > U(x + M),\ \forall x \in \mathbb{R}. \tag{4.3}$$

Note that both $u_1(x,t) := \hat{U}(x - M_1 - \hat{c}t)$ and $u_2(x,t) := U(x + M - ct)$ are solutions of the following spatially continuous version of (1.1):

$$u_t(x,t) = D[u(x+1,t) + u(x-1,t) - 2u(x,t)] - du(x,t) + \sum_{i\in\mathbb{Z}} J(i)b(u(x-i,t)).$$

By (4.3), we have $u_1(\cdot,0) \ge u_2(\cdot,0)$. The comparison principle implies that $u_1(\cdot,t) \ge u_2(\cdot,t)$ for all $t \ge 0$. Writing $u_1(x,t) \ge u_2(x,t)$ by

$$\hat{U}(x - (c+\hat{c})t/2 - M_1 + (c-\hat{c})t/2) \ge U(x - (c+\hat{c})t/2 + M - (c-\hat{c})t/2) \tag{4.4}$$

fixing $\xi := x - (c+\hat{c})t/2$ and letting $t \to \infty$ in (4.4), it follows from $c > \hat{c}$ that $0 = \hat{U}(\infty) \ge U(-\infty) = 1$, a contradiction to (1.7). Therefore, the proof is completed. □

5. Uniqueness

This section is devoted to the uniqueness of the traveling wave solution. We shall follow the method developed in [5]. For a smooth function ϕ, we let

$$\mathbb{L}[\phi](x) := -c\phi'(x) - \mathcal{D}_2[\phi](x) - \sum_{i=-2}^{2} J(i)[b(\phi(x-i)) - d\phi(x)].$$

First, we define the notion of super-sub-solutions as follows.

Definition 5.1. *A non-constant smooth function $\phi : [a-2, b+2] \to (0,1)$ is called a supersolution (subsolution, resp.) of* (1.6) *on $[a,b]$ for a wave speed c, if $\mathbb{L}[\phi](x) \ge 0$ ($\mathbb{L}[\phi](x) \le 0$, resp.) for $x \in (a,b)$.*

Definition 5.2. *A non-constant smooth function $\phi : [a-2, \infty) \to (0,1)$ is called a supersolution (subsolution, resp.) of* (1.6) *on $[a,\infty)$ for a wave speed c, if $\mathbb{L}[\phi](x) \ge 0$ ($\mathbb{L}[\phi](x) \le 0$, resp.) for $x \in (a,\infty)$.*

Lemma 5.1. *Assume* (**H1**). *Let (c,U) be a solution of* (P) *and $V(x)$ be a subsolution (supersolution, resp.) of* (1.6) *on $[a,b]$ for the same speed c, where $a < b$. If $V(x) < U(x)$ ($V(x) > U(x)$, resp.) for $x \in [a-2, a) \cup (b, b+2]$, then $V(x) < U(x)$ ($V(x) > U(x)$, resp.) for $x \in [a,b]$.*

Proof . Since the case for supersolution is similar, we only consider the case when $V(x)$ is a subsolution. We introduce

$$g(t) := \max_{x\in[a-2,b+2]}\{V(x) - U(x-t)\}.$$

Since $U(\infty) = 0$ and $U(-\infty) = 1$, we can choose $\zeta \in \mathbb{R}$ such that $g(\zeta) = 0$. Let $y \in [a-2, b+2]$ be the maximum value in $[a-2, b+2]$ such that $V(y) - U(y-\zeta) = 0$. We claim that $y \in [a-2, a) \cup (b, b+2]$.

Suppose on the contrary that $y \in [a, b]$. Then we have

$$\begin{aligned}&V(y) = U(y-\zeta),\ V'(y) = U'(y-\zeta),\ V(y-1) \le U(y-1-\zeta),\\ &V(y-2) \le U(y-2-\zeta),\ V(y+1) < U(y+1-\zeta),\ V(y+2) < U(y+2-\zeta).\end{aligned}$$

Hence we have $\mathbb{L}[V](y) > \mathbb{L}[U](y-\zeta)$. By the strictly inequality, without loss of generality we may assume that $y \in (a, b)$. This contradicts that $U(x)$ is a solution of (1.6) and $V(x)$ is a subsolution of (1.6) on $[a, b]$. Therefore, $y \in [a-2, a) \cup (b, b+2]$.

By hypothesis, we have $U(y) > V(y) = U(y-\zeta)$. It follows from the monotonicity of U that $\zeta < 0$. Hence $U(x-\zeta) < U(x)$ for all $x \in \mathbb{R}$. Since $g(\zeta) = 0$, we deduce that $V(x) < U(x)$ for all $x \in [a, b]$. The proof is completed. □

Lemma 5.2. *Assume* (**H1**)*. Let* (c, U) *be a solution of* (P) *and* $\phi(x)$ *be a subsolution (or supersolution) of (1.6) with the same speed* c *on* $[a, \infty)$ *for some constant* a*. If*

$$\lim_{x\to\infty}\frac{\phi'(x)}{\phi(x)} = \lim_{x\to\infty}\frac{U'(x)}{U(x)} = \Lambda < 0,$$

then there exists $A \in [-\infty, \infty]$ *such that*

$$\lim_{x\to\infty} W(\xi, x) = A + \Lambda\xi,\ \forall \xi \in \mathbb{R},$$

where $W(\xi, x) := \ln[U(x+\xi)] - \ln[\phi(x)]$*.*

Proof. Given a subsolution $\phi(x)$ of (1.6) on $[a, \infty)$ for some constant a. By the definition of $W(\xi, x)$, we obtain

$$\lim_{x\to\infty}[W(\xi, x) - W(0, x)] = \lim_{x\to\infty}\{\ln[U(x+\xi)] - \ln[U(x)]\} = \lim_{x\to\infty}\int_x^{x+\xi}\frac{U'(t)}{U(t)}dt = \Lambda\xi \tag{5.1}$$

for all $\xi \in \mathbb{R}$. It follows from (5.1) that either the limit $\lim_{x\to\infty} W(\xi, x)$ exists for all $\xi \in \mathbb{R}$ or it does not exist for all $\xi \in \mathbb{R}$.

Suppose that the limit $\lim_{x\to\infty} W(\xi, x)$ does not exist for all $\xi \in \mathbb{R}$. By (5.1), we can choose an appropriate ξ such that

$$A := \limsup_{x\to\infty} W(\xi, x) > 0 > B := \liminf_{x\to\infty} W(\xi, x). \tag{5.2}$$

Indeed, we can divide into the following three cases.

Case 1. If both $\limsup_{x\to\infty} W(0,x)$ and $\liminf_{x\to\infty} W(0,x)$ are finite, then ξ can be chosen as

$$\xi = -[\limsup_{x\to\infty} W(0,x) + \liminf_{x\to\infty} W(0,x)]/(2\Lambda).$$

For this choice of ξ, we have

$$\begin{aligned}\limsup_{x\to\infty} W(\xi,x) &\geq \limsup_{x\to\infty} W(0,x) - \limsup_{x\to\infty}[W(0,x) - W(\xi,x)]\\ &= \limsup_{x\to\infty} W(0,x) + \Lambda\xi\\ &= \frac{1}{2}[\limsup_{x\to\infty} W(0,x) - \liminf_{x\to\infty} W(0,x)] > 0.\end{aligned}$$

Similarly, we can show that $\liminf_{x\to\infty} W(\xi,x) < 0$.

Case 2. Either

$$\limsup_{x\to\infty} W(0,x) = \infty \text{ and } \liminf_{x\to\infty} W(0,x) \text{ is finite,}$$

or

$$\liminf_{x\to\infty} W(0,x) = -\infty \text{ and } \limsup_{x\to\infty} W(0,x) \text{ is finite.}$$

We only treat the former case. The latter case is similar. For the former case, we take

$$\xi = -\liminf_{x\to\infty} W(0,x)/\Lambda + 1.$$

Then, recalling that $\Lambda < 0$, we have

$$\begin{aligned}\liminf_{x\to\infty} W(\xi,x) &\leq \liminf_{x\to\infty} W(0,x) - \liminf_{x\to\infty}[W(0,x) - W(\xi,x)]\\ &= \liminf_{x\to\infty} W(0,x) + \Lambda\xi\\ &= \Lambda < 0.\end{aligned}$$

It is clear that $\limsup_{x\to\infty} W(\xi,x) = \infty$.

Case 3. If $\limsup_{x\to\infty} W(0,x) = \infty$ and $\liminf_{x\to\infty} W(0,x) = -\infty$, then we can take $\xi = 0$. Hence (5.2) holds in any case.

Now take α, β such that $B < \beta < 0 < \alpha < A$. Then we may choose two sequences $\{x_i\}$ and $\{y_i\}$ such that

$$\lim_{i\to\infty} x_i = \infty,\ a-2 \leq x_i < y_i < x_{i+1},\ W(\xi,x_i) = \alpha,\ W(\xi,y_i) = \beta,\ \forall i \in \mathbb{N}.$$

Since

$$\lim_{x\to\infty} W_x(\xi,x) = \lim_{x\to\infty} \left\{ \frac{U'(x+\xi)}{U(x+\xi)} - \frac{\phi'(x)}{\phi(x)} \right\} = 0,$$

we can choose a fixed integer i large enough such that

$$W(\xi,x) > 0,\ \text{if } x \in [x_i - 2, x_i] \cup [x_{i+1}, x_{i+1}+2],$$

i.e., $U(x+\xi) > \phi(x)$ for all $x \in [x_i - 2, x_i] \cup [x_{i+1}, x_{i+1}+2]$. Since $W(\xi, y_i) = \beta < 0$, we obtain that $U(y_i+\xi) < \phi(y_i)$. But, by Lemma 5.1, it is impossible, since y_i is between x_i and x_{i+1}. Therefore, the limit $\lim_{x\to\infty} W(\xi, x)$ exists for all $\xi \in \mathbb{R}$. We conclude that $\lim_{x\to\infty} W(\xi, x) = A + \Lambda\xi$ for all $\xi \in \mathbb{R}$, where $A := \lim_{x\to\infty} W(0, x)$.

The case when $\phi(x)$ is a supersolution is similar, the lemma follows. □

With this lemma, we are ready to prove Theorem 5.

Proof of Theorem 5. Suppose that the roots of $\Phi(\cdot\,; c) = 0$ are given by Λ and λ with $\Lambda > \lambda$. We choose $\omega < 0$ such that $\max\{\lambda, (1+\alpha)\Lambda\} < \omega < \Lambda$, where the constant α is defined in (**H2**). Since $\Phi(\lambda; c)$ is a convex function in λ, we have $\Phi(\omega; c) < 0$. Following [5], we define

$$\phi_\pm(x; \epsilon, \delta) := \delta((1 \mp \epsilon)e^{\Lambda x} \pm \epsilon e^{\omega x}),$$

where $\delta > 0, \epsilon \in (0, e^\omega], x \geq -2$. We may easily check

$$0 < \phi_\pm(x; \epsilon, \delta) < 2\delta e^{\Lambda x}, \text{ if } \delta > 0, \epsilon \in (0, e^\omega], x \geq 0. \tag{5.3}$$

By a simple computation, we have

$$\frac{\phi'_\pm}{\phi_\pm} = \frac{\Lambda(1 \mp \epsilon)e^{\Lambda x} \pm \omega\epsilon e^{\omega x}}{(1 \mp \epsilon)e^{\Lambda x} \pm \epsilon e^{\omega x}} = \Lambda + \frac{\pm(\omega - \Lambda)\epsilon}{(1 \mp \epsilon)e^{(\Lambda-\omega)x} \pm \epsilon}.$$

It follows that

$$\max_{-2\leq x\leq 0} \frac{\phi'_+}{\phi_+} = \Lambda + \epsilon(\omega - \Lambda), \ \min_{-2\leq x\leq 0} \frac{\phi'_-}{\phi_-} = \Lambda - \epsilon(\omega - \Lambda), \ \lim_{x\to\infty} \frac{\phi'_\pm}{\phi_\pm} = \Lambda. \tag{5.4}$$

Next, we compute

$$\begin{aligned}
&\mathbb{L}[\phi_+](x)\\
= \ & -\{\delta(1-\epsilon)e^{\Lambda x}[c\Lambda + D(e^\Lambda + e^{-\Lambda} - 2) - d] + \delta\epsilon e^{\omega x}[c\omega + D(e^\omega + e^{-\omega} - 2) - d]\\
&+ \sum_{i=-2}^{2} J(i) b(\phi_+(x+i))\}\\
= \ & -\{\delta(1-\epsilon)e^{\Lambda x}\Phi(\Lambda; c) + \delta\epsilon e^{\omega x}\Phi(\omega; c) + \sum_{i=-2}^{2} J(i)[b(\phi_+(x+i)) - b'(0)\phi_+(x+i)]\}\\
= \ & -\delta\epsilon e^{\omega x}\Phi(\omega; c) - \sum_{i=-2}^{2} J(i)[b(\phi_+(x+i)) - b'(0)\phi_+(x+i)].
\end{aligned}$$

On the other hand, by (5.3), we have

$$0 < \phi_+(x+i; \epsilon, \delta) < 2\delta e^{\Lambda(x+i)} \leq 2\delta e^{-2\Lambda} e^{\Lambda x},$$

if $x \geq 2$, $\delta > 0$, $\epsilon \in (0, e^{\omega}]$, $i \in \{\pm 1, \pm 2, 0\}$. Hence, if $2\delta e^{-2\Lambda} e^{\Lambda x} \leq 1$, then, by (**H2**), we have

$$\begin{aligned}\mathbb{L}[\phi_+](x) &\geq -\delta\epsilon e^{\omega x}\Phi(\omega; c) - M\sum_{i=-2}^{2} J(i)[\phi_+(x+i)]^{1+\alpha} \\ &\geq -\delta\epsilon e^{\omega x}\Phi(\omega; c) - M(2\delta e^{-2\Lambda}e^{\Lambda x})^{1+\alpha} \\ &= \delta e^{\omega x}\{-\epsilon\Phi(\omega; c) - 2^{1+\alpha}\delta^{\alpha} M e^{-2\Lambda(1+\alpha)} e^{[(1+\alpha)\Lambda - \omega]x}\}.\end{aligned}$$

Therefore, we can easily deduce the following facts.

(a1) For every $\epsilon \in (0, e^{\omega}]$, there exists $\delta_\epsilon > 0$ such that $\phi_+(x; \epsilon, \delta)$ is a supersolution on $[2, \infty)$, if $\delta \in (0, \delta_\epsilon]$.

(a2) For every $\epsilon \in (0, e^{\omega}]$ and $\delta = 1$, there exists $x_\epsilon \geq 0$ such that $\phi_+(x; \epsilon, 1)$ is a supersolution on $[x_\epsilon, \infty)$.

Now, we consider $\phi(x) := \phi_+(x; \epsilon, \delta)$ with $\epsilon = e^{\omega}$ and $\delta = 1$. By Lemma 5.2, there exists $A \in [-\infty, \infty]$ such that

$$\lim_{x\to\infty}\{\ln[U(x+\xi)] - \ln[\phi(x)]\} = A + \Lambda\xi,\ \forall \xi \in \mathbb{R}.$$

We claim that $A > -\infty$. Suppose not. Then we have

$$\lim_{x\to\infty}\{\ln[U(x+\xi)] - \ln[\phi(x)]\} = -\infty,\ \forall \xi \in \mathbb{R}. \tag{5.5}$$

Fix $\epsilon = e^{\omega} > 0$, let δ_ϵ be the constant defined in (a1). Then it follows from Theorem 2 and $U(\infty) = 0$ that there exists $\eta > 0$ such that $U(\eta) < \delta_\epsilon$ and

$$\frac{U'(x)}{U(x)} > \Lambda + \epsilon(\omega - \Lambda),\ \text{if } x \geq \eta - 2. \tag{5.6}$$

By the fact (a1), the function $\hat{\phi}(x) := \phi_+(x; \epsilon, U(\eta))$ is a supersolution on $[2, \infty)$.

Note that $\hat{\phi}(0) = \phi_+(0; \epsilon, U(\eta)) = U(\eta)$. Also, from (5.4) and (5.6) it follows that

$$\frac{\hat{\phi}'(x)}{\hat{\phi}(x)} \leq \Lambda + \epsilon(\omega - \Lambda) < \frac{U'(x+\eta)}{U(x+\eta)},\ \forall x \in [-2, 0].$$

By an integration, we obtain $\hat{\phi}(x) > U(x+\eta)$, if $x \in [-2, 0)$. On the other hand, since $[\hat{\phi}(x)/\phi(x)] \equiv U(\eta)$, it follows from (5.5) that

$$\lim_{x\to\infty}\{\ln[U(x+\eta)] - \ln[\hat{\phi}(x)]\} = -\infty.$$

So we can choose $T > 0$ such that $\hat{\phi}(x) > U(x+\eta)$, $\forall x \in [T, \infty)$. Therefore, by Lemma 5.1, we have $\hat{\phi}(x) > U(x+\eta)$ for all $x \in [-2, \infty)$. This contradicts $\hat{\phi}(0) = U(\eta)$. Therefore, we conclude that $A > -\infty$. Thus

$$\lim_{x\to\infty}\ln\left[\frac{U(x)}{\phi(x)}\right] = \lim_{x\to\infty}\{\ln[U(x)] - \ln[\phi(x)]\} = A.$$

It follows from the definition of $\phi(x)$ that the limit $L := \lim_{x\to\infty}[U(x)e^{-\Lambda x}]$ exists and $L > 0$.

Moreover, using the function ϕ_-, (5.4) and (**H2**), we can prove, by a similar reasoning as above, that $L < \infty$. This proves the theorem. □

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