TRAVELING WAVES IN DISCRETE PERIODIC MEDIA FOR BISTABLE DYNAMICS

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Abstract. This paper concerns about the existence, uniqueness, and global stability of traveling waves in discrete periodic media for a system of ordinary differential equations exhibiting bistable dynamics. Main tools used to prove the uniqueness and asymptotic stability of traveling waves are comparison principle, spectrum analysis, and constructions of super/subsolutions. For the existence of traveling waves, the system is converted to an integral equation which is common in the study of monostable dynamics but quite rare in the study of bistable dynamics. The main purpose of this paper is to introduce a general framework for the study of traveling waves in discrete periodic media.

1. Introduction

In many mathematical models, evolution of chemical or biological substances are quite often described by reaction-diffusion-convection equations. In the one space dimensional setting, a typical example is

$$u_t = (a(x)u_x)_x + b(x)u_x + f(u,x), \quad x \in \mathbb{R}, t > 0, \tag{1.1}$$

where $u = u(x,t)$ is related to a phase indicator in phase transition models or a density in population genetics with x and t being the space and time coordinates respectively. For example, (1.1) arises in the model of thermo-diffusive premixed flame propagation in which u represents the temperature; see [31, 46] for more physical background of this equation.

A semi-discretization of (1.1) takes the form, for $\mathbf{u}(t) = \{u_i(t)\}_{i\in\mathbb{Z}}$,

$$\dot{u}_i := \frac{du_i}{dt} = \frac{a_{i+1/2}[u_{i+1} - u_i] - a_{i-1/2}[u_i - u_{i-1}]}{h^2} + \frac{b_i[u_{i+1} - u_{i-1}]}{2h} + f_i(u_i) \tag{1.2}$$

where h is the mesh size, $a_{i+1/2} = a([i+1/2]h), b_i = b(ih)$ and $f_i(s) = f(s, ih)$. Indeed, the system (1.2) can also be derived directly from many biological models such as that in a patch environment (cf. [40, 42]). In population genetics, for example, $u_i(t)$ represents a gene fraction or a population density at time t at position i of a habitat. The habitat is divided into discrete regions or niches. Migration moves individuals to new niches

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according to the law (1.2) which can be interpreted as follows. The rate of change at time t at position i is equal to the sum of fluxes, transports, and sources. Under the assumption that only nearest neighbors can interact each other, the flux towards the position i from the right is proportional to the gradient $u_{i+1} - u_i$ with proportional constant $a_{i+1/2}/h^2$. Similarly, the flux from the left is $[u_{i-1} - u_i]a_{i-1/2}/h^2$. The transport is represented by the term $b_i[u_{i+1} - u_{i-1}]/h^2$ and the source by $f_i(u_i)$. See, e.g., the work of Weinberger [44] for a general framework including continuous models and fully discrete models. Additional references will be mentioned below.

Based on this prototype, we shall consider a general system, for $\mathbf{u}(\cdot) := \{u_i(\cdot)\}_{i\in\mathbb{Z}}$,

$$\dot{u}_i(t) = \sum_k a_{i,k}\, u_{i+k}(t) + f_i(u_i(t)), \quad t > 0, i \in \mathbb{Z}, \tag{1.3}$$

under the following assumptions.

(A1) Periodicity: *There exists a positive integer n such that*

$$a_{i+n,k} = a_{i,k}\ , \quad f_{i+n}(\cdot) = f_i(\cdot) \in C^2(\mathbb{R}) \quad \forall\, i, k \in \mathbb{Z}.$$

(A2) Existence of ordered steady states: *There are $\Phi^{\pm} = \{\phi_i^{\pm}\}_{i\in\mathbb{Z}}$ satisfying*

$$\sum_k a_{i,k}\phi^{\pm}_{i+k} + f_i(\phi_i^{\pm}) = 0, \quad \phi^{\pm}_{i+n} = \phi^{\pm}_i, \quad \phi_i^- < \phi_i^+ \quad \forall\, i \in \mathbb{Z}.$$

It can be shown that after a normalization, i.e., a change of variable, the ordered states take the canonical form

$$\Phi^+ = \mathbf{1} := \{1\}_{i\in\mathbb{Z}}, \quad \Phi^- = \mathbf{0} := 0\mathbf{1}.$$

(A3) Ellipticity:

$$a_{i,k} \geqslant 0 \quad \forall\, k \neq 0, \quad a_{i,0} := -\sum_{k\neq 0} a_{i,k} < 0 \quad \forall\, i \in \mathbb{Z}.$$

Since a linear function can always be added to the function $f_i(\cdot)$, the constant $a_{i,0}$ can be set at any preferable value. The particular choice of $a_{i,0}$ that we set here is to obtain the identity

$$\sum_k a_{i,k}[u_{i+k} - u_i] = \sum_k a_{i,k}u_{i+k} = \sum_k a_{i,k-i}\, u_k.$$

Without loss of generality, we shall assume that the system (1.3) cannot be decoupled into independent subsystems. This amounts to the following

(A4) Non-decoupledness: *For each integer $i \neq j$, there exist integers $i_0, \cdots, i_m$ such that $i_0 = i$, $i_m = j$ and*

$$\prod_{s=0}^{m-1} a_{i_s, i_{s+1}-i_s} > 0.$$

The following technical condition will also be assumed.

(A5) Finite Range Interaction: *There exists an integer* $k_0 \geqslant 1$ *such that*

$$a_{i,k} = 0 \quad \textit{if} \;\; |k| > k_0.$$

We are interested in solutions that connect the two steady states in the sense that

$$\lim_{i \to \pm\infty} [u_i(t) - \phi_i^{\pm}] = 0 \qquad \forall t > 0. \tag{1.4}$$

Among all solutions, we are looking for **traveling waves**, i.e., those solutions that satisfy, for some constants $c \in \mathbb{R}$ and $T > 0$,

$$cT \in n\mathbb{Z}, \quad u_i(T) = u_{i-cT}(0) \quad \forall\, i \in \mathbb{Z}. \tag{1.5}$$

For such a solution, we call c the **wave speed** since it represents the average number of grid points that the wave marches per unit time. A traveling wave $\mathbf{u}(\cdot)$ defined on $[0, \infty)$ can be uniquely extended to be a traveling wave on $\mathbb{R}$ such that

$$u_i(t + kT) = u_{i-ckT}(t) \quad \forall\, i, k \in \mathbb{Z},\, t \in \mathbb{R}.$$

Among all the positive T's that satisfy (1.5), there is one that is minimal, which we shall call the **period**. Later on we shall show that the period of a wave with speed $c \neq 0$ is indeed $T = n/|c|$.

The study of traveling waves in reaction-diffusion equations can be traced back to the pioneering works of Fisher [20] and Kolmogorov, Petrovsky, and Piskunov [29] in 1937. The main concerns are the existence, uniqueness, and stability of traveling waves. For this, we refer the reader to, for example, [1, 2, 3, 9, 10, 17, 18, 19, 26, 27, 28, 35, 37, 43] and the references cited therein. For the related spatial and/or temporal discrete versions, we refer the reader to, for example, [4, 11, 12, 13, 14, 15, 32, 33, 44, 47, 53, 54, 55] and the references cited in these papers.

In the above mentioned papers, the media in which the waves propagate are homogeneous. But, in many natural environments, such as noise effect in biology and non-homogeneous porous media in transport theory, we often encounter heterogeneous media; see, for example, [40, 42]. A typical heterogeneous medium that attracts research interest is the one with certain periodicity. This can be seen in the earlier papers by Gärtner & Freidlin [22], Freidlin [21], Shigesada, Kawasaki, & Teramoto [41], etc. In a series of papers [48, 49, 50, 51, 52], Xin studied traveling waves in periodic media for a reaction-diffusion-convection equation. See also the recent papers by Nakamura [36], Shen [38, 39], Weinberger [45], Berestycki & Hamel [5], Berestycki, Hamel, & Nadirachvili [6], Berestycki, Hamel, & Roques [7, 8], and Matano [34] for various heterogeneous media.

For the discrete version of a generalized Fisher's equation in periodic media, the existence of traveling waves was first obtained by Hudson & Zinner [24, 25]. Recently, one of the authors and Hamel [23] provided a different proof for the existence of traveling waves for all speeds $c \geqslant c_{\min}$ for some positive minimal speed $c_{\min}$. Moreover, it was shown in [23] that the condition $c \geqslant c_{\min}$ is not only a sufficient condition but also a necessary condition for the existence of traveling waves.

In this paper, we are mainly concerned with the existence, uniqueness, and stability of traveling waves in the **bistable** case, i.e., under the assumption that both steady states Φ^+ and Φ^- are stable under the dynamics of (1.3). A general study of traveling waves for bistable dynamics which possesses a comparison principle can be found in [10]. Although the method presented in [10] (see also [1]) is quite general, translation invariance (equivalent to homogeneous media) is used, thereby providing a full usage of the comparison principle. Here the main difficulty is the lack of spatial (grid) translation invariance. In this article we intend to develop a general theory for the traveling wave problem for the general dynamics (1.3).

The paper is organized as follows.

In §2, we provide a few basics for the ode system (1.3). First we state and prove a comparison principle to be used throughout this paper. After introducing steady states and their stability, we study an eigenvalue problem associated with a linearization of (1.3) about a generic n-periodic steady state $\Phi = \{\phi_i\}$. The problem can be stated as follows: Given $\lambda \in \mathbb{R}$, find $(\mu, \{\psi_i\}) \in \mathbb{R} \times \mathbb{R}^{\mathbb{Z}}$ such that

$$\begin{cases} \mu\psi_i = \sum_k \, a_{i,k} e^{k\lambda} \, \psi_{i+k} + L_i \psi_i \quad \forall i \in \mathbb{Z}, \\ \psi_{i+n} = \psi_i \geqslant 0 \quad \forall i \in \mathbb{Z}, \quad \max_{i\in\mathbb{Z}}\{\psi_i\} = 1, \end{cases}$$

where $L_i = f_i'(\phi_i)$ for all i. The existence of a unique solution follows from a standard Krein-Rutman theorem [30]. Our main result, **Theorem 1**, states that *as a function of λ, $\mu(\lambda)$ is strictly convex*. This result is surprising since we do not need any specific information about the (infinite) matrix $\{a_{i,k}\}$ and the vector $\{L_i\}$. Such a convexity result provides critical information about the real roots of the characteristic equation $\mu(z) + cz = 0$ for any given $c \in \mathbb{R}$. Also, in §2, we introduce and study attraction basins of stable steady states. A few examples are also provided for illustration.

The rest of the sections can be grouped into two parts.

In part I, consisting of §3–§5, we establish the uniqueness and stability of traveling waves, under the key assumption that $\Phi^+ = \mathbf{1}$ and $\Phi^- = \mathbf{0}$ are stable steady states and that there exists at least one traveling wave, i.e., a solution to (1.3)–(1.5).

First, in §3, we show (**Theorem 2**) that *a traveling wave must have exponential tails:*

$$\lim_{i-ct\to-\infty} \frac{u_i(t)}{\psi_i^{\mathbf{0}} e^{(i-ct)\Lambda^{\mathbf{0}}}} = h^-, \quad \lim_{i-ct\to\infty} \frac{1-u_i(t)}{\psi_i^{\mathbf{1}} e^{(i-ct)\Lambda^{\mathbf{1}}}} = h^+,$$

where h^+ and h^- are certain positive constants, Λ^0 is the unique positive root and $\Lambda^{\mathbf{1}}$ is the unique negative root of the characteristic equations associated with $\Psi^- = \mathbf{0}$ and $\Psi^+ = \mathbf{1}$ respectively, and $\{\psi_i^{\mathbf{0}}\}$ and $\{\psi_i^{\mathbf{1}}\}$ are the solutions to the eigenvalue problem with $(\lambda, \{L_i\}) = (\Lambda^{\mathbf{0}}, \{f_i'(0)\})$ and $\{\lambda, \{L_i\}) = (\Lambda^{\mathbf{1}}, \{f_i'(1)\})$ respectively. The basic idea of the proof is to construct sub-super solutions which are linear combinations of three vector functions: $\{\psi_i(0)\}, \{\psi_i(\Lambda)e^{\Lambda(i-ct)}\}$ and $\{\psi_i(2\Lambda)e^{2\Lambda(i-ct)}\}$, where $\{\psi_i(\lambda)\}$ is the solution to the eigenvalue problem. Here the first one $\{\psi_i(0)\}$ is to control the tail, the second one $\{\psi_i(\Lambda)e^{\Lambda(i-ct)}\}$ is the expected asymptotic behavior, and the last one $\{\psi(2\Lambda)e^{2\Lambda(i-ct)}\}$ is introduced to control the non-linearity of f_i. A crucial key here is the basic information about the sign of the characteristic function $P(\lambda)=\mu(\lambda)+c\lambda$ at the three exponents: $P(0)<0=P(\Lambda)<P(2\Lambda)$.

Next in §4 we prove the uniqueness result, **Theorem 3**, stating that *non-zero speed traveling waves are unique up to a time translation;* namely, if $(c,\mathbf{U})$ and $(\tilde{c},\tilde{\mathbf{U}})$ are two traveling waves, then $c=\tilde{c}$ and if $c\neq 0$, then $\mathbf{U}(\cdot)=\tilde{\mathbf{U}}(\cdot+\tau)$ for some $\tau\in\mathbb{R}$. The main idea is to use the exponential tail to show that a traveling wave is monotonic in t; more precisely, if $c\neq 0$, then $c\dot{\mathbf{U}}(t)<0$ for all $t\in\mathbb{R}$.

Finally, in §5, we show (**Theorem 4**) that *non-zero speed traveling waves are globally exponentially stable*. More precisely, if the left (near $i=-\infty$) tail of an initial data $\mathbf{u}(0)$ is in the attraction basin of $\mathbf{0}$ and its right tail (near $i=\infty$) is in the attraction basin of $\mathbf{1}$, then for some constants K and τ^*

$$\|\mathbf{u}(t)-\mathbf{U}(t+\tau^*)\|_\infty\leq Ke^{-\nu t}\quad \forall t>0,$$

where ν is a positive constant depending only on $\{a_{i,k}\}$ and $\{f_i\}$. The proof follows from the original idea developed in [10], with new techniques introduced. In particular, from a dynamical system point of view we introduced a new proof showing that a vaguely resembling wave front (e.g. both tails in the attraction basin of the corresponding steady states) will evolve to an asymptotically resembling wave front (e.g. asymptotically close to the steady states at both tails).

Part II, consisting of §6-§8, is devoted to the existence of traveling waves.

In §6, we convert the traveling wave problem into an integral equation, which is quite commonly used for monostable dynamics, but rare in the study of bistable dynamics. Then in §7, we show under very general conditions, the existence of non-trivial solutions to the integral equation (**Theorem 5**). Finally, in §8, we establish the existence of a traveling wave under the assumption that $\mathbf{1}$ and $\mathbf{0}$ are the only stable steady states (**Theorem 6**). Also, we establish in **Theorem 7** the existence of a traveling wave for the following special case:

$$\dot{u}_j=a_{j+1/2}[u_{j+1}-u_j]-a_{j-1/2}[u_j-u_{j-1}]+b_j[u_{j+1}-u_{j-1}]+f(u_j),$$

where f is a bistable nonlinearity.

Typically for bistable dynamics, the existence of a traveling wave is proven by a homotopy method [16] or by taking the asymptotical limit, as $t\to\infty$, of a solution to (1.3) with an appropriate initial data [10]. Both methods are difficult to apply in this case, since here spatial translation invariance is lost and spatial monotonicity of the solution is not guaranteed. As one shall see in §6, the traveling wave problem indeed corresponds to a system of equations with n unknown functions, instead of only a scalar function for the homogeneous case. For monostable dynamics, Guo and Hamel [23] used the sub-super-solution approach and a monotonic iteration for the differential equation. For bistable dynamics, this approach faces the obvious challenge since the speed is unknown. Here in this paper, we develop a framework (§§6,7) that unifies in a certain sense the existence proof for both monostable and bistable dynamics.

2. Basics

In this section, we define and study linear stability of periodic steady states of (1.3). This is equivalent to the study of an eigenvalue problem and a characteristic equation. For reader's convenience, we begin with one of the most important tools in the study of parabolic equations—the comparison principle.

Throughout this paper, $\mathbb{R}^{\mathbb{Z}}$ is the normed space consisting of real bounded sequences of the form $\mathbf{v} = \{v_i\}_{i\in\mathbb{Z}}$ equipped with the norm $\|\mathbf{v}\|_\infty := \sup_i |v_i|$.

2.1. The Comparison Principle.

We shall use the following comparison principle for the dynamical system (1.3). For convenience, we use the notation, for $\mathbf{u}(\cdot) = \{u_i(\cdot)\}_{i\in\mathbb{Z}}$,

$$\mathcal{N}\mathbf{u} := \{\mathcal{N}_i\mathbf{u}\}_{i\in\mathbb{Z}}, \quad \mathcal{N}_i\mathbf{u}(t) := \dot{u}_i(t) - \sum_k a_{i,k}\, u_{i+k}(t) - f_i(u_i(t)).$$

Lemma 2.1. *Let $t_0 \in \mathbb{R}$, $c \in \mathbb{R}$ and $j \in \mathbb{Z} \cup \{\infty\}$. Assume that $\mathbf{u}(t) = \{u_i(t)\}$ and $\mathbf{v}(t) = \{v_i(t)\}$ are bounded and continuous in the set*

$$\{(i,t) \in \mathbb{Z}\times\mathbb{R} \mid t \geqslant t_0, i \leqslant j + c\,t + k_0\}$$

and satisfy

$$\begin{aligned} \mathcal{N}_i\mathbf{u}(t) &\geqslant \mathcal{N}_i\mathbf{v}(t) && \text{when} \quad t > t_0,\ i < j + c\,t, \\ u_i(t_0) &\geqslant v_i(t_0) && \text{when} \quad i < j + c\,t_0, \\ u_i(t) &\geqslant v_i(t) && \text{when} \quad t \geqslant t_0, j + ct \leqslant i \leqslant j + ct + k_0. \end{aligned}$$

Then $u_i(t) \geqslant v_i(t)$ for all $t > t_0$ and integers $i < j + ct$.

In addition, if $u_{i_0}(t_0) > v_{i_0}(t_0)$ for at least one integer $i_0 < j + c\,t_0$, then

$$u_i(t) > v_i(t) \quad \forall\, t > t_0,\ i < j + c\,t.$$

Proof. Although this is a well-known result, for completeness, we provide a proof. Set

$$M = \sup_{t\geqslant t_0,\, i\leqslant j+c\,t} \max\{|u_i(t)|,\ |v_i(t)|\}, \quad L = \max_{1\leqslant i\leqslant n} \max_{s\in[-M,M]} |f_i'(s)|.$$

For each fixed ε satisfying

$$0 < \varepsilon < \min_{1\leqslant i\leqslant n} \frac{1}{\sum_k a_{i,k}k^2 + (\sum_k a_{i,k}k)^2},$$

let

$$T_\varepsilon = \sup\{\tau \geqslant t_0 \mid u_i(t) + \varepsilon e^{(L+1)t}(1+\varepsilon i^2) \geqslant v_i(t) \quad \forall t \in [t_0,\tau], i \leqslant j + ct\}.$$

We claim that $T_\varepsilon = \infty$. Suppose not. Since both $\|\mathbf{u}\|_\infty$ and $\|\mathbf{v}\|_\infty$ are bounded and continuous, there exists a finite integer $l \leqslant j + cT_\varepsilon$ such that

$$u_l(T_\varepsilon) + \varepsilon e^{(L+1)T_\varepsilon}(1+\varepsilon l^2) - v_l(T_\varepsilon) = 0 \leqslant u_i(t) + \varepsilon e^{(L+1)t}(1+\varepsilon i^2) - v_i(t)$$

for all $t \in [t_0, T_\varepsilon]$ and $i \leqslant j + c\,t + k_0$. The initial and boundary values of $\mathbf{u}$ and $\mathbf{v}$ indicate that $T_\varepsilon > t_0$ and $l < j + cT_\varepsilon$. Also,

$$\dot{u}_l(T_\varepsilon) - \dot{v}_l(T_\varepsilon) = \lim_{t \nearrow T_\varepsilon} \frac{u_l(T_\varepsilon) - v_l(T_\varepsilon) - [u_l(t) - v_l(t)]}{T_\varepsilon - t} \leqslant -(L+1)\varepsilon e^{(L+1)T_\varepsilon}(1+\varepsilon l^2).$$

Hence

$$\begin{aligned}
0 &\leqslant \mathcal{N}_l\mathbf{u}(T_\varepsilon) - \mathcal{N}_l\mathbf{v}(T_\varepsilon) \\
&= \dot{u}_l - \dot{v}_l + \sum_k a_{l,k}\{[v_{l+k} - v_l] - [u_{l+k} - u_l]\} + f_l(v_l) - f_l(u_l)\Big|_{t=T_\varepsilon} \\
&\leqslant -(L+1)\varepsilon e^{(L+1)T_\varepsilon}(1+\varepsilon l^2) + \sum_k a_{l,k}\{2lk + k^2\}\varepsilon^2 e^{(L+1)T_\varepsilon} + L\varepsilon e^{(L+1)T_\varepsilon}(1+\varepsilon l^2) \\
&= \varepsilon e^{(L+1)T_\varepsilon}\Big\{ -1 - \varepsilon l^2 + 2\varepsilon l \sum_k a_{l,k}k + \varepsilon \sum_k a_{l,k}k^2 \Big\} \\
&\leqslant \varepsilon e^{(L+1)T_\varepsilon}\Big\{ -1 + \varepsilon\Big(\sum_k a_{l,k}k\Big)^2 + \varepsilon \sum_k a_{l,k}k^2 \Big\} < 0
\end{aligned}$$

and we obtain a contradiction. Thus, we must have $T_\varepsilon = \infty$; i.e.,

$$u_i(t) + \varepsilon e^{Lt}(1+\varepsilon i^2) \geqslant v_i \quad \forall\, t \geqslant t_0, i \leqslant j + c\,t.$$

Sending $\varepsilon \searrow 0$ we obtain $u_i \geqslant v_i$ for all $t \geqslant t_0, i \leqslant j + c\,t$.

Now suppose $u_{i_0}(t_0) > v_{i_0}(t_0)$ for some $i_0 < j + ct_0$. Then using $u_i \geqslant v_i$ and the non-negativity of $a_{i,k}$ for $k \neq 0$, we have

$$\begin{aligned}
\frac{d}{dt}(u_{i_0} - v_{i_0}) &= \mathcal{N}_{i_0}\mathbf{u} - \mathcal{N}_{i_0}\mathbf{v} + \sum_k a_{i_0,k}[u_{i_0+k} - v_{i_0+k}] + f_{i_0}(u_{i_0}) - f_{i_0}(v_{i_0}) \\
&\geqslant -(|a_{i_0,0}| + L)\,(u_{i_0} - v_{i_0}) \quad \text{when } \ t > t_0, i_0 < j + ct.
\end{aligned}$$

Gronwall's Inequality then implies that

$$u_{i_0}(t) - v_{i_0}(t) \geq (u_{i_0}(t_0) - v_{i_0}(t_0))e^{-(|a_{i_0,0}|+L)(t-t_0)}$$

for all t satisfying $t > t_0$ and $i_0 < j + ct$. Now let i_1 be an integer such that $a_{i_1,i_0-i_1} > 0$. A similar calculation as above gives

$$\frac{d}{dt}(u_{i_1} - v_{i_1}) \ \geqslant\ -(|a_{i_1,0}| + L)\,(u_{i_1} - v_{i_1}) + a_{i_1,i_0-i_1}[u_{i_0} - v_{i_0}].$$

This implies that $u_{i_1} > v_{i_1}$ for every t satisfying $t > t_0$ and $i_1 < j + ct$. Using the non-decoupledness assumption, we can inductively show that $u_i > v_i$ for all $t > t_0$ and integer i satisfying $i < j + ct$. This completes the proof. □

Remark 2.1. *The proof of the strong comparison can be refined to show that there exists a sequence of positive functions $\{\delta_i(t)\}_{i\in\mathbb{Z}}$ on $(0,\infty)$ that depends only on M, $\{a_{i,k}\}$ and $\{f_i\}$ such that under the assumption of Lemma 2.1 with $j=\infty$,*

$$u_{l+i}(t)-v_{l+i}(t)\geqslant [u_l(0)-v_l(0)]\delta_i(t)\quad \forall\, l,i\in\mathbb{Z}, t>0.$$

2.2. **Steady States and Their Stability.** A **steady state** (or equilibrium) of (1.3) is a vector $\Phi=\{\phi_i\}_{i\in\mathbb{Z}}$ satisfying

$$\sum_k a_{i,k}\phi_{i+k}+f_i(\phi_i)=0\quad \forall\, i\in\mathbb{Z}. \tag{2.1}$$

It is called **periodic** (or more precisely n-periodic) if $\phi_{i+n}=\phi_i$ for every $i\in\mathbb{Z}$.

The linearization of (1.3) around a steady state $\Phi=\{\phi_i\}$ is, for $\mathbf{v}=\mathbf{v}(t)=\{v_i(t)\}$,

$$\dot v_i(t)=\sum_k a_{i,k}\, v_{i+k}(t)+f_i'(\phi_i)v_i(t),\quad t>0, i\in\mathbb{Z}. \tag{2.2}$$

In the sequel, a vector $\Psi=\{\psi_i\}_{i\in\mathbb{Z}}\in\mathbb{R}^{\mathbb{Z}}$ is called bounded if $\|\Psi\|_\infty:=\sup_i|\psi_i|<\infty$.

Definition 2.1. *A steady state $\Phi=\{\phi_i\}_{i\in\mathbb{Z}}$ is called* **stable** *if any solution to* (2.2) *with bounded initial data approaches zero as $t\to\infty$. It is called* **unstable** *if* (2.2) *admits an unbounded solution for some bounded initial data.*

To investigate the stability, we consider solutions of (2.2) that have the form $\mathbf{v}(t)=\{\psi_i e^{\mu t+i\lambda}\}$ and satisfy $v_{i+cT}(T)=v_i(0)\geqslant 0$ for all $i\in\mathbb{Z}$, where μ is a real number, T is a positive constant, and c is a nonzero constant such that cT is an integer multiple of n. If λ and μ are related by $\mu+c\lambda=0$, then $\{\psi_i\}$ is periodic. This leads to the following eigenvalue problem: Given $\lambda\in\mathbb{R}$, find $(\mu,\{\psi_i\})\in\mathbb{R}\times\mathbb{R}^{\mathbb{Z}}$ such that

$$\begin{cases} \mu\psi_i=\sum_k\ a_{i,k}e^{k\lambda}\ \psi_{i+k}+L_i\psi_i \quad \forall i\in\mathbb{Z},\\ \psi_{i+n}=\psi_i\geqslant 0\quad \forall i\in\mathbb{Z},\quad \max_{i\in\mathbb{Z}}\{\psi_i\}=1, \end{cases} \tag{2.3}$$

where $L_i=f_i'(\phi_i)$. The following result will play a fundamental role in our analysis.

Theorem 1. *Suppose $L_{i+n}=L_i\in\mathbb{R}$ for every $i\in\mathbb{Z}$. Then for each $\lambda\in\mathbb{R}$,* (2.3) *admits a unique solution $(\mu=\mu(\lambda),\Psi=\Psi(\lambda))$. In addition, as a function of $\lambda\in\mathbb{R}$, $\mu(\cdot)$ and $\Psi(\cdot)$ are analytic, $\mu(\cdot)$ is strictly convex,*

$$\min_i L_i\leqslant \mu(0)\leqslant \max_i L_i,$$

and $\liminf_{|\lambda|\to\infty}\mu(\lambda)\ e^{-|\lambda|}>0$. *Consequently, for any $c\in\mathbb{R}$, there are at most two real roots to the* **characteristic equation**

$$P(c,\cdot)=0\quad \textit{where } P(c,\lambda):=c\lambda+\mu(\lambda). \tag{2.4}$$

Furthermore, suppose $\Phi=\{\phi_i\}$ is an n-periodic steady state and $L_i=f_i'(\phi_i)$ for all i. Then Φ is stable if and only if $\mu(0)<0$, and Φ is unstable if and only if $\mu(0)>0$.

Proof. We divide the proof into several steps.

1. Decomposing $i+k$ as $j+mn$ where $j \in \{1,\cdots,n\}$ and $m \in \mathbb{Z}$ and using $\psi_{i+k} = \psi_j$, we obtain

$$\sum_k a_{i,k}\, e^{k\lambda}\psi_{i+k} \;=\; \sum_{j=1}^{n}\sum_{m\in\mathbb{Z}} a_{i,j+mn-i}e^{(j+mn-i)\lambda}\psi_j =: \sum_{j=1}^{n} A_{ij}\psi_j,$$

where

$$A_{ij} = A_{ij}(\lambda) := \sum_{m\in\mathbb{Z}} a_{i,j+mn-i}e^{(j+mn-i)\lambda}.$$

Thus, (2.3) is equivalent to, for unknown $\mu \in \mathbb{R}$ and $\mathbf{x} = (x_1,\cdots,x_n)^T \in \mathbb{R}^n$,

$$\mu\,\mathbf{x} = \mathbf{B}\mathbf{x}, \quad \mathbf{x} \geqslant 0, \quad \|\mathbf{x}\|_\infty = 1, \tag{2.5}$$

where

$$\mathbf{B} = \mathbf{B}(\lambda) := (\ A_{ij}(\lambda)\ )_{n\times n} + \mathrm{diag}(L_1,\cdots,L_n).$$

Here $\mathbf{x} = (x_1,\cdots,x_n)^T \geqslant 0$ or (>0) means that $x_i \geqslant 0$ (or >0) for all $i = 1,\cdots,n$. Also, $\|\mathbf{x}\|_\infty = \max_i\{|x_i|\}$. One can verify that all the non-diagonal entries of the matrix $\mathbf{B}$ are non-negative. In addition, for $\kappa := \max_{1\leqslant i\leqslant n}\{|A_{ii}| + |L_i|\} + 1$, the non-decoupledness condition implies that

$$\mathbf{x} \geqslant 0, \mathbf{x} \neq 0 \quad \Rightarrow \quad [\kappa\,\mathbf{I} + \mathbf{B}]^n\mathbf{x} > 0.$$

2. The eigen-problem (2.5) can be solved as follows. Define

$$\mathbf{X} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \geqslant 0, \ \ \|\mathbf{x}\|_\infty = 1\}, \quad \mu^* = \sup_{\mathbf{x}\in\mathbf{X}}\ \sup\ \{h \in \mathbb{R} \mid h\mathbf{x} \leqslant \mathbf{B}\mathbf{x}\}.$$

One can show that there exists $\mathbf{x}^* \in \mathbf{X}$ such that $\mu^*\mathbf{x}^* \leqslant \mathbf{B}\mathbf{x}^*$. We claim that $\mu^*\mathbf{x}^* = \mathbf{B}\mathbf{x}^*$. Indeed, if $\mathbf{B}\mathbf{x}^* \neq \mu^*\mathbf{x}^*$, then, since $\mathbf{B}\mathbf{x}^* - \mu^*\mathbf{x}^* \geqslant 0$,

$$0 < [\kappa\,\mathbf{I} + \mathbf{B}]^n[\mathbf{B}\mathbf{x}^* - \mu^*\mathbf{x}^*] = \mathbf{B}[\kappa\,\mathbf{I} + \mathbf{B}]^n\mathbf{x}^* - \mu^*[\kappa\,\mathbf{I} + \mathbf{B}]^n\mathbf{x}^*.$$

This implies, for $\mathbf{y} := [\kappa\,\mathbf{I} + \mathbf{B}]^n\mathbf{x}^*/\|[\kappa\,\mathbf{I} + \mathbf{B}]^n\mathbf{x}^*\|_\infty$, that $\mathbf{y} \in \mathbf{X}$ and $\mu^*\mathbf{y} < \mathbf{B}\mathbf{y}$, i.e., $(\mu^* + \varepsilon)\mathbf{y} \leqslant \mathbf{B}\mathbf{y}$ for some small positive ε, contradicting the definition of μ^*. Thus, $\mu^*\mathbf{x}^* = \mathbf{B}\mathbf{x}^*$ and we obtain a solution to (2.5).

3. For uniqueness, suppose $(\mu, \mathbf{x})$ is an arbitrary solution to (2.5). Then

$$[\kappa + \mu]^n\mathbf{x} = [\kappa\,\mathbf{I} + \mathbf{B}]^n\mathbf{x} > 0,$$

i.e., $\mathbf{x} > 0$. Also, by the definition of μ^*, $\mu \leqslant \mu^*$. Set $K = \min\{k > 0 \mid \mathbf{x}^* \leqslant k\mathbf{x}\}$. Then $\mathbf{x}^* \leqslant K\mathbf{x}$. We claim that $\mathbf{x}^* = K\mathbf{x}$. Indeed, if it is not true, then, since $K\mathbf{x} - \mathbf{x}^* \geqslant 0$,

$$\begin{aligned} 0 &< [\kappa\mathbf{I} + \mathbf{B}]^n(K\mathbf{x} - \mathbf{x}^*) \\ &= K(\kappa + \mu)^n\mathbf{x} - (\kappa + \mu^*)^n\mathbf{x}^* \\ &\leqslant K(\kappa + \mu^*)^n\mathbf{x} - (\kappa + \mu^*)^n\mathbf{x}^* \\ &= (\kappa + \mu^*)^n(K\mathbf{x} - \mathbf{x}^*) \end{aligned}$$

which implies that $K\mathbf{x} > \mathbf{x}^*$. This contradicts to the minimality of K. Thus, $\mathbf{x}^* = K\mathbf{x}$, which implies that $K = 1$ and $\mu = \mu^*$.

4. Using an idea similar to that in the previous step one can show that μ is a simple eigenvalue of $\mathbf{B}$, so that μ is a simple zero of the determinant of the matrix $\mu\mathbf{I}-\mathbf{B}$. Since $\mathbf{B}$ is analytic in λ, we see that $\mu=\mu(\lambda)$ is also an analytic function of $\lambda\in\mathbb{R}$.

Let $\psi_j=1$. Using $a_{j,k}\geqslant 0$ for $k\neq 0$ and (2.3), we have

$$\mu(0)\leqslant\sum_k a_{j,k}+L_j=L_j\leqslant\max_i L_i.$$

Set $\psi_l=\min_i\psi_i$. Note that $\psi_l>0$. Then from (2.3) it follows that

$$\mu(0)\psi_l\geqslant\sum_k a_{l,k}\psi_l+L_l\psi_l=L_l\psi_l\quad\Rightarrow\quad\mu(0)\geqslant\min_i L_i.$$

Finally, one derives from the equation

$$(\mu-a_{i,0}-L_i)\psi_i=\sum_{k\neq 0}a_{i,k}e^{k\lambda}\psi_{i+k}$$

and the non-decoupledness condition that μ has an order at least $e^{|\lambda|}$ as $\pm\lambda\to\infty$; that is, $\liminf_{|\lambda|\to\infty}\mu(\lambda)e^{-|\lambda|}\in(0,\infty)\cup\{\infty\}$.

5. We now show that $\mu(\lambda)$ is strictly convex. Let λ_1 and λ_2 be any different real numbers and $t\in(0,1)$. Consider $\lambda=t\lambda_1+(1-t)\lambda_2$. We shall use the inequality

$$x^tz^{1-t}<tx+(1-t)z\quad\forall\, x>0,z>0,x\neq z.$$

For all $i\in\mathbb{Z}$, set

$$\tilde\psi_i(\lambda_2):=\Big[\frac{\psi_i(\lambda)}{\psi_i^t(\lambda_1)}\Big]^{1/(1-t)}\qquad\Rightarrow\qquad\psi_i(\lambda)=[\psi_i(\lambda_1)]^t[\tilde\psi_i(\lambda_2)]^{1-t}.$$

Then

$$\begin{aligned}
\mu(\lambda)-a_{i,0}-L_i&=\frac{1}{\psi_i(\lambda)}\sum_{k\neq 0}a_{i,k}e^{k\lambda}\psi_{i+k}(\lambda)\\
&=\sum_{k\neq 0}a_{i,k}\Big[e^{k\lambda_1}\frac{\psi_{i+k}(\lambda_1)}{\psi_i(\lambda_1)}\Big]^t\Big[e^{k\lambda_2}\frac{\tilde\psi_{i+k}(\lambda_2)}{\tilde\psi_i(\lambda_2)}\Big]^{1-t}\\
&\leqslant t\sum_{k\neq 0}a_{i,k}e^{k\lambda_1}\frac{\psi_{i+k}(\lambda_1)}{\psi_i(\lambda_1)}+(1-t)\sum_{k\neq 0}a_{i,k}e^{k\lambda_2}\frac{\tilde\psi_{i+k}(\lambda_2)}{\tilde\psi_i(\lambda_2)}\\
&=t\{\mu(\lambda_1)-a_{i,0}-L_i\}+(1-t)\sum_{k\neq 0}a_{i,k}e^{k\lambda_2}\frac{\tilde\psi_{i+k}(\lambda_2)}{\tilde\psi_i(\lambda_2)}.
\end{aligned}$$

The inequality must be strict for at least one i, since otherwise we would have

$$e^{k\lambda_1}\frac{\psi_{i+k}(\lambda_1)}{\psi_i(\lambda_1)}=e^{k\lambda_2}\frac{\tilde\psi_{i+k}(\lambda_2)}{\tilde\psi_i(\lambda_2)}\quad\text{whenever}\quad a_{i,k}>0.$$

By non-decoupledness condition, this implies that

$$\psi_j(\lambda_1)=\frac{\psi_0(\lambda_1)}{\tilde\psi_0(\lambda_2)}e^{(\lambda_2-\lambda_1)j}\tilde\psi_j(\lambda_2)$$

for all j, which is impossible since both $\{\psi_i(\lambda_1)\}$ and $\{\tilde\psi_i(\lambda_2)\}$ are n-periodic.

Thus, for $\mathbf{x} := (\tilde{\psi}_1(\lambda_2), \cdots, \tilde{\psi}_n(\lambda_2))^T$, $(1-t)\mathbf{B}(\lambda_2)\mathbf{x} \geqslant (\neq)\{\mu(\lambda) - t\mu(\lambda_1)\}\mathbf{x}$. Consequently, by the characterization of the eigenvalue, we see that $(1-t)\mu(\lambda_2) > \mu(\lambda) - t\mu(\lambda_1)$, i.e., $\mu(\lambda) < t\mu(\lambda_1) + (1-t)\mu(\lambda_2)$. Hence $\mu(\cdot)$ is strictly convex.

6. Suppose $L_i = f_i'(\phi_i)$ where $\Phi = \{\phi_i\}$ is periodic. We show that Φ is stable if and only if $\mu(0) < 0$.

(i) Suppose $\mu(0) < 0$. Let $\mathbf{v}(\cdot) = \{v_i(\cdot)\}$ be a solution to (2.2) with bounded initial data. Set $M = \sup_i |v_i(0)|/\psi_i(0)$. Then by the comparison principle, $-M\psi_i(0)e^{\mu(0)t} \leqslant v_i(t) \leqslant M\psi_i(0)e^{\mu(0)t}$ for all $i \in \mathbb{Z}$ and all $t \geqslant 0$. It then follows that, uniformly in $i \in \mathbb{Z}$, $v_i(t)$ decays to zero exponentially fast as $t \to \infty$. Thus Φ is stable.

(ii) Suppose $\mu(0) \geqslant 0$. Then $\mathbf{v} := \{\psi_i(0)e^{\mu(0)t}\}$ is a solution to (2.2) with bounded initial data, so that Φ is not stable.

The above two arguments imply that Φ is stable if and only if $\mu(0) < 0$. In a similar manner, one can show that Φ is unstable if and only if $\mu(0) > 0$. This completes the proof of Theorem 1. $\square$

2.3. **An Example for Steady State.** Suppose there exist constants $M^{\pm}$ such that

$$f_i(\cdot) < 0 \quad \text{in} \quad (M^+, \infty), \qquad f_i(\cdot) > 0 \quad \text{in} \quad (-\infty, M^-) \quad \forall\, i \in \mathbb{Z}.$$

Pick an arbitrary constant $M > M^+$. Consider the initial value problem of the ode system (1.3) subject to the initial condition

$$u_i(0) = M \quad \forall\, i \in \mathbb{Z}.$$

Simple comparison shows that $u_{i+n}(\cdot) \equiv u_i(\cdot)$ and

$$\dot{u}_i(t) \leqslant 0, \quad M^- \leqslant u_i(t) \leqslant M \quad \forall\, i \in \mathbb{Z}.$$

It follows that for each $i \in \mathbb{Z}$, there exists the limit $\phi_i^* := \lim_{t\to\infty} u_i(t)$. One can show that $\Phi^* := \{\phi_i^*\}$ is an equilibrium of (1.3). This equilibrium is n-periodic and is maximal in the sense that if $\Phi = \{\phi_i\}$ is a bounded equilibrium, then $\Phi \leqslant \Phi^*$, i.e., $\phi_i \leqslant \phi_i^*$ for every $i \in \mathbb{Z}$.

Similarly, one can show that there exists an equilibrium $\Phi_* = \{\phi_{*i}\}$ that is n-periodic and is minimal in the sense that if Φ is a bounded equilibrium, then $\Phi \geqslant \Phi_*$.

Notice that if $\mathbf{u}(t) = \{u_i(t)\}, t \geqslant 0$, is a solution to (1.3) with bounded initial data, then, by the comparison principle,

$$\phi_{*i} \leqslant \liminf_{t\to\infty} u_i(t) \leqslant \limsup_{t\to\infty} u_i(t) \leqslant \phi_i^* \quad \text{uniformly in} \quad i \in \mathbb{Z}.$$

2.4. **Normalization.** Suppose $\Phi^\pm = \{\phi_i^\pm\}$ are two ordered periodic steady states.

Introduce a change of variable $\mathbf{u} = \{u_i\} \mapsto \Theta = \{\theta_i\}$:

$$u_i = \theta_i\, \phi_i^+ + [1-\theta_i]\, \phi_i^- \quad \forall\, i \in \mathbb{Z}.$$

Then the system (1.3) is equivalent to the new system, for $\Theta(t) = \{\theta_i(t)\}$,

$$\dot\theta_i = \sum_{k\neq 0} \tilde a_{i,k}[\theta_{i+k} - \theta_i] + \tilde f_i(\theta_i)$$

where

$$\begin{aligned} \tilde f_i(s) &= \frac{f_i(s\phi_i^+ + [1-s]\phi_i^-) - s f_i(\phi_i^+) - [1-s] f_i(\phi_i^-)}{\phi_i^+ - \phi_i^-} \quad \forall\, i \in \mathbb{Z}, \\ \tilde a_{i,k} &= \frac{\phi_{i+k}^+ - \phi_{i+k}^-}{\phi_i^+ - \phi_i^-}\, a_{i,k} \quad \forall\, i \in \mathbb{Z},\ k \neq 0. \end{aligned}$$

Notice that $\tilde f_{i+n}(\cdot) = \tilde f_i(\cdot)$, $\tilde a_{i+n,k} = \tilde a_{i,k}$ and $\tilde f_i(0) = 0 = \tilde f_i(1)$ for all i, k.

Observe that the transformation does not change the stability of any equilibrium. In particular, the stability of the equilibria Φ^+ and Φ^- of the original system (1.3) are the same as that of $\mathbf{1} := \{1\}_{i\in\mathbb{Z}}$ and $\mathbf{0} := 0\mathbf{1}$ of the new system. This can be verified by using the transformation from $\tilde{\mathbf{v}}(t) = \{\tilde v_i(t)\}$ to $\mathbf{v}(t) = \{v_i(t)\}$ via $v_i(t) = \phi_i^- + (\phi_i^+ - \phi_i^-)\tilde v_i(t)$ for the corresponding linearized equations.

2.5. **An Example for Stability.**

Lemma 2.2. *Suppose $f_i = f$ for all $i \in \mathbb{Z}$, $f \in C^1(\mathbb{R})$, and $f(a) = 0$. Then $a\mathbf{1}$ is a stable steady state if and only if $f'(a) < 0$; it is unstable if and only if $f'(a) > 0$.*

Proof. The assertion follows from Theorem 1 by observing that $L_i = f_i'(a) = f(a)$ for all $i \in \mathbb{Z}$. $\square$

2.6. **Attraction Basin of Stable Steady States.** Note that our Definition 2.1 of stability for steady states is in the sense of linear stability. Here, we shall show that the linear stability implies a form of non-linear stability. For this purpose, we denote by $\mathbf{u}(t) = \mathbb{S}^t\mathbf{v}$ the solution to (1.3) with initial value $\mathbf{u}(0) = \mathbf{v}$.

Suppose $\Phi \in \mathbb{R}^{\mathbb{Z}}$ is an n-periodic steady state of (1.3). Its **attraction basin** consists of all vectors $\mathbf{v} \in \mathbb{R}^{\mathbb{Z}}$ such that the solution $\mathbf{u}$ to (1.3) with initial value $\mathbf{u}(0) = \mathbf{v}$ satisfies $\lim_{t\to\infty} \|\mathbf{u}(t) - \Phi\|_\infty = 0$. More precisely,

$$\mathcal{A}(\Phi) := \{\mathbf{v} \in \mathbb{R}^{\mathbb{Z}} \mid \lim_{t\to\infty} \|\mathbb{S}^t\mathbf{v} - \Phi\|_\infty = 0\}.$$

In the sequel, we shall assume that

$$|f_i'(s)| \leqslant L, \quad |f_i''(s)| \leqslant M \quad \forall s \in \mathbb{R},\ i \in \mathbb{Z}.$$

Lemma 2.3. *Assume that $\Phi = \{\phi_i\}$ is an n-periodic steady state and is stable in the sense of Definition 2.1. Then the following statements hold:*

(i) Φ *is* **non-linearly stable** *in the sense that there exists $\eta > 0$ satisfying*

$$\mathcal{A}(\Phi) \supset \{\mathbf{v} \in \mathbb{R}^{\mathbb{Z}} \mid \|\mathbf{v} - \Phi\|_\infty \leqslant \eta\}.$$

(ii) *Suppose $\mathbf{v} = \{v_i\} \in \mathcal{A}(\Phi)$. Then for every $\varepsilon > 0$, there exist positive constants δ and τ such that the solutions $\mathbf{w}^+$ and $\mathbf{w}^-$ to*

$$\mathcal{N}\mathbf{w}^\pm(t) = \pm\delta\mathbf{1} \quad \forall t > 0, \quad \mathbf{w}^\pm(0) = \mathbf{v} \pm \delta\mathbf{1} \tag{2.6}$$

satisfy

$$\Phi - \varepsilon\mathbf{1} \leqslant \mathbf{w}^-(t) < \mathbf{w}^+(t) \leqslant \Phi + \varepsilon\mathbf{1} \quad \forall t \geqslant \tau.$$

Proof. (i) Let $\Psi = \{\psi_i\}$ be the solution to (2.3) with $L_i = f_i'(\phi_i)$, $\lambda = 0$ and $\mu = \mu(0)$. Since Φ is a stable steady state, $\mu < 0$. Consider for $\eta > 0$ the function

$$\mathbf{u}^\pm(t) = \Phi \pm \eta e^{\mu t/2}\Psi.$$

We can calculate

$$\begin{aligned}\mathcal{N}_i\mathbf{u}^+(t) &= \eta e^{\mu t/2}\Big\{\tfrac{1}{2}\mu\psi_i - \sum_k a_{i,k}\psi_{i+k} - f_i'(\phi_i)\psi_i\Big\} \\ &\quad + \Big\{f_i(\phi_i) + f_i'(\phi_i)\eta\psi_i e^{\mu t/2} - f_i(\phi_i + \eta\psi_i e^{\mu t/2})\Big\}.\end{aligned}$$

Using the equation for Ψ and the boundedness of f_i'', we obtain

$$\mathcal{N}_i\mathbf{u}^+(t) \geqslant -\tfrac{1}{2}\mu\eta e^{\mu t/2}\psi_i - M\eta^2 e^{\mu t}\psi_i^2 = \tfrac{1}{2}\eta\psi_i e^{\mu t/2}\Big\{-\mu - 2M\eta e^{\mu t/2}\psi_i\Big\} > 0$$

provided $\eta < |\mu|/(2M)$. Thus $\mathbf{u}^+$ is a supersolution. Similarly, we can show that $\mathbf{u}^-(t)$ is a subsolution. Thus, whenever $-\eta\Psi \leqslant \mathbf{v} - \Phi \leqslant \eta\Psi$, $\mathbf{u}^-(t) \leqslant \mathbb{S}^t\mathbf{v} \leqslant \mathbf{u}^+(t)$ for all $t > 0$. Consequently, $\|\mathbb{S}^t\mathbf{v} - \Phi\|_\infty \leqslant \eta e^{\mu t/2}$ for all $t > 0$. This proves the assertion (i).

(ii) Let $\varepsilon \in (0, \eta]$ be an arbitrarily fixed small positive number. It is clear that $\mathbf{w}^-(t) < \mathbf{w}^+(t)$ for all $t \geqslant 0$. We prove the assertion (ii) in two steps. In the first step we consider a special $\mathbf{v}$ and in the second step a general $\mathbf{v}$.

1. Since $\Phi + \eta\mathbf{1} \in \mathcal{A}(\Phi)$, there exists $\tau_1 > 0$ such that $\|\mathbb{S}^t(\Phi + \eta\mathbf{1}) - \Phi\|_\infty < \varepsilon/2$ for all $t \geqslant \tau_1$. Now by continuity, there exists $\delta_1 > 0$ such that the solution $\mathbf{w} = \{w_i\}$ to

$$\mathcal{N}\mathbf{w}(t) = \delta_1\mathbf{1} \quad \forall\, t > 0, \quad \mathbf{w}(0) = \Phi + \eta\mathbf{1}$$

satisfies $\mathbf{w}(t) \leq \mathbb{S}^t(\Phi + \eta\mathbf{1}) + \frac{1}{2}\varepsilon\mathbf{1}$ for all $t \in [0, 2\tau_1]$. Consequently, this implies that $\mathbf{w}(t) \leqslant \Phi + \varepsilon\mathbf{1}$ for all $t \in [\tau_1, 2\tau_1]$; in particular, as $\varepsilon \leqslant \eta$, $\mathbf{w}(\tau_1) \leqslant \Phi + \eta\mathbf{1} = \mathbf{w}(0)$, so that by the comparison principle, $\mathbf{w}(t + \tau_1) \leqslant \mathbf{w}(t)$ for all $t \geqslant 0$. Starting from $\mathbf{w}(t) \leq \Phi + \varepsilon\mathbf{1}$ for all $t \in [\tau_1, 2\tau_1]$, an induction then implies that $\mathbf{w}(t) \leqslant \Phi + \varepsilon\mathbf{1}$ for all $t \in [k\tau_1, (k+1)\tau_1]$ and for all positive integer k. Hence $\mathbf{w}(t) \leqslant \Phi + \varepsilon\mathbf{1}$ for all $t \geqslant \tau_1$.

2. Now suppose $\mathbf{v} \in \mathcal{A}(\Phi)$. Then there exists $\tau_2 > 0$ such that $\mathbb{S}^t\mathbf{v} \leq \Phi + \frac{1}{2}\eta\mathbf{1}$ for all $t \geqslant \tau_2$. Consequently, there exists $\delta \in (0, \min\{\eta, \delta_1\}]$ such that the solution $\mathbf{w}^+$ to (2.6) satisfies $\mathbf{w}^+(\tau_2) \leqslant \Phi + \eta\mathbf{1}$. As $\delta \leqslant \delta_1$, we then have $\mathbf{w}^+(t + \tau_2) \leqslant \mathbf{w}(t)$ for all $t > 0$. Thus, $\mathbf{w}^+(t) \leqslant \mathbf{w}(t - \tau_2) \leqslant \Phi + \varepsilon\mathbf{1}$ for all $t \geqslant \tau_1 + \tau_2 =: \tau$.

Similarly, we can estimate the lower bound. This completes the proof. □

The following special case provides us an idea about the size of $\mathcal{A}(\Phi)$.

Lemma 2.4. *Assume that for some $a \in \mathbb{R}$ and positive constants α and β,*

$$f_i(z) > 0 = f_i(a) > f_i(s) \quad \forall z \in [a-\beta, a), s \in (a, a+\alpha], i \in \mathbb{Z}.$$

Then $\{\mathbf{v} \in \mathbb{R}^{\mathbb{Z}} \mid -\beta\mathbf{1} \leqslant \mathbf{v} - a\mathbf{1} \leqslant \alpha\mathbf{1}\} \subset \mathcal{A}(a\mathbf{1})$.

Proof. Set

$$f^+(s) = \max_{1\leqslant i\leqslant n} f_i(s), \quad f^-(z) = \min_{1\leqslant i\leqslant n} f_i(z).$$

Then $f^+(s) < 0 = f^\pm(a) < f^-(z)$ for all $z \in [a-\beta, a)$ and $s \in (a, a+\alpha]$. Denote by $w^\pm(t)$ the solution to

$$\dot{w}^\pm(t) = f^\pm(w^\pm(t)), \quad w^+(0) = a+\alpha, \quad w^-(0) = a-\beta.$$

It is easy to show that $\lim_{t\to\infty} w^\pm(t) = a$ and that $w^+(t)\mathbf{1}$ is a supersolution and $w^-(t)\mathbf{1}$ is a subsolution to (1.3). Hence, if $\mathbf{v} \in \mathbb{R}^{\mathbb{Z}}$ satisfies $(a-\beta)\mathbf{1} \leqslant \mathbf{v} \leqslant (a+\alpha)\mathbf{1}$, then $w^-(t)\mathbf{1} \leqslant \mathbb{S}^t\mathbf{v} \leqslant \mathbf{w}^+(t)\mathbf{1}$ for all $t > 0$. This implies that $\lim_{t\to\infty} \|\mathbb{S}^t\mathbf{v} - a\mathbf{1}\|_\infty = 0$, thereby completing the proof. □

The rest of the paper is divided into two parts. In the first part, consisting of §§3–5, we shall study the uniqueness and globally asymptotic stability of traveling waves, under the assumption that the steady states Φ^+ and Φ^- are stable. In the second part, remaining sections, we consider the existence of traveling waves.

Part I: Uniqueness and Global Stability

3. Exponential Tail Near Stable States

For a traveling wave, its asymptotic behavior near $i = \pm\infty$ determines its stability and other important properties of the wave. In this section, we investigate the rate of approach to steady state as $i \to \infty$ or $i \to -\infty$ of a generic traveling wave. We shall only consider the case that the steady state is stable.

3.1. **The Exponential Tails.** Suppose $(c, \mathbf{u})$ solves (1.3)–(1.5). Since $\mathbf{u}$ is a traveling wave, it is natural to postulate that, for some positive constants $h^\pm$,

$$\begin{aligned} u_i(t) - \phi_i^- &\approx h^- e^{\lambda^-(i-ct)}\psi_i^- \quad \text{as } i \to -\infty, \\ \phi_i^+ - u_i(t) &\approx h^+ e^{\lambda^+(i-ct)}\psi_i^+ \quad \text{as } i \to \infty \end{aligned}$$

where $\lambda^+ < 0 < \lambda^-$ and $\{\psi_i^\pm\}$ are n-periodic. This leads to the system (2.3), together with the characteristic equation $\mu(\lambda) + c\lambda = 0$.

By the normalization mentioned in §2.4, in the sequel, we assume without loss of generality that $\Phi^+ = \mathbf{1}$ and $\Phi^- = \mathbf{0}$. For each $\lambda \in \mathbb{R}$, we denote by $(\mu^{\mathbf{0}}(\lambda), \Psi^{\mathbf{0}}(\lambda))$ the solution to (2.3) with $L_i := f_i'(0)$, and by $(\mu^{\mathbf{1}}(\lambda), \Psi^{\mathbf{1}}(\lambda))$ the solution with $L_i = f_i'(1)$. The characteristic equations associated with $\mathbf{0}$ and $\mathbf{1}$ are, respectively,

$$P^{\mathbf{0}}(c,\lambda) := \mu^{\mathbf{0}}(\lambda) + c\lambda = 0, \quad \lambda \geqslant 0; \tag{3.1}$$

$$P^{\mathbf{1}}(c,\lambda) := c\lambda + \mu^{\mathbf{1}}(\lambda) = 0, \quad \lambda \leqslant 0. \tag{3.2}$$

If $\mathbf{0}$ is a stable steady state, then (3.1) has exactly one root, which is positive, denoted by $\Lambda^{\mathbf{0}}$. We denote by $\{\psi_i^{\mathbf{0}}\}$ the corresponding eigenvector to (2.3). Analogously, if $\mathbf{1}$ is a stable steady state, then (3.2) has exactly one root, which is negative, denoted by $\Lambda^{\mathbf{1}}$. The corresponding eigenvector to (2.3) will be denoted by $\{\psi_i^{\mathbf{1}}\}$. We shall prove the following:

Theorem 2. *Assume that $(c, \mathbf{u})$ is a solution to (1.3)–(1.5) with $\Phi^+ = \mathbf{1}$ and $\Phi^- = \mathbf{0}$.*
If $\mathbf{0}$ is a stable steady state, then there exists a positive constant h^- such that

$$\lim_{i-ct\to-\infty} \frac{u_i(t)}{\psi_i^{\mathbf{0}} e^{(i-ct)\Lambda^{\mathbf{0}}}} = h^-.$$

Similarly, if $\mathbf{1}$ is a stable steady state, there exists a positive constant h^+ such that

$$\lim_{i-ct\to\infty} \frac{1-u_i(t)}{\psi_i^{\mathbf{1}} e^{(i-ct)\Lambda^{\mathbf{1}}}} = h^+.$$

This theorem will be proved in the subsequent subsections, by the comparison principle and construction of sub/super solutions.

3.2. **Sub/Supersolutions.** Assume that $\Phi = \{\phi_i\}$ is a stable n-periodic steady state. For each $\lambda \in \mathbb{R}$, denote by $(\mu(\lambda), \Psi(\lambda))$ the unique solution to (2.3) with $L_i = f_i'(\phi_i)$ for all $i \in \mathbb{Z}$. Let $P(c,\lambda) = c\lambda + \mu(\lambda)$ be the characteristic function. That Φ is a stable equilibrium means that $\mu(0) < 0$, so that $P(c,\lambda) = 0$ has exactly two roots, one positive and the other negative. For definiteness, we shall study the solution as $i \to -\infty$. For this, we denote by Λ the positive root to $P(c,\cdot) = 0$.

Consider the function $\mathbf{u}^\pm = \{u_i^\pm\}$ defined by

$$u_i^\pm(\varepsilon_1, \theta, \varepsilon_3, t) := \phi_i \pm \varepsilon_1 \psi_{1i} + \theta \psi_{2i} e^{(i-ct)\Lambda} \mp \varepsilon_3 \psi_{3i} e^{2(i-ct)\Lambda} \tag{3.3}$$

where $\varepsilon_1 \geqslant 0, \theta \in \mathbb{R}, \varepsilon_3 \geqslant 0$ are parameters and

$$\psi_{1i} = \psi_i(0), \quad \psi_{2i} = \psi_i(\Lambda), \quad \psi_{3i} = \psi_i(2\Lambda).$$

Since $P(c,\Lambda) = 0$ and $\sum_k a_{i,k}\phi_{i+k} + f_i(\phi_i) = 0$,

$$\begin{aligned}
\mathcal{N}_i \mathbf{u}^\pm(t) &:= \dot{u}_i^\pm - \sum_k a_{i,k} u_{i+k}^\pm - f_i(u_i^\pm) \\
&= \dot{u}_i^\pm - \sum_k a_{i,k}[u_{i+k}^\pm - \phi_{i+k}] - f_i'(\phi_i)[u_i^\pm - \phi_i] - R_i^\pm \\
&= \mp\varepsilon_1 P(c,0)\psi_{1i} \pm \varepsilon_3\, P(c,2\Lambda)\psi_{3i} e^{2(i-ct)\Lambda} - R_i^\pm
\end{aligned}$$

where $R_i^\pm := f_i(u_i^\pm) - f_i(\phi_i) - f_i'(\phi_i)[u_i^\pm - \phi_i]$ and can be estimated, when $i \leqslant ct$ and $\varepsilon_3\psi_{3i} \leqslant |\theta|\psi_{2i}$, by

$$|R_i^\pm| \leqslant M|u_i^\pm - \phi_i|^2 \leqslant M[\varepsilon_1\psi_{1i} + 2|\theta|\psi_{2i}e^{(i-ct)\Lambda}]^2 \leqslant 2M\varepsilon_1^2\psi_{1i}^2 + 8M\theta^2 e^{2(i-ct)\Lambda}\psi_{2i}^2.$$

It then follows that, when $i \leqslant ct$ and $\varepsilon_3\psi_{3i} \leqslant |\theta|\psi_{2i}$,

$$\pm\mathcal{N}_i\mathbf{u}^\pm(t) \geqslant \varepsilon_1\psi_{1i}[-P(c,0) - 2M\varepsilon_1\psi_{1i}] + e^{2(i-ct)\Lambda}[P(c,2\Lambda)\varepsilon_3\psi_{3i} - 8M\theta^2\psi_{2i}^2].$$

As $P(c,0) < 0 = P(c,\Lambda) < P(c,2\Lambda)$, $\pm\mathcal{N}_i\mathbf{u}^\pm \geqslant 0$ if

$$0 \leqslant \varepsilon_1 \leqslant E_1, \quad \varepsilon_3 = E_3\theta^2, \quad |\theta| \leqslant E_2, \tag{3.4}$$

where

$$E_1 := \min_i \frac{-P(c,0)}{2M\psi_{1i}(0)}, \quad E_2 := \frac{1}{E_3}\min_i \frac{\psi_{2i}}{\psi_{3i}}, \quad E_3 := \max_i \frac{8M\psi_{2i}^2}{P(c,2\Lambda)\psi_{3i}}.$$

Therefore, we have proved the following lemma.

Lemma 3.1. *Assume that $\Phi = \{\phi_i\}$ is a stable n-periodic steady state. Let $\mathbf{u}^\pm$ be defined as* (3.3)*. Then there exist positive constants E_1, E_2, E_3 such that, in the parameter range* (3.4)*,*

$$\mathcal{N}_i\mathbf{u}^+(t) \geqslant 0 \geqslant \mathcal{N}_i\mathbf{u}^-(t) \quad \forall\, t \in \mathbb{R},\ i \leqslant ct.$$

3.3. **Proof of Theorem 2.** Let $(c,\mathbf{u})$ be a solution to (1.3)-(1.5) with $\Phi^+ = \mathbf{1}$ and $\Phi^- = \mathbf{0}$. Using the transformation

$$\tilde{u}_i(t) = 1 - u_{-i}(t) \quad \forall\, t \in \mathbb{R}, i \in \mathbb{Z},$$

if necessary, we need only consider the behavior of the solution as $i \to -\infty$. Hence we assume that $\mathbf{0}$ is a stable steady state. For simplicity, we write $(\mu^{\mathbf{0}}, \Psi^{\mathbf{0}}, \{\psi_i^{\mathbf{0}}\}, \Lambda^{\mathbf{0}})$ as $(\mu, \Psi, \{\psi_i\}, \Lambda)$. For each $m \in \mathbb{Z}$, we define

$$\delta_m := \max_{-k_0 \leqslant i-ct \leqslant 0,\ t\in[0,T]} \frac{u_{mn+i}(t)}{\psi_i(\Lambda)e^{\Lambda(i-ct)}} = \max_{-k_0 \leqslant i-ct \leqslant 0,\ t\in\mathbb{R}} \frac{u_{mn+i}(t)}{\psi_i(\Lambda)e^{\Lambda(i-ct)}}.$$

Here the second equality is derived from the fact that $u_{i+kcT}(t+kT) = u_i(t)$ for all integers i, k and $t \in \mathbb{R}$. Since $\lim_{j\to-\infty} u_j(t) = 0$, we have $\lim_{m\to-\infty}\delta_m = 0$.

When $\delta_m \leqslant E_2/4$, we define

$$\theta_m = \frac{2\delta_m}{1 + \sqrt{1 - 4\delta_m/E_2}}, \quad \epsilon_{3m} = E_3\theta_m^2.$$

Then $\delta_m = \theta_m(1 - \theta_m/E_2)$ so that, when $-k_0 \leqslant i - ct \leqslant 0$,

$$\begin{aligned} u_{mn+i}(t) &\leqslant \delta_m\psi_i(\Lambda)e^{(i-ct)\Lambda} = \theta_m[1 - \theta_m/E_2]\psi_i(\Lambda)e^{(i-ct)\Lambda} \\ &\leqslant \theta_m\psi_i(\Lambda)e^{(i-ct)\Lambda} - E_3\theta_m^2\psi_i(2\Lambda)e^{2(i-ct)\Lambda} \\ &= u_i^+(0, \theta_m, \epsilon_{3m}, t). \end{aligned}$$

Now define

$$\varepsilon_{1m} = \min\{\varepsilon \geqslant 0 \mid u_{mn+i}(0) \leqslant u_i^+(\varepsilon, \theta_m, \varepsilon_{3m}, 0) \text{ for all } i \leqslant 0\}.$$

Since $u_i^+(\varepsilon,\theta_m,\varepsilon_{3m},0)\geqslant \varepsilon\psi_i(0)$ and $\lim_{j\to-\infty}u_j=0$, there exists $m_0\ll -1$ such that for every integer $m\leqslant m_0$, there holds $0\leqslant \varepsilon_{1m}<E_1,\ 0<\theta_m<E_2$. This implies, by the comparison principle, that for each integer $m\leqslant m_0$,

$$u_{mn+i}(t)\leqslant u_i^+(\varepsilon_{1m},\theta_m,\varepsilon_{3m},t)\quad \forall t\geqslant 0,\quad i\leqslant c\,t.$$

We claim that $\varepsilon_{1m}=0$. Suppose not. Then, by the strong comparison principle,

$$u_{mn+i}(T)<u_i^+(\varepsilon_{1m},\theta_m,\varepsilon_{3m},T)\quad \forall\, i\leqslant cT.$$

Since $\lim_{i\to-\infty}u_i(T)=0$, these strict inequalities imply the existence of $\varepsilon\in(0,\varepsilon_{1m})$ such that

$$u_{mn+i}(T)<u_i^+(\varepsilon,\theta_m,\varepsilon_{3m},T)\quad \forall\, i\leqslant cT.$$

Since $u_i(0)=u_{i+cT}(T)$ and $u_i^+(0)=u_{i+cT}^+(T)$ for all $i\in\mathbb{Z}$, the above inequality implies that $u_{mn+i}(0)\leqslant u_i^+(\varepsilon,\theta_m,\varepsilon_{3m},0)$ for all $i\leqslant 0$, contradicting the definition of ε_{1m}. Thus, $\varepsilon_{1m}=0$ when $m\leqslant m_0$. Consequently, by the comparison principle, $u_{mn+i}(t)\leqslant u_i^+(0,\theta_m,\varepsilon_{3m},t)$ for all $t\geqslant 0$ and integers $i\leqslant ct$. By the definition of u_i^+ and periodicity, we then obtain

$$u_{mn+i}(t)\leqslant \frac{2\delta_m\,\psi_i(\Lambda)\,e^{(i-ct)\Lambda}}{1+\sqrt{1-4\delta_m/E_2}}\quad \forall\, t\in\mathbb{R},\ i\leqslant ct.$$

In a similar manner, define

$$\hat\delta_m\ :=\ \min_{-k_0\leqslant i-ct\leqslant 0,\ t\in[0,T]}\ \frac{u_{mn+i}(t)}{\psi_i(\Lambda)e^{(i-ct)\Lambda}}\ =\ \min_{-k_0\leqslant i-ct\leqslant 0,\ t\in\mathbb{R}}\ \frac{u_{mn+i}(t)}{\psi_i(\Lambda)e^{(i-ct)\Lambda}}.$$

We can show that, when $m\leqslant m_0$,

$$u_{mn+i}(t)\geqslant \frac{2\hat\delta_m\psi_i(\Lambda)\,e^{(i-ct)\Lambda}}{1+\sqrt{1+4\hat\delta_m/E_2}}\quad \forall\, t\in\mathbb{R},\ i\leqslant ct.$$

To tighten the upper and lower bounds, we consider the function

$$w_i(t):=u_i(t)/[\psi_i(\Lambda)e^{(i-ct)\Lambda}].$$

Note that

$$w_i(T)=w_{i-cT}(0),\quad u_i(t)=w_i(t)\psi_i(\Lambda)e^{(i-ct)\Lambda}.$$

Then the established upper and lower bound estimates can be written as

$$\frac{2\hat h_m}{1+\sqrt{1+4\hat\delta_m/E_2}}\leqslant w_{mn+i}(t)\leqslant \frac{2h_m}{1+\sqrt{1-4\delta_m/E_2}}\quad \forall\, t\in\mathbb{R},\ i\leqslant ct, m\leqslant m_0, \tag{3.5}$$

where

$$h_m:=e^{-mn\Lambda}\delta_m=\max_{-k_0\leqslant i-ct\leqslant 0,\ t\in[0,T]}w_{mn+i}(t),$$
$$\hat h_m:=e^{-mn\Lambda}\hat\delta_m=\min_{-k_0\leqslant i-ct\leqslant 0,\ t\in[0,T]}w_{mn+i}(t).$$

The estimates (3.5) and the definitions of $\hat{h}_m$ and h_m imply that

$$0 < h := \liminf_{i-ct\to-\infty} w_i(t) \leqslant \limsup_{i-ct\to-\infty} w_i(t) =: H < \infty.$$

We now show that $h = H$. Writing $\psi_i(\Lambda)$ as ψ_i, we have

$$\begin{aligned}\mathcal{L}_i\mathbf{w} &:= \psi_i\,\dot{w}_i(t) - \sum_j a_{i,j}e^{j\Lambda}\psi_{i+j}\,[w_{i+j} - w_i]\\ &= e^{-(i-ct)\Lambda}\{f_i(u_i) - f_i'(0)u_i\}\\ &\geqslant -Mu_i^2e^{-(i-ct)\Lambda} = -Mw_i^2\psi_i^2e^{(i-ct)\Lambda}.\end{aligned}$$

Since $\max_i \psi_i = 1$ and $w_i(t) \leqslant 2h_{m+1}$ when $i \leqslant ct + mn + 1$ (and $m \leqslant m_0 - 1$), we have

$$\mathcal{L}_{mn+i}\mathbf{w} \geqslant -\eta_m\psi_i \quad \forall i \leqslant ct+1, \quad \eta_m := 4Mh_{m+1}^2e^{n(m+1)\Lambda}.$$

We shall construct subsolutions with respect to the linear operator $\mathcal{L}$. For this, we define $\underline{\mathrm{w}}_m := \inf_{i-ct\leqslant k_0+1} w_{mn+i}(t)$. Then $\underline{\mathrm{w}}_m \leqslant h$ and $\lim_{m\to-\infty}\underline{\mathrm{w}}_m = h$. Also, for each $\tau \in [0,T]$ and each integer k in $[-|c|T-n, -|c|T]$, we define an auxiliary function $\mathbf{W}^{k,\tau}(t) := \{W_i^{k,\tau}(t)\}_{i\in\mathbb{Z}}$ as the solution to the initial "boundary" value problem

$$\begin{aligned}&\mathcal{L}_i\mathbf{W}^{k,\tau}(t) = 0 \quad \forall\, t > \tau,\ i < ct+1,\\ &W_k^{k,\tau}(\tau) = 1, \quad W_i^{k,\tau}(\tau) = 0 \quad \forall i \neq k,\\ &W_i^{k,\tau}(t) = 0 \quad \forall\, t \geqslant \tau,\ i \geqslant ct+1.\end{aligned}$$

One can show that $W_i^{k,\tau}(t) > 0$ for all $t > \tau, i < ct+1$.

Now suppose that $h < H$. Then there exist an integer $k \in [-n-|c|T, -|c|T]$, a sequence $\{t_l\}_{l=0}^\infty$ in $[0,T]$, and an integer sequence $\{m_l\}_{l=0}^\infty$ such that $\lim_{l\to\infty} m_l = -\infty$ and $w_{m_ln+k}(t_l) \geqslant \frac{1}{2}(H+h)$. For each integer $l \geqslant 0$, consider the function $\mathbf{W} = \{W_i\}$ defined by

$$W_i(t) = \underline{\mathrm{w}}_{m_l} - (t-t_l)\eta_{m_l} + \tfrac{1}{2}(H-h)W_i^{k,t_l}(t).$$

We have

$$\begin{aligned}&\mathcal{L}_i\mathbf{W}(t) = -\eta_{m_l}\psi_i \leqslant \mathcal{L}_{m_ln+i}\mathbf{w}(t) \quad \forall\, t \geqslant t_l, i < ct+1,\\ &W_i(t_l) \leqslant w_{m_ln+i}(t_l) \quad \forall\, i \leqslant c\,t_l + 1,\\ &W_i(t) \leqslant w_{m_ln+i}(t) \quad \forall t \geqslant t_l, ct+1 \leqslant i \leqslant ct+k_0+1.\end{aligned}$$

It then follows by the comparison principle that $W_i(t) \leqslant w_{m_ln+i}(t)$ when $t \geqslant t_l$ and $i \leqslant ct + k_0 + 1$. In particular,

$$w_{m_ln+i}(t) \geqslant \underline{\mathrm{w}}_{m_l} + \frac{(H-h)\delta}{2} - 3T\eta_{m_l} \quad \forall\, t \in [2T, 3T], -k_0 \leqslant i - ct \leqslant 0,$$

where

$$\delta := \min\{W_i^{k,\tau}(t) \mid -k_0 \leqslant i - ct \leqslant 0, -n \leqslant k + |c|T \leqslant 0, \tau \in [0,T], t \in [2T, 3T]\}.$$

This implies that

$$\hat{h}_{m_l} \geqslant \underline{\mathrm{w}}_{m_l} - 3T\eta_{m_l} + \frac{(H-h)\delta}{2}.$$

Consequently, it follows from (3.5) that

$$h \geqslant \limsup_{l\to-\infty} \hat{h}_{m_l} \geqslant h + \frac{(H-h)\delta}{2},$$

since $\lim_{m\to-\infty} \underline{\mathrm{w}}_m = h$ and $\lim_{m\to-\infty}[\eta_m + \hat{\delta}_m] = 0$. This contradicts the assumption that $H > h$. Hence $H = h > 0$. This completes the proof of Theorem 2. □

4. Uniqueness of Traveling Waves

It is well-known that standing waves (i.e., traveling waves with $c = 0$) are not necessarily unique (cf. [10]). Hence we consider the uniqueness problem only when $c \neq 0$. We shall prove the following.

Theorem 3. *Assume that $\Phi^- = \mathbf{0}$ and $\Phi^+ = \mathbf{1}$ are stable steady states of (1.3). Suppose $(c, \mathbf{u})$ is a traveling wave with $c \neq 0$. Then it is unique in the sense that if $(\tilde{c}, \tilde{\mathbf{u}})$ is another traveling wave, then $\tilde{c} = c$ and, for some $\tau > 0$, $\tilde{\mathbf{u}}(t) = \mathbf{u}(t+\tau)$ for all $t \in \mathbb{R}$. In addition, $c\dot{u}_i(t) < 0$ for all $i \in \mathbb{Z}$ and $t > 0$. Furthermore, its period is $n/|c|$; that is,*

$$u_i\Big(\frac{n}{c} + t\Big) = u_{i-n}(t) \quad \forall\, t \in \mathbb{R}, i \in \mathbb{Z}.$$

This theorem implies that if (1.3)–(1.5) admits a solution $(c, \mathbf{u})$ with $c = 0$, then any other solution $(\tilde{c}, \tilde{\mathbf{u}})$ to (1.3)-(1.5) must satisfy $\tilde{c} = 0$. Hence, for bistable dynamics, traveling wave speeds are unique.

The theorem will be proven in the following subsections.

4.1. Monotonicity in t.

4.1. Monotonicity in t. One key to show the uniqueness of the traveling waves is the monotonicity in t of the wave, stated in the following lemma.

Lemma 4.1. *Assume that $\Phi^- = \mathbf{0}$ and $\Phi^+ = \mathbf{1}$ are stable steady states and $(c, \mathbf{u})$ is a solution to (1.3)–(1.5). The following statements hold:*

1. *If $c < 0$, then $\dot{u}_i(t) > 0$ for all $i \in \mathbb{Z}, t \in \mathbb{R}$;*
2. *If $c > 0$, then $\dot{u}_i(t) < 0$ for all $i \in \mathbb{Z}, t \in \mathbb{R}$;*
3. *If $c = 0$, then there exists a stationary wave $\mathbf{U} = \{U_i\} \in \mathbb{R}^{\mathbb{Z}}$ satisfying*

$$\sum_k a_{i,k} U_{i+k} + f_i(U_i) = 0 \quad \forall\, i \in \mathbb{Z}, \quad \lim_{i\to\infty} U_i = 1, \quad \lim_{i\to-\infty} U_i = 0.$$

Remark 4.1. We do not know if a zero speed wave has to be stationary. We did not find any evidence against the existence of such a zero speed non-stationary wave.

Proof. Applying Theorem 2, we have

$$\lim_{i\to-\infty} \frac{u_{i-n}(0)}{u_i(0)} = e^{-n\Lambda^{\mathbf{0}}} \lim_{i\to-\infty} \frac{u_{i-n}(0)}{\psi_i^{\mathbf{0}} e^{(i-n)\Lambda^{\mathbf{0}}}} \lim_{i\to-\infty} \frac{\psi_i^{\mathbf{0}} e^{i\Lambda^{\mathbf{0}}}}{u_i(0)} = e^{-n\Lambda^{\mathbf{0}}} < 1,$$

$$\lim_{i\to\infty} \frac{1 - u_{i-n}(0)}{1 - u_i(0)} = e^{-n\Lambda^{\mathbf{1}}} \lim_{i\to\infty} \frac{1 - u_{i-n}(0)}{\psi_i^{\mathbf{1}} e^{(i-n)\Lambda^{\mathbf{1}}}} \lim_{i\to\infty} \frac{\psi_i^{\mathbf{1}} e^{i\Lambda^{\mathbf{1}}}}{1 - u_i(0)} = e^{-n\Lambda^{\mathbf{1}}} > 1.$$

It then follows that there exists a large integer i_1 such that

$$u_{i-n}(0) < u_i(0) \quad \forall\, |i| \geqslant i_1.$$

Consequently, as $\lim_{i\to-\infty} u_i(0) = 0$ and $\lim_{i\to\infty} u_i(0) = 1$, there is an integer K such that

$$u_{i-mn}(0) \leqslant u_i(0) \quad \forall\, i \in \mathbb{Z}, m \geqslant K.$$

Now we consider the case $c > 0$. By (1.5), there exists $T > 0$ such that cT is a multiple of n and $u_{i+cT}(T) = u_i(0)$. Set $k = cT/n$. We have $u_j(T) = u_{j-kn}(0)$ for every integer j. Thus,

$$u_i(KT) = u_{i-kKn}(0) \leqslant u_i(0) \quad \forall\, i \in \mathbb{Z}.$$

We now define

$$\tau^* = \inf\{\tau \leqslant KT \mid u_i(t) \leqslant u_i(0) \ \ \forall\, i \in \mathbb{Z}, \ \ t \in [\tau, KT]\}.$$

Since $\mathbf{u}$ is differentiable in t, we see that $u_i(t) \leqslant u_i(0)$ for all $i \in \mathbb{Z}$ and $t \in [\tau^*, KT]$.

We claim that $\tau^* = 0$. Suppose not. Then $\tau^* > 0$. Since $c > 0$, we cannot have $\mathbf{u}(0) = \mathbf{u}(\tau^*)$. Thus, by the strong comparison principle, $\mathbf{u}(T) > \mathbf{u}(\tau^* + T)$, which, after using $u_i(t+T) = u_{i-cT}(t)$, implies that $\mathbf{u}(0) > \mathbf{u}(\tau^*)$.

Note that

$$\lim_{i\to-\infty} \frac{u_i(t)}{u_i(0)} = e^{-c\Lambda^{\mathbf{0}}t} \lim_{i-ct\to-\infty} \frac{u_i(t)}{\psi_i^{\mathbf{0}} e^{(i-ct)\Lambda^{\mathbf{0}}}} \lim_{i\to-\infty} \frac{\psi_i e^{i\Lambda^{\mathbf{0}}}}{u_i(0)} = e^{-c\Lambda^{\mathbf{0}}t}$$

uniformly in $t \in [0, KT]$. It then follows that there exists a large integer i_2 such that

$$u_i(t) < u_i(0) \quad \forall i \leqslant -i_2, t \in [\tfrac{1}{2}\tau^*, KT].$$

Similarly, using the limiting behavior of $u_i(t)$ as $i \to \infty$, we have

$$u_i(t) < u_i(0) \quad \forall i \geqslant i_2, t \in [\tfrac{1}{2}\tau^*, KT].$$

When $i \in [-i_2, i_2]$, using continuity and $u_i(\tau^*) < u_i(0)$, we see that there exists $\varepsilon \in (0, \frac{1}{2}\tau^*]$ such that $u_i(t) < u_i(0)$ for all $t \in [\tau^* - \varepsilon, \tau^*]$ for every integer $i \in [-i_2, i_2]$. In conclusion, $u_i(t) < u_i(0)$ for all $i \in \mathbb{Z}$ and $t \in [\tau^* - \varepsilon, KT]$, contradicting the definition of τ^*. Thus we must have $\tau^* = 0$.

That $\tau^* = 0$ implies that $u_i(t) \leqslant u_i(0)$ for all $t \in [0, KT]$. Consequently, $\dot{u}_i(0) \leqslant 0$, for every $i \in \mathbb{Z}$. Note that $\dot{\mathbf{u}}$ satisfies

$$\frac{d}{dt}\dot{u}_i = \sum_j a_{i,j}[\dot{u}_{i+j} - \dot{u}_i] + f_i'(u_i)\dot{u}_i \quad \forall\, i \in \mathbb{Z}, t > 0.$$

A strong maximum principle then implies that $\dot{u}_i < 0$ for all $i \in \mathbb{Z}$ and $t > 0$. The relation $u_i(t+T) = u_{i-cT}(t)$ then also implies that $\dot{u}_i < 0$ for all $t \in \mathbb{R}$ and $i \in \mathbb{Z}$.

The case $c < 0$ can be treated similarly.

Finally, we consider the case $c = 0$. This implies by (1.5) that for some $T > 0$,

$$\mathbf{u}(T) = \mathbf{u}(0) \quad \text{ or } \quad u_i(t+T) = u_i(t) \quad \forall\, i \in \mathbb{Z}, t \in \mathbb{R}.$$

By Theorem 2 and the limit of $u_i(0)$ as $i \to \pm\infty$, there exists a large integer m such that

$$u_i(t) < u_{i+mn}(0) \quad \forall\, i \in \mathbb{Z}, t \in [0,T].$$

Now consider the initial value problem

$$\dot{w}_i = \sum_k a_{i,k} w_{i+k}(t) + f_i(w_i(t)) \quad \forall i \in \mathbb{Z}, t > 0,$$

with the initial data $w_i(0) = \max_{0 \le \tau \le T} u_i(\tau)$ for $i \in \mathbb{Z}$. By the comparison principle, one can show that

$$u_{i+mn}(t) \geqslant w_i(t) \geqslant u_i(t+\tau) \quad \forall\, i \in \mathbb{Z}, t > 0, \tau \in [0,T].$$

It then follows that for any $\varepsilon > 0$, $w_i(\varepsilon) \geqslant \max_{0 \le \tau \le T} u_i(\varepsilon + \tau) = w_i(0)$ for all $i \in \mathbb{Z}$. Consequently, by the comparison principle, $\mathbf{w}(t+\varepsilon) \geqslant \mathbf{w}(t)$ for all $t > 0$. Hence, $\dot{\mathbf{w}}(t) \geqslant 0$ for all $t > 0$. Also, note that

$$\max_{\tau \in [0,T]} u_{i+mn}(\tau) \geqslant w_i(t) \geqslant \max_{\tau \in [0,T]} u_i(\tau) \quad \forall i \in \mathbb{Z}, t \in \mathbb{R}.$$

Now, set $\mathbf{U} = \lim_{t \to \infty} \mathbf{w}(t)$. It is not difficult to show that $\mathbf{U}$ is a stationary wave. This completes the proof. □

4.2. **Uniqueness of Traveling Waves.** We are now ready to show the uniqueness of traveling waves. Theorem 3 follows from Lemma 4.1 and the following lemma.

Lemma 4.2. *Suppose $(c, \mathbf{u})$ is a traveling wave with $c \neq 0$, where $\Phi^- = \mathbf{0}$ and $\Phi^+ = \mathbf{1}$ are stable. If $(\tilde{c}, \tilde{\mathbf{u}})$ is another traveling wave, then $\tilde{c} = c$ and, for some $\tau > 0$, $\tilde{\mathbf{u}}(t) = \mathbf{u}(t+\tau)$ for all $t \in \mathbb{R}$. In addition, the period is $T = n/|c|$, i.e.,*

$$u_i\Big(\frac{n}{c} + t\Big) = u_{i-n}(t) \quad \forall\, t \in \mathbb{R}, i \in \mathbb{Z}.$$

Proof. Suppose $\tilde{\mathbf{u}}$ is a solution to (1.3)-(1.4) and for some $\tilde{T} > 0, \tilde{c} \in \mathbb{R}$,

$$\tilde{c}\tilde{T} \in n\mathbb{Z}, \quad \tilde{u}_i(\tilde{T}) = \tilde{u}_{i-\tilde{c}\tilde{T}}(0) \quad \forall i \in \mathbb{Z}.$$

First we show that $c = \tilde{c}$. Suppose not. Then, by exchanging the roles of c and $\tilde{c}$ if necessary, we may assume that $c > \tilde{c}$. Since $\mu^{\mathbf{0}}(\cdot)$ is convex and $\mu^{\mathbf{0}}(0) < 0$, we see that the unique positive roots to

$$\mu^{\mathbf{0}}(\Lambda) = -c\Lambda, \quad \mu^{\mathbf{0}}(\tilde{\Lambda}) = -\tilde{c}\tilde{\Lambda}$$

satisfy $0 < \Lambda < \tilde{\Lambda}$. Thus, near $i = -\infty$, $u_i(0)$ decays to zero slower than $\tilde{u}_i$ does. Namely, $u_i(0) > \tilde{u}_i(0)$ for all $i \ll -1$.

Similarly, the unique negative roots to

$$\mu^{\mathbf{1}}(\Lambda) = -c\Lambda, \quad \mu^{\mathbf{1}}(\tilde{\Lambda}) = -\tilde{c}\tilde{\Lambda}$$

satisfy $\Lambda < \tilde{\Lambda} < 0$, so that $1 - u_i(0)$ decays to zero faster than $1 - \tilde{u}_i(0)$ does. Thus, $u_i(0) > \tilde{u}_i(0)$ for all sufficiently large i. Hence, after a sufficiently large translation, there exists an integer m such that

$$u_{mn+i}(0) > \tilde{u}_i(0) \quad \forall i.$$

This implies by comparison that $u_{mn+i}(t) > \tilde{u}_i(t)$ for all $t > 0$ and $i \in \mathbb{Z}$. Intuitively this is impossible, since $\mathbf{u}$ travels faster than $\tilde{\mathbf{u}}$ does. Here is the detail. Setting $i = kcT$ and $t = kT$ we obtain

$$u_{mn}(0) = u_{mn+kcT}(kT) \geqslant \tilde{u}_{kcT}(kT) = \tilde{u}_{kcT-\tilde{k}\tilde{c}\tilde{T}}(kT - \tilde{k}\tilde{T}) \quad \forall \tilde{k} \in \mathbb{Z}.$$

Take $\tilde{k} \in \mathbb{Z}$ such that $kT - \tilde{k}\tilde{T} := \tau_k \in [0, \tilde{T})$. Then

$$u_{mn}(0) \geqslant \tilde{u}_{(c-\tilde{c})kT+\tilde{c}\tau_k}(\tau_k).$$

Sending $k \to \infty$, we obtain $u_{mn}(0) \geqslant \lim_{k\to\infty} \tilde{u}_{(c-\tilde{c})kT+\tilde{c}\tau_k}(\tau_k) = 1$, a contradiction. Thus we must have $c = \tilde{c}$.

Next we prove that $\mathbf{u}$ differs from $\tilde{\mathbf{u}}$ by a time translation. We have to assume that $c \neq 0$. Consider, for definiteness, that $c < 0$. From the tail expansion, there exists a large integer K such that

$$u_i(KT) \geqslant \tilde{u}_i(0) \quad \forall i \in \mathbb{Z}.$$

Set

$$\tau^* = \inf\{\tau \leqslant KT \mid u_i(t) \leqslant \tilde{u}_i(0) \quad \forall i \in \mathbb{Z}, t \in [\tau, \infty)\}.$$

Following the same proof as before, we can show that $\mathbf{u}(\tau^*) = \tilde{\mathbf{u}}(0)$. This completes the uniqueness proof.

Finally, we show that the period is $n/|c|$, namely, $u_{i+n}(n/c) = u_i(0)$. Since $\tilde{\mathbf{u}} = \{u_{i+n}\}_{i\in\mathbb{Z}}$ is a traveling wave, we see that for some τ, $u_{i+n}(\tau + t) = u_i(t)$. This also implies that $u_i(m\tau + t) = u_{i-mn}(t)$ for any $m \in \mathbb{Z}$. By (1.5), there exists an integer k such that $cT = kn$ and $u_{i-kn}(t) = u_i(T+t)$; that is, $u_i(T+t) = u_i(k\tau+t)$. Since $\dot{u}_i > 0$, we derive that $k\tau = T$, i.e., $\tau = T/k = n/c$. Thus, the period is $n/|c|$. The case $c < 0$ can be proven in a similar manner. This completes the proof. □

5. Globally Exponential Stability

In this section, we shall study the asymptotic stability of non-zero speed traveling waves. For this purpose, let $(c, \mathbf{U})$, where $c \neq 0$ and $\mathbf{U} = \{U_i\}$, be a traveling wave solution. For definiteness, we assume $\Phi^+ = \mathbf{1}$ and $\Phi^- = \mathbf{0}$ so that $(c, \mathbf{U})$ satisfies

$$\text{(5.1)} \qquad \begin{cases} \mathcal{N}_i\mathbf{U} := \dot{U}_i(t) - \sum_k a_{i,k}U_{i+k}(t) - f_i(U_i(t)) = 0 & \forall\, i \in \mathbb{Z},\ t \in \mathbb{R}, \\ U_i(t + n/c) = U_{i-n}(t) & \forall\, i \in \mathbb{Z}, t \in \mathbb{R}, \\ \lim_{i\to-\infty} U_i(t) = 0, \quad \lim_{i\to\infty} U_i(t) = 1 & \forall\, t \in \mathbb{R}. \end{cases}$$

We shall show that any solution $\mathbf{u}$ to (1.3) with an initial data that "vaguely resembles" a wave profile (to be defined in §5.1) will approach exponentially fast to a time translation of the traveling wave.

5.1. **Initial Datum Resembles Vaguely A Wave Front.** We recall from §2.6 that the attraction basin $\mathcal{A}(\Phi)$ of a steady state Φ is composed of all those $\mathbf{v} \in \mathbb{R}^{\mathbb{Z}}$ such that with initial data $\mathbf{u}(0) = \mathbf{v}$, the solution $\mathbf{u} := \mathbb{S}^t\mathbf{v}$ to (1.3) satisfies $\lim_{t\to\infty} \|\mathbb{S}^t\mathbf{v} - \Phi\|_\infty = 0$.

Definition 5.1. *Assume that* $\mathbf{0}$ *and* $\mathbf{1}$ *are two stable steady states. A vector* $\mathbf{v} = \{v_i\} \in \mathbb{R}^{\mathbb{Z}}$ *is* **vaguely resembling** *a wave front if there exist a positive integer* N *and vectors* $\Psi_0^\pm = \{\psi_{0i}^\pm\} \in \mathcal{A}(\mathbf{0})$ *and* $\Psi_1^\pm = \{\psi_{1i}^\pm\} \in \mathcal{A}(\mathbf{1})$ *such that*

$$v_i \leqslant \psi_{0i}^+ \quad \forall i \leqslant -N, \quad v_i \geqslant \psi_{1i}^- \quad \forall i \geqslant N, \quad \psi_{0i}^- \leqslant v_i \leqslant \psi_{1i}^+ \quad \forall\, i \in \mathbb{Z}.$$

Example. Assume that $f_{i+n} = f_i \in C^2(\mathbb{R})$ for all $i \in \mathbb{Z}$ and that for some $\alpha, \beta \in (0,1)$ there holds

$$f_i(z) > f_i(0) = 0 = f_i(1) > f_i(s) \quad \forall z \in (-\infty, 0) \cup [\beta, 1), s \in (0, \alpha] \cup (1, \infty), i \in \mathbb{Z}.$$

If $\mathbf{v} = \{v_i\} \in \mathbb{R}^{\mathbb{Z}}$ is a vector satisfying

$$\infty > \limsup_{i\to\infty} v_i \geqslant \liminf_{i\to\infty} v_i > \beta \geqslant \alpha > \limsup_{i\to-\infty} v_i \geqslant \liminf_{i\to-\infty} v_i > -\infty,$$

then $\mathbf{v}$ resembles vaguely a wave front; see Lemma 2.4 for a proof.

5.2. **Evolution from A Vaguely Resembling Front to An Asymptotic Front.** In this subsection we shall prove the following:

Lemma 5.1. *Assume that* $\mathbf{0}$ *and* $\mathbf{1}$ *are stable steady states of* (1.3) *and that* $\mathbf{v} \in \mathbb{R}^{\mathbb{Z}}$ *resembles vaguely a wave front. Then for every* ε*, there exist a positive integer* $\bar{N}$ *and a positive constant* τ *such that the solution* $\mathbf{u} = \{u_i\}$ *of* (1.3) *with initial value* $\mathbf{u}(0) = \mathbf{v}$ *satisfies*

$$-\varepsilon\mathbf{1} \leqslant \mathbf{u}(t) \leqslant (1+\varepsilon)\mathbf{1} \quad \forall t \geqslant \tau,$$
$$u_i(\tau) \leqslant \varepsilon \quad \forall i \leqslant -\bar{N}, \quad u_i(\tau) \geqslant 1 - \varepsilon \quad \forall\, i \geqslant \bar{N}.$$

Proof. Let $\varepsilon > 0$ be given. We divide the proof into two steps.

1. First we establish global upper and lower bounds. By the assumption, there exist $\Psi_0^- \in \mathcal{A}(\mathbf{0})$ and $\Psi_1^+ \in \mathcal{A}(\mathbf{1})$ such that $\Psi_0^- \leqslant \mathbf{v} \leqslant \Psi_1^+$. By the comparison principle, $\mathbb{S}^t\Psi_0^- \leq \mathbf{u}(t) \leq \mathbb{S}^t\Psi_1^+$ for all $t \geq 0$. It then follows from the definition of $\Psi_0^- \in \mathcal{A}(\mathbf{0})$ and $\Psi_1^+ \in \mathcal{A}(\mathbf{1})$ that there exists $\tau_0 > 0$ such that $-\varepsilon\mathbf{1} \leqslant \mathbf{u}(t) \leqslant (1+\varepsilon)\mathbf{1}$ for all $t \geqslant \tau_0$.

2. Next we establish an upper bound for $u_i(t)$ for all sufficiently large negative i. By the assumption, there exist $\Psi_0^+ \in \mathcal{A}(\mathbf{0})$ and integer N such that $v_i \leqslant \psi_{0i}^+$ for all $i \leqslant -N$. Also, we have $\mathbf{v} \leqslant \Psi_1^+ \in \mathcal{A}(\mathbf{1})$.

First of all, by Lemma 2.3(ii), there exist $\delta > 0$ and $\tau_1 > 0$ such that the solutions to

$$\begin{aligned}\mathcal{N}\mathbf{w}_0(t) &= \delta\mathbf{1} \quad \forall t > 0, \quad \mathbf{w}_0(0) = \delta\mathbf{1} + \{\min\{\psi_{1i}^+, \psi_{0i}^+\}\}_{i\in\mathbb{Z}},\\ \mathcal{N}\mathbf{w}_1(t) &= \delta\mathbf{1} \quad \forall t > 0, \quad \mathbf{w}_1(0) = \delta\mathbf{1} + \Psi_1^+\end{aligned}$$

satisfy

$$\mathbf{w}_0(t) \leqslant \frac{1}{2}\varepsilon\mathbf{1}, \quad \mathbf{w}_1(t) \leqslant (1+\varepsilon)\mathbf{1} \quad \forall t \geqslant \tau_1.$$

In addition, $\mathbf{w}_0(t) < \mathbf{w}_1(t)$ for all $t > 0$.

We define

$$L := \max_i \|f_i'\|_\infty, \quad M_1 := \sup_{t\geq 0} \|\mathbf{w}_0(t) - \mathbf{w}_1(t)\|_\infty, \quad \zeta(s) := \frac{1}{2}\Big(1 + \tanh\frac{s}{2}\Big).$$

Note that $\zeta' = \zeta(1-\zeta)$ on $\mathbb{R}$. For a small positive constant η and a large positive constant N_1 to be defined later, we consider the function $\mathbf{w} := \{w_i\}$ defined by

$$w_i(t) := \zeta_i(t)\, w_{1i}(t) + [1-\zeta_i(t)]w_{0i}(t), \quad \zeta_i(t) := \zeta(N_1 + i\eta + 2Lt).$$

Denote $\zeta_i' = \zeta'(N_1 + i\eta + 2Lt)$. We can calculate

$$\begin{aligned}\mathcal{N}_i\mathbf{w}(t) &= \zeta_i\mathcal{N}_i\mathbf{w}_1 + (1-\zeta_i)\mathcal{N}_i\mathbf{w}_0 + 2L\zeta_i'[w_{1i} - w_{0i}] + h_i + g_i\\ &= \delta + 2L\zeta_i'[w_{1i} - w_{0i}] + h_i + g_i,\end{aligned}$$

where

$$\begin{aligned}h_i &= -\sum_k a_{i,k}[w_{1\,i+k} - w_{0\,i+k}]\Big\{\zeta(N_1 + [i+k]\eta + 2Lt) - \zeta(N_1 + i\eta + 2Lt)\Big\},\\ g_i &= -f_i(\zeta_i w_{1i} + [1-\zeta_i]w_{0i}) + \zeta_i f_i(w_{1i}) + [1-\zeta_i]f_i(w_{0i}).\end{aligned}$$

First of all, since $0 \leqslant \zeta' \leqslant 1$, by the mean value theorem,

$$|h_i| \leqslant A_1 M_1 \eta, \quad A_1 := \sup_{1\leq i\leq n}\sum_k |a_{i,k}k|.$$

Secondly, writing

$$\begin{aligned}g_i &= \zeta_i\Big\{f_i(w_{1i}) - f_i(w_{1i} - [1-\zeta_i][w_{1i} - w_{0i}])\Big\}\\ &+ (1-\zeta_i)\Big\{f_i(w_{0i}) - f_i(w_{0i} + \zeta_i[w_{1i} - w_{0i}])\Big\},\end{aligned}$$

we see that

$$|g_i| \leq 2L\zeta_i(1-\zeta_i)[w_{1i} - w_{0i}] = 2L\zeta_i'[w_{1i} - w_{0i}].$$

It then follows that $\mathcal{N}_i\mathbf{w}(t) \geq \delta - A_1M_1\eta$. Taking $\eta := \delta/(A_1M_1)$, we obtain $\mathcal{N}_i\mathbf{w} \geqslant 0$.

Finally, notice that

$$\begin{aligned}w_i(0) - v_i &\geq \min\{w_{1i}(0), w_{0i}(0)\} - v_i \geqslant \delta + \min\{\psi_{1i}^+, \psi_{0i}^+\} - v_i > 0 \quad \forall i \leq -N,\\ w_i(0) - v_i &= \{w_{1i}(0) - v_i\} - [1-\zeta_i][w_{1i}(0) - w_{0i}(0)]\\ &\geq \delta - [1-\zeta_i]M_1 > 0 \quad \forall i \geq -N,\end{aligned}$$

provided that we take N_1 satisfying $1 - \zeta(N_1 - N\eta)] = \min\{1/2, \delta/M_1\}$.

Thus, with such η and N_1, we have, by the comparison principle, $\mathbf{u}(t) = \mathbb{S}^t\mathbf{v} \leqslant \mathbf{w}(t)$ for all $t > 0$. Since $\lim_{i\to-\infty}[w_i(t) - w_{0i}(t)] = 0$ for all t and $\sup_i w_{0i}(t) \le \frac{1}{2}\varepsilon$ for all $t \ge \tau_1$, we then see that for every $t \geqslant \tau_1$, there exists $\bar N(t)$ such that $u_i(t) \le \varepsilon$ for all $i \le -\bar N(t)$.

The lower bound can be proven similarly. This completes the proof. $\square$

5.3. **Sub/super Solutions Based on a Traveling Wave.** Let $(\mu^{\mathbf{0}}(0), \Psi^{\mathbf{0}} = \{\psi_i^{\mathbf{0}}\})$ $((\mu^{\mathbf{1}}(0), \Psi^{\mathbf{1}} = \{\psi_i^{\mathbf{1}}\}))$ be the unique solution to the linearized equation (2.3) with $\lambda = 0$ and $\{L_i\} = \{f_i'(0)\}$ ($\{L_i\} = \{f_i'(1)\}$, respectively). We set

$$\beta := \tfrac{1}{2}\min\{-\mu^{\mathbf{1}}(0), -\mu^{\mathbf{0}}(0)\} \min_{1\le i\le n}\Big\{\min\{\psi_i^{\mathbf{0}}, \psi_i^{\mathbf{1}}\}\Big\}.$$

As $\mathbf{0}$ and $\mathbf{1}$ are assumed to be stable, $\beta > 0$.

For a small positive constant η to be chosen later, we define

$$\Psi(\eta, t) = \{\psi_i(\eta, t)\}, \quad \psi_i(\eta,t) = \zeta_i(t)\psi_i^{\mathbf{1}} + [1-\zeta_i(t))]\psi_i^{\mathbf{0}}, \quad \zeta_i(t) := \zeta([i-ct]\eta).$$

Since both $\Psi^{\mathbf{1}}$ and $\Psi^{\mathbf{0}}$ are n-periodic, we see that

$$\psi_i(\eta, t + n/c) = \psi_{i-n}(\eta, t) \quad \forall i \in \mathbb{Z}, t \in \mathbb{R}.$$

Assume that $(c, \mathbf{U})$ is a traveling wave with $c \ne 0$. For definiteness, we assume that $c < 0$ so that $\dot{\mathbf{U}} > 0$. For a fixed (large) positive constant K_0 to be chosen, we consider, for every small positive constant ε and every $\xi \in \mathbb{R}$, the function

$$\mathbf{U}^{\pm}(\varepsilon, \xi, t) = \mathbf{U}(\xi + t \mp K_0\varepsilon e^{-\beta t}) \pm \varepsilon e^{-\beta t}\Psi(\eta, \xi + t \mp K_0\varepsilon e^{-\beta t}), \quad t \geqslant 0. \tag{5.2}$$

We shall show that $\mathbf{U}^+$ is a supersolution and $\mathbf{U}^-$ is a subsolution.

Write $t_1 = \xi + t - K_0\varepsilon e^{-\beta t}$, $\mathbf{U} = \mathbf{U}(t_1)$, $\dot{\mathbf{U}} = \dot{\mathbf{U}}(t_1)$, and $\Psi = \Psi(\eta, t_1)$. For $t \geqslant 0$, we can use $\mathscr{N}\mathbf{U}(t_1) = 0$ to estimate

$$\begin{aligned} \mathscr{N}\mathbf{U}^+(t) &\geqslant K_0\varepsilon\beta e^{-\beta t}\dot{\mathbf{U}} + \varepsilon e^{-\beta t}\{-\beta\Psi + \mathcal{L}\Psi\} - M\varepsilon^2 e^{-2\beta t}\mathbf{1} \\ &\geqslant \varepsilon e^{-\beta t}\Big\{K_0\beta\dot{\mathbf{U}} + \mathcal{L}\Psi - \beta\Psi - M\varepsilon\mathbf{1}\Big\}, \end{aligned}$$

where $M = \max_i \|f_i''\|_\infty$ and $\mathcal{L}\Psi = \{\mathcal{L}_i\Psi\}$,

$$\mathcal{L}_i\Psi := [1 + K_0\varepsilon\beta e^{-\beta t}]\dot\psi_i(t_1) - \sum_k a_{i,k}\psi_{i+k}(t_1) - f_i'(U_i(t_1))\psi_i(t_1).$$

Denoting $\zeta_i' = \zeta'([i - ct_1]\eta)$, we can write $\mathcal{L}_i\Psi = h_{i1} - h_{i2} - h_{i3} - h_{i4}$, where

$$\begin{aligned} h_{i1} &:= [1 + K_0\varepsilon\beta e^{-\beta t}]\dot\psi_i(t_1) = c\eta[1 + \varepsilon K_0\beta e^{-\beta t}]\zeta_i'[\psi_i^{\mathbf{0}} - \psi_i^{\mathbf{1}}], \\ h_{i2} &:= \zeta_i\Big\{\sum_k a_{i,k}\psi_{i+k}^{\mathbf{1}} + f_i'(1)\psi_i^{\mathbf{1}}\Big\} + [1-\zeta_i]\Big\{\sum_k a_{i,k}\psi_{i+k}^{\mathbf{0}} + f_i'(0)\psi_i^{\mathbf{0}}\Big\}, \\ h_{i3} &:= \sum_k a_{i,k}[\psi_{i+k}^{\mathbf{1}} - \psi_{i+k}^{\mathbf{0}}][\zeta_{i+k} - \zeta_i], \\ h_{i4} &:= f_i'(U_i)\psi_i - \zeta_i f_i'(1)\psi_i^{\mathbf{1}} - [1-\zeta_i]f_i'(0)\psi_i^{\mathbf{0}}. \end{aligned}$$

First of all, by the definition of $\Psi^{\mathbf{1}}$ and $\Psi^{\mathbf{0}}$,

$$-h_{i2} \;=\; -\zeta_i\psi_i^{\mathbf{1}}\mu^{\mathbf{1}}(0) - [1-\zeta_i]\psi_i^{\mathbf{0}}\mu^{\mathbf{0}}(0) \geq 2\beta \geqslant \beta\psi_i + \beta.$$

Next, the "bad term" h_{i1} and h_{i3} can be estimated by

$$|h_{i1}| + |h_{i3}| \leq C(\varepsilon K_0\beta)\eta$$

where $C(\cdot)$ is an increasing positive linear function on $[0,\infty)$. The "bad term" h_{i4} can be estimated by

$$|h_{i4}| \leq \zeta_i|f_i'(U_i) - f_i'(1)| + [1-\zeta_i]|f_i'(U_i) - f_i'(0)| \leq M\Big\{\zeta_i[1-U_i] + [1-\zeta_i]U_i\Big\}.$$

Thus,

$$\mathcal{N}\mathbf{U}^+(t) \geq \varepsilon e^{-\beta t}\Big\{K_0\beta\dot{\mathbf{U}} + [\beta - M\varepsilon - C(\varepsilon K_0\beta)\eta]\mathbf{1} - M\{\zeta[\mathbf{1}-\mathbf{U}] + (1-\zeta)\mathbf{U}\}\Big\}.$$

Note that there exists an integer N such that when $t_1 \in \mathbb{R}$ and $|i - ct_1| > N$,

$$\zeta_i[1-U_i] + [1-\zeta_i]U_i < \beta/(4M).$$

Now we fix K_0 by

$$K_0 = \frac{2M}{\beta \min_{t_1\in\mathbb{R},|i-ct_1|<N} \dot{U}_i(t_1)}.$$

It then follows that

$$M\Big\{\zeta[\mathbf{1}-\mathbf{U}] + [1-\zeta]\mathbf{U}\Big\} \leq \tfrac{1}{4}\beta + K_0\beta\dot{\mathbf{U}}.$$

Finally, taking $\eta = \beta/(4C(1))$ we conclude that there exists $\varepsilon_0 \in (0, 1/(4K_0\beta)]$ such that $\mathcal{N}\mathbf{U}^+(t) > 0$ for all $t \geq 0$, $\varepsilon \in (0, 2\varepsilon_0]$, and $\xi \in \mathbb{R}$.

Similarly, we can consider the function $\mathbf{U}^-$. Thus, we have the following lemma.

Lemma 5.2. *Suppose* $\mathbf{1}$ *and* $\mathbf{0}$ *are two stable steady states and that* $(c,\mathbf{U})$ *is a traveling wave with* $c < 0$*. Then there exist positive constants* η, β, K_0 *and* ε_0 *such that for every* $\varepsilon \in (0, 2\varepsilon_0]$ *and* $\xi \in \mathbb{R}$*, the function* $\mathbf{U}^\pm(\varepsilon,\xi,\cdot)$ *defined in* (5.2) *is a super/subsolution.*

5.4. **Exponential Stability.** Now we can state and prove our main result.

Theorem 4. *Suppose* $\mathbf{1}$ *and* $\mathbf{0}$ *are two stable steady states and that* $(c,\mathbf{U})$ *is a traveling wave with* $c \neq 0$*. Then it is globally exponentially stable; that is, for any initial data* $\mathbf{v} = \mathbf{u}(0)$ *that resembles a wave front (cf. Definition* 5.1*), there exist constants* K *and* τ^* *such that the solution* $\mathbf{u}(t) = \mathbb{S}^t\mathbf{v}$ *to* (1.3) *satisfies*

$$\|\mathbf{u}(t) - \mathbf{U}(t+\tau^*)\|_\infty \leq Ke^{-\nu t} \quad \forall\, t \geqslant 0$$

where ν *is a positive constant depending only on* $\{a_{i,k}\}$ *and* $\{f_i\}$*.*

Proof. The proof is based on the techniques developed in [10].

Without lose of generality, we assume that $c < 0$ so that $\dot{\mathbf{U}} > \mathbf{0}$.

1. First of all, by using Lemma 5.1, we need only consider those initial data satisfying

$$-\tfrac{1}{2}\varepsilon_0\mathbf{1} \le \mathbf{v} = \mathbf{u}(0) \le (1 + \tfrac{1}{2}\varepsilon_0)\mathbf{1}, \quad v_i \leqslant \tfrac{1}{2}\varepsilon_0 \quad \forall i < -N, \quad v_i \ge 1 - \tfrac{1}{2}\varepsilon_0 \quad \forall i \ge N.$$

This implies that there exist $\xi_0 \ll -1$ and $\tau_0 \gg 1$ such that

$$\mathbf{U}^-(\varepsilon_0, \xi_0, 0) \leqslant \mathbf{u}(0) \leqslant \mathbf{U}^+(\varepsilon_0, \xi_0 + \tau_0, 0).$$

By taking smaller ε_0 if necessary, we can assume that $4K_0\varepsilon_0 \leqslant 1$.

2. For each $t \geqslant 0$, we denote

$$D(t) := \Big\{(\varepsilon, \xi, \tau) \in (0, 2\varepsilon_0] \times \mathbb{R} \times [0, \infty) \mid \mathbf{U}^-(\varepsilon, \xi, 0) \le \mathbf{u}(t) \le \mathbf{U}^+(\varepsilon, \xi + \tau, 0)\Big\}.$$

If $(\varepsilon, \xi, \tau) \in D(t)$, then by the comparison principle,

$$\mathbf{U}^-(\varepsilon, \xi, \hat{t}) \leqslant \mathbf{u}(t + \hat{t}) \leqslant \mathbf{U}^+(\varepsilon, \xi + \tau, \hat{t}) \quad \forall \hat{t} \geqslant 0.$$

Upon noting that

$$\mathbf{U}^\pm(\varepsilon, \xi, \hat{t}) = \mathbf{U}^\pm(\varepsilon e^{-\beta\hat{t}}, \xi + \hat{t}, 0) \quad \forall \varepsilon \in (0, \varepsilon_0], \xi \in \mathbb{R}, \hat{t} \geqslant 0,$$

we conclude that

$$(\varepsilon, \xi, \tau) \in D(t) \quad \Longrightarrow \quad (\varepsilon e^{-\beta\hat{t}}, \xi + \hat{t}, \tau) \in D(t + \hat{t}) \quad \forall \hat{t} \geqslant 0.$$

Since $D(0)$ is non-empty, so is $D(t)$ for each $t \geqslant 0$. Also, one can show that for each $t \geqslant 0$, the set $D(t)$ is closed.

3. Next we define a distance between $\mathbf{u}(t)$ and the manifold $\{\mathbf{U}(s)\}_{s\in\mathbb{R}}$ by

$$d(t) := \inf_{(\varepsilon,\xi,\tau)\in D(t)} \Big\{K_0\varepsilon + 2\tau\Big\} \quad \forall\, t \geqslant 0.$$

By continuity, for every $t \geqslant 0$, there exists $(\varepsilon(t), \xi(t), \tau(t)) \in D(t)$ such that

$$d(t) = K_0\varepsilon(t) + 2\tau(t).$$

4. Suppose $\varepsilon(t) \geqslant \min\{\varepsilon_0, \tau(t)/(4K_0)\}$. Since $(\varepsilon(t)e^{-\beta}, \xi(t) + 1, \tau(t)) \in D(t + 1)$,

$$d(t + 1) \leqslant K_0\varepsilon(t)e^{-\beta} + 2\tau(t) = d(t) - [1 - e^{-\beta}]K_0\varepsilon(t).$$

5. Suppose $\varepsilon(t) \leqslant \min\{\varepsilon_0, \tau(t)/(4K_0)\}$. For simplicity, we write $(\varepsilon(t), \xi(t), \tau(t))$ as (ε, ξ, τ). Let $j = j(t)$ be the integer such that

$$-\tfrac{1}{2} \leqslant j - c\xi < \tfrac{1}{2}.$$

At least one of the following holds:

$$\text{(i) } u_j(t) \geqslant U_j(\xi + \tau/2), \quad \text{(ii) } u_j(t) \leqslant U_j(\xi + \tau/2).$$

Suppose (i) holds. Let $\tau_1 := \min\{1, \tau/4)$. Note that, when $\tau \geqslant 4$, we have $\tau_1 = 1 \geqslant K_0\varepsilon_0 \geqslant K_0\varepsilon$; when $\tau < 4$, we have $\tau_1 = \tau/4 > K_0\varepsilon$. Hence, using $\dot{\mathbf{U}} > 0$, $|j - c\xi| \leqslant 1/2$, and $0 \leqslant K_0\varepsilon \leqslant \tau_1 \leqslant 1$, we obtain

$$\begin{aligned} u_j(t) - U_j^-(\varepsilon, \xi, 0) &\geqslant U_j(\xi + 2\tau_1) - U_j^-(\varepsilon, \xi, 0) \\ &\geqslant U_j(\xi + 2\tau_1) - U_j(\xi + K_0\varepsilon) + \varepsilon \\ &\geqslant \kappa_0\Big\{2\tau_1 - K_0\varepsilon\Big\} \geqslant \kappa_0\tau_1, \end{aligned}$$

where

$$\kappa_0 := \min_{s\in\mathbb{R}, |l-cs|\leqslant 3} \dot{U}_l(s).$$

Since $\mathbf{U}^-(\varepsilon, \xi, \cdot)$ is a subsolution and $\mathbf{U}^-(\varepsilon, \xi, 0) \leqslant \mathbf{u}(t)$, by the strong comparison principle (cf. §2), we have

$$u_{j+i}(t+1) \geqslant U_{j+i}^-(\varepsilon, \xi, 1) + \tau_1\delta_i = U_{j+i}^-(\varepsilon e^{-\beta}, \xi + 1, 0) + \kappa_0\tau_1\delta_i \quad \forall\, i \in \mathbb{Z},$$

where $\{\delta_i\}_{i\in\mathbb{Z}}$ is positive and depends only on $\{a_{i,k}\}$ and $\max_i \|f_i'\|_\infty$.

For all $s \in (0, \varepsilon_0]$, we calculate

$$\frac{d}{ds}U_{j+i}^-(\varepsilon e^{-\beta} + s, \xi + 1 + 2K_0 s, 0) = -\psi_{j+i} + O(1)\dot{U}_{j+i} + O(1)\eta\zeta_{j+i}' < 0$$

when $|i| > N_2$ for some universal constant N_2, since $\min_l \psi_l(\eta, 0) > 0$ and

$$\lim_{N\to\infty} \max_{z\in\mathbb{R}, |l-cz|>N} \Big\{|\dot{U}_l(z)| + \zeta'(\eta(l - cz))\Big\} = 0.$$

Hence

$$U_{j+i}^-(\varepsilon e^{-\beta}, \xi + 1, 0) > U_{j+i}^-(\varepsilon e^{-\beta} + s, \xi + 2K_0 s + 1, 0) \quad \forall |i| > N_2, s \in (0, \varepsilon_0].$$

Since $\min_{|i|<N_2} \delta_i > 0$, there exists a universal constant $\kappa_1 > 0$ such that

$$U_{j+i}^-(\varepsilon e^{-\beta}, \xi + 1, 0) + \kappa_0\tau_1\delta_i \geqslant U_{j+i}^-(\varepsilon e^{-\beta} + \kappa_1\tau_1, \xi + 1 + 2K_0\kappa_1\tau_1, 0) \quad \forall\, i \in \mathbb{Z}.$$

Taking smaller κ_1 if necessary, we can assume that $K_0\kappa_1 \leqslant 1$. Thus

$$\mathbf{u}(t+1) \geqslant \mathbf{U}^-(\varepsilon e^{-\beta} + \kappa_1\tau_1, \xi + 1 + 2K_0\kappa_1\tau_1, 0).$$

On the other hand,

$$\mathbf{u}(t+1) \leqslant \mathbf{U}^+(\varepsilon e^{-\beta}, \xi + \tau + 1, 0) \leqslant \mathbf{U}^+(\varepsilon e^{-\beta} + \kappa_1\tau_1, \xi + 1 + \tau + K_0\kappa_1\tau_1, 0),$$

where the first inequality follows from the comparison principle and the second one from the definition of $\mathbf{U}^+$. Hence, by the definition of $D(t+1)$,

$$(\varepsilon e^{-\beta} + \kappa_1\tau_1, \xi + 1 + 2K_0\kappa_1\tau_1, \tau - K_0\kappa_1\tau_1) \in D(t+1).$$

This implies that

$$\begin{aligned} d(t+1) &\leqslant K_0[\varepsilon e^{-\beta} + \kappa_1\tau_1] + 2(\tau - K_0\kappa_1\tau_1) \\ &= d(t) - K_0[1 - e^{-\beta}]\varepsilon(t) - K_0\kappa_1 \min\{1, \tau(t)/4\}. \end{aligned}$$

The same inequality holds also for the case (ii). That is, the estimate holds whenever $\varepsilon(t) \leqslant \min\{\varepsilon_0, \tau(t)/(4K_0)\}$.

6. Finally, combining the conclusion from the previous two steps, we see that for every $t \geqslant 0$,

$$\begin{aligned} d(t+1) &\leqslant d(t) - \begin{cases} (1-e^{-\beta})K_0\varepsilon_0 & \text{if } \ \varepsilon(t) \geqslant \varepsilon_0, \\ K_0\kappa_1 & \text{if } \ \varepsilon(t) < \varepsilon_0, \tau(t) \geqslant 4, \\ \frac{1}{9}(1-e^{-\beta})\Big\{K_0\varepsilon(t) + 2\tau(t)\Big\} & \text{if } \ \varepsilon(t) < \varepsilon_0, \ \tau(t) \leqslant 4K_0\varepsilon(t), \\ \frac{K_0\kappa_1}{4}\tau(t) + K_0(1-e^{-\beta})\varepsilon(t) & \text{if } \ \varepsilon(t) < \varepsilon_0, 4K_0\varepsilon(t) < \tau(t) < 4, \end{cases} \\ &\leqslant d(t) - \min\{\delta_0, \delta_1 d(t)\}, \end{aligned}$$

where

$$\delta_0 := \min\{(1-e^{-\beta})K_0\varepsilon_0, K_0\kappa_1\}, \quad \delta_1 := \frac{\delta_0}{9}.$$

From this, it is then easy to show that

$$d(t) \leqslant Ke^{-\nu t} \quad \forall t \geq 0,$$

where $\nu = -\ln(1-\delta_1)$. Finally, one can show that there exists τ^* such that

$$\xi(t) - t = \tau^* + O(1)e^{-\nu t}.$$

See [10] for the details and we omit it here.

As $\mathbf{U}^-(\varepsilon, \xi, 0) - \mathbf{U}(\xi) = O(\varepsilon)$ and $\mathbf{U}^+(\varepsilon, \xi+\tau, 0) - \mathbf{U}(\xi) = O(\varepsilon) + O(\tau)$, we then obtain the assertion of Theorem 4. □

Part II: Existence

In this part, we shall establish the existence of traveling waves under the following basic assumptions, for all $i \in \mathbb{Z}$,

$$\begin{cases} a_{i+n,k} = a_{i,k} \ \forall\, k \in \mathbb{Z}; \quad \sum_k a_{i,k} = 0; \quad a_{i,k} \geqslant 0 \ \forall\, k \neq 0; \\ \quad a_{i,1} > 0, \ a_{i,-1} > 0; \quad \sum_k a_{i,k}(e^k + e^{-k} - 2) < \infty, \\ f_{i+n} = f_i \in C^1([0,1]), \quad f_i(0) = f_i(1) = 0. \end{cases}$$

Since these assumptions are not sufficient, more technical assumptions will be given later.

6. An Integral Formulation

Here in this section, we shall reformulate the problem of finding traveling waves. For the existence of traveling waves, we pay attention to only those "special" ones that have periods $n/|c|$ when $c \neq 0$ and are stationary when $c = 0$. Note that stationary solutions are merely vectors in $\mathbb{R}^{\mathbb{Z}}$. Sometimes it is necessary to discuss the cases $c = 0$ and $c \neq 0$ separately.

6.1. **The case $c \neq 0$.** When $c \neq 0$, our uniqueness result reveals that a traveling wave with speed c should have a period $T = n/|c|$. That is,

$$u_i(t + n/c) = u_{i-n}(t). \tag{6.1}$$

For such a wave, it is convenient to introduce

$$w_i(x) := u_i(t)\Big|_{ct=i-x} = u_i\Big(\frac{i-x}{c}\Big) \quad \forall\, x \in \mathbb{R}, i \in \mathbb{Z}.$$

It is easy to derive from (6.1) that $w_{i+n}(x) = w_i(x)$ for all $x \in \mathbb{R}$ and $i \in \mathbb{Z}$. Hence, we have a total of n unknown functions.

Also, the limits in (1.4) (with $\Phi^+ = \mathbf{1}$ and $\Phi^- = \mathbf{0}$) imply that for each $i \in \mathbb{Z}$

$$\begin{aligned} w_i(\infty) &:= \lim_{x\to\infty} w_i(x) = \lim_{ct\to-\infty} u_i(t) = 1, \\ w_i(-\infty) &:= \lim_{x\to-\infty} w_i(x) = \lim_{ct\to\infty} u_i(t) = 0. \end{aligned}$$

Thus $\mathbf{w} := \{w_i\} \in \mathbf{X}$, where

$$\mathbf{X} \;:=\; \Big\{\{w_i\} \;\Big|\; w_i(-\infty) = 0 \leqslant w_i(x) = w_{i+n}(x) \leqslant 1 = w_i(\infty) \;\; \forall\, x \in \mathbb{R}, i \in \mathbb{Z}\Big\}.$$

Finally, equation (1.3) becomes

$$-c\, w_i'(x) = \sum_k a_{i,k} w_{i+k}(x+k) + f_i(w_i(x)), \quad w_i(x) = w_{i+n}(x) \quad \forall x \in \mathbb{R}. \tag{6.2}$$

Instead of studying the differential equation, we shall work on its integral form. For this, we let

$$\nu := 1 + \max_i |a_{i,0}| + \max_i \max_{s\in[0,1]} |f_i'(s)|.$$

Via the integrating factor $e^{-\nu x/c}$, (6.2) can be re-written as

$$\begin{aligned} w_i(x) &= \int_{c\cdot\infty}^{x} \frac{e^{\nu(x-y)/c}}{-c}\Big\{\nu w_i(y) + f_i(w_i(y)) + \sum_k a_{i,k} w_{i+k}(y+k)\Big\} dy \\ &= \int_{-\infty}^{0} e^{\nu t}\Big\{\nu\, w_i(x-ct) + f_i(w_i(x-ct)) + \sum_k a_{i,k} w_{i+k}(x-ct+k)\Big\} dt. \end{aligned}$$

Here the factor $e^{\nu t}$ is introduced for two purposes: (i) the integral is uniformly convergent; (ii) the integrand is monotonically increasing in each w_{i+k}, $k \in \mathbb{Z}, i \in \mathbb{Z}$.

For convenience, we use the following notation:

$$\mathbf{w} = \{w_i\}, \quad \mathbb{W}[\mathbf{w}] = \{\mathbb{W}_i[\mathbf{w}]\}, \quad \mathbb{T}^c[\mathbf{w}] = \{\mathbb{T}_i^c[\mathbf{w}]\},$$

$$\begin{aligned} \mathbb{W}_i[\mathbf{w}](x) &= \nu\, w_i(x) + f_i(w_i(x)) + \sum_k a_{i,k} w_{i+k}(x+k), \\ \mathbb{T}_i^c[\mathbf{w}](x) &= \int_{-\infty}^{0} \mathbb{W}_i[\mathbf{w}](x-ct) e^{\nu t} dt. \end{aligned}$$

Therefore, to find a traveling wave, it is sufficient to find $(c, \mathbf{w}) \in \mathbb{R} \times \mathbf{X}$ such that

$$\mathbf{w} = \mathbb{T}^c[\mathbf{w}]. \tag{6.3}$$

Lemma 6.1. *A pair* $(c, \mathbf{w}) \in \mathbb{R} \times \mathbf{X}$ *solves* (6.2) *if and only if it solves* (6.3)*. In addition, if* $(c, \mathbf{w}) \in \mathbb{R} \times \mathbf{X}$ *solves* (6.2)*, then the function* $\mathbf{u} = \{u_i\}$ *defined by* $u_i(t) = w_i(i - ct)$ *for all* $i \in \mathbb{Z}$ *and* $t \in \mathbb{R}$ *satisfies* (1.3), (1.4)*, as well as* (6.1) *when* $c \neq 0$*.*

The proof follows from direct calculation.

6.2. **The Case** $c = 0$**.** In this case, (6.2) and (6.3) are identical. Also, we have the following observations.

(i) If $\mathbf{w}$ satisfies (6.2) with $c = 0$, then for each $z \in [0, n)$, the vector $\Phi^z = \{\phi_i^z\} \in \mathbb{R}^{\mathbb{Z}}$ defined by $\phi_i^z = w_i(i + z)$, $i \in \mathbb{Z}$, is a steady state of (1.3) (i.e., a solution to (2.1)). Note that if $w_0(\cdot)$ is non-decreasing and $w_0(0) < w_0(z) < w_0(n)$, then Φ^0, Φ^z are two different steady states that cannot be obtained from one to the other by a grid translation.

(ii) If $\Phi \equiv \{\phi_i\}$ is a steady state of (1.3), then the following $\mathbf{w} = \{w_i\}$ is a solution to (6.2) with $c = 0$:

$$w_i(x) = \phi_{i+n[(x-i)/n]} \quad \forall\, x \in \mathbb{R},\, i \in \mathbb{Z}.$$

Here $[y]$ denotes the maximum integer no larger than y.

(iii) It is a well-known fact that stationary solutions to (1.3) and (1.4), if they exist, are in general not unique (modulo the grid translation invariance).

For steady states, it is convenient to work on the following equations

$$\sum_k a_{i,k} w_{i+k}(x + k) + f_i(w_i(x)) = 0, \quad w_{i+n}(x) = w_i(x) \quad \text{a.e. } \; x \in \mathbb{R}. \tag{6.4}$$

Lemma 6.2. *Suppose* $\mathbf{w} = \{w_i\}$ *satisfies* (6.4)*. Then for a.e.* $z \in \mathbb{R}$*, the vector* $\Phi^z = \{\phi_i^z\}$*, where* $\phi_i^z = w_i(i + z)$ *for all* $i \in \mathbb{Z}$*, is a steady state of* (1.3)*.*

6.3. **Wave Profile.** In the sequel, for $c \in \mathbb{R}$, a solution $\mathbf{w}(\cdot) = \{w_i(\cdot)\}$ to (6.2) in the set $\mathbf{X}$ will be called a **wave profile**. By the periodicity, a wave profile consists of n unknown functions. A traveling wave $(c, \mathbf{u})$ can be obtained from a wave profile $\mathbf{w}$ via

$$u_i(t) := w_i(z + i - ct) \quad \forall i \in \mathbb{Z}, t \in \mathbb{R}$$

where $z \in \mathbb{R}$ is any fixed constant. Note that in general when $c = 0$ wave profiles are not continuous, whereas when $c \neq 0$, wave profiles are always smooth.

In the sequel, we shall frequently use the following Helly's Lemma:

Lemma 6.3. *Let* $\{w_l(\cdot)\}_{l \in \mathbb{N}}$ *be a sequence of uniformly bounded non-decreasing functions on* $\mathbb{R}$*. Then there exist a non-decreasing function* w *and a sequence of integers* $\{l_i\}_{i \in \mathbb{N}}$ *such that*

$$\lim_{i \to \infty} l_i = \infty, \quad \lim_{i \to \infty} w_{l_i}(x) = w(x) \quad \forall\, x \in \mathbb{R}.$$

7. Existence of a Non-trivial Solution

Without any structural assumption on the $\{f_i(\cdot)\}$, in this section, we shall prove the following:

Theorem 5. *Let $\mathbf{a} = \{a_i\}$ be a vector satisfying $\mathbf{0} < \mathbf{a} < \mathbf{1}$. Then there exist $c \in \mathbb{R}$ and $\mathbf{w}(\cdot) = \{w_i(\cdot)\}$ that satisfies* (6.2) *if $c \neq 0$ and* (6.4) *if $c = 0$, as well as*

$$\mathbf{0} < \mathbf{w}(x) \leqslant \mathbf{w}(y) < \mathbf{1} \quad \forall x \leqslant y, \tag{7.1}$$

$$\begin{cases} \text{either} & c \geqslant 0, \ \ \max\{w_i(0) - a_i\} = 0, \\ \text{or} & c \leqslant 0, \ \ \min\{w_i(0) - a_i\} = 0. \end{cases} \tag{7.2}$$

Note that if $\mathbf{a}$ is not a steady state, then the solution in the theorem is non-trivial (i.e. non-constant vector). In the next section, a sufficient condition on $\{f_i\}$ will be supplied to guarantee the limits $\mathbf{w}(-\infty) = \mathbf{0}$ and $\mathbf{w}(\infty) = \mathbf{1}$.

7.1. A Truncation.

We approximate (6.2) by the related problem on finite interval

$$\begin{cases} -cw_i'(x) = \sum_k a_{i,k} w_{i+k}(x+k) + f_i(w_i(x)), \quad -m < x < m, \ i \in \mathbb{Z}, \\ w_{i+n}(x) = w_i(x) \quad \forall\, x \in \mathbb{R}, \ i \in \mathbb{Z}, \\ w_i(x) = 0 \quad \forall x < -m, i \in \mathbb{Z}, \quad w_i(x) = 1 \quad \forall\, x > m, \ i \in \mathbb{Z}. \end{cases} \tag{7.3}$$

Here $m \in \mathbb{N} := \{1, 2, 3, \cdots\}$ is a parameter. Eventually, we shall take the limit as $m \to \infty$. We remark that the solution is discontinuous at $x = \pm m$. Due to this, it is convenient to consider (7.3) in its integral forms. For this we define a projection operator $\mathbb{P}_m$ by

$$\mathbb{P}_m \mathbf{w}(x) \ := \ \begin{cases} \mathbf{0} & \text{if } \ x < -m, \\ \mathbf{w}(x) & \text{if } \ x \in [-m, m], \\ \mathbf{1} & \text{if } \ x > m. \end{cases}$$

Note that for every $c \in \mathbb{R}$, $\mathbf{w}$ solve (7.3) if and only if

$$\mathbf{w} \in \mathbf{X}, \quad \mathbf{w} = \mathbb{P}_m \mathbb{T}^c[\mathbf{w}]. \tag{7.4}$$

Since $\mathbb{P}_m^2 = \mathbb{P}_m$, the solution automatically satisfies $\mathbf{w} = \mathbb{P}_m \mathbf{w}$.

Lemma 7.1. *Let $m \in \mathbb{N}$.*

(1) *For every $c \neq 0$,* (7.4) *admits a unique solution $\mathbf{w} = \mathbf{w}^{m,c}$. In addition, the solution satisfies $\mathbf{0} < \mathbf{w}(x) < \mathbf{1}$ for all $x \in [-m, m]$ and $\mathbf{w}'(x) > 0$ for all $x \in (-m, m)$.*

(2) *When $c = 0$,* (7.4) *admits a minimal solution, denoted by $\mathbf{w}_*^m$, and a maximal solution, denoted by $\mathbf{w} = \mathbf{w}^{*m}$, in the sense that any solution $\mathbf{w}$ to* (7.4) *with $c = 0$ satisfies*

$$\mathbf{0} < \mathbf{w}_*^m(x) \leqslant \mathbf{w}(x) \leqslant \mathbf{w}^{*m}(x) < \mathbf{1} \quad \forall\, x \in [-m, m].$$

In addition, both $\mathbf{w}_*^m(\cdot)$ and $\mathbf{w}^{*m}(\cdot)$ are non-decreasing on $\mathbb{R}$ and are constant (n-periodic vectors) on each interval $(l, l+1)$ for all $l \in \mathbb{Z}$.

Proof. **1. Existence.** Fix $c \in \mathbb{R}$. The operator $\mathbb{P}_m\mathbb{T}^c$ is monotonic on $\mathbf{X}$:

$$\mathbf{w}, \tilde{\mathbf{w}} \in \mathbf{X}, \quad \mathbf{w} \leqslant \tilde{\mathbf{w}} \quad \Longrightarrow \quad \mathbb{P}_m\mathbb{T}^c[\mathbf{w}] \leqslant \mathbb{P}_m\mathbb{T}^c[\tilde{\mathbf{w}}].$$

Consider the sequences $\{\mathbf{w}_{*j}\}_{j=0}^{\infty}$ and $\{\mathbf{w}_j^*\}_{j=0}^{\infty}$ defined by

$$\mathbf{w}_{*0} := \mathbf{0}, \quad \mathbf{w}_{*j} := \mathbb{P}_m\mathbb{T}^c[\mathbf{w}_{*\,j-1}] \quad \forall\, j \in \mathbb{N},$$
$$\mathbf{w}_0^* := \mathbf{1}, \quad \mathbf{w}_j^* := \mathbb{P}_m\mathbb{T}^c[\mathbf{w}_{j-1}^*] \quad \forall\, j \in \mathbb{N}.$$

Since $\mathbb{T}^c[\mathbf{1}] = \mathbf{1}$ and $\mathbb{T}^c[\mathbf{0}] = \mathbf{0}$, an induction shows that

$$\mathbf{0} \leqslant \mathbf{w}_{*\ j-1}(x) \leqslant \mathbf{w}_{*j}(x) \ \leqslant \ \mathbf{w}_j^*(x) \leqslant \mathbf{w}_{j-1}^*(x) \leqslant \mathbf{1} \quad \forall j \in \mathbb{N},\ x \in \mathbb{R}.$$

Thus, there exist the limits

$$\mathbf{w}_*(x) := \lim_{j\to\infty} \mathbf{w}_{*j}(x), \quad \mathbf{w}^*(x) := \lim_{j\to\infty} \mathbf{w}_j^*(x) \quad \forall\, x \in \mathbb{R}.$$

Using Lebesgue's dominated convergence theorem, it is easy to show that both $\mathbf{w} = \mathbf{w}_*$ and $\mathbf{w} = \mathbf{w}^*$ are solutions to (7.4).

2. Basic Properties of the Solution.

First of all, if $\mathbf{w}$ is a solution to (7.4), then $\mathbf{0} \leqslant \mathbf{w} \leqslant \mathbf{1}$. Inductively, we derive that $\mathbf{w}_{*j} \leqslant \mathbf{w} \leqslant \mathbf{w}_j^*$ for all $j \in \mathbb{N}$ so that $\mathbf{w}_* \leqslant \mathbf{w} \leqslant \mathbf{w}^*$.

Next we show that

$$\mathbf{0} < \mathbf{w}_*(x) \leqslant \mathbf{w}^*(x) < \mathbf{1} \quad \forall\, x \in [-m, m].$$

Consider $\mathbf{w} = \mathbf{w}^*$. Suppose, for a contradiction argument, that there exists $x \in [-m, m]$ such that $w_i(x) = 1$ for some $i \in \mathbb{Z}$. By taking smaller x if necessary, we may assume that $\mathbf{w}(y) < \mathbf{1}$ for all $y \leqslant x - 1/2$. As $a_{i,k} > 0$ for at least one integer $k < 0$, we see that $\mathbb{W}_i[\mathbf{w}](z) < \mathbb{W}_i[\mathbf{1}] \equiv \nu$ for every $z < x + 1/2$. Consequently,

$$1 = \mathbb{T}_i^c[\mathbf{w}](x) = \int_{-\infty}^{0} e^{\nu t}\mathbb{W}_i[\mathbf{w}](x - ct)dt < \int_{-\infty}^{0} \nu e^{\nu t} dt = 1,$$

which is impossible. Thus, we must have $\mathbf{w}^* < \mathbf{1}$ on $(-\infty, m]$. In a similar manner, we can show that $\mathbf{w}_* > \mathbf{0}$ on $[-m, \infty)$.

Note that for each $j \in \mathbb{N}$, both w_{*j} and w_j^* are non-decreasing. It follows that $\mathbf{w}_*' \geqslant 0$ and $\mathbf{w}^{*'} \geqslant 0$ on $\mathbb{R}$ in the distribution sense. Also, $\mathbf{w}'(m) = (\mathbf{1} - \mathbf{w}(m))\delta$ and $\mathbf{w}'(-m) = \mathbf{w}(-m)\delta$ for any solution $\mathbf{w}$ of (7.4), where δ is the Dirac mass.

When $c = 0$, $\mathbb{T}^c[\mathbf{w}](x) = \frac{1}{\nu}\mathbb{W}[\mathbf{w}](x)$. One can inductively show that both $\mathbf{w}_{*j}(\cdot)$ and $\mathbf{w}_j^*(\cdot)$ are constant vectors on each interval $(l, l+1)$ for all $l \in \mathbb{Z}$. It follows that both $\mathbf{w}_*$ and $\mathbf{w}^*$ are constant vectors on each interval $(l, l+1), l \in \mathbb{Z}$.

When $c \neq 0$, for either $\mathbf{w} = \mathbf{w}_*$ or $\mathbf{w} = \mathbf{w}^*$, $\mathbf{w}$ is Lipschitz continuous on $[-m, m]$. At any $x \in (-m, m)$, differentiating $\mathbf{w} = \mathbb{P}_m\mathbb{T}^c[\mathbf{w}] = \mathbb{T}^c[\mathbf{w}]$ and using the definition of ν, one derives that

$$\mathbf{w}'(x) \geqslant \int_{-\infty}^{0} \mathbf{w}'(x - ct)e^{\nu t}dt \geqslant \begin{cases} [\mathbf{1} - \mathbf{w}(m)]e^{\nu(x-m)/c} & \text{if } \ c > 0, \\ \mathbf{w}(-m)e^{\nu(x+m)/c} & \text{if } \ c < 0. \end{cases}$$

Thus $\mathbf{w}' > \mathbf{0}$ on $(-m, m)$.

3. Uniqueness. Suppose $c \neq 0$. We want to show that $\mathbf{w}^* = \mathbf{w}_*$. For this, let

$$h := \inf\{\xi > 0 \mid \mathbf{w}_*(\cdot + \xi) \geqslant \mathbf{w}^*(\cdot)\}.$$

Since $\mathbf{w}_*(x+2m+\varepsilon) \geqslant \mathbf{w}^*(x) \geqslant \mathbf{w}_*(x)$ for every $\varepsilon > 0$ and $x \in \mathbb{R}$, h is well-defined and $h \in [0, 2m]$. We claim that $h = 0$. Suppose, on the contrary, that $h > 0$. Since $\mathbf{w}^*$ and $\mathbf{w}_*$ are continuous on $\mathbb{R} \setminus \{m, -m\}$, by the definition of h, we have

$$\mathbf{0} \leqslant \limsup_{\xi \to h} \Big\{\mathbb{T}^c[\mathbf{w}_*](x+\xi) - \mathbb{T}^c[\mathbf{w}^*](x)\Big\} = \mathbb{T}^c[\mathbf{w}_*](x+h) - \mathbb{T}^c[\mathbf{w}^*](x)$$

for every $x \in \mathbb{R}$. After projection, we obtain $\mathbf{w}_*(\cdot + h) \geqslant \mathbf{w}^*(\cdot)$ on $\mathbb{R}$. Consequently, by the definition of ν, $\mathbb{W}[\mathbf{w}_*](x+h) - \mathbb{W}[\mathbf{w}^*](x) \geqslant \mathbf{w}_*(x+h) - \mathbf{w}^*(x) \geqslant 0$ for all $x \in \mathbb{R}$. It then follows that for every $x \in \mathbb{R}$,

$$\mathbf{w}_*(x+h) - \mathbf{w}^*(x) \geqslant \int_{-\infty}^0 e^{\nu t}\{\mathbf{w}_*(x+h-ct) - \mathbf{w}^*(x-ct)\}dt.$$

Consider $x \in [-h/2 - m, m - h/2]$. Since $\mathbf{w}^*(y) = \mathbf{0} < \mathbf{w}_*(y+h)$ when $y = x - ct \in [-h-m, -m)$ and $\mathbf{w}^*(y) < \mathbf{1} = \mathbf{w}_*(y+h)$ when $y = x - ct \in (m-h, m]$, the integrand is positive when $ct \in (x+m, x+m+h] \cup (x-m, x-m+h]$. As $c \neq 0$, we see that $\mathbf{w}_*(x+h) > \mathbf{w}^*(x)$ for all $x \in [-h/2 - m, m - h/2]$. Hence, by continuity, there exists $\varepsilon \in (0, h/2)$ such that $\mathbf{w}_*(x+h-\varepsilon) - \mathbf{w}^*(x) \geqslant 0$ for all $x \in [-h/2 - m, m - h/2]$ and therefore also for all $x \in \mathbb{R}$, contradicting the definition of h. This contradiction shows that $h = 0$. Since $\mathbb{T}^c$ is continuous from $L^\infty(\mathbb{R})$ to $C(\mathbb{R})$, we also have, for every $x \in [-m, m]$,

$$\mathbf{0} \leqslant \limsup_{\xi \searrow 0} \Big\{\mathbb{T}^c[\mathbf{w}_*](x+\xi) - \mathbb{T}^c[\mathbf{w}^*](x)\Big\} = \mathbb{T}^c[\mathbf{w}_*](x) - \mathbb{T}^c[\mathbf{w}^*](x).$$

This completes the proof. □

7.2. Monotonicity in c.

Lemma 7.2. *Let $m \in \mathbb{N}$, $c_1 < c_2$, and $\mathbf{w}^{m,c_i}$, $i = 1, 2$, be a solution to (7.4) with $c = c_i$. Then $\mathbf{w}^{m,c_1}(x) < \mathbf{w}^{m,c_2}(x)$ for all $x \in [-m, m]$. In addition, denoting by $\mathbf{w}^{*m}$ and $\mathbf{w}_*^m$ the maximal and minimal solution to (7.4) with $c = 0$ respectively, we have*

$$\lim_{c \nearrow 0} \mathbf{w}^{m,c}(x) = \mathbf{w}_*^m(x), \quad \lim_{c \searrow 0} \mathbf{w}^{m,c} = \mathbf{w}^{*m}(x) \quad \forall x \in \mathbb{R} \setminus \mathbb{Z}. \tag{7.5}$$

Proof. **1.** First consider the case $c_2 > c_1 \neq 0$. Since $\mathbf{w}^{m,c_1}(\cdot)$ is strictly increasing on $[-m, m]$, $\sum_{k>0} a_{i,k} > 0$ and $\sum_{k<0} a_{i,k} > 0$, we have, for every $x \in [-m, m]$,

$$\begin{aligned}
\mathbb{T}^{c_2}[\mathbf{w}^{m,c_1}](x) &= \int_{-\infty}^0 e^{\nu t}\mathbb{W}[\mathbf{w}^{m,c_1}](x - c_2 t)dt \\
&> \int_{-\infty}^0 e^{\nu t}\mathbb{W}[\mathbf{w}^{m,c_1}](x - c_1 t)dt \\
&= \mathbb{T}^{c_1}[\mathbf{w}^{m,c_1}](x) = \mathbf{w}^{m,c_1}(x).
\end{aligned}$$

Now starting from $\mathbf{w}_0 = \mathbf{w}^{m,c_1}$ and defining $\mathbf{w}_{j+1} := \mathbb{P}_m\mathbb{T}^{c_2}[\mathbf{w}_j]$ for all $j \geqslant 0$, one can show that $\{\mathbf{w}_j\}$ is an increasing sequence and approaches a limit $\hat{\mathbf{w}}$, being a solution to $\hat{\mathbf{w}} = \mathbb{P}_m\mathbb{T}^{c_2}[\hat{\mathbf{w}}]$. When $c_2 \neq 0$, we have $\hat{\mathbf{w}} = \mathbf{w}^{m,c_2}$ so that $\mathbf{w}^{m,c_1} < \mathbf{w}_1 \leqslant \mathbf{w}^{m,c_2}$ on $[-m,m]$. When $c_2 = 0$, one can show directly by the sliding method used in the Step 3 in the previous subsection to show that $\mathbf{w}^{m,c_1} < \mathbf{w}_*$; here the key for the sliding method to work is the continuity of $\mathbf{w}^{m,c_1}$ on $[-m,m]$.

In a similar manner, we can consider the case $c_1 < c_2 \neq 0$.

2. The monotonicity of $\mathbf{w}^{m,c}$ in c implies the existence of the limit

$$\tilde{\mathbf{w}}(x) := \lim_{c\searrow 0} \mathbf{w}^{m,c}(x).$$

Since $\mathbf{w}^{m,c} > \mathbf{w}^{*m}$ for each $c > 0$, we have $\tilde{\mathbf{w}} \geqslant \mathbf{w}^{*m}$. Furthermore, since $\mathbf{w}^{m,c}(\cdot)$ is non-decreasing, $\tilde{\mathbf{w}}(\cdot)$ is also non-decreasing. Thus, there is a countable set Σ_0 such that $\tilde{\mathbf{w}}$ is continuous on $\mathbb{R}\setminus\Sigma_0$. Consequently,

$$\limsup_{y\to x, c\searrow 0} |\tilde{\mathbf{w}}(x) - \mathbf{w}^{m,c}(y)| = 0 \quad \text{for all} \quad x \in \mathbb{R}\setminus\Sigma_0.$$

Define $\Sigma = \{k + x \mid k \in \mathbb{Z}, x \in \Sigma_0\}$. Note that Σ is a countable and grid-translation invariant set. Also for every $x \in [-m,m]\setminus\Sigma$,

$$\begin{aligned} \tilde{\mathbf{w}}(x) &= \lim_{c\searrow 0} \mathbf{w}^{m,c}(x) = \lim_{c\searrow 0} \int_{-\infty}^{0} \mathbb{W}[\mathbf{w}^{m,c}](x - ct)e^{\nu t}dt \\ &= \mathbb{W}[\tilde{\mathbf{w}}](x) \int_{-\infty}^{0} e^{\nu t}dt. \end{aligned}$$

This implies, writing $\tilde{\mathbf{w}} = \{\tilde{w}_i\}$, that $\sum_k a_{i,k}\tilde{w}_{i+k}(x+k) + f_i(\tilde{w}_i(x)) = 0$ for every $x \in [-m,m]\setminus\Sigma$. Finally, define

$$\hat{\mathbf{w}}(x) := \begin{cases} \tilde{\mathbf{w}}(x) & x \in \mathbb{R}\setminus\Sigma, \\ \mathbf{w}^{*m}(x) & x \in \Sigma. \end{cases}$$

One can verify that $\hat{\mathbf{w}}$ is a solution to (7.4) with $c = 0$. By the maximality, we must have $\hat{\mathbf{w}} \leqslant \mathbf{w}^{*m}$. Since $\tilde{\mathbf{w}} \geqslant \mathbf{w}^{*m}$, we have that $\tilde{\mathbf{w}} = \mathbf{w}^{*m}$ on $\mathbb{R}\setminus\Sigma$. Finally, since $\tilde{\mathbf{w}}$ is non-decreasing and $\mathbf{w}^{*m}$ is constant on each interval $(l, l+1), l \in \mathbb{Z}$, we conclude that $\tilde{\mathbf{w}} = \mathbf{w}^{*m}$ on $\mathbb{R}\setminus\mathbb{Z}$. That is, $\lim_{c\searrow 0}\mathbf{w}^{m,c} = \mathbf{w}^{*m}$ on $\mathbb{R}\setminus\mathbb{Z}$.

In a similar manner, we can show that $\lim_{c\nearrow 0}\mathbf{w}^{m,c} = \mathbf{w}_*^{m}$ on $\mathbb{R}\setminus\mathbb{Z}$. This completes the proof. ☐

7.3. **Two Useful Bounds.** For convenience, we use notation, for $\mathbf{w} = \{w_i\}$,

$$\mathcal{N}_i^c[\mathbf{w}](x) := -cw_i'(x) - \textstyle\sum_k a_{i,k}w_{i+k}(x+k) - f_i(w_i(x)).$$

Introduce the following constants

$$c^* = L + \sum_{k<0} a_{i,k}(e^{-k} - 1), \quad c_* := -L - \sum_{k>0} a_{i,k}(e^{k} - 1), \quad L := \max_i \max_{s\in[0,1]} |f_i'(s)|.$$

For each $m \in \mathbb{N}$, we consider $\overline{\mathbf{w}}(x) := \overline{w}(x)\mathbf{1}$ with $\overline{w}(x) := \min\{1, e^{x-m}\}$. When $x > m$, we have $\overline{w}(x) = 1 \geqslant \overline{w}(y)$ for all $y \in \mathbb{R}$. Thus,

$$\mathcal{N}_i^c[\overline{\mathbf{w}}](x) = -\sum_k a_{i,k}\overline{w}(x+k) = \sum_k a_{i,k}[\overline{w}(x) - \overline{w}(x+k)] \geqslant 0 \quad \forall x > m.$$

When $c \leqslant c_*$ and $x < m$, using $|f_i(s)| \leqslant L|s|$ and $-a_{i,k}\overline{w}(x+k) \geqslant -a_{i,k}e^{x+k-m}$ for $k \neq 0$, we obtain

$$\mathcal{N}_i^c[\overline{\mathbf{w}}](x) \quad \geqslant \quad e^{x-m}\Big\{ -c_* - \sum_k a_{i,k}(e^k - 1) - L\Big\} \geqslant 0.$$

Hence, after integrating $e^{-\nu y/c}\mathcal{N}_i^c[\overline{\mathbf{w}}](y) \geqslant 0$ over $(-\infty, x]$, we obtain

$$\overline{\mathbf{w}}(x) \geqslant \mathbb{T}^c[\overline{\mathbf{w}}](x) \quad \forall\, x \in \mathbb{R},\ c \leqslant c_*.$$

Moreover, since $\mathbf{0} \leq \overline{\mathbf{w}}$ and $\overline{\mathbf{w}}(x) = \mathbf{1}$ for all $x \geq m$, we have

$$P_m\mathbb{T}^c[\mathbf{0}](x) \leq P_m\mathbb{T}^c[\overline{\mathbf{w}}](x) \leq P_m\overline{\mathbf{w}}(x) \leq \overline{\mathbf{w}}(x)$$

for $c \leqslant c_*$. By a similar argument as the **Step 1** of Lemma 7.1, we have $\mathbf{w}^{m,c}(x) \leq \overline{\mathbf{w}}(x)$ for all $x \in \mathbb{R}$, if $c \leqslant c_*$. Hence we obtain

$$\mathbf{w}^{m,c_*}(0) \leq e^{-m}\mathbf{1} \quad \forall m \in \mathbb{N}. \tag{7.6}$$

Similarly, by considering $\underline{\mathbf{w}}(x) := \underline{w}(x)\mathbf{1}$ with $\underline{w}(x) := \max\{0, 1 - e^{-(x+m)}\}$, we obtain the following lower bound

$$\mathbf{w}^{m,c^*}(0) \geqslant (1 - e^{-m})\mathbf{1} \quad \forall m \in \mathbb{N}. \tag{7.7}$$

7.4. **Proof of Theorem 5.** Fix a vector $\mathbf{a}$ satisfying $\mathbf{0} < \mathbf{a} < \mathbf{1}$. There are three possibilities:

(1) $\liminf_{m\to\infty} \mathbf{w}^{*m}(0) \leqslant \mathbf{a}$;
(2) $\limsup_{m\to\infty} \mathbf{w}_*^m(0) \geqslant \mathbf{a}$;
(3) $\liminf_{m\to\infty} \max_i\{w_i^{*m}(0) - a_i\} > 0 > \limsup_{m\to\infty} \min_i\{w_{*i}^m(0) - a_i\}$.

First, we consider the case (1). Suppose $\liminf_{m\to\infty} \mathbf{w}^{*m}(0) \leqslant \mathbf{a}$. Then, by (7.7), there exists a sequence of positive integers $\{m_l\}_{l\in\mathbb{N}}$ such that for each $l \in \mathbb{N}$

$$m_l > l, \quad \mathbf{w}^{*m_l}(0) < \mathbf{a} + \frac{\mathbf{1}}{l}, \quad \mathbf{w}^{m_l,c^*}(0) \geqslant (1 - e^{-m_l})\mathbf{1} > \mathbf{a} + \frac{\mathbf{1}}{l}.$$

By continuity, there exists $c_l \in (0, c^*)$ such that $\max_i\{w_i^{m_l,c_l}(0) - (a_i + \frac{1}{l})\} = 0$.

Now consider the sequence $\{(c_l, \mathbf{w}^{m_l,c_l})\}_{l=1}^\infty$. Since each $\mathbf{w}^{m,c}(\cdot)$ is monotonic, by taking a subsequence if necessary, we have the limits

$$c := \lim_{l\to\infty} c_l \in [0, c^*], \qquad \mathbf{w}(x) = \{w_i(x)\} := \lim_{l\to\infty} \mathbf{w}^{m_l,c_l}(x) \quad \forall x \in \mathbb{R}.$$

By Lebesgue's dominated convergence theorem, $\mathbf{w}$ satisfies (6.2) when $c > 0$. It satisfies (6.4) when $c = 0$, cf. Step **2** in the proof in §7.2. Finally, by the definition of c_l and also by the strong maximum principle, $\mathbf{w}$ has the property (7.1) and (7.2).

Next, we consider the case (2). Suppose $\limsup_{m\to\infty} \mathbf{w}_*^m(0) \geqslant \mathbf{a}$. Using (7.6), this case can be treated similarly as above to obtain the existence of a solution to either (6.2) with $c < 0$ or (6.4) with $c = 0$.

Finally, we consider the case (3):

$$\liminf_{m\to\infty} \max_i \{w_i^{*m}(0) - a_i\} > 0 > \limsup_{m\to\infty} \min\{w_{*i}^m(0) - a_i).$$

This implies that there exists $m_0 \gg 1$ such that

$$\max_i \{w_i^{*m}(0) - a_i\} > 0 > \min_i \{w_{*i}^m(0) - a_i\}$$

for all integer $m \geqslant m_0$. Set

$$\begin{aligned} x_m &:= \sup\{x \mid \mathbf{w}^{*m}(x) \leqslant \mathbf{a}\}, \quad \tilde{\mathbf{w}}^{*m}(x) := \mathbf{w}^{*m}(x_m + x), \\ y_m &:= \sup\{x \mid \mathbf{w}_*^m(x) \geqslant \mathbf{a}\}, \quad \tilde{\mathbf{w}}_*^m(x) := \mathbf{w}_*^m(y_m + x). \end{aligned}$$

Since $\mathbf{w}_*^m$ and $\mathbf{w}^{*m}$ are constant on each interval $(l, l+1), l \in \mathbb{Z}$, we see that when $m \geqslant m_0$, x_m is a non-positive integer and y_m is a positive integer. There are three cases:

- **(i)** $\limsup_{m\to\infty}\{m + x_m\} = \infty$;
- **(ii)** $\limsup_{m\to\infty}\{m - y_m\} = \infty$;
- **(iii)** $\limsup_{m\to\infty}\{m + x_m\} < \infty$ and $\limsup_{m\to\infty}\{m - y_m\} < \infty$.

(i) Suppose $\limsup_{m\to\infty}\{m + x_m\} = \infty$. Since each $\tilde{\mathbf{w}}^{*m}$ is monotonic, there exists a sequence of integers $\{m_l\}_{l\in\mathbb{N}}$ tending to infinity and a function $\mathbf{w} = \{w_i\}$ such that

$$\lim_{l\to\infty}\{m_l + x_{m_l}\} = \infty, \quad \lim_{l\to\infty} \tilde{\mathbf{w}}^{*m_l}(x) = \mathbf{w}(x) \quad \forall\, x \in \mathbb{R}.$$

From the definition of x_m we see that $\mathbf{w}$ satisfies (7.2). In addition, since $\tilde{\mathbf{w}}^{*m}$ satisfies (6.4) for all $x \in [-m - x_m, m - x_m]$ with $x_m \leqslant 0$, we can derive that $\mathbf{w}$ satisfies (6.4) for all $x \in \mathbb{R}$. The strong maximum principle gives (7.1).

(ii) Suppose $\limsup_{m\to\infty}\{m - y_m\} = \infty$. Similar to the above case, we can show that along a subsequence $\tilde{\mathbf{w}}_*^m$ approaches a limit which satisfies (6.4), (7.1), and (7.2).

(iii) Finally suppose $\limsup_{m\to\infty}\{m + x_m\} < \infty$ and $\limsup_{m\to\infty}\{m - y_m\} < \infty$.

Since both $m + x_m$ and $m - y_m$ are integers, there exist finite integers $A, B \geqslant 0$ and a sequence of integers $\{m_l\}_{l\in\mathbb{N}}$ such that $\lim_{l\to\infty} m_l = \infty$, $m_l + x_{m_l} = A$ and $m_l - y_{m_l} = B$ for all $l \in \mathbb{N}$. Same as before, by taking a subsequence if necessary, we have, for some $\mathbf{w}^1$ and $\mathbf{w}^2$,

$$\lim_{l\to\infty} \tilde{\mathbf{w}}^{*m_l}(x) = \mathbf{w}^1(x), \quad \lim_{l\to\infty} \tilde{\mathbf{w}}_*^{m_l}(x) = \mathbf{w}^2(x) \quad \forall\, x \in \mathbb{R}.$$

Since $\tilde{\mathbf{w}}^{*m_l}$ satisfies (6.4) on $[-A, 2m_l - A]$ and $\tilde{\mathbf{w}}_*^{m_l}$ satisfies (6.4) on $[-2m_l + B, B]$ respectively, we see that

$$\begin{gathered}\sum a_{i,k} w^1_{i+k}(x+k) + f_i(w^1_i(x)) = 0 \quad \forall\ x \geqslant -A;\\ \sum a_{i,k} w^2{}_{i+k}(x+k) + f_i(w^2_i(x)) = 0 \quad \forall x \leqslant B;\\ \mathbf{w}^1(x) = \mathbf{0} \quad \forall x < -A, \quad \mathbf{w}^2 = \mathbf{1} \quad \forall x > B.\\ \mathbf{w}^1(x) \leqslant \mathbf{a}, \quad \min_i\{w^2_i(x) - a_i\} \leqslant 0 \quad \forall x < 0,\\ \max_i\{w^1_i(x) - a_i\} \geqslant 0, \quad \mathbf{w}^2(x) \geqslant \mathbf{a} \quad \forall x > 0.\end{gathered}$$

That is to say, $\mathbf{w}^2$ is a supersolution and $\mathbf{w}^1$ is a subsolution to the stationary problem (6.3) with $c = 0$. In addition, $\mathbf{w}^2(\cdot) \geqslant \mathbf{w}^1(\cdot - A - B - 1)$ since $\mathbf{w}^2(x) = \mathbf{1}$ when $x > B$ and $\mathbf{w}^1(x - A - B - 1) = \mathbf{0}$ for $x < B + 1$.

Finally, consider $\{\mathbf{w}_l\}_{l \in \mathbb{N}}$ defined by

$$\mathbf{w}_0 = \mathbf{w}^2, \quad \mathbf{w}_l = \mathbb{T}^0 \mathbf{w}_{l-1} \quad \forall l \in \mathbb{N}.$$

One can derive that $\mathbf{w}^2(x) \geqslant \mathbf{w}_{l-1}(x) \geqslant \mathbf{w}_l(x) \geqslant \mathbf{w}^1(x - A - B - 1)$ for all $l \in \mathbb{N}$ and $x \in \mathbb{R}$. It then follows that $\mathbf{w} := \lim_{l\to\infty} \mathbf{w}_l$ is a solution to $\mathbf{w} = \mathbb{T}^0\mathbf{w}$, and hence also a solution to (6.4).

By the strong comparison principle, we have $\mathbf{w}^2(x) > \mathbf{w}(x) > \mathbf{w}^1(x - A - B - 1)$, and so $\mathbf{w}(\cdot)$ is not a constant vector. Hence $\mathbf{w}(x) \leqslant \mathbf{w}(y)$ for all $x < y$ and

$$\mathbf{w}(x) < \mathbf{a} \quad \forall x < 0, \quad \mathbf{w}(x) > \mathbf{a} \quad \forall x > A + B + 1.$$

After a translation, $\mathbf{w}$ becomes an n-periodic solution to (6.4), (7.1) and (7.2). This completes the proof of Theorem 5.

8. A Bistable Case

In this section, we shall first construct a traveling wave under the following assumption

(BS) *The constant vectors* $\mathbf{0}$ *and* $\mathbf{1}$ *are steady states of (1.3). Any other* n*-periodic steady state, if exists, is unstable.*

Then we shall consider a special example where although the condition (**BS**) might be very hard to check, we can still modify the method presented in subsection 8.1 to show the existence of a traveling wave.

8.1. **Existence of A Traveling Wave.** Set $\mathbf{a} = \frac{1}{2}\mathbf{1}$. Let $\mathbf{w}$ be a solution established in Theorem 5 with the given $\mathbf{a}$. As $\mathbf{w}(x)$ is non-decreasing in x, there exist the limits $\mathbf{w}(\infty)$ and $\mathbf{w}(-\infty)$. From (6.2), one can show that both $\mathbf{w}(\infty)$ and $\mathbf{w}(-\infty)$ are steady states. In addition, by (7.2), we have $\mathbf{w}(\infty) > \mathbf{0}$ and $\mathbf{w}(-\infty) < \mathbf{1}$.

It remains to show that $\mathbf{w}(\infty) = \mathbf{1}$ and $\mathbf{w}(-\infty) = \mathbf{0}$. To do this, we assume, by contradiction, that Φ $(= \mathbf{w}(\infty)$ or $\mathbf{w}(-\infty))$ is a steady state satisfying $\mathbf{0} < \Phi < \mathbf{1}$.

Denote by $(\mu, \Psi = \{\psi_i\})$ the unique solution to eigen-problem (2.3) with $L_i = f_i'(\phi_i)$. Since $\sum_k a_{i,k} = 0$, we have

$$\mu\psi_i = \sum_k a_{i,k}[\psi_{i+k} - \psi_i] + f_i'(\phi_i)\psi_i, \quad \psi_i = \psi_{i+n} \quad \forall\, i \in \mathbb{Z}.$$

Let

$$\beta := \frac{\mu}{2M}, \quad M := \max_i \max_{s\in[0,1]} |f_i''(s)|.$$

For the solution $\mathbf{w}^{m,c}$ of (7.3), where it is taken either $\mathbf{w}^{*m}$ or $\mathbf{w}_*^m$ when $c = 0$, we set

$$\begin{aligned} x^{m,c} &:= \min\{x \mid \mathbf{w}^{m,c}(x) \geqslant \Phi - \beta\mathbf{1}\}; \\ y^{m,c} &:= \max\{x \mid \mathbf{w}^{m,c}(x) \leqslant \Phi + \beta\mathbf{1}\}. \\ \ell &:= \limsup_{m\to\infty} \sup_{c\in[c_*,c^*]} (y^{m,c} - x^{m,c}). \end{aligned}$$

Then the following lemma holds.

Lemma 8.1. *Assume that Φ is an unstable steady state in the sense that the unique solution to* (2.3) *with $L_i = f_i'(\phi_i)$ has the property that $\mu > 0$. Then $\ell < \infty$.*

Proof. Suppose on the contrary that $\ell = \infty$. Then, in view of (7.5), there exists a sequence $\{m_l, c_l\}_{l\in\mathbb{N}}$ satisfying $c_l \in [c_*, 0) \cup (0, c^*]$ for all $l \in \mathbb{N}$, and

$$\lim_{l\to\infty} m_l = \infty, \quad \lim_{l\to\infty} [y^{m_l,c_l} - x^{m_l,c_l}] = \infty.$$

By taking a subsequence if necessary, we may assume that there exist the limits

$$\begin{aligned} c &:= \lim_{l\to\infty} c_l \ \in [c_*, c^*], \\ \mathbf{w}^1(x) &:= \lim_{l\to\infty} \mathbf{w}^{m_l,c_l}(x + x^{m_l,c_l}) \quad \forall x \in \mathbb{R}, \\ \mathbf{w}^2(x) &:= \lim_{l\to\infty} \mathbf{w}^{m_l,c_l}(x + y^{m_l,c_l}) \quad \forall x \in \mathbb{R}, \\ A &:= \lim_{l\to\infty} (x^{m_l,c_l} + m_l) \in (0,\infty) \cup \{\infty\}, \\ B &:= \lim_{l\to\infty} (m_l - y^{m_l,c_l}) \in (0,\infty) \cup \{\infty\}. \end{aligned}$$

Since $\ell = \infty$, taking the limit of the corresponding equation (7.4) we have

$$\begin{aligned} &\mathbf{w}^1 = \mathbb{T}^c\mathbf{w}^1 \quad \text{a.e.} \quad x > -A, && \mathbf{w}^1 = \mathbf{0} \quad \forall x < -A; \\ &\mathbf{w}^2 = \mathbb{T}^c\mathbf{w}^2 \quad \text{a.e.} \quad x < B, && \mathbf{w}^2 = \mathbf{1} \quad \forall x > B, \\ &\|\mathbf{w}^1(x) - \Phi\|_\infty \leqslant \beta \quad \forall\, x \geqslant 0, && \|\mathbf{w}^1(x) - \Phi\|_\infty \geqslant \beta \quad \forall\, x < 0, \\ &\|\mathbf{w}^2(x) - \Phi\|_\infty \leqslant \beta \quad \forall\, x \leqslant 0, && \|\mathbf{w}^2(x) - \Phi\|_\infty \geqslant \beta \quad \forall\, x > 0. \end{aligned}$$

This implies that $\mathbf{w}^1$ is strictly increasing in $(-A, \infty)$ and $\mathbf{w}^2$ is strictly increasing in $(-\infty, B)$. Hence there exists a positive constant η such that

$$\begin{aligned} \mathbf{w}^1(x+1) - \mathbf{w}^1(x) &\geqslant 2\eta\mathbf{1} \quad \forall\, x \in [0, k_0], \\ \mathbf{w}^2(x+1) - \mathbf{w}^2(x) &\geqslant 2\eta\mathbf{1} \quad \forall\, x \in [-k_0 - 1, -1]. \end{aligned}$$

Consequently, for all sufficiently large l, writing (m_l, c_l) simply as (m, c),

$$\mathbf{w}^{m,c}(x+1) - \mathbf{w}^{m,c}(x) \geqslant \eta \mathbf{1}$$

for all $x \in [x^{m,c}, x^{m,c} + k_0] \cup [y^{m,c} - k_0 - 1, y^{m,c} - 1]$. Define

$$z_i(x) = \frac{w_i^{m,c}(x+1) - w_i^{m,c}(x)}{\psi_i}, \quad z(x) = \min_i z_i(x).$$

Note that $z_{i+n} = z_i$. Also, as $c = c^l \neq 0$, $\mathbf{w}^{m,c}$ is continuous in $[x^{m,c}, y^{m,c}]$. There exist $j \in \mathbb{Z}$ and $x_* \in [x^{m,c}, y^{m,c} - 1]$ such that

$$\hat{\eta} := \min\{z(x) \mid x^{m,c} \leqslant x \leqslant y^{m,c} - 1\} = z_j(x_*).$$

There are two cases:

(1) $x^{m,c} + k_0 < x_* < y^{m,c} - k_0 - 1$;

(2) $x_* \in [x^{m,c}, x^{m,c} + k_0] \cup [y^{m,c} - k_0 - 1, y^{m,c} - 1]$.

Suppose $x^{m,c} + k_0 < x_* < y^{m,c} - k_0 - 1$. Then x_* is a local minimum point of $z_j(\cdot)$ so that $cz_j'(x_*) = 0$. Thus, using (6.2), it implies that

$$\begin{aligned} 0 &= -cz_j'(x_*)\psi_j = -c[w_j^{m,c\,\prime}(x_*+1) - w_j^{m,c\,\prime}(x_*)] \\ &= \sum_k a_{j,k}[\psi_{j+k} z_{j+k}(x_*+k) - \psi_j z_j(x_*)] + [f_j(w_j(x_*+1)) - f_j(w_j(x_*))] \\ &\geqslant z_j(x_*)\Big\{\sum_k a_{j,k}[\psi_{j+k} - \psi_j] + f_j'(\xi)\psi_j\Big\} \end{aligned}$$

by the assumption that $z_{j+k}(x_*+k) \geqslant z_j(x_*)$ for all $k \in [-k_0, k_0]$ and the mean value theorem. Note that $\xi \in [w_j(x_*), w_j(x_*+1)]$ so that $|\xi - \phi_j| \leqslant \beta$. Hence we have

$$0 \geqslant \sum_k a_{j,k}[\psi_{j+k} - \psi_j] + f_j'(\xi)\psi_j = \Big\{\mu + f_j'(\xi) - f_j'(\phi_j)\Big\}\psi_j > 0$$

which is impossible. Here we used the fact that $|f_j'(\xi) - f_j'(\phi_j)| \leqslant M|\xi - \phi_j| \leqslant M\beta < \mu$. Thus, case (1) cannot happen.

Hence, we have case (2) so that $\hat{\eta} \geqslant \eta$. Consequently, taking i such that $\psi_i = 1$ we have

$$\eta \leqslant \frac{\sum_{k=0}^K \{w_i^{m,c}(x^{m,c}+k+1) - w_i^{m,c}(x^{m,c}+k)\}}{\sum_{k=0}^K 1} \leqslant \frac{2\beta}{y^{m,c} - x^{m,c} - 1},$$

where $K := [y^{m,c} - x^{m,c} - 1]$ is the maximum integer no larger than $y^{m,c} - x^{m,c} - 1$. This implies that $y^{m_l,c_l} - x^{m_l,c_l} \leqslant 1 + (2\beta)/\eta$ for all sufficiently large l, contradicting the assumption that $\lim_{l\to\infty}(y^{m_l,c_l} - x^{m_l,c_l}) = \infty$. This completes the proof of the lemma. $\square$

With this lemma, we can prove the following theorem.

Theorem 6. *Assume* **(BS)**. *Then the problem* (6.2) *admits a solution* $(c, \mathbf{w})$ *satisfying*

$$\mathbf{w}(-\infty) = \mathbf{0} < \mathbf{w}(x) < \mathbf{1} = \mathbf{w}(\infty) \quad \forall\, x \in \mathbb{R}. \tag{8.1}$$

Proof. Let $(c, \mathbf{w})$ be a solution established in Theorem 5 with $\mathbf{a} = \mathbf{1}/2$. If $\mathbf{w}(\infty) \neq \mathbf{1}$ (or $\mathbf{w}(-\infty) \neq \mathbf{0}$), then $\Phi := \mathbf{w}(\infty)$ (or $\Phi := \mathbf{w}(-\infty)$) is a steady state satisfying $\mathbf{0} < \Phi < \mathbf{1}$.

Since $w_i(x) \uparrow \phi_i$ as $x \to \infty$ for all i, there exists $x_0 \gg 1$, independent of i (by the periodicity), such that $\phi_i - \beta/2 \leqslant w_i(x) \leqslant \phi_i$ for all $x \geqslant x_0$ for all $i \in \mathbb{Z}$. Since $w_i^{m_l,c_l}(x) \to w_i(x)$ as $l \to \infty$ for all $x \in \mathbb{R}$, there exists $l_0 \gg 1$ such that

$$w_i^{m_l,c_l}(x_0) \geqslant w_i(x_0) - \beta/2 \geqslant \phi_i - \beta$$

for all $l \geqslant l_0$ for all $i \in \mathbb{Z}$. Then $w_i^{m_l,c_l}(x) \geqslant \phi_i - \beta$ for all $x \geqslant x_0$, $l \geqslant l_0$, $i \in \mathbb{Z}$. This implies that $x^{m_l,c_l} \leqslant x_0$ for all $l \geqslant l_0$.

On the other hand, for each i, since $w_i^{m_l,c_l}(x) \to w_i(x)$ as $l \to \infty$ for all $x \in \mathbb{R}$, we have $\mathbf{w}^{m_l,c_l}(x) \leqslant \Phi + \beta\mathbf{1}$ for each $x \in \mathbb{R}$ for some $l \gg 1$. This means that $y^{m_l,c_l} \to \infty$ as $l \to \infty$, a contradiction to Lemma 8.1. Hence $\mathbf{w}(-\infty) = \mathbf{0}$, $\mathbf{w}(\infty) = \mathbf{1}$, and the theorem is proven. $\square$

Remark 8.1. *Note that in* (**BS**), *no information about the stability of the steady states* $\mathbf{0}$ *and* $\mathbf{1}$ *is needed. For monostable dynamics, e.g.,* $f_i(0) = f_i(1) = 0 < f_i(s)$ *for all* $s \in (0, 1)$ *and all* $i \in \mathbb{Z}$*, one can show that if (1.3) is a discretization of a divergence operator (i.e.,* $b(x) \equiv 0$*), then there are no steady states other than* $\mathbf{0}$ *and* $\mathbf{1}$*, so that* (**BS**) *is automatically satisfied.*

8.2. **Example.** Consider the system

$$\dot{u}_j = a_{j+1/2}[u_{j+1} - u_j] - a_{j-1/2}[u_j - u_{j-1}] + b_j[u_{j+1} - u_{j-1}] + f(u_j), \tag{8.2}$$

where we assume the following.

(Ha) $a_{j+1/2} + b_j > 0, a_{j-1/2} - b_j > 0$, $a_{j+1/2+n} = a_{j+1/2}, b_{j+n} = b_j$ for all $j \in \mathbb{Z}$;

(Hf) There exists $a \in (0, 1)$ such that $f(0) = f(1) = f(a) = 0$ and

$$f > 0 \text{ in } (a, 1),\ f < 0 \text{ in } (0, a),\ f'(0) < 0,\ f'(1) < 0,\ f'(a) > 0.$$

We notice the following:

1. The system has three constant steady states: $\mathbf{0}, a\mathbf{1}$, and $\mathbf{1}$. The corresponding eigen-problems (2.3) with $\{L_i\} = f'(0)\mathbf{1}$, $\{L_i\} = f'(a)\mathbf{1}$, and $\{L_i\} = f'(1)\mathbf{1}$ admit explicit solutions given by $\Psi = \mathbf{1}$ and $\mu = f'(0), \mu = f'(a)$, and $\mu = f'(1)$ respectively. Since $f'(0) < 0, f'(1) < 0$ and $f'(a) > 0$, we conclude that $\mathbf{0}$ and $\mathbf{1}$ are stable steady states, whereas $a\mathbf{1}$ is an unstable steady state.

2. In general there are an unknown numbers of periodic steady states Φ satisfying $\mathbf{0} < \Phi < \mathbf{1}$. It may not be an easy task to show that these periodic steady states are unstable. On the other hand, without showing the instability of these steady states, we cannot directly apply Theorem 6. Nevertheless, we have the following observation.

3. Notice that if $\Phi = \{\phi_i\}$ is a non-constant periodic steady state and $\mathbf{0} < \Phi < \mathbf{1}$, then, as a sequence, $\{\phi_i\}$ oscillates. On the other hand, under (**Ha**) and (**Hf**), we can show that the solution to (8.2) with a monotonic initial data remains monotonic (in the grid index i for each fixed t). This special property will be sufficient for the existence of a traveling wave, which, in particular, is monotonic in the grid index.

Lemma 8.2. *Consider the dynamics* (8.2).

(1) *If* $\mathbf{u}(t) = \{u_i(t)\}$, $t \geqslant 0$, *is a solution to* (8.2) *with initial data satisfying* $0 \leqslant u_i(0) \leqslant u_{i+1}(0) \leqslant 1$ *for all* $i \in \mathbb{Z}$, *then*

$$u_i(t) \leqslant u_{i+1}(t) \quad \forall t > 0, \quad i \in \mathbb{Z}.$$

(2) *Each solution* $\mathbf{w} = \mathbf{w}^{m,c}$ *to* (7.4) *satisfies*

$$w_i(i+x) \leqslant w_j(j+x) \quad \forall i \leqslant j, \quad x \in \mathbb{R}.$$

Proof. (1) Suppose $\mathbf{u} = \{u_i\}$ is a solution to (8.2) with initial data satisfying $0 \leqslant u_i(0) \leqslant u_{i+1}(0) \leqslant 1$ for all $i \in \mathbb{Z}$. Set $v_i(t) = [u_{i+1}(t) - u_i(t)]e^{Mt}$ where

$$M = \|f'\|_\infty + \max_i |b_{i+1} - b_i| + 2\max_i |a_{i+1/2}|.$$

Writing $f(u_{i+1}(t)) - f(u_i(t)) = d_i(t)[u_{i+1}(t) - u_i(t)]$, where

$$d_i(t) = \int_0^1 f'(su_i(t) + [1-s]u_{i+1}(t))ds.$$

Then $\mathbf{v} = \{v_i\}$ satisfies

$$\dot{v}_i = [a_{i+3/2} + b_{i+1}]v_{i+1} + [a_{i-1/2} - b_i]v_{i-1} + [M + d_i + b_{i+1} - b_i - 2a_{i+1/2}]v_i$$

for all $t > 0$ and $i \in \mathbb{Z}$. Note that the coefficients of v_i, v_{i+1}, and v_{i-1} on the right-hand side are all positive. Since $\mathbf{v}(0) \geqslant \mathbf{0}$, one can use a comparison to show that $\mathbf{v}(t) \geqslant \mathbf{0}$ for all $t > 0$. In addition, if $\mathbf{v}(0) \neq \mathbf{0}$, then $\mathbf{v}(t) > \mathbf{0}$ for all $t > 0$. This proves the first assertion.

(2) To prove the second assertion, we work on the space

$$\hat{\mathbf{X}} := \{\mathbf{w} = \{w_i\} \in \mathbf{X} \mid w_i(i+x) \leqslant w_j(j+x) \quad \forall\, i < j,\ x \in \mathbb{R}\}.$$

We need only show that $\mathbb{P}_m\mathbb{T}^c$ maps $\hat{\mathbf{X}}$ to $\hat{\mathbf{X}}$.

For this, suppose $\mathbf{w} = \{w_i\} \in \hat{\mathbf{X}}$ and let $\mathbf{v} := \mathbb{P}_m\mathbb{T}^c[\mathbf{w}]$. We want to show that $\mathbf{v} = \{v_i\} \in \hat{\mathbf{X}}$, i.e., $v_i(i+x) \leqslant v_{i+1}(i+1+x)$. This is equivalent to showing that $v_i(x) \leqslant v_{i+1}(x+1)$ for all $i \in \mathbb{Z}$ and $x \in \mathbb{R}$.

(a) If $x > m-1$, we have, by definition, $v_{i+1}(x+1) = 1$ so that $v_{i+1}(x+1) \geqslant v_i(x)$.

(b) If $x < -m$, then $v_i(x) = 0$ and so we also have $v_i(x) \leqslant v_{i+1}(x+1)$.

(c) Suppose $-m \leqslant x \leqslant m-1$. Then $x, x+1 \in [-m, m]$. Again, writing

$$f(w_{i+1}(x-ct+1)) - f(w_i(x-ct)) = d_i(x-ct)[w_{i+1}(x-ct+1) - w_i(x-ct)],$$

by choosing ν large enough it follows that

$$\begin{aligned} & v_{i+1}(x+1) - v_i(x) = \mathbb{T}^c_{i+1}[\mathbf{w}](x+1) - \mathbb{T}^c_i[\mathbf{w}](x) \\ = & \int_{-\infty}^0 e^{\nu t}\Big\{[a_{i+3/2}+b_{i+1}][w_{i+2}(x-ct+2) - w_{i+1}(x-ct+1)] \\ & \quad +[a_{i-1/2} - b_i][w_i(x-ct) - w_{i-1}(x-ct-1)] \\ & \quad +[\nu - 2a_{i+1/2} + d_i(x-ct) + b_{i+1} - b_i][w_{i+1}(x-ct+1) - w_i(x-ct)]\Big\}dt \\ \geqslant & \ 0, \end{aligned}$$

since $\mathbf{w} \in \hat{\mathbf{X}}$. Thus, $\mathbf{v} \in \hat{\mathbf{X}}$. Namely, $\hat{\mathbf{X}}$ is invariant under the operator $\mathbb{P}_m\mathbb{T}^c$.

Following the proof of the existence of $\mathbf{w}^{m,c}$, we see that $\mathbf{w}^{m,c} \in \hat{\mathbf{X}}$. This completes the proof. □

Theorem 7. *Assume* **(Ha)** *and* **(Hf)** *and consider* (1.3) *in the special form of* (8.2). *Then there is a traveling wave satisfying* (1.3), (1.4) *and* (1.5) *for some* $c \in \mathbb{R}$. *In addition, the wave satisfies*

$$u_i(t) < u_{i+1}(t) \quad \forall i \in \mathbb{Z}, t \in \mathbb{R}. \tag{8.3}$$

Proof. The solution $\mathbf{w} = \{w_i\}$ established in Theorem 5 has the additional property that $\mathbf{w} \in \hat{\mathbf{X}}$; that is, $w_{i+1}(x+1) \geqslant w_i(x)$ for all $x \in \mathbb{R}$ and $i \in \mathbb{Z}$.

Note that for each $1 \leqslant i < j \leqslant n$, $w_i(x+n) = w_{i+n}(x+n) \geqslant w_j(x+j) \geqslant w_i(x+i)$ for every $x \in \mathbb{R}$. Sending $x \to \infty$, we see that $\mathbf{w}(\infty) = \alpha\mathbf{1}$ for some constant $\alpha \in (0,1]$. Consequently, if $\mathbf{w}(\infty) \neq \mathbf{1}$, we must have $\mathbf{w}(\infty) = a\mathbf{1}$. Since $a\mathbf{1}$ is unstable, following the same proof as that of Theorem 6, we see that is impossible. Hence, $\mathbf{w}(\infty) = \mathbf{1}$. Similarly, $\mathbf{w}(-\infty) = \mathbf{0}$. Finally, (8.3) follows from Lemma 8.2. Hence the theorem is proven. □

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