ENTIRE SOLUTIONS ORIGINATING FROM MONOTONE FRONTS TO THE ALLEN-CAHN EQUATION

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Abstract. In this paper, we study entire solutions of the Allen-Cahn equation in one-dimensional Euclidean space. This equation is a scalar reaction-diffusion equation with a bistable nonlinearity. It is well-known that this equation admits three different types of traveling fronts connecting two of its three constant states. Under certain conditions on the wave speeds, the existence of entire solutions originating from three and four fronts is shown by constructing some suitable pairs of super-sub-solutions. Moreover, we show that there are no entire solutions originating from more than four fronts.

1. Introduction

In this paper, we consider the following reaction-diffusion equation

$$u_t = u_{xx} + f(u), \quad x \in \mathbb{R},\ t \in \mathbb{R}, \tag{1.1}$$

where the function $f(u) \in C^2(\mathbb{R})$ satisfies

$$f(0) = f(1) = 0, \quad f'(0),\ f'(1) < 0, \tag{1.2}$$

$$f(a) = 0,\ f'(a) > 0,\ a \in (0,1), \quad f(u) \neq 0 \text{ for } u \in (0,a) \cup (a,1), \tag{1.3}$$

$$\int_0^1 f(u)\,du > 0. \tag{1.4}$$

A typical example of f is $f(u) = u(1-u)(u-a)$, where $a \in (0, 1/2)$. This equation is often called the Allen-Cahn equation or the Nagumo equation. It is easy to see that the constant states $u = 0$ and $u = 1$ are stable and the constant state $u = a$ is unstable for the kinetic equation (i.e., (1.1) without diffusion term), since $f'(0) < 0$, $f'(1) < 0$ and $f'(a) > 0$.

Due to the rich dynamics of this prototype equation (1.1), there have been a lot of research on the dynamical behaviors of (1.1). One of the main concerns on the dynamics of (1.1) is the existence of entire solutions. Here an entire solution means a classical solution defined for all $(x,t) \in \mathbb{R}^2$. One of typical examples of entire solutions is the traveling wave solution. A solution u of (1.1) is called a traveling wave solution, if $u(x,t) = \Phi(x+vt)$ for some constant v (the wave speed) and some function Φ (the wave profile). A traveling wave solution is called

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a *traveling front*, if it connects two different constant states. In fact, (1.1) admits three different kinds of traveling fronts connecting states $\{0,1\}$, $\{0,a\}$, $\{a,1\}$, respectively. The first one is the bistable connection and the latter two cases are the monostable connections. For the reader's convenience, we recall in the followings the traveling front connecting states $\{0,1\}$, ($\{0,a\}$, $\{a,1\}$, resp.) with wave profile denoted by ψ_0, (ψ_1,ψ_2, resp.) and admissible wave speed denoted by v_0, (v_1,v_2, resp.)

By [1, 2], there exists a unique (up to translations) traveling front $u(x,t)=\psi_0(x+v_0t)$ of (1.1) connecting $\{0,1\}$ with the unique speed v_0. Note that, by setting $z=x+v_0t$, ψ_0 satisfies

$$\begin{aligned}&\psi_0''(z)-v_0\,\psi_0'(z)+f(\psi_0(z))=0,\ \psi_0'(z)>0,\quad z\in\mathbb{R},\\&\psi_0(-\infty)=0,\quad \psi_0(\infty)=1.\end{aligned}$$

and the speed v_0 is given by

$$v_0=\frac{\displaystyle\int_0^1 f(\psi_0)\,d\psi_0}{\displaystyle\int_{-\infty}^{\infty}(\psi_0'(z))^2dz}>0.$$

By [1, 13], there exists a constant $c_{1,max}\le -2\sqrt{f'(a)}$ such that a traveling front $u(x,t)=\psi_1(x+v_1t)$ of (1.1) connecting $\{0,a\}$ with speed v_1 exists for each $v_1\le c_{1,max}$. Set $z=x+v_1t$. Then $\psi_1(z)$ satisfies

$$\begin{aligned}&\psi_1''(z)-v_1\,\psi_1'(z)+f(\psi_1(z))=0,\ \psi_1'(z)>0,\quad z\in\mathbb{R},\\&\psi_1(-\infty)=0,\quad \psi_1(\infty)=a.\end{aligned}\tag{1.5}$$

Similarly, there exists a constant $c_{2,min}\ge 2\sqrt{f'(a)}$ such that a traveling front $u(x,t)=\psi_2(x+v_2t)$ of (1.1) connecting $\{a,1\}$ with speed v_2 exists for each $v_2\ge c_{2,min}$. Set $z=x+v_2t$. Then $\psi_2(z)$ satisfies

$$\begin{aligned}&\psi_2''(z)-v_2\,\psi_2'(z)+f(\psi_2(z))=0,\ \psi_2'(z)>0,\quad z\in\mathbb{R},\\&\psi_2(-\infty)=a,\quad \psi_2(\infty)=1.\end{aligned}$$

Note that $v_0>0$, $v_1<0<v_2$, and $0<\psi_0<1$, $0<\psi_1<a$, $a<\psi_2<1$ in $\mathbb{R}$.

In 1999, Hamel and Nadirashvili [11] constructed a new type of entire solutions originating from two fronts (at $t=-\infty$) for the Fisher-KPP equation (see also [12]). Since then, there have been many works devoted to the construction of entire solutions originating from two fronts for the scalar reaction-diffusion equations (see, e.g., [19, 8, 9, 3, 4, 14]). In particular, Yagisita [19] derived the existence of entire solutions which behave as two traveling fronts $\psi_0(-x+ct)$ and $\psi_0(x+ct)$ on the left x-axis and right x-axis as $t\to-\infty$, respectively. Then, Fukao, Morita and Ninomiya [8] provided a simple proof for the results shown in [19] for the Allen-Cahn equation.

For the function $f(u)$ satisfying (1.2), (1.3), according to the results shown in [11, 9], for any $c_{11},c_{12}\le c_{1,max}$, there exists an entire solution of (1.1) which converges to $\psi_1(x+c_{11}t)$ and $\psi_1(-x+c_{12}t)$ on the left x-axis and right x-axis, respectively, as $t\to-\infty$. Similarly, the existence of an entire solution of (1.1) which converges to $\psi_2(-x+c_{21}t)$ and $\psi_2(x+c_{22}t)$ on the left x-axis and right x-axis, respectively, as $t \to -\infty$ can be shown for any $c_{21}, c_{22} \geq c_{2,min}$. Later, in [14], Morita and Ninomiya proposed a unified method to construct all types of entire solutions including entire solutions with two merging fronts mentioned above.

Extending these works to multiple fronts, we shall study entire solutions u *originating* from k fronts $\{(c_j, \phi_j), j = 1, 2, \cdots, k\}$ $(k \geq 2)$ satisfying the condition

$$c_1 < c_2 < \cdots < c_k \tag{1.6}$$

such that

$$\lim_{t\to-\infty} \left\{ \sum_{1\leq j\leq k} \sup_{w_{j-1}(t)<x<w_j(t)} |u(x,t) - \phi_j(x + c_j t + \theta_j)| \right\} = 0$$

for some constants $\theta_1, \cdots, \theta_k$, where $w_j(t) := -(c_j + c_{j+1})t/2$, $w_0(t) := -\infty$ and $w_k(t) := \infty$.

Note that the condition (1.6) is quite natural, since two adjacent waves must intersect at some negative time, if $c_j > c_{j+1}$ for some $j \in \{1, \cdots, k-1\}$. We do not consider the case when $c_j = c_{j+1}$ for some j. It is a delicate case which is left for open. Also, in this paper, the new terminology "originating" is used, since we mainly focus on the behavior at $t = -\infty$. Entire solutions originating from two fronts may be merging to a single front or a constant state (which is called annihilating) as $t \to \infty$.

For an entire solution u originating from k fronts, there is the sequence $\{\alpha_1, \omega_1, \cdots, \alpha_k, \omega_k\}$ satisfying $\phi_j(-\infty) = \alpha_j$, $\phi_j(\infty) = \omega_j$ for $j = 1, \cdots, k$. We call it the *sequence of* u. Due to the continuity of entire solutions, we have $\alpha_{j+1} = \omega_j$ for $j = 1, \cdots, k-1$.

We can easily check that the following sequences

$$\{0, 1, 1, 0\},\ \{0, 1, 1, a\},\ \{a, 0, 0, a\},\ \{a, 1, 1, a\},\ \{a, 1, 1, 0\} \tag{1.7}$$

cannot be the sequences of entire solutions originating from two fronts. For example, for the case $\{0, 1, 1, 0\}$, the speeds of the corresponding traveling fronts satisfy $c_1 > 0 > c_2$. The condition (1.6) is violated. The other cases can be checked similarly. Therefore, entire solutions originating from two fronts consist of the following seven types:

$$\{0, a, a, 0\},\ \{0, a, a, 1\},\ \{a, 0, 0, 1\},\ \{1, 0, 0, a\},\ \{1, 0, 0, 1\},\ \{1, a, a, 0\},\ \{1, a, a, 1\}.$$

The first, the fifth and the seventh cases were constructed in [19, 8, 9] and the others are done in [14]. The entire solutions also exhibit the "universal" transient behavior when two fronts collide each other. In fact, since the dynamics of solutions depend on the choice of initial data, the collision depends on where we put these two fronts at the initial time. By "universal", we are looking for the collision which does not depend on the initial data.

Hence we encounter the natural question: *are there entire solutions originating from three or more fronts for* (1.1)*?* The main purpose of this work is to construct entire solutions originating from k fronts for $k \geq 3$ for equation (1.1). This would give us more varieties of dynamics of equation (1.1).

The following theorem is the first main result of this paper.

Theorem 1.1. *Let (v_0, ψ_0), (v_1, ψ_1) and (v_2, ψ_2) be traveling fronts described as above such that*

$$-v_0 < v_1 \le c_{1,max}. \tag{1.8}$$

Then there exists an entire solution of (1.1) such that

$$\lim_{t\to-\infty} \Big\{ \sup_{x\le w_1(t)} |u(x,t) - \psi_0(-x + v_0 t + \theta)| + \sup_{w_1(t)\le x\le w_2(t)} |u(x,t) - \psi_1(x + v_1 t + \theta)| + \sup_{x\ge w_2(t)} |u(x,t) - \psi_2(x + v_2 t + \theta)| \Big\} = 0 \tag{1.9}$$

for some constant θ, where

$$w_1(t) := \frac{-(-v_0 + v_1)t}{2}, \quad w_2(t) := \frac{-(v_1 + v_2)t}{2}.$$

Moreover, it holds

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |u(x,t) - 1| = 0. \tag{1.10}$$

This theorem shows us a new type of entire solution originating from three fronts with sequence $\{1, 0, 0, a, a, 1\}$. Moreover, three fronts of this entire solution are annihilated as $t \to \infty$. Note that the condition (1.8) on the speeds can be realized when we take the constant a such that $v_0 > 2\sqrt{f'(a)}$. For example, when $f(u) = u(1-u)(u-a)$, we have $v_0 = \sqrt{2}(1/2 - a) > 2\sqrt{a(1-a)} = 2\sqrt{f'(a)}$ if we consider $0 < a < (3 - \sqrt{6})/6$.

For notational convenience, in the sequel we shall use $\tilde{\psi}_1$ to denote another traveling front connecting $\{0, a\}$ with speed $\tilde{v}_1 \le c_{1,max}$. Similar to Theorem 1.1, we can construct entire solutions originating from three fronts with sequence $\{1, 0, 0, a, a, 0\}$ as follows.

Theorem 1.2. *Let (v_0, ψ_0), (v_1, ψ_1) and $(\tilde{v}_1, \tilde{\psi}_1)$ be traveling fronts described as above such that (1.8) holds. Then there exists an entire solution of (1.1) such that*

$$\lim_{t\to-\infty} \Big\{ \sup_{x\le w_1(t)} |u(x,t) - \psi_0(-x + v_0 t + \theta_1)| + \sup_{w_1(t)\le x\le w_2(t)} |u(x,t) - \psi_1(x + v_1 t + \theta_1)| + \sup_{x\ge w_2(t)} |u(x,t) - \tilde{\psi}_1(-x + \tilde{v}_1 t - \theta_2)| \Big\} = 0 \tag{1.11}$$

for some constants θ_1 and θ_2, where

$$w_1(t) := \frac{-(-v_0 + v_1)t}{2}, \quad w_2(t) := \frac{-(v_1 - \tilde{v}_1)t}{2}.$$

Moreover, it holds

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |u(x,t) - \psi_0(-x + v_0 t + \theta)| = 0$$

for some constant θ.

Notice that three fronts of the entire solution constructed in Theorem 1.2 are merging to a single front as $t \to \infty$.

Next, we have the following existence theorem for entire solutions originating from four fronts such that these four fronts are annihilated as $t \to \infty$.

Theorem 1.3. *Let* (v_0, ψ_0) *and* (v_1, ψ_1) *be traveling fronts described as in Theorem 1.1 such that* (1.8) *holds. Then there exists a symmetric* (*with respect to* $x = 0$) *entire solution of* (1.1) *such that*

$$\begin{aligned}(1.12)\ \lim_{t\to-\infty}\Big\{ &\sup_{x\le w_1(t)} |u(x,t) - \psi_0(-x + v_0 t + \theta_1)| + \sup_{w_1(t)\le x\le 0} |u(x,t) - \psi_1(x + v_1 t + \theta_2)| \\ &+ \sup_{0\le x\le -w_1(t)} |u(x,t) - \psi_1(-x + v_1 t + \theta_2)| + \sup_{x\ge -w_1(t)} |u(x,t) - \psi_0(x + v_0 t + \theta_1)| \Big\} = 0\end{aligned}$$

for some constants θ_1 *and* θ_2, *where*

$$w_1(t) := -\frac{(-v_0 + v_1)t}{2}.$$

Moreover, the asymptotic behavior (1.10) *holds.*

Finally, we have the following nonexistence theorem for entire solutions originating from k fronts for $k \ge 5$.

Theorem 1.4. *Under the condition* (1.6), *there are no entire solutions originating from* k *fronts if* $k \ge 5$.

The existence of these entire solution originating from three or four fronts gives us the "universal" transient behavior of solutions when three or four fronts collide each other. Since the comparison principle is available for (1.1), it is well-known that an entire solution exists if we can find a suitable pair of super-sub-solutions (see, e.g., [8, 9, 14]). Therefore, the main task of finding entire solutions originating from multiple fronts is to construct some suitable super-sub-solutions with the desired properties. One of the main ideas in [14] is to find an auxiliary rational function with certain properties in order to construct a suitable pair of super-sub-solutions. The form of this auxiliary function depends on the equilibrium states which are connected by those two traveling fronts under consideration. Although the method of finding an auxiliary function can be applied to the construction of entire solutions originating from three fronts, finding this useful rational function is by no means trivial.

Due to the increase of the number of fronts, we were unable to construct a suitable auxiliary function for an entire solution originating from four fronts. Instead, the super-sub-solutions constructed for deriving entire solutions originating from three fronts are used effectively to construct an entire solution originating from four fronts.

Theorem 1.4 implies that there is no "universal" transient behavior of solutions when more than four fronts collide each other. To show Theorem 1.4, we introduce the notion of terminated sequence and the non-extendable sequence. According to the proof of Theorem 1.4, we can check that the possible sequences with $k = 3$ are essentially $\{1, 0, 0, a, a, 1\}$ and $\{1, 0, 0, a, a, 0\}$ and that the possible sequence with $k = 4$ is essentially $\{1, 0, 0, a, a, 0, 0, 1\}$. See Remark 1. Entire solutions corresponding to all possible sequences for $k = 3, 4$ are constructed in Theorems 1.1–1.3.

Besides the works on the scalar equations, there are many works on the entire solutions originating from two fronts for systems of two reaction-diffusion equations. We refer the reader to, for examples, [15, 10, 17, 18, 16, 20]. For the discrete version of (1.1), the same results as in Theorem 1.1 and Theorem 1.2 have been shown in [5].

The rest of this paper is organized as follows. First, a proof of Theorem 1.4 is given in §2. In §3, we first give an auxiliary function linking three traveling fronts of (1.1). Then we provide some useful properties of this auxiliary function and derive the key estimates (see Lemma 3.4) for the later construction of super-sub-solutions. In §4, we use this auxiliary function to construct a pair of super-sub-solutions and give a proof of Theorem 1.1 on the existence of entire solutions originating from three fronts with sequence $\{1, 0, 0, a, a, 1\}$. For the sequence $\{1, 0, 0, a, a, 0\}$, since the proof of Theorem 1.2 is similar to that of Theorem 1.1, we only point out the main differences in §5. Finally, in §6, we give a proof of Theorem 1.3.

2. Proof of Theorem 1.4

First we recall a sequence $\{\alpha_1, \omega_1, \cdots, \alpha_k, \omega_k\}$ with $\alpha_1 \in \{0, a, 1\}$, $\alpha_{i+1} = \omega_i$ and $\omega_{i+1} \in \{0, a, 1\} \setminus \{\omega_i\}$ for $i = 1, \cdots, k-1$ in Section 1. The sequence $\{\alpha_1, \omega_1, \cdots, \alpha_k, \omega_k\}$ is called a *terminated sequence*, if $\{\alpha_{k-1}, \omega_{k-1}, \alpha_k, \omega_k\}$ is one of the cases in (1.7). This means that, for a terminated sequence $\{\alpha_1, \omega_1, \cdots, \alpha_k, \omega_k\}$, there is no entire solution originating from k fronts with this sequence $\{\alpha_1, \omega_1, \cdots, \alpha_k, \omega_k\}$. Similarly the sequence $\{\alpha_1, \omega_1, \cdots, \alpha_k, \omega_k\}$ is called a *non-extendable sequence*, if k is the first number among $\{j \in \{1, \cdots, k\} \mid \{\alpha_j, \omega_j\}$ is one of $\{0, 1\}$ or $\{a, 1\}\}$. We also say that the sequence is *terminated* (resp. *non-extendable*) if it is a terminated sequence (resp. a non-extendable sequence). Note that any sequence $\{\alpha_1, \omega_1, \cdots, \alpha_{k+1}, \omega_{k+1}\}$ becomes terminated if $\{\alpha_1, \omega_1, \cdots, \alpha_k, \omega_k\}$ is a non-extendable sequence. Namely, $\{0, 1\}$ and $\{a, 1\}$ are non-extendable. For example, if a sequence starts from a, then the possible sequences are $\{a, 0\}$ or $\{a, 1\}$. But, the latter is already non-extendable. In fact, the possible non-extendable sequences starting from a are the following two:

$$\{a, 1\}, \ \{a, 0, 0, 1\},$$

because $\{a, 0, 0, a\}$ is a terminated sequence. Thus the longest non-extendable sequence starting from a is $\{a, 0, 0, 1\}$. Using this argument, we can prove Theorem 1.4.

Proof of Theorem 1.4. Let u be an entire solution of (1.1) originating from k fronts and let $\{\alpha_1, \omega_1, \cdots, \alpha_k, \omega_k\}$ be the sequence of u. As stated above, the longest non-extendable sequence starting from a is $\{a, 0, 0, 1\}$. This means that there are no entire solutions originating from k fronts for $k \geq 5$ if $\alpha_1 = a$.

For the case where the sequence starts from 0, $\{0, 1\}$ is non-extendable, while $\{0, a\}$ is not. By combing $\{0, a\}$ and the longest non-extendable sequence starting from a, we can conclude that the longest non-extendable sequence starting from 0 is $\{0, a, a, 0, 0, 1\}$.

Similarly, let us consider the case where the sequence starts from 1. If it starts from $\{1, a\}$, then the longest non-extendable one is $\{1, a, a, 0, 0, 1\}$. If it starts from $\{1, 0\}$, then combing $\{0, a, a, 0, 0, 1\}$, we can obtain the longest non-extendable sequence $\{1, 0, 0, a, a, 0, 0, 1\}$. Thus the longest non-extendable sequence starting from 1 is $\{1, 0, 0, a, a, 0, 0, 1\}$. Therefore, there are no sequences for entire solutions originating from k fronts for $k \geq 5$. ∎

Remark 1. Taking the symmetry into account, we can check that the only possible sequences with $k = 3$ are $\{1, 0, 0, a, a, 1\}$ and $\{1, 0, 0, a, a, 0\}$. Moreover, the only possible sequence with $k = 4$ is $\{1, 0, 0, a, a, 0, 0, 1\}$.

3. Some function linking three-front dynamics

Set $c_1 := -v_0$, $c_2 := v_1 \le c_{1,max}$ and $c_3 := v_2 \ge c_{2,min}$. Let $\phi_i = \phi_i(x + c_i t)$, $i = 1, 2, 3$, be traveling fronts of (1.1) that satisfy

$$\begin{cases} \phi_i''(s) - c_i\phi_i'(s) + f(\phi_i(s)) = 0, \quad s \in \mathbb{R}, \\ \phi_i(-\infty) = \alpha_i, \quad \phi_i(\infty) = \omega_i, \end{cases} \tag{3.1}$$

where $(\alpha_1, \omega_1, \alpha_2, \omega_2, \alpha_3, \omega_3) = (1, 0, 0, a, a, 1)$. Here the prime denotes the derivative with respect to s. Note that $\phi_1(z) = \psi_0(-z)$ and $\phi_i = \psi_{i-1}$, $i = 2, 3$. In this and next sections, we assume

$$\phi_1(0) = \frac{a}{2}, \quad \phi_2(0) = \frac{a}{2}, \quad \phi_3(0) = \frac{1+a}{2}. \tag{3.2}$$

By the nondegenerate condition on f, for $p \le 0$, there are positive constants β_i,γ_i, $i = 1, 2, 3$, and $K > 0$ such that (cf. [9])

$$\begin{cases} |\phi_i'(x+p)| \le K\exp(\beta_i(x+p)), \ x \le -p, \\ |\phi_i'(x+p)| \le K\exp(-\gamma_i(x+p)), \ x \ge -p. \end{cases} \tag{3.3}$$

In addition, there is a constant $\tau > 0$ such that

$$\begin{cases} \dfrac{|\phi_1(x-p)-1|}{|\phi_1'(x-p)|} \le \tau, \ x \le p, & \dfrac{|\phi_1(x-p)-0|}{|\phi_1'(x-p)|} \le \tau, \ x \ge p, \\ \dfrac{|\phi_2(x+p)-0|}{|\phi_2'(x+p)|} \le \tau, \ x \le -p, & \dfrac{|\phi_2(x+p)-a|}{|\phi_2'(x+p)|} \le \tau, \ x \ge -p, \\ \dfrac{|\phi_3(x+p)-a|}{|\phi_3'(x+p)|} \le \tau, \ x \le -p, & \dfrac{|\phi_3(x+p)-1|}{|\phi_3'(x+p)|} \le \tau, \ x \ge -p. \end{cases} \tag{3.4}$$

The key auxiliary function we found for linking three fronts is as follows.

Lemma 3.1. *Set*

$$Q(y, z, w) = z + (1-z)\frac{(1-y)z(w-a) + y(a-z)(1-w)}{(1-y)z(1-a) + (a-z)(1-w)}, \tag{3.5}$$

where $(y, z, w) \in [0, 1] \times [0, a] \times [a, 1] \backslash (\{(1, a, w)|\, a \le w \le 1\} \cup \{(1, z, 1)|\, 0 \le z \le a\} \cup \{(y, 0, 1)|\, 0 \le y \le 1\})$. *Then the following three statements hold:*

(i) Q *can be rewritten as*

$$Q(y, z, w) = \begin{cases} y + (1-y)z\dfrac{(1-a)(w-y)}{(1-y)z(1-a) + (a-z)(1-w)}, \\ w + (a-z)(1-w)\dfrac{y-w}{(1-y)z(1-a) + (a-z)(1-w)}. \end{cases} \tag{3.6}$$

(ii) *There exist functions Q_i, $i = 1, 2, 3$, such that*

$$\begin{aligned} Q_y(y,z,w) &= (a-z)(1-w)Q_1(y,z,w), \\ Q_z(y,z,w) &= (1-y)(1-w)Q_2(y,z,w), \\ Q_w(y,z,w) &= (1-y)zQ_3(y,z,w). \end{aligned}$$

(iii) *There exist functions R_j, $j = 1, \cdots, 16$, such that*

$$\begin{aligned} Q_{yy}(y,z,w) &= zR_1(y,z,w) = (a-z)R_2(y,z,w) = (1-w)R_3(y,z,w), \\ Q_{zz}(y,z,w) &= (1-y)R_4(y,z,w) = (1-w)R_5(y,z,w) \\ &= yR_6(y,z,w) + (w-a)R_7(y,z,w), \\ Q_{ww}(y,z,w) &= (1-y)R_8(y,z,w) = zR_9(y,z,w) = (a-z)R_{10}(y,z,w), \\ Q_{yz}(y,z,w) &= (1-w)R_{11}(y,z,w), \quad Q_{zw}(y,z,w) = (1-y)R_{12}(y,z,w), \\ Q_{yw}(y,z,w) &= (1-y)R_{13}(y,z,w) = zR_{14}(y,z,w) \\ &= (a-z)R_{15}(y,z,w) = (1-w)R_{16}(y,z,w). \end{aligned}$$

Proof. Obviously, the function $Q(y,z,w)$ defined by (3.5) allows the expression as (3.6).

By a simple calculation, we can derive

$$\begin{aligned} Q_y(y,z,w) &= \frac{a(1-z)(a-z)(1-w)^2}{[(1-y)z(1-a)+(a-z)(1-w)]^2}, \\ Q_z(y,z,w) &= \frac{(1-a)a(1-y)(1-w)(w-y)}{[(1-y)z(1-a)+(a-z)(1-w)]^2}, \\ Q_w(y,z,w) &= \frac{a(1-a)(1-y)^2z(1-z)}{[(1-y)z(1-a)+(a-z)(1-w)]^2}. \end{aligned}$$

Hence, the conclusion (ii) holds.

For the statement (iii), we compute the second derivative of function Q and obtain that

$$\begin{aligned} Q_{yy}(y,z,w) &= \frac{2(1-a)az(a-z)(1-z)(1-w)^2}{[(1-y)z(1-a)+(a-z)(1-w)]^3}, \\ Q_{zz}(y,z,w) &= -\frac{2(1-a)a(1-y)(1-w)(w-y)[w-a-y(1-a)]}{[(1-y)z(1-a)+(a-z)(1-w)]^3}, \\ Q_{ww}(y,z,w) &= \frac{2(1-a)a(1-y)^2z(a-z)(1-z)}{[(1-y)z(1-a)+(a-z)(1-w)]^3}, \\ Q_{yz}(y,z,w) &= -\frac{(1-a)a(1-w)^2[(y-w)z+a(1-2y-z+w+yz)]}{[(1-y)z(1-a)+(a-z)(1-w)]^3}, \\ Q_{yw}(y,z,w) &= -\frac{2(1-a)a(1-y)z(a-z)(1-z)(1-w)}{[(1-y)z(1-a)+(a-z)(1-w)]^3}, \\ Q_{zw}(y,z,w) &= -\frac{(1-a)a(1-y)^2[(w-y)z+a(-1+w+z-2wz+yz)]}{[(1-y)z(1-a)+(a-z)(1-w)]^3}. \end{aligned}$$

Thus, we get the conclusion (iii) and the lemma is proved. ∎

With this auxiliary function Q, we can construct a suitable pair of super-sub-solutions. For this, we put $u(x,t) = U(\xi,t)$ with $\xi := x + \bar{c}t$ and $\bar{c} = (c_1 + c_2)/2 = (-v_0 + v_1)/2$. Then (1.1) becomes

$$U_t = U_{\xi\xi} - \bar{c}U_\xi + f(U), \quad \xi \in \mathbb{R}. \tag{3.7}$$

We can easily check that (3.7) has traveling wave solutions

$$U = \phi_1(\xi - s_1 t),\ \phi_2(\xi + s_1 t),\ \phi_3(\xi + s_2 t),$$

where $s_1 := (c_2 - c_1)/2 = (v_1 + v_0)/2 > 0$, by (1.8), and

$$s_2 := c_3 - \bar{c} = (2c_3 - c_1 - c_2)/2 = (2v_2 + v_0 - v_1)/2 > (v_0 - v_1)/2 > s_1.$$

Now we consider

$$U(\xi,t) = Q(\phi_1,\phi_2,\phi_3),\ \phi_1 = \phi_1(\xi - q_1(t)),\ \phi_2 = \phi_2(\xi + q_2(t)),\ \phi_3 = \phi_3(\xi + q_3(t)),$$

where $q_i(t) < 0$, $i = 1,2,3$, and $-q_2(t) < -q_3(t)$. Set

$$\mathcal{T}[U] := U_t - U_{\xi\xi} + \bar{c}U_\xi - f(U).$$

Then

$$\begin{aligned}\mathcal{T}[Q(\phi_1,\phi_2,\phi_3)] \;\; = \;\; & -Q_y\phi_1'(q_1' - s_1) + Q_z\phi_2'(q_2' - s_1) + Q_w\phi_3'(q_3' - s_2) \\ & -G(\phi_1,\phi_2,\phi_3) - H(\phi_1,\phi_2,\phi_3)\end{aligned}$$

where

$$\begin{aligned}G(\phi_1,\phi_2,\phi_3) &:= Q_{yy}\{\phi_1'\}^2 + Q_{zz}\{\phi_2'\}^2 + Q_{ww}\{\phi_3'\}^2 + 2[Q_{yz}\phi_1'\phi_2' + Q_{yw}\phi_1'\phi_3' + Q_{zw}\phi_2'\phi_3'], \\ H(\phi_1,\phi_2,\phi_3) &:= f(Q) - Q_y f(\phi_1) - Q_z f(\phi_2) - Q_w f(\phi_3).\end{aligned}$$

From (3.6) and Lemma 3.1 we see that

$$\begin{aligned}H(1,z,w) &= f(Q(1,z,w)) - Q_y(1,z,w)f(1) - Q_z(1,z,w)f(z) - Q_w(1,z,w)f(w) = 0, \\ H(y,0,w) &= f(Q(y,0,w)) - Q_y(y,0,w)f(y) - Q_z(y,0,w)f(0) - Q_w(y,0,w)f(w) = 0, \\ H(y,a,w) &= f(Q(y,a,w)) - Q_y(y,a,w)f(y) - Q_z(y,a,w)f(a) - Q_w(y,a,w)f(w) = 0, \\ H(y,z,1) &= f(Q(y,z,1)) - Q_y(y,z,1)f(y) - Q_z(y,z,1)f(z) - Q_w(y,z,1)f(1) = 0,\end{aligned}$$

which implies that there is a smooth function H_1 satisfying

$$H(y,z,w) = (1-y)z(a-z)(1-w)H_1(y,z,w).$$

Since $Q(0,z,a) = z$ and $Q_z(0,z,a) = 1$, we have

$$H(0,z,a) = f(Q(0,z,a)) - Q_z(0,z,a)f(z) = 0,$$

which implies $H_1(0,z,a) = 0$. Applying the mean value theorem to H_1 yields

$$\begin{aligned}H_1(y,z,w) \;\; = \;\; & \int_0^1 H_{1y}(\theta y, z, \theta w + (1-\theta)a)d\theta \cdot y \\ & + \int_0^1 H_{1w}(\theta y, z, \theta w + (1-\theta)a)d\theta \cdot (w - a).\end{aligned}$$

Thus we obtain

$$\begin{cases} H(y,z,w) = (1-y)z[yH_{11}(y,z,w) + (w-a)H_{12}(y,z,w)], \\ H(y,z,w) = (1-w)(a-z)[yH_{21}(y,z,w) + (w-a)H_{22}(y,z,w)] \end{cases} \tag{3.8}$$

for some functions H_{ij}, $i,j = 1,2$.

Lemma 3.2. *For $q_1, q_2, q_3 \le -\delta < 0$, there exist positive constants ϵ_1, ϵ_2 and ϵ_3 such that*

$$\begin{aligned} &Q_y(\phi_1(\xi - q_1), \phi_2(\xi + q_2), \phi_3(\xi + q_3)) \ge \epsilon_1 \text{ for } \xi \le -q_2, \\ &Q_z(\phi_1(\xi - q_1), \phi_2(\xi + q_2), \phi_3(\xi + q_3)) \ge \epsilon_2 \text{ for } q_1 \le \xi \le -q_3, \\ &Q_w(\phi_1(\xi - q_1), \phi_2(\xi + q_2), \phi_3(\xi + q_3)) \ge \epsilon_3 \text{ for } \xi \ge -q_2. \end{aligned}$$

Proof. Recall (3.2). Then

$$\begin{aligned} &\frac{a}{2} \le \phi_1(\xi - q_1) \le 1,\ 0 \le \phi_2(\xi + q_2) \le \frac{a}{2},\ a \le \phi_3(\xi + q_3) \le \frac{1+a}{2} \text{ for } \xi \le q_1, \\ &0 \le \phi_1(\xi - q_1) \le \frac{a}{2},\ 0 \le \phi_2(\xi + q_2) \le \frac{a}{2},\ a \le \phi_3(\xi + q_3) \le \frac{1+a}{2} \text{ for } q_1 \le \xi \le -q_2, \\ &0 \le \phi_1(\xi - q_1) \le \frac{a}{2},\ \frac{a}{2} \le \phi_2(\xi + q_2) \le a,\ a \le \phi_3(\xi + q_3) \le \frac{1+a}{2} \text{ for } -q_2 \le \xi \le -q_3, \\ &0 \le \phi_1(\xi - q_1) \le \frac{a}{2},\ \frac{a}{2} \le \phi_2(\xi + q_2) \le a,\ \frac{1+a}{2} \le \phi_3(\xi + q_3) \le 1 \text{ for } \xi \ge -q_3, \end{aligned}$$

when $q_1, q_2, q_3 \le -\delta$. Then we have

$$\frac{a(1-a)}{4} \le (1-a)(1-\phi_1)\phi_2 + (a-\phi_2)(1-\phi_3) \le \frac{3a(1-a)}{2} \tag{3.9}$$

for $\xi \in \mathbb{R}$, $q_1, q_2, q_3 \le -\delta$.

By Lemma 3.1, for $q_1, q_2, q_3 < -\delta$, we derive that

$$\begin{aligned} &Q_y(\phi_1(\xi - q_1), \phi_2(\xi + q_2), \phi_3(\xi + q_3)) \\ = \;& \frac{a(1-\phi_2)(a-\phi_2)(1-\phi_3)^2}{[(1-\phi_1)\phi_2(1-a) + (a-\phi_2)(1-\phi_3)]^2} \\ \ge \;& \frac{a(1-a/2)(a/2)(1-(a+1)/2))^2}{[3a(1-a)/2]^2} = \frac{2-a}{36} \end{aligned}$$

for $\xi \le -q_2$,

$$\begin{aligned} &Q_z(\phi_1(\xi - q_1), \phi_2(\xi + q_2), \phi_3(\xi + q_3)) \\ = \;& \frac{(1-a)a(1-\phi_1)(1-\phi_3)(\phi_3-\phi_1)}{[(1-\phi_1)\phi_2(1-a) + (a-\phi_2)(1-\phi_3)]^2} \\ \ge \;& \frac{[(1-a)a(1-a/2)(1-(a+1)/2)(a-a/2)}{[3a(1-a)/2]^2} = \frac{2-a}{18} \end{aligned}$$

for $q_1 \le \xi \le -q_3$, and

$$\begin{aligned} & Q_w(\phi_1(\xi - q_1), \phi_2(\xi + q_2), \phi_3(\xi + q_3)) \\ = \ & \frac{a(1-a)(1-\phi_1)^2\phi_2(1-\phi_2)}{[(1-\phi_1)\phi_2(1-a) + (a-\phi_2)(1-\phi_3)]^2} \\ \ge \ & \frac{a(1-a)(1-a/2)^2(a/2)(1-a)}{[3a(1-a)/2]^2} = \frac{(2-a)^2}{18} \end{aligned}$$

for $\xi \ge -q_2$. Therefore, the lemma follows. ∎

From Lemma 3.1, it is easy to check that there exists a positive constant C such that

$$\begin{aligned} & |R_j(\phi_1(\xi - q_1), \phi_2(\xi + q_2), \phi_3(\xi + q_3))| \le C, \\ & |H_{mn}(\phi_1(\xi - q_1), \phi_2(\xi + q_2), \phi_3(\xi + q_3))| \le C, \end{aligned}$$

for $\xi \in \mathbb{R}$, $q_1, q_2, q_3 < -\delta$, $j = 1, \cdots, 16$, and $m, n = 1, 2$. Now, we define a function $F(\phi_1, \phi_2, \phi_3)$ as follows

$$\begin{aligned} & F(\phi_1(\xi - q_1), \phi_2(\xi + q_2), \phi_3(\xi + q_3)) \\ := \ & -Q_y(\phi_1, \phi_2, \phi_3)\phi_1'(\xi - q_1) + Q_z(\phi_1, \phi_2, \phi_3)\phi_2'(\xi + q_2) + Q_w(\phi_1, \phi_2, \phi_3)\phi_3'(\xi + q_3). \end{aligned}$$

Then the function F is bounded above for $\xi \in \mathbb{R}$, $q_1, q_2, q_3 < -\delta$, since Q_y, Q_z, Q_w, $-\phi_1'$, ϕ_2' and ϕ_3' are bounded above for $\xi \in \mathbb{R}$, $q_1, q_2, q_3 < -\delta$.

The next lemma shows that the function F has a positive lower bound for $\xi \in \mathbb{R}$ and $q_1, q_2, q_3 < -\delta$, if δ is sufficiently large.

Lemma 3.3. *There exists a sufficiently large constant δ such that*

$$F(\phi_1(\xi - q_1), \phi_2(\xi + q_2), \phi_3(\xi + q_3)) > 0 \text{ for } \xi \in \mathbb{R},\ q_1, q_2, q_3 \le -\delta.$$

Moreover, $F(\phi_1, \phi_2, \phi_3) = F(\phi_1(\xi - q_1), \phi_2(\xi + q_2), \phi_3(\xi + q_3))$ satisfies

$$F(\phi_1, \phi_2, \phi_3) \ge \frac{1}{2} Q_y |\phi_1'(\xi - q_1)| \quad \text{for } \xi \le q_1, \tag{3.10}$$

$$F(\phi_1, \phi_2, \phi_3) \ge \frac{1}{2}\Big[Q_y |\phi_1'(\xi - q_1)| + Q_z |\phi_2'(\xi + q_2)|\Big] \quad \text{for } q_1 \le \xi \le -q_2, \tag{3.11}$$

$$F(\phi_1, \phi_2, \phi_3) \ge \frac{1}{2}\Big[Q_z |\phi_2'(\xi + q_2)| + Q_w |\phi_3'(\xi + q_3)|\Big] \quad \text{for } -q_2 \le \xi \le -q_3, \tag{3.12}$$

$$F(\phi_1, \phi_2, \phi_3) \ge \frac{1}{2} Q_w |\phi_3'(\xi + q_3)| \quad \text{for } \xi \ge -q_3, \tag{3.13}$$

when $q_1, q_2, q_3 \le -\delta$.

Proof. Since $\phi_1(-\infty) = 1$, $\phi_3(-\infty) = a$, $\phi_1(\xi - q_1)$ is decreasing and $\phi_3(\xi + q_3)$ is increasing for $\xi \in \mathbb{R}$, there exists a $q_0 < q_1$ such that $\phi_1(q_0 - q_1) = \phi_3(q_0 + q_3)$, $\phi_1(\xi - q_1) > \phi_3(\xi + q_3)$ for $\xi < q_0$ and $\phi_1(\xi - q_1) < \phi_3(\xi + q_3)$ for $q_0 < \xi \le q_1$. For $q_0 \le \xi \le q_1$, we have $Q_z(\phi_1(\xi - q_1), \phi_2(\xi + q_2), \phi_3(\xi + q_3)) \ge 0$ and

$$F(\phi_1, \phi_2, \phi_3) = -Q_y\phi_1' + Q_z\phi_2' + Q_w\phi_3' \ge -Q_y\phi_1' \ge \frac{1}{2} Q_y |\phi_1'|$$

by $Q_w, \phi_2', \phi_3' \geq 0$. If $\xi < q_0$, we know that $Q_z(\phi_1(\xi - q_1), \phi_2(\xi + q_2), \phi_3(\xi + q_3)) < 0$. From (3.3)-(3.4), we have $|\phi_1'| \geq (1-\phi_1)/\tau$ and $|\phi_2'| \leq Ke^{\beta_2 q_2}$. Then we compute that

$$
\begin{aligned}
F(\phi_1,\phi_2,\phi_3) - \frac{1}{2}Q_y|\phi_1'| = \frac{1}{2}Q_y|\phi_1'| + Q_z\phi_2' + Q_w\phi_3' \geq \frac{1}{2}Q_y|\phi_1'| + Q_z\phi_2' \\
\geq \quad & \frac{a(1-\phi_2)(a-\phi_2)(1-\phi_3)^2(1-\phi_1)}{2\tau[(1-\phi_1)\phi_2(1-a)+(a-\phi_2)(1-\phi_3)]^2} \\
& + \frac{(1-a)a(1-\phi_3)(1-\phi_1)(\phi_3-\phi_1)}{[(1-\phi_1)\phi_2(1-a)+(a-\phi_2)(1-\phi_3)]^2}Ke^{\beta_2 q_2} \\
\geq \quad & \frac{a(1-\phi_1)(1-\phi_3)}{2\tau[3a(1-a)/2]^2}\left[\left(1-\frac{a}{2}\right)\left(a-\frac{a}{2}\right)\left(1-\frac{a+1}{2}\right) + 2\tau(1-a)(a-1)Ke^{-\beta_2\delta}\right] \\
\geq \quad & 0
\end{aligned}
$$

for δ sufficiently large. Therefore, (3.10) holds for $\xi \leq q_1$ and $q_1, q_2, q_3 \leq -\delta$.

For $\xi \geq q_1$, since $Q_y, Q_z, Q_w \geq 0$, $\phi_1' < 0$ and $\phi_2', \phi_3' > 0$, we get the conclusion. ∎

With Lemma 3.3, we now state and prove the following key lemma on the estimates to be used later in verifying super-sub-solutions.

Lemma 3.4. *There is a positive constant M such that*

$$
\left|\frac{H(\phi_1,\phi_2,\phi_3)+G(\phi_1,\phi_2,\phi_3)}{F(\phi_1,\phi_2,\phi_3)}\right| \leq \begin{cases} M(|\phi_2'|+|\phi_3'|) & \text{for } \xi \leq 0, \\ M(|\phi_1'|+|\phi_3'|) & \text{for } 0 \leq \xi \leq -\dfrac{q_3+q_2}{2}, \\ M(|\phi_1'|+|\phi_2'|) & \text{for } \xi \geq -\dfrac{q_3+q_2}{2}, \end{cases} \tag{3.14}
$$

for $q_1, q_2, q_3 < -\delta$ with $\delta \gg 1$.

Proof. For the simplicity of notation, we denote the functions $H_{ij}(\phi_1,\phi_2,\phi_3)$ $(i,j=1,2)$ by H_{ij}. Similarly we also omit (ϕ_1,ϕ_2,ϕ_3) for $H(\phi_1,\phi_2,\phi_3)$, $G(\phi_1,\phi_2,\phi_3)$, $Q_y(\phi_1,\phi_2,\phi_3)$ and so on.

First, we estimate $|H/F|$. For $\xi \leq q_1$, by (3.8), Lemma 3.2, (3.4) and (3.10), we have

$$
\begin{aligned}
\left|\frac{H}{F}\right| \quad \leq \quad & \left|\frac{2(1-\phi_1)\phi_2[\phi_1 H_{11} + (\phi_3 - a)H_{12}]}{Q_y|\phi_1'|}\right| \\
\leq \quad & \frac{2\tau}{\epsilon_1}[C|\phi_2| + C|\phi_3 - a|] \leq \frac{2C\tau^2}{\epsilon_1}(|\phi_2'|+|\phi_3'|).
\end{aligned} \tag{3.15}
$$

For $q_1 \leq \xi \leq 0$, by (3.8), Lemma 3.2, (3.4) and (3.11), we have

$$
\begin{aligned}
\left|\frac{H}{F}\right| \quad \leq \quad & \left|\frac{2(1-\phi_1)\phi_2[\phi_1 H_{11} + (\phi_3 - a)H_{12}]}{Q_y|\phi_1'| + Q_z|\phi_2'|}\right| \\
\leq \quad & \frac{2|1-\phi_1||\phi_2||\phi_1||H_{11}|}{Q_y|\phi_1'|} + \frac{2|1-\phi_1||\phi_2||\phi_3-a||H_{12}|}{Q_z|\phi_2'|} \\
\leq \quad & \frac{2C\tau^2|\phi_2'|}{\epsilon_1} + \frac{2C\tau^2|\phi_3'|}{\epsilon_2}.
\end{aligned} \tag{3.16}
$$

From (3.15)-(3.16), we obtain that

$$\left|\frac{H}{F}\right| \le M_1(|\phi_2'(\xi+q_2)|+|\phi_3'(\xi+q_3)|) \quad \text{for} \quad \xi \le 0. \tag{3.17}$$

For $0 \le \xi \le -q_2$, by (3.8), Lemma 3.2, (3.4) and (3.11), we have

$$\begin{aligned} \left|\frac{H}{F}\right| &\le \left|\frac{2(1-\phi_1)\phi_2[\phi_1 H_{11}+(\phi_3-a)H_{12}]}{Q_z|\phi_2'|}\right| \\ &\le \frac{2|1-\phi_1||\phi_2||\phi_1||H_{11}|}{Q_z|\phi_2'|}+\frac{2|1-\phi_1||\phi_2||\phi_3-a||H_{12}|}{Q_z|\phi_2'|} \\ &\le \frac{2C\tau^2|\phi_1'|}{\epsilon_2}+\frac{2C\tau^2|\phi_3'|}{\epsilon_2}. \end{aligned} \tag{3.18}$$

For $-q_2 \le \xi \le (-q_3-q_2)/2$, by (3.8), Lemma 3.2, (3.4) and (3.12), we have

$$\begin{aligned} \left|\frac{H}{F}\right| &\le \left|\frac{2(1-\phi_3)(a-\phi_2)[\phi_1 H_{21}+(\phi_3-a)H_{22}]}{Q_z|\phi_2'|}\right| \\ &\le \frac{2|1-\phi_3||a-\phi_2||\phi_1||H_{21}|}{Q_z|\phi_2'|}+\frac{2|1-\phi_3||a-\phi_2||\phi_3-a||H_{22}|}{Q_z|\phi_2'|} \\ &\le \frac{2C\tau^2|\phi_1'|}{\epsilon_2}+\frac{2C\tau^2|\phi_3'|}{\epsilon_2}. \end{aligned} \tag{3.19}$$

From (3.18)-(3.19), we obtain that

$$\left|\frac{H}{F}\right| \le M_2(|\phi_1'(\xi-q_1)|+|\phi_3'(\xi+q_3)|) \quad \text{for} \quad 0 \le \xi \le -\frac{q_3+q_2}{2}. \tag{3.20}$$

For $(-q_3-q_2)/2 \le \xi \le -q_3$, by (3.8), Lemma 3.2, (3.4) and (3.12), we have

$$\begin{aligned} \left|\frac{H}{F}\right| &\le \left|\frac{2(1-\phi_3)(a-\phi_2)[\phi_1 H_{21}+(\phi_3-a)H_{22}]}{Q_z|\phi_2'|+Q_w|\phi_3'|}\right| \\ &\le \frac{2|1-\phi_3||a-\phi_2||\phi_1||H_{21}|}{Q_z|\phi_2'|}+\frac{2|1-\phi_3||a-\phi_2||\phi_3-a||H_{22}|}{Q_w|\phi_3'|} \\ &\le \frac{2C\tau^2|\phi_1'|}{\epsilon_2}+\frac{2C\tau^2|\phi_2'|}{\epsilon_3}. \end{aligned} \tag{3.21}$$

For $\xi \ge -q_3$, by (3.8), Lemma 3.2, (3.4) and (3.13), we have

$$\begin{aligned} \left|\frac{H}{F}\right| &\le \left|\frac{2(1-\phi_3)(a-\phi_2)[\phi_1 H_{21}+(\phi_3-a)H_{22}]}{Q_w|\phi_3'|}\right| \\ &\le \frac{2\tau}{\epsilon_3}[C|\phi_1|+C|a-\phi_2|] \le \frac{2C\tau^2}{\epsilon_3}(|\phi_1'|+|\phi_2'|). \end{aligned} \tag{3.22}$$

From (3.21)-(3.22), we obtain that

$$\left|\frac{H}{F}\right| \le M_3(|\phi_1'(\xi-q_1)|+|\phi_2'(\xi+q_2)|) \quad \text{for} \quad \xi \ge -\frac{q_3+q_2}{2}. \tag{3.23}$$

Next, we estimate $|G/F|$. For $\xi \le q_1$, by Lemma 3.1(ii), Lemma 3.2, (3.4) and (3.10), we have

$$\begin{aligned}
\left|\frac{G}{F}\right| &\le 2\frac{|Q_{yy}(\phi_1')^2+Q_{zz}(\phi_2')^2+Q_{ww}(\phi_3')^2+2Q_{yz}\phi_1'\phi_2'+2Q_{yw}\phi_1'\phi_3'+2Q_{zw}\phi_2'\phi_3'|}{Q_y|\phi_1'|}\\
&\le 2\left[\frac{|\phi_2||R_1||\phi_1'|}{\epsilon_1}+\frac{|1-\phi_1||R_4||\phi_2'|^2}{\epsilon_1|\phi_1'|}+\frac{|1-\phi_1||R_8||\phi_3'|^2}{\epsilon_1|\phi_1'|}\right]\\
&\quad+\left[4\left(\frac{|Q_{yz}||\phi_2'|}{\epsilon_1}+\frac{|Q_{yw}||\phi_3'|}{\epsilon_1}+\frac{|1-\phi_1||R_{12}||\phi_2'||\phi_3'|}{\epsilon_1|\phi_1'|}\right)\right]\\
&\le 2\left[\frac{C\tau K}{\epsilon_1}|\phi_2'|+\frac{C\tau K}{\epsilon_1}|\phi_2'|+\frac{C\tau K}{\epsilon_1}|\phi_3'|+2\left(\frac{C}{\epsilon_1}|\phi_2'|+\frac{C}{\epsilon_1}|\phi_3'|+\frac{C\tau}{\epsilon_1}|\phi_2'||\phi_3'|\right)\right]\\
&\le M_4(|\phi_2'|+|\phi_3'|).
\end{aligned}$$

For $q_1 \le \xi \le 0$, by Lemma 3.1(ii), Lemma 3.2, (3.4) and (3.11), we have

$$\begin{aligned}
\left|\frac{G}{F}\right| &\le 2\frac{|Q_{yy}(\phi_1')^2+Q_{zz}(\phi_2')^2+Q_{ww}(\phi_3')^2+2Q_{yz}\phi_1'\phi_2'+2Q_{yw}\phi_1'\phi_3'+2Q_{zw}\phi_2'\phi_3'|}{Q_y|\phi_1'|+Q_z|\phi_2'|}\\
&\le 2\left(\frac{|\phi_2||R_1||\phi_1'|}{\epsilon_1}+\frac{|Q_{zz}||\phi_2'|}{\epsilon_2}+\frac{|\phi_2||R_9||\phi_3'|^2}{\epsilon_2|\phi_2'|}\right)\\
&\quad+4\left(\frac{|Q_{yz}||\phi_2'|}{\epsilon_1}+\frac{|Q_{yw}||\phi_3'|}{\epsilon_1}+\frac{|Q_{zw}||\phi_3'|}{\epsilon_2}\right)\\
&\le 2\left[\frac{C\tau K}{\epsilon_1}|\phi_2'|+\frac{C}{\epsilon_2}|\phi_2'|+\frac{C\tau K}{\epsilon_2}|\phi_3'|+2\left(\frac{C}{\epsilon_1}|\phi_2'|+\frac{C}{\epsilon_1}|\phi_3'|+\frac{C}{\epsilon_2}|\phi_3'|\right)\right]\\
&\le M_5(|\phi_2'|+|\phi_3'|).
\end{aligned}$$

Then we obtain that

$$\left|\frac{G}{F}\right| \le M_6(|\phi_2'(\xi+q_2)|+|\phi_3'(\xi+q_3)|) \quad \text{for} \quad \xi \le 0. \tag{3.24}$$

For $0 \le \xi \le -q_2$, by (3.8), Lemma 3.2, (3.4) and (3.11), we have

$$\begin{aligned}
\left|\frac{G}{F}\right| &\le 2\frac{|Q_{yy}(\phi_1')^2+Q_{zz}(\phi_2')^2+Q_{ww}(\phi_3')^2+2Q_{yz}\phi_1'\phi_2'+2Q_{yw}\phi_1'\phi_3'+2Q_{zw}\phi_2'\phi_3'|}{Q_y|\phi_1'|+Q_z|\phi_2'|}\\
&\le 2\left(\frac{|Q_{yy}||\phi_1'|}{\epsilon_1}+\frac{(|\phi_1||R_6|+|\phi_3-a||R_7|)|\phi_2'|}{\epsilon_2}+\frac{|\phi_2||R_9||\phi_3'|^2}{\epsilon_2|\phi_2'|}\right)\\
&\quad 4\left(\frac{|Q_{yz}||\phi_1'|}{\epsilon_2}+\frac{|Q_{yw}||\phi_3'|}{\epsilon_1}+\frac{|Q_{zw}||\phi_3'|}{\epsilon_2}\right)\\
&\le 2\left[\frac{C}{\epsilon_1}|\phi_1'|+\frac{C\tau K}{\epsilon_2}(|\phi_1'|+|\phi_3'|)+\frac{C\tau K}{\epsilon_2}|\phi_3'|+2\left(\frac{C}{\epsilon_2}|\phi_1'|+\frac{C}{\epsilon_1}|\phi_3'|+\frac{C}{\epsilon_2}|\phi_3'|\right)\right]\\
&\le M_7(|\phi_1'|+|\phi_3'|).
\end{aligned}$$

For $-q_2 \le \xi \le (-q_3 - q_2)/2$, by Lemma 3.1(ii), Lemma 3.2, (3.4) and (3.12), we have

$$
\begin{aligned}
\left|\frac{G}{F}\right| &\le 2\frac{|Q_{yy}(\phi_1')^2 + Q_{zz}(\phi_2')^2 + Q_{ww}(\phi_3')^2 + 2Q_{yz}\phi_1'\phi_2' + 2Q_{yw}\phi_1'\phi_3' + 2Q_{zw}\phi_2'\phi_3'|}{Q_z|\phi_2'| + Q_w|\phi_3'|} \\
&\le 2\left(\frac{|a-\phi_2||R_2||\phi_1'|^2}{\epsilon_2|\phi_2'|} + \frac{(|\phi_1||R_6| + |\phi_3 - a||R_7|)|\phi_2'|}{\epsilon_2} + \frac{|Q_{ww}||\phi_3'|}{\epsilon_3}\right) \\
&\quad +4\left(\frac{|Q_{yz}||\phi_1'|}{\epsilon_2} + \frac{|Q_{yw}||\phi_1'|}{\epsilon_3} + \frac{|Q_{zw}||\phi_3'|}{\epsilon_2}\right) \\
&\le 2\left[\frac{C\tau K}{\epsilon_2}|\phi_1'| + \frac{C\tau K}{\epsilon_2}(|\phi_1'| + |\phi_3'|) + \frac{C}{\epsilon_3}|\phi_3'| + 2\left(\frac{C}{\epsilon_2}|\phi_1'| + \frac{C}{\epsilon_3}|\phi_1'| + \frac{C}{\epsilon_2}|\phi_3'|\right)\right] \\
&\le M_8(|\phi_1'| + |\phi_3'|).
\end{aligned}
$$

Then we obtain that

$$
\left|\frac{G}{F}\right| \le M_9(|\phi_1'(\xi - q_1)| + |\phi_3'(\xi + q_3)|) \ \text{ for } 0 \le \xi \le -\frac{q_3 + q_2}{2}. \tag{3.25}
$$

For $(-q_3 - q_2)/2 \le \xi \le -q_3$, by Lemma 3.1(ii), Lemma 3.2, (3.4) and (3.12), we have

$$
\begin{aligned}
\left|\frac{G}{F}\right| &\le 2\frac{|Q_{yy}(\phi_1')^2 + Q_{zz}(\phi_2')^2 + Q_{ww}(\phi_3')^2 + 2Q_{yz}\phi_1'\phi_2' + 2Q_{yw}\phi_1'\phi_3' + 2Q_{zw}\phi_2'\phi_3'|}{Q_z|\phi_2'| + Q_w|\phi_3'|} \\
&\le 2\left(\frac{|a-\phi_2||R_2||\phi_1'|^2}{\epsilon_2|\phi_2'|} + \frac{|Q_{zz}||\phi_2'|}{\epsilon_2} + \frac{|a-\phi_2||R_{10}||\phi_3'|}{\epsilon_3}\right) \\
&\quad +4\left(\frac{|Q_{yz}||\phi_1'|}{\epsilon_2} + \frac{|Q_{yw}||\phi_1'|}{\epsilon_3} + \frac{|Q_{zw}||\phi_2'|}{\epsilon_3}\right) \\
&\le 2\left[\frac{C\tau K}{\epsilon_2}|\phi_1'| + \frac{C}{\epsilon_2}|\phi_2'| + \frac{C\tau K}{\epsilon_3}|\phi_2'| + 2\left(\frac{C}{\epsilon_2}|\phi_1'| + \frac{C}{\epsilon_3}|\phi_1'| + \frac{C}{\epsilon_3}|\phi_2'|\right)\right] \\
&\le M_{10}(|\phi_1'| + |\phi_2'|).
\end{aligned}
$$

For $\xi \ge -q_3$, by Lemma 3.1(ii), Lemma 3.2, (3.4) and (3.13), we have

$$
\begin{aligned}
\left|\frac{G}{F}\right| &\le 2\frac{|Q_{yy}(\phi_1')^2 + Q_{zz}(\phi_2')^2 + Q_{ww}(\phi_3')^2 + 2Q_{yz}\phi_1'\phi_2' + 2Q_{yw}\phi_1'\phi_3' + 2Q_{zw}\phi_2'\phi_3'|}{Q_w|\phi_3'|} \\
&\le 2\left[\frac{|1-\phi_3||R_3||\phi_1'|^2}{\epsilon_3|\phi_3'|} + \frac{|1-\phi_3||R_5||\phi_2'|^2}{\epsilon_3|\phi_3'|} + \frac{|a-\phi_2||R_{10}||\phi_3'|}{\epsilon_3}\right] \\
&\quad +4\left(\frac{|1-\phi_3||R_{11}||\phi_1'||\phi_2'|}{\epsilon_3|\phi_3'|} + \frac{|Q_{yw}||\phi_1'|}{\epsilon_3} + \frac{|Q_{zw}||\phi_2'|}{\epsilon_3}\right) \\
&\le 2\left[\frac{C\tau K}{\epsilon_3}|\phi_1'| + \frac{C\tau K}{\epsilon_3}|\phi_2'| + \frac{C\tau K}{\epsilon_3}|\phi_2'| + 2\left(\frac{C\tau}{\epsilon_3}|\phi_1'||\phi_2'| + \frac{C}{\epsilon_3}|\phi_1'| + \frac{C}{\epsilon_3}|\phi_2'|\right)\right] \\
&\le M_{11}(|\phi_1'| + |\phi_2'|).
\end{aligned}
$$

Then we obtain that

$$
\left|\frac{G}{F}\right| \le M_{12}(|\phi_1'(\xi - q_1)| + |\phi_2'(\xi + q_2)|) \ \text{ for } \ \xi \ge -\frac{q_3 + q_2}{2}. \tag{3.26}
$$

The lemma is proved by combining (3.17), (3.20), (3.23), (3.24), (3.25) and (3.26). ∎

Therefore, from (3.3) and (3.14), we have

$$
\begin{aligned}
&|G(\phi_1,\phi_2,\phi_3)+H(\phi_1,\phi_2,\phi_3)| \\
\le\ & F(\phi_1,\phi_2,\phi_3)KM\{e^{\beta_2(\xi+q_2)}+e^{\beta_3(\xi+q_3)}\} \\
\le\ & F(\phi_1,\phi_2,\phi_3)KM\{e^{\beta_2 q_2}+e^{\beta_3 q_3}\}
\end{aligned} \tag{3.27}
$$

for $\xi \le 0$;

$$
\begin{aligned}
&|G(\phi_1,\phi_2,\phi_3)+H(\phi_1,\phi_2,\phi_3)| \\
\le\ & F(\phi_1,\phi_2,\phi_3)KM\{e^{-\gamma_1(\xi-q_1)}+e^{\beta_3(\xi+q_3)}\} \\
\le\ & F(\phi_1,\phi_2,\phi_3)KM\{e^{\gamma_1 q_1}+e^{\beta_3(q_3-q_2)/2}\}
\end{aligned} \tag{3.28}
$$

for $0 \le \xi \le (-q_3-q_2)/2$; and

$$
\begin{aligned}
&|G(\phi_1,\phi_2,\phi_3)+H(\phi_1,\phi_2,\phi_3)| \\
\le\ & F(\phi_1,\phi_2,\phi_3)KM\{e^{-\gamma_1(\xi-q_1)}+e^{-\gamma_2(\xi+q_2)}\} \\
\le\ & F(\phi_1,\phi_2,\phi_3)KM\{e^{\gamma_1 q_1}+e^{\gamma_2(q_3-q_2)/2}\}
\end{aligned} \tag{3.29}
$$

for $\xi \ge (-q_3-q_2)/2$.

4. Existence of Entire solutions - Proof of Theorem 1.1

In this section, we always assume (1.8) holds. Then $-v_0 < v_1 < 0 < v_2$ and $s_2 > s_1 > 0$.

To construct the functions q_i, $i = 1,2,3$, in §2, we consider the following initial value problems (cf. [9, 14]):

$$p_1' = s_1 + Le^{\kappa p_1}, \quad -\infty < t < 0, \quad p_1(0) = p_0; \tag{4.1}$$

$$p_2' = s_2 + Le^{\kappa p_1}, \quad -\infty < t < 0, \quad p_2(0) = p_0; \tag{4.2}$$

$$r_1' = s_1 - Le^{\kappa r_1}, \quad -\infty < t < 0, \quad r_1(0) = r_0; \tag{4.3}$$

$$r_2' = s_2 - Le^{\kappa r_1}, \quad -\infty < t < 0, \quad r_2(0) = r_0, \tag{4.4}$$

where $L > 2KM$ is a positive constant and

$$\kappa := \min\left\{\gamma_1,\gamma_2,\beta_2,\beta_3,\frac{(s_2-s_1)\gamma_2}{4s_1},\frac{(s_2-s_1)\beta_3}{4s_1}\right\}.$$

In fact, the solutions can be written explicitly as

$$
\begin{aligned}
p_1(t) &= s_1 t - \frac{1}{\kappa}\log\left[e^{-\kappa p_0} + \frac{L(1-e^{\kappa s_1 t})}{s_1}\right], \\
p_2(t) &= s_2 t - \frac{1}{\kappa}\log\left[e^{-\kappa p_0} + \frac{L(1-e^{\kappa s_1 t})}{s_1}\right], \\
r_1(t) &= s_1 t - \frac{1}{\kappa}\log\left[e^{-\kappa r_0} - \frac{L(1-e^{\kappa s_1 t})}{s_1}\right], \\
r_2(t) &= s_2 t - \frac{1}{\kappa}\log\left[e^{-\kappa r_0} - \frac{L(1-e^{\kappa s_1 t})}{s_1}\right].
\end{aligned}
$$

Now, we take p_0 and r_0 satisfying

$$p_0 = -\frac{1}{\kappa}\log\left(e^{-\kappa r_0} - \frac{2L}{s_1}\right) < -\delta, \quad r_0 < -\frac{1}{\kappa}\log\left(\frac{2L}{s_1} + e^{\kappa\delta}\right),$$

where δ is defined as in Lemma 3.3. Then we have

$$\begin{aligned} &\lim_{t\to-\infty}(p_1(t) - r_1(t)) = \lim_{t\to-\infty}(p_2(t) - r_2(t)) = 0, \\ &\lim_{t\to-\infty}(p_1(t) - s_1 t) = \lim_{t\to-\infty}(p_2(t) - s_2 t) = -\frac{1}{\kappa}\log\left(e^{-\kappa p_0} + \frac{L}{s_1}\right), \\ &\lim_{t\to-\infty}(r_1(t) - s_1 t) = \lim_{t\to-\infty}(r_2(t) - s_2 t) = -\frac{1}{\kappa}\log\left(e^{-\kappa r_0} - \frac{L}{s_1}\right). \end{aligned}$$

Also, there exists a positive constant N such that

$$0 < p_1(t) - r_1(t) = p_2(t) - r_2(t) \le N e^{\kappa s_1 t} \quad \text{for all } \ t \le 0, \tag{4.5}$$

and $p_1(t), p_2(t), r_1(t), r_2(t) \le -\delta$ for all $t \le 0$.

The next lemma shows the existence of super-sub-solutions of (3.7).

Lemma 4.1. *Define the functions $\overline{U}(\xi, t)$ and $\underline{U}(\xi, t)$ by*

$$\begin{aligned} \overline{U}(\xi, t) &:= Q(\phi_1(\xi - p_1(t)), \phi_2(\xi + p_1(t)), \phi_3(\xi + p_2(t))), \\ \underline{U}(\xi, t) &:= Q(\phi_1(\xi - r_1(t)), \phi_2(\xi + r_1(t)), \phi_3(\xi + r_2(t))). \end{aligned}$$

Then $(\overline{U}, \underline{U})(\xi, t)$ is a pair of super-sub-solutions of (3.7) for $t \le t_0$ with some $t_0 < 0$. Moreover,

$$\overline{U}(\xi, t) \ge \underline{U}(\xi, t) \quad \textit{for} \quad \xi \in \mathbb{R}, \ t \le t_0, \tag{4.6}$$

$$\sup_{\xi\in\mathbb{R}}\{\overline{U}(\xi, t) - \underline{U}(\xi, t)\} \le \mu e^{\kappa s_1 t} \quad \textit{for} \quad t \le t_0, \tag{4.7}$$

for some constant $\mu > 0$.

Proof. First, we prove $\overline{U}(\xi, t)$ is a super-solution of (3.7) for $t \le t_0$ with some $t_0 < 0$. By (3.27)-(3.29), we have

$$\begin{aligned} &|G(\phi_1, \phi_2, \phi_3) + H(\phi_1, \phi_2, \phi_3)| \\ &\le \begin{cases} F(\phi_1, \phi_2, \phi_3) KM(e^{\beta_2 p_1} + e^{\beta_3 p_2}) & \text{for } \xi \le 0, \\ F(\phi_1, \phi_2, \phi_3) KM(e^{\gamma_1 p_1} + e^{\beta_3(p_2 - p_1)/2}) & \text{for } 0 \le \xi \le -\dfrac{p_1 + p_2}{2}, \\ F(\phi_1, \phi_2, \phi_3) KM(e^{\gamma_1 p_1} + e^{\gamma_2(p_2 - p_1)/2})) & \text{for } \xi \ge -\dfrac{p_1 + p_2}{2}. \end{cases} \end{aligned}$$

Moreover, we have

$$p_2(t) - p_1(t) = r_2(t) - r_1(t) = (s_2 - s_1)t \to -\infty$$

as $t \to -\infty$. Hence, by the choice of κ, there exists a $t_0 < 0$ such that

$$\frac{\beta_3(p_2(t) - p_1(t))}{2} < \kappa p_1(t) < 0, \quad \frac{\gamma_2(p_2(t) - p_1(t))}{2} < \kappa p_1(t) < 0, \tag{4.8}$$

$$\frac{\beta_3(r_2(t) - r_1(t))}{2} < \kappa r_1(t) < 0, \quad \frac{\gamma_2(r_2(t) - r_1(t))}{2} < \kappa r_1(t) < 0 \tag{4.9}$$

for all $t \le t_0$. Thus, by (4.8), we get

$$|G(\phi_1,\phi_2,\phi_3)+H(\phi_1,\phi_2,\phi_3)| \le 2F(\phi_1,\phi_2,\phi_3)KMe^{\kappa p_1}. \tag{4.10}$$

Then we obtain

$$\begin{aligned}\mathcal{T}[\overline{U}] &= -Q_y\phi_1'(p_1'-s_1)+Q_z\phi_2'(p_1'-s_1)+Q_w\phi_3'(p_2'-s_2)-G(\phi_1,\phi_2,\phi_3)-H(\phi_1,\phi_2,\phi_3)\\ &\ge F(\phi_1,\phi_2,\phi_3)(L-2KM)e^{\kappa p_1} \ge 0\end{aligned}$$

by (4.1), (4.2), (4.10) and Lemma 3.3. Hence $\overline{U}(\xi,t)$ is a super-solution of (3.7) for $t \le t_0$.

Next, we prove $\underline{U}(\xi,t)$ is a sub-solution of (3.7) for $t \le t_0$. By (3.27)-(3.29) and (4.9), we have

$$|G(\phi_1,\phi_2,\phi_3)+H(\phi_1,\phi_2,\phi_3)| \le 2F(\phi_1,\phi_2,\phi_3)KMe^{\kappa r_1}. \tag{4.11}$$

Then we obtain

$$\begin{aligned}\mathcal{T}[\underline{U}] &= -Q_y\phi_1'(r_1'-s_1)+Q_z\phi_2'(r_1'-s_1)+Q_w\phi_3'(r_2'-s_2)-G(\phi_1,\phi_2,\phi_3)-H(\phi_1,\phi_2,\phi_3)\\ &\le -F(\phi_1,\phi_2,\phi_3)(L-2KM)e^{\kappa r_1} \le 0\end{aligned}$$

by (4.3), (4.4), (4.11) and Lemma 3.3. Hence $\underline{U}(\xi,t)$ is a sub-solution of (3.7) for $t \le t_0$.

Finally, by (4.5), Lemma 3.3, the function F is bounded above and

$$\begin{aligned}&\overline{U}(\xi,t)-\underline{U}(\xi,t)\\ &\quad= Q(\phi_1(\xi-p_1(t)),\phi_2(\xi+p_1(t)),\phi_3(\xi+p_2(t)))\\ &\qquad\quad -Q(\phi_1(\xi-r_1(t)),\phi_2(\xi+r_1(t)),\phi_3(\xi+r_2(t)))\\ &\quad= \int_0^1 F(\phi_1(\xi-\theta p_1-(1-\theta)r_1),\phi_2(\xi+\theta p_1+(1-\theta)r_1),\phi_3(\xi+\theta p_2+(1-\theta)r_2))d\theta\\ &\qquad \times(p_1-r_1),\end{aligned}$$

(4.6) and (4.7) hold. Hence the lemma is proved. ∎

Now, we have a pair of super-sub-solutions of (3.7) satisfying (4.6). By using the same method as in [8, 9], the existence and uniqueness of entire solutions of (1.1) can be shown as follows.

Theorem 4.2. *There exists a unique entire solution $u(x,t)$ of (1.1) such that*

$$\underline{U}(x+\bar{c}t,t) \le u(x,t) \le \overline{U}(x+\bar{c}t,t)$$

for all $x\in\mathbb{R}$ and $t\le t_0$ where the functions $\underline{U}$ and $\overline{U}$ are defined as in Lemma 4.1.

Finally, we consider the asymptotic behavior of the entire solution in Theorem 4.2 as $t\to\pm\infty$. Since $r_2(t)-(s_1+s_2)t/2\to-\infty$ and $(s_1+s_2)t/2-r_1(t)\to-\infty$ as $t\to-\infty$, there exists a constant $T<0$ such that

$$r_2(t) < \frac{s_1+s_2}{2}t < r_1(t)$$

for $t<T$. Define

$$\theta := -\frac{1}{\kappa}\log\left(e^{-\kappa r_0}-\frac{L}{s_1}\right). \tag{4.12}$$

By a simple computation, there exists a constant $\rho > 0$ such that

$$-\rho e^{\kappa s_1 t} < r_1(t) - s_1 t - \theta = r_2(t) - s_2 t - \theta \leq 0. \tag{4.13}$$

The next theorem shows the asymptotic behavior, as $t \to -\infty$, of the entire solution obtained in Theorem 4.2.

Theorem 4.3. *Let $u(x,t)$ be an entire solution obtained in Theorem 4.2. Then* (1.9) *holds for the constant θ defined by* (4.12).

Proof. Recall that $\xi := x + \bar{c}t$. For $x \leq -(c_1 + c_2)t/2$, we have $\xi \leq 0 \leq -r_1(t)$. By (4.7), (3.3), (3.4), (3.6), (3.9) and (4.13), we derive that

$$\begin{aligned}
&|u(x,t) - \phi_1(x + c_1 t - \theta)| = |U(\xi,t) - \phi_1(\xi - s_1 t - \theta)| \\
&\quad \leq \ |U(\xi,t) - \underline{U}(\xi,t)| + |\underline{U}(\xi,t) - \phi_1(\xi - s_1 t - \theta)| \\
&\quad \leq \ |\overline{U}(\xi,t) - \underline{U}(\xi,t)| + |\phi_1(\xi - r_1(t)) - \phi_1(\xi - s_1 t - \theta)| + |\phi_2(\xi + r_1(t))| \cdot \\
&\qquad \left| \frac{(1-a)(1 - \phi_1(\xi - r_1(t)))(\phi_3(\xi + r_2(t)) - \phi_1(\xi - r_1(t)))}{(1 - \phi_1(\xi - r_1(t)))\phi_2(\xi + r_1(t))(1-a) + (a - \phi_2(\xi + r_1(t)))(1 - \phi_3(\xi + r_2(t)))} \right| \\
&\quad \leq \ |\overline{U}(\xi,t) - \underline{U}(\xi,t)| + |\phi_1(\xi - r_1(t)) - \phi_1(\xi - s_1 t - \theta)| + \eta_1 |\phi_2(\xi + r_1(t))| \\
&\quad \leq \ \mu e^{\kappa s_1 t} + \sup_{\zeta \in \mathbb{R}} |\phi_1'(\zeta)| |r_1(t) - s_1 t - \theta| + \eta_1 \tau K e^{\beta_2(\xi + r_1(t))} \\
&\quad \leq \ \mu e^{\kappa s_1 t} + K\rho e^{\kappa s_1 t} + \eta_1 \tau K e^{\beta_2 r_1(t)}
\end{aligned}$$

for $t \leq \min\{t_0, T\}$, where $\eta_1 = 8/a$.

Now we consider $-(c_1 + c_2)t/2 \leq x \leq -(c_2 + c_3)t/2$. This implies that $0 \leq \xi \leq -(s_1 + s_2)t/2$. Recall that $-(s_1 + s_2)t/2 \leq -r_2(t)$ for $t \leq T$. By (4.7), (3.3), (3.4), (3.5), (3.9) and (4.13), we have

$$\begin{aligned}
&|u(x,t) - \phi_2(x + c_2 t + \theta)| = |U(\xi,t) - \phi_2(\xi + s_1 t + \theta)| \\
&\quad \leq \ |U(\xi,t) - \underline{U}(\xi,t)| + |\underline{U}(\xi,t) - \phi_2(\xi + s_1 t + \theta)| \\
&\quad \leq \ |\overline{U}(\xi,t) - \underline{U}(\xi,t)| + |\phi_2(\xi + r_1(t)) - \phi_2(\xi + s_1 t + \theta)| + |\phi_3(\xi + r_2(t)) - a| \cdot \\
&\qquad \left| \frac{(1 - \phi_2(\xi + r_2(t)))(1 - \phi_1(\xi - r_1(t)))\phi_2(\xi + r_2(t))}{(1 - \phi_1(\xi - r_1(t)))\phi_2(\xi + r_1(t))(1-a) + (a - \phi_2(\xi + r_1(t)))(1 - \phi_3(\xi + r_2(t)))} \right| \\
&\qquad + |\phi_1(\xi - r_1(t))| \cdot \\
&\qquad \left| \frac{(1 - \phi_2(\xi + r_1(t)))(a - \phi_2(\xi + r_1(t)))(1 - \phi_3(\xi + r_2(t)))}{(1 - \phi_1(\xi - r_1(t)))\phi_2(\xi + r_1(t))(1-a) + (a - \phi_2(\xi + r_1(t)))(1 - \phi_3(\xi + r_2(t)))} \right| \\
&\quad \leq \ |\overline{U}(\xi,t) - \underline{U}(\xi,t)| + |\phi_2(\xi + r_1(t)) - \phi_2(\xi + s_1 t + \theta)| \\
&\qquad\quad + \eta_2 (|\phi_3(\xi + r_2(t)) - a| + |\phi_1(\xi - r_1(t))|) \\
&\quad \leq \ \mu e^{\kappa s_1 t} + \sup_{\zeta \in \mathbb{R}} |\phi_2'(\zeta)| |r_1(t) - s_1 t - \theta| + \eta_2 (|\phi_3(\xi + r_2(t)) - a| + |\phi_1(\xi - r_1(t))|) \\
&\quad \leq \ \mu e^{\kappa s_1 t} + K\rho e^{\kappa s_1 t} + \eta_2 \tau K (e^{\beta_3(\xi + r_2(t))} + e^{-\gamma_1(\xi - r_1(t))}) \\
&\quad \leq \ \mu e^{\kappa s_1 t} + K\rho e^{\kappa s_1 t} + \eta_2 \tau K (e^{\beta_3(-(s_1+s_2)t/2 + r_2(t))} + e^{\gamma_1 r_1(t)})
\end{aligned}$$

for $t \leq \min\{t_0, T\}$, where $\eta_2 = 4/(1-a)$.

For the case $x \geq -(c_2+c_3)t/2$, we have $\xi \geq -(s_1+s_2)t/2$. Also, for $t \leq T$, we know that $-(s_1+s_2)t/2 \geq -r_1(t)$. From (4.7), (3.3), (3.4), (3.6), (3.9) and (4.13), we show that

$$\begin{aligned}
&|u(x,t)-\phi_3(x+c_3t+\theta)| = |U(\xi,t)-\phi_3(\xi+s_2t+\theta)| \\
&\quad\leq\; |U(\xi,t)-\underline{U}(\xi,t)| + |\underline{U}(\xi,t)-\phi_3(\xi+s_2t+\theta)| \\
&\quad\leq\; |\overline{U}(\xi,t)-\underline{U}(\xi,t)| + |\phi_3(\xi+r_2(t))-\phi_3(\xi+s_2t+\theta)| + |a-\phi_2(\xi+r_1(t))| \cdot \\
&\qquad \left|\frac{(1-\phi_3(\xi+r_2(t)))(\phi_1(\xi-r_1(t))-\phi_3(\xi+r_2(t)))}{(1-\phi_1(\xi-r_1(t)))\phi_2(\xi+r_1(t))(1-a)+(a-\phi_2(\xi+r_1(t)))(1-\phi_3(\xi+r_2(t)))}\right| \\
&\quad\leq\; |\overline{U}(\xi,t)-\underline{U}(\xi,t)| + |\phi_3(\xi+r_2(t))-\phi_3(\xi+s_2t+\theta)| + \eta_3|a-\phi_2(\xi+r_1(t))| \\
&\quad\leq\; \mu e^{\kappa s_1 t} + \sup_{\zeta\in\mathbb{R}}|\phi_3'(\zeta)||r_2(t)-s_2t-\theta| + \eta_3\tau K e^{-\gamma_2(\xi+r_1(t))} \\
&\quad\leq\; \mu e^{\kappa s_1 t} + K\rho e^{\kappa s_1 t} + \eta_3\tau K e^{\gamma_2((s_1+s_2)t/2-r_1(t))}
\end{aligned}$$

for $t \leq \min\{t_0, T\}$, where $\eta_3 = 8/a$.

Therefore, the theorem is proved. ∎

Finally, the asymptotic behavior, as $t \to \infty$, of the entire solution obtained in Theorem 4.2 follows directly by a result in [6]. This completes the proof of Theorem 1.1.

5. Proof of Theorem 1.2

Since the proof of Theorem 1.2 is quite similar to that of Theorem 1.1, we only point out the main differences in this section.

First, as in §3, (3.1) holds for $c_1 = -v_0,\ c_2 = v_1,\ c_3 = -\tilde{v}_1$ and $\phi_1(s) = \psi_0(-s),\ \phi_2(s) = \psi_1(s),\ \phi_3(s) = \tilde{\psi}_1(-s)$, where $(\alpha_1,\omega_1,\alpha_2,\omega_2,\alpha_3,\omega_3) = (1,0,0,a,a,0)$. The key auxiliary function we found for linking these three fronts is as follows

$$\widetilde{Q}(y,z,w) = z + \frac{(1-y)z(a-w)(-z)+y(a-z)w(1-z)}{(1-y)za+(a-z)w}, \tag{5.1}$$

where $(y,z,w) \in [0,1]\times[0,a]\times[0,a]\setminus(\{(1,a,w)|\,0\leq w\leq a\}\cup\{(1,z,0)|\,0\leq z\leq a\}\cup\{(y,0,0)|\,0\leq y\leq 1\})$. Similar to Lemma 3.1, we have the following lemma on some properties of this function.

Lemma 5.1. *The following three statements hold:*

(i) $\widetilde{Q}$ *can be rewritten as*

$$\widetilde{Q}(y,z,w) = \begin{cases} y+(1-y)z\dfrac{a(w-y)}{(1-y)za+(a-z)w}, \\ w+(a-z)w\dfrac{y-w}{(1-y)za+(a-z)w}. \end{cases}$$

(ii) *There exist functions* $\widetilde{Q}_i$, $i=1,2,3$, *such that*

$$\begin{aligned}
\widetilde{Q}_y(y,z,w) &= (a-z)w\widetilde{Q}_1(y,z,w), \\
\widetilde{Q}_z(y,z,w) &= (1-y)w\widetilde{Q}_2(y,z,w), \\
\widetilde{Q}_w(y,z,w) &= (1-y)z\widetilde{Q}_3(y,z,w).
\end{aligned}$$

(iii) *There exist functions R_j, $j = 1, \cdots, 14$, such that*

$$
\begin{aligned}
&\widetilde{Q}_{yy}(y,z,w) = zR_1(y,z,w) = (a-z)R_2(y,z,w) = wR_3(y,z,w),\\
&\widetilde{Q}_{zz}(y,z,w) = (1-y)R_4(y,z,w) = wR_5(y,z,w)\\
&\qquad\qquad\quad = yR_6(y,z,w) + (w-a)R_7(y,z,w),\\
&\widetilde{Q}_{ww}(y,z,w) = (1-y)R_8(y,z,w) = zR_9(y,z,w) = (a-z)R_{10}(y,z,w),\\
&\widetilde{Q}_{yz}(y,z,w) = wR_{11}(y,z,w), \quad \widetilde{Q}_{zw}(y,z,w) = (1-y)R_{12}(y,z,w),\\
&\widetilde{Q}_{yw}(y,z,w) = zR_{13}(y,z,w) = (a-z)R_{14}(y,z,w).
\end{aligned}
$$

Now, set $\bar{c} = (c_1 + c_2)/2 = (-v_0 + v_1)/2$, $s_1 = (c_2 - c_1)/2 = (v_0 + v_1)/2 > 0$ and $s_2 = c_3 - \bar{c} = (-2\tilde{v}_1 + v_0 - v_1)/2 > s_1$. Replacing Q by $\widetilde{Q}$, we have the same conclusion as in Lemma 3.2. However, to get the positivity of $\widetilde{Q}_z$, we need $\phi_3 - \phi_1 > 0$ for $q_1 < \xi < -q_3$. Therefore, we replace (3.2) by

$$\phi_1(0) = \frac{a}{4}, \quad \phi_2(0) = \phi_3(0) = \frac{a}{2}.$$

Next, for Lemma 3.3 we need to change the definition of F as follows:

$$
\begin{aligned}
&F(\phi_1(\xi - q_1), \phi_2(\xi + q_2), \phi_3(\xi + q_3))\\
&\quad := \; -\widetilde{Q}_y(\phi_1,\phi_2,\phi_3)\phi_1'(\xi - q_1) + \widetilde{Q}_z(\phi_1,\phi_2,\phi_3)\phi_2'(\xi + q_2) - \widetilde{Q}_w(\phi_1,\phi_2,\phi_3)\phi_3'(\xi + q_3).
\end{aligned}
$$

Then the proof of all estimates in Lemma 3.3 is similar, except the proof of (3.13). Indeed, for this $\widetilde{Q}$, we do not have the positivity of $\widetilde{Q}_z$ for $\xi \geq -q_3$ because of $0 \leq w \leq a$. Thus we assume the extra condition

$$q_3 - q_2 < -\delta \tag{5.2}$$

to guarantee this lemma. More precisely, when $\xi \geq -q_3$, we know that $|\phi_3'| \geq \phi_3/\tau$ and $|\phi_2'| \leq Ke^{\gamma_2(q_3 - q_2)} \leq Ke^{-\gamma_2\delta}$ by using (5.2), (3.3) and

$$
\begin{cases}
\dfrac{|\phi_1(s) - 1|}{|\phi_1'(s)|} \leq \tau, \ s \leq 0, & \dfrac{|\phi_1(s) - 0|}{|\phi_1'(s)|} \leq \tau, \ s \geq 0,\\
\dfrac{|\phi_2(s) - 0|}{|\phi_2'(s)|} \leq \tau, \ s \leq 0, & \dfrac{|\phi_2(s) - a|}{|\phi_2'(s)|} \leq \tau, \ s \geq 0,\\
\dfrac{|\phi_3(s) - a|}{|\phi_3'(s)|} \leq \tau, \ s \leq 0, & \dfrac{|\phi_3(s) - 0|}{|\phi_3'(s)|} \leq \tau, \ s \geq 0,
\end{cases}
\tag{5.3}
$$

for some positive constant τ. Then we obtain that

$$
\begin{aligned}
&F(\phi_1,\phi_2,\phi_3) - \frac{1}{2}\widetilde{Q}_w|\phi_3'|\\
&= \; \widetilde{Q}_y|\phi_1'| + \widetilde{Q}_z\phi_2' + \frac{1}{2}\widetilde{Q}_w|\phi_3'| \geq \widetilde{Q}_z\phi_2' + \frac{1}{2}\widetilde{Q}_w|\phi_3'|\\
&\geq \; \frac{a^2(1-\phi_1)\phi_3(\phi_3 - \phi_1)}{[(1-\phi_1)\phi_2 a + (a-\phi_2)\phi_3]^2}\phi_2' + \frac{1}{2\tau}\frac{(4-a)^2}{144}\phi_3\\
&\geq \; \phi_3\left[\frac{a^2(1-0)(0-a/4)}{(a/4)^2}Ke^{-\gamma_2\delta} + \frac{(4-a)^2}{288\tau}\right] \geq 0
\end{aligned}
$$

and (3.13) follows. With Lemma 3.3, the same proof as before can lead to Lemma 3.4. In particular, we also have the estimates (3.27)-(3.29).

Now, we consider the the functions $\overline{U}(\xi,t)$ and $\underline{U}(\xi,t)$ by

$$\overline{U}(\xi,t) := \widetilde{Q}(\phi_1(\xi-\tilde{p}_1(t)),\phi_2(\xi+\tilde{p}_1(t)),\phi_3(\xi+\tilde{p}_2(t))),$$
$$\underline{U}(\xi,t) := \widetilde{Q}(\phi_1(\xi-\tilde{r}_1(t)),\phi_2(\xi+\tilde{r}_1(t)),\phi_3(\xi+\tilde{r}_2(t))).$$

Here $\tilde{p}_i$ and $\tilde{r}_i$, $i=1,2$, are the solutions of the following initial value problems:

$$\begin{aligned}
\tilde{p}_1' &= s_1 + Le^{\kappa\tilde{p}_1}, \quad -\infty<t<0, \quad \tilde{p}_1(0)=\tilde{p}_0,\\
\tilde{r}_1' &= s_1 - Le^{\kappa\tilde{r}_1}, \quad -\infty<t<0, \quad \tilde{r}_1(0)=\tilde{r}_0,\\
\tilde{p}_2' &= s_2 - Le^{\kappa\tilde{p}_1}, \quad -\infty<t<0, \quad \tilde{p}_2(0)=\tilde{r}_0,\\
\tilde{r}_2' &= s_2 + Le^{\kappa\tilde{r}_1}, \quad -\infty<t<0, \quad \tilde{r}_2(0)=\tilde{p}_0,
\end{aligned}$$

where $\tilde{p}_0=p_0,\ \tilde{r}_0=r_0$ are the same as in §4, $L>2KM$ is a positive constant and

$$\kappa := \min\left\{\gamma_1,\gamma_2,\beta_2,\beta_3,\frac{(s_2-s_1)\gamma_2}{4s_1},\frac{(s_2-s_1)\beta_3}{4s_1}\right\}.$$

It is easy to show that

$$\begin{aligned}
\tilde{p}_1(t) &= s_1t - \frac{1}{\kappa}\log\left[e^{-\kappa\tilde{p}_0}+\frac{L(1-e^{\kappa s_1 t})}{s_1}\right],\\
\tilde{p}_2(t) &= s_2t + \frac{1}{\kappa}\log\left[e^{-\kappa\tilde{p}_0}+\frac{L(1-e^{\kappa s_1 t})}{s_1}\right]+\tilde{p}_0+\tilde{r}_0,\\
\tilde{r}_1(t) &= s_1t - \frac{1}{\kappa}\log\left[e^{-\kappa\tilde{r}_0}-\frac{L(1-e^{\kappa s_1 t})}{s_1}\right],\\
\tilde{r}_2(t) &= s_2t + \frac{1}{\kappa}\log\left[e^{-\kappa\tilde{r}_0}-\frac{L(1-e^{\kappa s_1 t})}{s_1}\right]+\tilde{p}_0+\tilde{r}_0.
\end{aligned}$$

Notice that, since we have $\tilde{p}_2(t)-\tilde{p}_1(t)\to-\infty$ and $\tilde{r}_2(t)-\tilde{r}_1(t)\to-\infty$ as $t\to-\infty$, there exists a $t_0<0$ such that

$$\tilde{p}_2(t)-\tilde{p}_1(t)<-\delta \quad\text{and}\quad \tilde{r}_2(t)-\tilde{r}_1(t)<-\delta$$

for all $t\le t_0$. Also, there exists a positive constant N such that

$$0<\tilde{p}_1(t)-\tilde{r}_1(t)=\tilde{r}_2(t)-\tilde{p}_2(t)\le Ne^{\kappa s_1 t} \text{ for all } t\le 0.$$

Then Lemma 4.1 can be easily proved.

With Lemma 4.1, we then have the existence of entire solution connecting three fronts ϕ_i, $i=1,2,3$ as that of Theorem 4.2. The asymptotic behavior, as $t\to-\infty$, of this entire solution (namely, (1.11)) can be proved similarly as that of Theorem 4.3 by defining

$$\theta_1 := -\frac{1}{\kappa}\log\left(e^{-\kappa\tilde{r}_0}-\frac{L}{s_1}\right), \quad \theta_2 := \frac{1}{\kappa}\log\left(e^{-\kappa\tilde{r}_0}-\frac{L}{s_1}\right)+\tilde{p}_0+\tilde{r}_0.$$

Note that the fact that

$$|\tilde{r}_1(t)-s_1t-\theta_1|\le\rho e^{\kappa s_1 t}, \quad |\tilde{r}_2(t)-s_2t-\theta_2|\le\rho e^{\kappa s_1 t}$$

for some positive constant ρ was used. Finally, the asymptotic behavior, as $t \to \infty$, of this entire solution follows directly by a result of [6]. This completes the proof of our second main theorem, Theorem 1.2.

6. Proof of Theorem 1.3

First, recall the rational function Q defined by (3.5) and $\tilde{p}_1$ defined as in §5. Let

$$\overline{c} = \frac{-v_0 + v_1}{2} < 0$$

and take $\psi_0(0) = a/4$ and $\psi_1(0) = a/2$. Then, according to [14, Proposition 3.2 and §4] or a proof similar to Theorem 1.1, the function

$$\overline{U}_1(x,t) := Q(\psi_0(-x - \overline{c}t + \tilde{p}_1(t)), \psi_1(x + \overline{c}t + \tilde{p}_1(t)), a)$$

is a supersolution of (1.1) for $t \ll -1$. Note that

$$Q(y,z,a) = z + \frac{y(1-z)(a-z)}{a - yz}.$$

Lemma 6.1. *Set $\overline{U}(x,t) := \overline{U}_1(-|x|,t)$. Then $\overline{U}$ is a supersolution of* (1.1) *for $t \ll -1$.*

Proof. We have

$$\overline{U}_{1,x}(x,t) = -Q_y \cdot \psi_0'(-x - \overline{c}t + \tilde{p}_1(t)) + Q_z \cdot \psi_1'(x + \overline{c}t + \tilde{p}_1(t))$$

where

$$\begin{aligned} Q_y &= Q_y(\psi_0(-x - \overline{c}t + \tilde{p}_1(t)), \psi_1(x + \overline{c}t + \tilde{p}_1(t)), a), \\ Q_z &= Q_z(\psi_0(-x - \overline{c}t + \tilde{p}_1(t)), \psi_1(x + \overline{c}t + \tilde{p}_1(t)), a). \end{aligned}$$

Recall that $\psi_0(-\overline{c}t + \tilde{p}_1(t)) \to 0$, $\psi_1(\overline{c}t + \tilde{p}_1(t)) \to a$ as $t \to -\infty$. Using

$$Q_y(y,z,a) = \frac{a(a-z)(1-z)}{(a-yz)^2}, \quad Q_z(y,z,a) = \frac{a(a-y)(1-y)}{(a-yz)^2},$$

and (5.3), we have

$$\overline{U}_x(0^-,t) \geq 0$$

for $t \ll -1$. Therefore, $\overline{U}$ is a supersolution of (1.1) for $t \ll -1$. ∎

Proof of Theorem 1.3. Because we already have a supersolution, we need to construct a subsolution. To construct a subsolution, we borrow $\widetilde{Q}$ in the proof of Theorem 1.2, namely,

$$\widetilde{Q}(y,z,w) = y + (1-y)z\frac{a(w-y)}{(1-y)za + (a-z)w}$$

and define

$$\underline{U}(x,t) := \widetilde{Q}(\psi_0(-x - \overline{c}t + \tilde{r}_1), \psi_1(x + \overline{c}t + \tilde{r}_1), \psi_1(-x - \overline{c}t - \tilde{r}_3)),$$

where $\tilde{r}_3(t) := \tilde{r}_2(t) - \tilde{p}_0 - \tilde{r}_0 + 1$ with $\tilde{r}_i := \tilde{r}_i(t)$, $i = 1, 2$, defined as in §5 and here we take $c_3 = -c_2$, $\phi_3(s) = \phi_2(-s)$. Then this $\underline{U}$ is a subsolution of (1.1) for $t \ll -1$.

We claim that $\overline{U}(x,t) - \underline{U}(x,t) \geq 0$ for $x \in \mathbb{R}$ and $t < -T$ with some sufficiently large T.

First, we consider $x \le 0$. Because $Q(y,z,a) - \widetilde{Q}(y,z,w) > 0$ for $y \in (0,1)$, $z \in (0,a)$ and $w \in (0,a)$, we can easily check that

$$\begin{aligned} &Q(\psi_0(-x-\overline{c}t+\tilde{r}_1), \psi_1(x+\overline{c}t+\tilde{r}_1), a) \\ &\quad -\widetilde{Q}(\psi_0(-x-\overline{c}t+\tilde{r}_1), \psi_1(x+\overline{c}t+\tilde{r}_1), \psi_1(-x-\overline{c}t-\tilde{r}_3)) > 0. \end{aligned}$$

Then we obtain

$$\begin{aligned} &\overline{U}(x,t) - \underline{U}(x,t) \\ &\quad > \ Q(\psi_0(-x-\overline{c}t+\tilde{p}_1), \psi_1(x+\overline{c}t+\tilde{p}_1), a) - Q(\psi_0(-x-\overline{c}t+\tilde{r}_1), \psi_1(x+\overline{c}t+\tilde{p}_1), a) \\ &\qquad +Q(\psi_0(-x-\overline{c}t+\tilde{r}_1), \psi_1(x+\overline{c}t+\tilde{p}_1), a) - Q(\psi_0(-x-\overline{c}t+\tilde{r}_1), \psi_1(x+\overline{c}t+\tilde{r}_1), a) \\ &\quad = \ \int_0^1 J d\theta \cdot (\tilde{p}_1 - \tilde{r}_1) \end{aligned}$$

where

$$\begin{aligned} J \ := \ & Q_y(\psi_0(-x-\overline{c}t+\theta\tilde{p}_1+(1-\theta)\tilde{r}_1), \psi_1(x+\overline{c}t+\tilde{p}_1), a)\psi_0'(-x-\overline{c}t+\theta\tilde{p}_1+(1-\theta)\tilde{r}_1) \\ & +Q_z(\psi_0(-x-\overline{c}t+\tilde{r}_1), \psi_1(x+\overline{c}t+\theta\tilde{p}_1+(1-\theta)\tilde{r}_1), a)\psi_1'(x+\overline{c}t+\theta\tilde{p}_1+(1-\theta)\tilde{r}_1). \end{aligned}$$

Using this, we divide our discussion into two cases: $\psi_0(-x-\overline{c}t+\tilde{r}_1) \le a$ and $\psi_0(-x-\overline{c}t+\tilde{r}_1) > a$.

When $\psi_0(-x-\overline{c}t+\tilde{r}_1) \le a$, we have $Q_z \ge 0$. Then it is easy to see that $\overline{U} - \underline{U} \ge 0$, since we always have $Q_y \ge 0$.

Next we consider the case where $\psi_0(-x-\overline{c}t+\tilde{r}_1) > a$. Note that $\tilde{p}_1(t) > \tilde{r}_1(t)$ for $t \le 0$. Because $\psi_0(-x-\overline{c}t+\tilde{r}_1) > a > a/2 = \psi_0(0)$ and ψ_0 is increasing, we have

$$\begin{aligned} &0 < -x-\overline{c}t+\tilde{r}_1 \le -x-\overline{c}t+\theta\tilde{p}_1+(1-\theta)\tilde{r}_1 \le -x-\overline{c}t+\tilde{p}_1, \\ &x+\overline{c}t+\tilde{r}_1 \le x+\overline{c}t+\theta\tilde{p}_1+(1-\theta)\tilde{r}_1 \le x+\overline{c}t+\tilde{p}_1 \le \tilde{r}_1+\tilde{p}_1 < 0. \end{aligned}$$

for any $\theta \in [0,1]$. This implies that

$$\begin{aligned} &\psi_0(-x-\overline{c}t+\theta\tilde{p}_1+(1-\theta)\tilde{r}_1) \ge \frac{a}{2}, \\ &0 < \psi_1(x+\overline{c}t+\tilde{p}_1), \psi_1(x+\overline{c}t+\theta\tilde{p}_1+(1-\theta)\tilde{r}_1) < \frac{a}{2}. \end{aligned}$$

Thus we have

$$\begin{aligned} J \ = \ & \frac{a[a-\psi_1(x+\overline{c}t+\tilde{p}_1)][1-\psi_1(x+\overline{c}t+\tilde{p}_1)]}{[a-\psi_0(-x-\overline{c}t+\theta\tilde{p}_1+(1-\theta)\tilde{r}_1)\psi_1(x+\overline{c}t+\tilde{p}_1)]^2}\psi_0'(-x-\overline{c}t+\theta\tilde{p}_1+(1-\theta)\tilde{r}_1) \\ & +\frac{a[a-\psi_0(-x-\overline{c}t+\tilde{r}_1)][1-\psi_0(-x-\overline{c}t+\tilde{r}_1)]}{[a-\psi_0(-x-\overline{c}t+\tilde{r}_1)\psi_1(x+\overline{c}t+\theta\tilde{p}_1+(1-\theta)\tilde{r}_1)]^2}\psi_1'(x+\overline{c}t+\theta\tilde{p}_1+(1-\theta)\tilde{r}_1) \\ \ge \ & \frac{a(a-a/2)(1-a/2)}{a^2}\psi_0'(-x-\overline{c}t+\theta\tilde{p}_1+(1-\theta)\tilde{r}_1) \\ & -\frac{4a(1-a)[1-\psi_0(-x-\overline{c}t+\tilde{r}_1)]}{a^2}\psi_1'(x+\overline{c}t+\theta\tilde{p}_1+(1-\theta)\tilde{r}_1). \end{aligned}$$

Also, by the facts $\psi_0'(s)/(1-\psi_0(s)) \le \lambda$ for $s \ge 0$ for some positive constant λ and $\tilde{p}_1 - \tilde{r}_1 \le N$, we know

$$\frac{1-\psi_0(-x-\overline{c}t+\tilde{r}_1)}{1-\psi_0(-x-\overline{c}t+\theta\tilde{p}_1+(1-\theta)\tilde{r}_1)} = \exp\left\{\int_{-x-\overline{c}t+\tilde{r}_1}^{-x-\overline{c}t+\theta\tilde{p}_1+(1-\theta)\tilde{r}_1} \frac{\psi_0'(\zeta)}{1-\psi_0(\zeta)}d\zeta\right\} \le e^{N\lambda}.$$

Using (3.3) and (3.4) yields

$$\begin{aligned} J \;\ge\; & \frac{(a-a/2)(1-a/2)[1-\psi_0(-x-\overline{c}t+\theta\tilde{p}_1+(1-\theta)\tilde{r}_1)]}{a\tau} \\ & -\frac{4(1-a)[1-\psi_0(-x-\overline{c}t+\tilde{r}_1)]}{a}Ke^{-\beta_2\delta} \\ \ge\; & \frac{1-\psi_0(-x-\overline{c}t+\theta\tilde{p}_1+(1-\theta)\tilde{r}_1)}{a\tau}\left[\frac{a(2-a)}{4} - 4e^{N\lambda}\tau(1-a)Ke^{-\beta_2\delta}\right] > 0 \end{aligned}$$

by taking $\delta := -\tilde{r}_1(-T) - \tilde{p}_1(-T)$ sufficiently large. Hence we obtain that $\overline{U} - \underline{U} \ge 0$ when $\psi_0(-x-\overline{c}t+\tilde{r}_1) > a$ and $t < -T$.

Next, we show $\overline{U}(x,t) - \underline{U}(x,t) \ge 0$ for $x > 0$ and $t < -T$ with some sufficiently large T. For this, we write

$$\begin{aligned} & \overline{U}(x,t) - \underline{U}(x,t) \\ & = \psi_1(-x+\overline{c}t+\tilde{p}_1) - \psi_1(-x-\overline{c}t-\tilde{r}_3) \\ & \quad + \frac{\psi_0(x-\overline{c}t+\tilde{p}_1)[a-\psi_1(-x+\overline{c}t+\tilde{p}_1)][1-\psi_1(-x+\overline{c}t+\tilde{p}_1)]}{a-\psi_0(x-\overline{c}t+\tilde{p}_1)\psi_1(-x+\overline{c}t+\tilde{p}_1)} \\ & \quad - \frac{[a-\psi_1(x+\overline{c}t+\tilde{r}_1)]\psi_1(-x-\overline{c}t-\tilde{r}_3)[\psi_0(-x-\overline{c}t+\tilde{r}_1)-\psi_1(-x-\overline{c}t-\tilde{r}_3)]}{[1-\psi_0(-x-\overline{c}t+\tilde{r}_1)]a\psi_1(x+\overline{c}t+\tilde{r}_1)+[a-\psi_1(x+\overline{c}t+\tilde{r}_1)]\psi_1(-x-\overline{c}t-\tilde{r}_3)}. \end{aligned} \tag{6.1}$$

It is easy to check that

$$0 < -\overline{c}t - \tilde{r}_2 + \tilde{p}_0 + \tilde{r}_0 \le \overline{c}t + \tilde{p}_1 < \overline{c}t - \tilde{p}_1$$

for $t < -T$ with some sufficiently large T by the facts $\tilde{p}_1 < 0$,

$$-\overline{c}t - \tilde{r}_2 + \tilde{p}_0 + \tilde{r}_0 = -c_3t - \frac{1}{\kappa}\log\left[e^{-\kappa\tilde{r}_0} - \frac{L(1-e^{\kappa s_1 t})}{s_1}\right]$$

$$\overline{c}t + \tilde{p}_1 - (-\overline{c}t - \tilde{r}_2 + \tilde{p}_0 + \tilde{r}_0) = \frac{1}{\kappa}\log\left[\frac{e^{-\kappa\tilde{r}_0} - \frac{L(1-e^{\kappa s_1 t})}{s_1}}{e^{-\kappa\tilde{r}_0} - \frac{L(1+e^{\kappa s_1 t})}{s_1}}\right].$$

Also, since ψ_1 is increasing, we know that $\psi_1(-x+\overline{c}t+\tilde{p}_1) \ge \psi_1(-x-\overline{c}t-\tilde{r}_3)$.

For $0 \le x \le -\overline{c}t - \tilde{r}_2 + \tilde{p}_0 + \tilde{r}_0$, we have $\psi_1(-x-\overline{c}t-\tilde{r}_3) \ge \psi_1(-1) > 0$, $0 \le \psi_1 \le a$ and $\psi_0(-x-\overline{c}t+\tilde{r}_1) \to 0$ as $t \to -\infty$. Then it follows from (6.1) that $\overline{U}(x,t) - \underline{U}(x,t) \ge 0$ for $t < -T$ with some sufficiently large T.

For $-\overline{c}t - \tilde{r}_2 + \tilde{p}_0 + \tilde{r}_0 \le x \le \overline{c}t + \tilde{p}_1$, we have $\psi_1(-x+\overline{c}t+\tilde{p}_1) \ge \psi_1(0)$, $\psi_1(-x-\overline{c}t-\tilde{r}_3) < \psi_1(-1)$ and $\psi_0(-x-\overline{c}t+\tilde{r}_1) \to 0$ as $t \to -\infty$. Again, by (6.1), we obtain that $\overline{U} - \underline{U} \ge 0$ for $t < -T$ with some sufficiently large T.

For $x \geq \overline{c}t + \tilde{p}_1$, we know $\psi_1(-x+\overline{c}t+\tilde{p}_1) \leq \psi_1(0) = a/2$ and $\psi_0(x-\overline{c}t+\tilde{p}_1) \geq \psi_0(-x-\overline{c}t+\tilde{r}_1)$, by $x-\overline{c}t+\tilde{p}_1 \geq -x-\overline{c}t+\tilde{r}_1$ and ψ_0 and ψ_1 are increasing. Also, as $t \to -\infty$, we have $\psi_1(x+\overline{c}t+\tilde{r}_1) \to a$, since

$$\begin{aligned} x+\overline{c}t+\tilde{r}_1 &\geq 2\overline{c}t+\tilde{p}_1+\tilde{r}_1 \\ &= 2c_2t - \frac{1}{\kappa}\log\left[e^{-\kappa p_0} + \frac{L(1-e^{\kappa s_1 t})}{s_1}\right] - \frac{1}{\kappa}\log\left[e^{-\kappa\tilde{r}_0} - \frac{L(1-e^{\kappa s_1 t})}{s_1}\right] \to \infty. \end{aligned}$$

From (6.1), we obtain that

$$\begin{aligned} &\overline{U}(x,t) - \underline{U}(x,t) \\ &\geq \psi_1(-x+\overline{c}t+\tilde{p}_1) - \psi_1(-x-\overline{c}t-\tilde{r}_3) \\ &\quad +\psi_0(-x-\overline{c}t+\tilde{r}_1)\left[\frac{[a-\psi_1(-x+\overline{c}t+\tilde{p}_1)][1-\psi_1(-x+\overline{c}t+\tilde{p}_1)]}{a-\psi_0(x-\overline{c}t+\tilde{p}_1)\psi_1(-x+\overline{c}t+\tilde{p}_1)}\right. \\ &\quad \left. - \frac{[a-\psi_1(x+\overline{c}t+\tilde{r}_1)]\psi_1(-x-\overline{c}t-\tilde{r}_3)}{[1-\psi_0(-x-\overline{c}t+\tilde{r}_1)]a\psi_1(x+\overline{c}t+\tilde{r}_1)+[a-\psi_1(x+\overline{c}t+\tilde{r}_1)]\psi_1(-x-\overline{c}t-\tilde{r}_3)}\right] \\ &\quad + \frac{[a-\psi_1(x+\overline{c}t+\tilde{r}_1)]\psi_1(-x-\overline{c}t-\tilde{r}_3)\psi_1(-x-\overline{c}t-\tilde{r}_3)}{[1-\psi_0(-x-\overline{c}t+\tilde{r}_1)]a\psi_1(x+\overline{c}t+\tilde{r}_1)+[a-\psi_1(x+\overline{c}t+\tilde{r}_1)]\psi_1(-x-\overline{c}t-\tilde{r}_3)} \geq 0 \end{aligned}$$

for $t < -T$ with some sufficiently large T.

Since we have the supersolution $\overline{U}(x,t) = U_1(-|x|,t)$, the subsolution $\underline{U}(x,t)$ and $\overline{U}(x,t) \geq \underline{U}(x,t)$ for $x \in \mathbb{R}$ and $t < -T_0$ with some sufficiently large T_0. For $T > T_0$, let us denote the solution of (1.1) with the initial function u_0 by $u(x,t;u_0)$. Consider the following solution $u^T(x,t) := u(x,t+T;\overline{U}(\cdot,-T))$, we see that

$$\underline{U}(x,t) \leq u^T(x,t) \leq \overline{U}(x,t)$$

for any $x \in \mathbb{R}$ and $t \geq -T$. By the uniqueness and $u^T(x,-T) = \overline{U}(x,-T) = \overline{U}(-x,-T) = u^T(-x,-T)$, we have $u^T(-x,t) = u^T(x,t)$ any $x \in \mathbb{R}$ and $t \geq -T$. Note that, by comparison, $u^{T_1} \geq u^{T_2}$ if $T_1 < T_2$. Thus $u^\infty := \lim_{T\to\infty} u^T$ is well-defined. Then we obtain that u^∞ is an entire solution of (1.1) which satisfies

$$\underline{U}(x,t) \leq u^\infty(x,t) \leq \overline{U}(x,t)$$

and $u^\infty(-x,t) = u^\infty(x,t)$.

Finally, we claim that $u(x,t) = u^\infty(x,t)$ satisfies (1.12). For $x \leq 0$ and $t \leq 0$, we have

$$-x-\overline{c}t-\tilde{r}_3 \geq -\overline{c}t-(-v_1-\overline{c})t - \frac{1}{\kappa}\log\left[e^{-\kappa\tilde{r}_0} - \frac{L(1-e^{\kappa s_1 t})}{s_1}\right] - 1 \geq v_1 t + \tilde{r}_0 - 1$$

and

$$\begin{aligned} a-\psi_1(-x-\overline{c}t-\tilde{r}_3) &\leq \tau|\psi_1'(-x-\overline{c}t-\tilde{r}_3)| \\ &\leq \tau K\exp(-\gamma_2(-x-\overline{c}t-\tilde{r}_3)) \leq \tau K_1 e^{-\gamma_2 v_1 t}, \end{aligned}$$

where $K_1 = Ke^{-\gamma_2(\tilde{r}_0-1)}$. It follows that

$$\begin{aligned}
& Q(\psi_0(-x-\overline{c}t+\tilde{r}_1),\psi_1(x+\overline{c}t+\tilde{r}_1),a) \\
& \qquad -\widetilde{Q}(\psi_0(-x-\overline{c}t+\tilde{r}_1),\psi_1(x+\overline{c}t+\tilde{r}_1),\psi_1(-x-\overline{c}t-\tilde{r}_3)) \\
\leq \;& a(1-0)[a-\psi_1(-x-\overline{c}t-\tilde{r}_3)]\frac{(a-0)1+a(1-0)a}{(a^2/4)(a/2)} \\
\leq \;& K_2 e^{-\gamma_2 v_1 t},
\end{aligned}$$

where

$$K_2 := \frac{8(1+a)\tau K_1}{a}.$$

Now we choose $\kappa_1 = \min\{\kappa, -\gamma_2 v_1/s_1\}$. By the facts Q_y, Q_z, ψ_0', ψ_1' are bounded, (4.5) and the last inequality, we obtain that

$$\overline{U}(x,t) - \underline{U}(x,t) \leq K_3 e^{\kappa s_1 t} + K_2 e^{-\gamma_2 v_1 t} \leq (K_2+K_3)e^{\kappa_1 s_1 t}, \quad x \leq 0,\ t \leq 0,$$

for some positive constant K_3. By a similar argument as in the proof of Theorem 4.3, we obtain that

$$\lim_{t\to-\infty}\left\{\sup_{x\leq w_1(t)}|u(x,t)-\psi_0(-x+v_0t+\theta_1)| + \sup_{w_1(t)\leq x\leq 0}|u(x,t)-\psi_1(x+v_1t+\theta_2)|\right\} = 0.$$

Since $u(x,t) = u(-x,t)$, (1.12) is proved.

Finally, the asymptotic behavior (1.10) can be derived as before and thereby the proof of Theorem 1.3 is completed. ∎

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