EXISTENCE OF ROTATING SPOTS WITH SPATIALLY DEPENDENT FEEDBACK IN THE PLANE IN A WAVE FRONT INTERACTION MODEL

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Abstract. In this paper, we study the localized rotating spots created by the spatially inhomogeneous feedback. We adopt the wave front interaction model proposed by Zykov and Showalter in 2005. The existence of rotating spots in the plane are shown by choosing the spatially dependent feedbacks appropriately.

Keywords: rotating spot, localized spiral wave pattern, angular speed

1. Introduction

Spatially localized moving objects have attracted many researchers [6, 7, 10, 12, 9, 11, 2]. Especially, Sakurai et al. carried out experiments using the photosensitive BZ medium and controlled wave behavior by the interactive feedback of the illuminated images [10]. They designed wave behavior guided through a circle, a trochoid and a random walk.

In this paper, we focus on wave pattern in the plane moving along a (core) circle. Mathematically, a wave pattern is called a *rotating spot* if it rotates along a circle with a constant angular velocity. To formulate this setting, we borrow the wave front interaction model introduced by Zykov and Showalter [13] in which they treat the traveling segment patterns of the photosensitive BZ medium. The wave front interaction model is as follows:

$$V = a - bv_{\pm} - \kappa \tag{1.1}$$

where a, b are nonnegative constants and V, κ are the normal velocity and the curvature of the interface respectively. We note that a, b correspond to the speed of one dimensional traveling front and the feedback. The density of the inhibitor on the front and the back are represented by $v_{\pm}$. Here we set $v_+ = 0$. For the traveling segment, v_- can be taken as the distance from the front and there is a unique traveling spot with speed $c \in [0, a)$ for a given constant $a > 0$, provided that a positive constant b is chosen appropriately ([13, 4]).

Later, Zykov applied the approach of [13] to the existence of rotating waves [14, 15]. He treated two types of rotating waves: (i) rotating waves on a disk which is stuck to the boundary of the disk; (ii) rotating spirals in a plane. For these cases, v_- is assumed to be

Date: November 26, 2015.

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This work was supported in part by the grants MOST 102-2811-M-032-007 and MOST 102-2115-M-032-003-MY3 from the Ministry of Science and Technology of Taiwan (ROC), by a Research Project Grant (A) from the Institute of Science and Technology, Meiji University, and by JSPS KAKENHI Grant Numbers 25610036 and 26287024.

determined by the angular distance from the front. The mathematical proof for rotating waves on a disk and in the plane are obtained in [5, 1] respectively. For the case of a disk with radius R_D which is bigger than 1, there is a unique rotating spot with angular velocity $\omega \in [0, 1/R_D)$, provided that a constant b is chosen appropriately ([14, 5]). However, the feedback is not necessary to be a constant as seen in [10].

To obtain localized rotating spots in the plane, we need to introduce the space-dependent feedback as seen in [10]. In this paper, we consider the case where b depends on the space variable as follows

$$b(x,y) = \begin{cases} b_1, & x^2+y^2 < \widetilde{R}^2, \\ b_2, & x^2+y^2 \geq \widetilde{R}^2 \end{cases} \tag{1.2}$$

for some positive radius $\widetilde{R}$. This function (1.2) is the simplest case of inhomogeneous feedback. The main purpose of this paper is to show the existence of rotating spot in the wave front interaction model by choosing appropriate b_1 and b_2. In this paper, we show the existence of rotating spots for any angular velocity below a certain constant. The fact that the existence of the localized rotating waves comes from the inhomogeneity of the feedback can be viewed in a different way as the following. Since the feedback determines the velocity of the wave, we can expect that it begins to rotate clockwise if the velocity in right hand side of the wave is higher than the left hand side. Thus the inhomogeneity of the feedback is important for the existence of the rotating waves. The inhomogeneity is not necessary to be a step function as in (1.2). This result suggests that the function with local minima as well as the continuous function which is close to (1.2) may create the localized rotating waves for some appropriate angular velocity with some appropriate radius. Since the feedback can be regarded as the new component of the system, we can also expect the existence of localized rotating waves in the three component system (see also [8]).

Our method to derive the existence of rotating spots in the wave front interaction model is the same as that in the previous works ([4, 5, 1]), namely, to study both front and back parts. However, the analysis here is more delicate. First, for the back of a rotating spot, we need to treat two ordinary differential systems at the same time. By a shooting argument with suitable match, we need to analyze the relationship between the corresponding solutions starting from two tips of the spot for different parameters b_1 and b_2, respectively. However, it is not trivial that these two trajectories will meet together for certain choices of parameters b_1 and b_2. To overcome this difficulty, we utilize an idea from [1] to define two appropriate sets of parameters b_1 and b_2 (see Section 4 for details) to guarantee the intersection of two suitable trajectories shooting from both tips. But, this pair of trajectories may not match smoothly. To overcome this further difficulty, we introduce a special set of parameter (b_1, b_2) such that its corresponding trajectories meet together. Using some properties of this set with the help of the comparison theorem and the continuous dependence on b_1 and b_2, we are able to derive the existence of the back of a rotating spot. However, in comparing with the homogeneous case ([4, 5, 1]), our analysis here is much more involved and complicated. Lastly, due to some technical difficulty, we can only derive the existence of rotating spots with core radii in a certain range. We leave the problem of the existence of rotating spots with small core radii as an open problem.

This paper is organized as follows. In Section 2, we give the mathematical setting of the problem and present the main result. In Section 3, we prepare some auxiliary lemmas. Section 4 is devoted to the proof of the main existence theorem stated in Section 2.

2. The problem setting and main result

A planar curve can be described by the Euclidean coordinates (x, y) and the angle θ of the normal vector (right-hand to the tangent) measuring from the positive x-axis such that

$$\frac{dx}{ds} = -\sin\theta, \quad \frac{dy}{ds} = \cos\theta,$$

where s is the arc length parameter. Introducing the polar coordinates (r, γ), by the relation $(x, y) = (r\cos\gamma, r\sin\gamma)$, we have

$$\frac{dr}{ds} = \sin\varphi, \quad \frac{d\gamma}{ds} = \frac{1}{r}\cos\varphi, \tag{2.1}$$

where $\varphi := \gamma - \theta$. In the sequel, we let κ be the (signed) curvature defined by $\kappa := d\theta/ds$.

In this paper, we are interested in the existence of a bounded localized rotating wave pattern (or, *rotating spot*) in the plane which is rotating *counter-clockwise* with a constant positive angular speed ω. Here a wave pattern corresponds to the excited region in the media. Let the origin be the center of the rotation. Then we have the relations

$$r(s,t) = r(s), \quad \gamma(s,t) = \gamma(s) + \omega t, \quad \theta(s,t) = \theta(s) + \omega t. \tag{2.2}$$

Then, using (2.2), the normal velocity V can be computed by

$$V = -\omega r \sin\varphi. \tag{2.3}$$

Assume the radius function is monotone in s. This is the case if $\varphi \in [n\pi, (n+1)\pi]$ for some integer n. We call those points on the boundary of the excited region with vanishing normal velocities as *tips*. It is clear that either $\varphi = n\pi$ or $\varphi = (n+1)\pi$ for a tip. Note that the set of tips might contain a connected circular arc (with the same radius). In this case we call it a *tipped arc*. We shall call the tip with the largest angle among those points with the minimal radius as the *inner tip*, and the one with the maximal radius and the largest angle as the *outer tip*. Therefore, the radius function has its minimum at the inner tip and the maximum at the outer tip. So that the *front* and the *back* of a rotating spot are realized as dividing the wave boundary by its inner and outer tips. A rotating spot has two types (i) tips consist of only two points; (ii) tips consist of a point and a tipped arc. To specify two cases, we call them a *point-tipped rotating spot* and an *arc-tipped rotating spot*, respectively.

For clarity, the functions of the front curve and the back curve are denoted by

$$(r_+, \gamma_+, \theta_+, \varphi_+, V_+) \text{ and } (r_-, \gamma_-, \theta_-, \varphi_-, V_-),$$

respectively. For the front, we normalize $a = 1$ in (1.1) and the equation for the front becomes the following linear eikonal equation:

$$V_+ = 1 - \kappa_+.$$

Using (2.1) and (2.3), we obtain

$$\left\{\begin{array}{rcl}\dfrac{dr_+}{ds} & = & \sin\varphi_+,\\ \dfrac{d\varphi_+}{ds} & = & \dfrac{\cos\varphi_+}{r_+} - 1 - \omega r_+\sin\varphi_+.\end{array}\right. \tag{2.4}$$

We take the inner tip to be $(r_+,\gamma_+)|_{s=0} = (R_1, 3\pi/2)$ for some $R_1 > 0$. Also, we choose $\theta_+(0) = \pi/2$. Here the arc length is measured *backward* so that $s < 0$ for the front. Therefore, (2.4) is equipped with the terminal condition

$$r_+|_{s=0} = R_1, \quad \varphi_+|_{s=0} = \pi. \tag{2.5}$$

We denote the solution of (2.4) with (2.5) by $(r_+(s;\omega), \varphi_+(s;\omega))$, or simply $(r_+(s), \varphi_+(s))$.

Using the change of variables

$$X_+(s) := \omega r_+(s)\cos\varphi_+(s), \quad Y_+(s) := 1 + \omega r_+(s)\sin\varphi_+(s),$$

we end up with

$$\left\{\begin{array}{rcl}\dfrac{dX_+}{ds} & = & Y_+(Y_+ - 1), \quad \dfrac{dY_+}{ds} = \omega - X_+Y_+, \quad s < 0,\\ X_+(0) & = & -\omega R_1, \quad Y_+(0) = 1.\end{array}\right. \tag{2.6}$$

Note that Y_+ is the curvature function.

We look for solutions such that the radius function is monotone in s. Since $\varphi_+(0;\omega) = \pi$, $\theta_+(s;\omega) < \pi/2$, and $\gamma_+(s;\omega) > 3\pi/2$ for $0 < -s \ll 1$, we see from (2.4) that $\varphi_+ \in [\pi, 2\pi]$ for the front. This gives us the condition that $Y_+ < 1$ for the front, except when $\varphi_+ = \pi$ or 2π.

The following proposition for the existence of the front was shown in [3, Lemma 2.1 and Theorem 1].

Proposition 2.1 ([3, Lemma 2.1 and Theorem 1])**.** *There exists a positive constant ω_* such that for each $\omega \in (0, \omega_*)$ there exist a unique positive constant $R^* := R^*(\omega)$ and a unique solution (X_+, Y_+) of (2.6) with the following properties:*

$$X_+(0) = -\omega R^*, \; Y_+(0) = 1, \; \lim_{s\to-\infty} X_+(s) = \infty, \; \lim_{s\to-\infty} Y_+(s) = 0,$$

and $Y_+(s) < 1$ for all $s < 0$. Moreover, for any $0 < R_1 < R^*$ there exists a unique solution (X_+, Y_+) of (2.6) such that $Y_+(-\tau_1) = 1$ and $0 < Y_+(s) < 1$ for all $-\tau_1 < s < 0$ for some positive constant τ_1.

By Proposition 2.1, for any $0 < \omega < \omega_*$ and $0 < R_1 < R^*$, there exist a solution (r, φ) of (2.4) and a positive constant τ_1 such that

$$r_+(0) = R_1, \quad \varphi_+(0) = \pi,$$
$$r_+(-\tau_1) = R_2 := R_2(R_1) = \frac{X_+(-\tau_1)}{\omega}, \quad \varphi_+(-\tau_1) = 2\pi.$$

Moreover, due to

$$\frac{d}{ds}\sqrt{(X_+(s) - \omega)^2 + (Y_+(s) - 1)^2} = \frac{-\omega(Y_+(s) - 1)^2}{\sqrt{(X_+(s) - \omega)^2 + (Y_+(s) - 1)^2}} < 0,$$

we have $|\omega(R_2-1)| > \omega(R_1+1)$ which implies that $R_2 > 1$.

After solving (r_+, φ_+), we can solve γ_+ from (2.1). Since r_+ is strictly monotone in s, the function $\Gamma_+(r) := \gamma_+(s(r))$ is well-defined for $r \in [R_1, R_2]$ such that $\Gamma_+(R_1) = 3\pi/2$. In the sequel, we denote $\gamma^* := \gamma_+(-\tau_1) = \Gamma_+(R_2)$.

Next, we consider the back of a rotating spot. The back of a rotating spot is influenced by the front through the inhibitor of the excitable medium. In [14], this influence is given by

$$v_- = \Gamma_+(r_-) - \gamma_-. \tag{2.7}$$

Thus the equation (1.1) for the back is written by

$$V_- = 1 - \kappa_- - b(r_-)(\Gamma_+(r_-) - \gamma_-), \tag{2.8}$$

where b is a nonnegative function to be determined. In [14], b is a constant, but it corresponds to the light intensity. In [10], the light intensity depends on the position. Hence we may consider the case

$$b(r_-) = \begin{cases} b_1, & R_1 \le r_- < \widetilde{R}, \\ b_2, & \widetilde{R} \le r_- \le R_2. \end{cases}$$

where R_1 and R_2 are defined as in Proposition 2.1, and $\widetilde{R} \in [R_1, R_2]$ is an unknown constant to be determined.

For the case $b(r_-) = b_1$, from (2.1), (2.3) and (2.8), the equations for the back lead

$$\begin{cases} \dfrac{dr_{1,-}}{ds_1} &= \sin\varphi_{1,-}, \\ \dfrac{d\gamma_{1,-}}{ds_1} &= \dfrac{\cos\varphi_{1,-}}{r_{1,-}}, \\ \dfrac{d\varphi_{1,-}}{ds_1} &= \dfrac{\cos\varphi_{1,-}}{r_{1,-}} - 1 - \omega r_{1,-}\sin\varphi_{1,-} + b_1(\Gamma_+(r_{1,-}) - \gamma_{1,-}). \end{cases} \tag{2.9}$$

Here the arc length is measured forward so that $s_1 \ge 0$ and the initial condition is given by

$$r_{1,-}|_{s_1=0} = R_1, \quad \gamma_{1,-}|_{s_1=0} = \frac{3}{2}\pi, \quad \varphi_{1,-}|_{s_1=0} = \pi. \tag{2.10}$$

The solution of (2.9) with (2.10) is denoted by $(r_{1,-}(s_1; b_1), \gamma_{1,-}(s_1; b_1), \varphi_{1,-}(s_1; b_1))$.

Similarly, when $b(r_-) = b_2$, we consider the back from the terminal point of the front and the arc length is measured backward so that $s_2 \le 0$. Then the solution, denoted by $(r_{2,-}, \gamma_{2,-}, \varphi_{2,-})(s_2; b_2)$, should satisfy

$$\begin{cases} \dfrac{dr_{2,-}}{ds_2} &= \sin\varphi_{2,-}, \\ \dfrac{d\gamma_{2,-}}{ds_2} &= \dfrac{\cos\varphi_{2,-}}{r_{2,-}}, \\ \dfrac{d\varphi_{2,-}}{ds_2} &= \dfrac{\cos\varphi_{2,-}}{r_{2,-}} - 1 - \omega r_{2,-}\sin\varphi_{2,-} + b_2(\Gamma_+(r_{2,-}) - \gamma_{2,-}). \end{cases} \tag{2.11}$$

and the terminal condition is given by

$$r_{2,-}|_{s_2=0} = R_2, \quad \gamma_{2,-}|_{s_2=0} = \gamma_+(-\tau_1) := \gamma^*, \quad \varphi_{2,-}|_{s_2=0} = 2\pi. \tag{2.12}$$

To ensure the existence of the back of a rotating spot, we need to impose some restriction on the inner radius R_1. For this, we shall define two new special radii R_0, R_3 as follows.

When $b_1 = b_2 = 0$, we introduce

$$\begin{aligned} X_1(s_1) &:= \omega r_1(s_1)\cos\varphi_1(s_1), \quad Y_1(s_1) := 1 + \omega r_1(s_1)\sin\varphi_1(s_1), \\ X_2(s_2) &:= \omega r_2(s_2)\cos\varphi_2(s_2), \quad Y_2(s_2) := 1 + \omega r_2(s_2)\sin\varphi_2(s_2). \end{aligned}$$

Then

$$(X_1, Y_1)(0) = (-\omega R_1, 1), \quad (X_2, Y_2)(0) = (\omega R_2, 1),$$

and (X_1, Y_1) and (X_2, Y_2) satisfy

$$\begin{cases} \dfrac{dX}{ds} &= Y(Y-1), \\ \dfrac{dY}{ds} &= \omega - XY. \end{cases}$$

For each $R_1 \in (0, R^*)$, there exist $\tau_1, \tau_2 > 0$ such that $(X_1, Y_1)(\tau_1) = (\omega R_0, 1)$ and $(X_2, Y_2)(-\tau_2) = (-\omega R_3, 1)$ which means $\Phi_1(R_0; 0) = 0$ and $\Phi_2(R_3; 0) = 3\pi$ for some $R_0 := R_0(R_1)$ and $R_3 := R_3(R_1)$. Since

$$\begin{aligned} & \frac{1}{2}\frac{d}{ds}[(X-\omega)^2 + (Y-1)^2] = (X-\omega)\frac{dX}{ds} + (Y-1)\frac{dY}{ds} \\ = \ & (X-\omega)Y(Y-1) + (Y-1)(\omega - XY) = -\omega(Y-1)^2 < 0, \end{aligned}$$

we have $(R_0-1)^2 < (R_1+1)^2 < (R_2-1)^2 < (R_3+1)^2$. It is easy to see that $R_0 - R_1 < 2$ and $R_2 - R_3 < 2$.

Now, we give a sufficient condition for R_1 such that $R_0 < R_3$.

Lemma 2.2. *For each $\omega \in (0, \omega_*)$, there exists a $R_* := R_*(\omega)$ such that for any $R_1 \in [R_*, R^*)$, we have $R_0(R_1) < R_3(R_1)$.*

Proof. By Proposition 2.1, we know that $R_2(R_1) \to +\infty$ as $R_1 \to R^*$. Hence, there exists a $R_* \in (0, R^*)$ such that $R_2(R_*) - R_* > 4$. Also, from the result of the Lemma 2.5 (iii) in [3], we obtain that $R_2(R_1) - R_1$ is increasing in $R_1 \in (0, R^*)$. This implies that $R_2(R_1) - R_1 > 4$ for any $R_1 \in [R_*, R^*)$. So we have

$$\begin{aligned} R_0(R_1) - R_3(R_1) &= [R_0(R_1) - R_1] + [R_1 - R_2(R_1)] + [R_2(R_1) - R_3(R_1)] \\ &< 4 - [R_2(R_1) - R_1] < 0 \end{aligned}$$

for any $R_1 \in [R_*, R^*)$. The lemma is proved. □

The following main theorem of this paper gives the existence of a rotating spot (see Figure 1 and Figure 2).

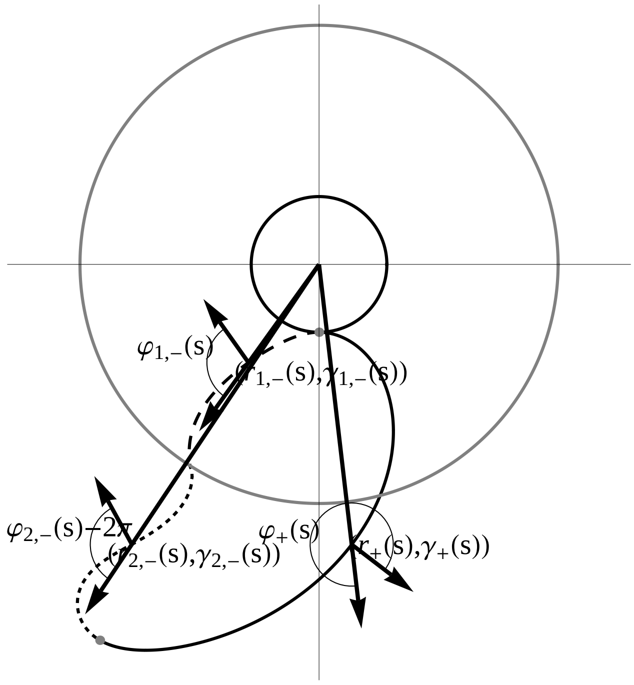

FIGURE 1. An example of point-tipped rotating spot: numerical solutions of system (2.4) with initial condition (2.5) (solid curve), system (2.9) with initial condition (2.10) (dashed curve) and system (2.11) with initial condition (2.12) (dotted curve).

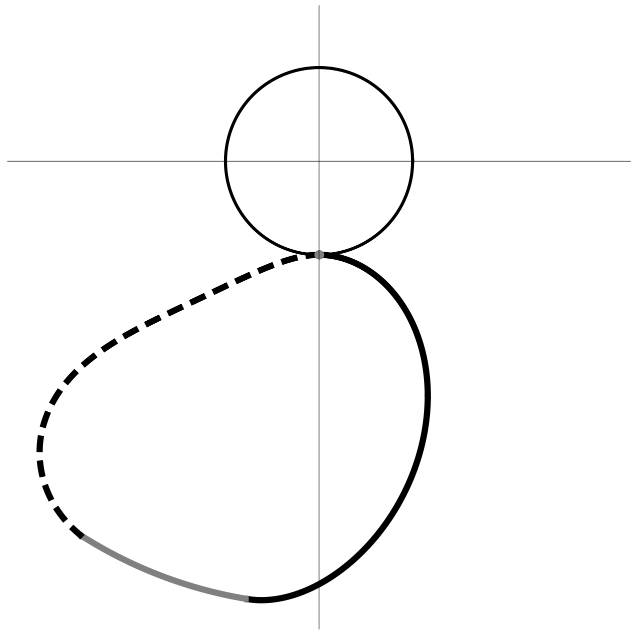

FIGURE 2. An example of arc-tipped rotating spot: numerical solutions of system (2.4) with initial condition (2.5) (solid curve) and system (2.9) with initial condition (2.10) (dashed curve) with a tipped arc (gray curve).

Theorem 1. *There exists a rotating spot for any $\omega \in (0, \omega_*)$ and $R_1 \in [R_*, R^*)$, where R^* and R_* are defined in Proposition 2.1 and Lemma 2.2, respectively.*

In fact, a point-tipped rotating spot can be derived, if two backs starting from the inner tip and the outer tip intersect such that

$$
\begin{aligned}
&r_{1,-}(\xi_1; b_1) = r_{2,-}(-\xi_2; b_2) = \widetilde{R}, \quad \gamma_{1,-}(\xi_1; b_1) = \gamma_{2,-}(-\xi_2; b_2),\\
&\varphi_{2,-}(-\xi_2; b_2) - \varphi_{1,-}(\xi_1; b_1) = 2\pi
\end{aligned}
$$

for some constants ξ_1, ξ_2, b_1, b_2 and $\widetilde{R} \in [R_1, R_2]$. If the back starting from the inner tip intersects the outer circle satisfying

$$
r_{1,-}(\xi_1; b_1) = R_2, \quad \gamma_{1,-}(\xi_1; b_1) < \gamma^*, \quad \varphi_{1,-}(\xi_1; b_1) = 0,
$$

for some positive constants b_1 and ξ_1, then the rotating spot has a tipped arc with radius R_2 for angles lying between $\gamma_{1,-}(\xi_1; b_1)$ and γ^*. Similarly, it becomes an arc-tipped rotating spot with a circular arc with radius R_1, if the solution of (2.11)-(2.12) for $s_2 \in [-\xi_2, 0]$ satisfies

$$
r_{2,-}(-\xi_2; b_2) = R_1, \quad \gamma_{2,-}(-\xi_2; b_2) < \frac{3\pi}{2}, \quad \varphi_{2,-}(-\xi_2; b_2) = 3\pi
$$

for some positive constants b_2 and ξ_2.

Remark 2.3. From the numerical simulation, we can find $(b_1, b_2, \widetilde{R}) = (1.105, 4.295, 4.6)$ and $(0.5, 4.04, 3)$ such that Proposition 2.1 and Theorem 1 hold for $\omega = 0.2$ and $R_1 = 1.305$ (see (a) and (b) in Figure 3). For $\omega = 0.1$ and $R_1 = 3$, we also observe that there are rotating spots for the different b_1, b_2 and $\widetilde{R}$ (see (c) and (d) in Figure 3). These results show us that the rotating spots are not unique even we fix the angular speed ω and core radius R_1.

3. Preliminaries

In this section, we provide some preliminaries in order to show the existence of a rotating spot in the next section. By Proposition 2.1, we have the front of a rotating spot. For convenience, from now on we ignore the subscript minus sign for the back of a rotating spot. Since all results of this section can be found in [5], we only state them (without proof) here for the reader's convenience.

First, we consider the solution $(r_1, \gamma_1, \varphi_1)(s_1) = (r_1, \gamma_1, \varphi_1)(s_1; b_1)$ of (2.9)-(2.10) for a given $b_1 \geq 0$.

Lemma 3.1 ([5, Lemma 4.2])**.** *Let $b_1 > 0$. Then the following statements hold:*

(i) *If $\varphi_1(s_*) = 0$ for some $s_* > 0$ and $0 < \varphi_1(s) < \pi$ for $s_* > s > 0$, then $\varphi_1(s) < 0$ for $s > s_*$ with $s - s_*$ small.*
(ii) *If $\varphi_1(s_*) = \pi$ for some $s_* > 0$ and $0 < \varphi_1(s) < \pi$ for $s_* > s > 0$, then $\varphi_1(s) > \pi$ for $s > s_*$ with $s - s_*$ small.*

Let $Q_1 := (R_1, R_2) \times \mathbb{R} \times (0, \pi)$. It follows from Lemma 3.1 that the following *exit-length* $S_1 = S_1(b_1)$ and the *exit-point* $(r_1^e, \gamma_1^e, \varphi_1^e)(b_1)$ are well-defined.

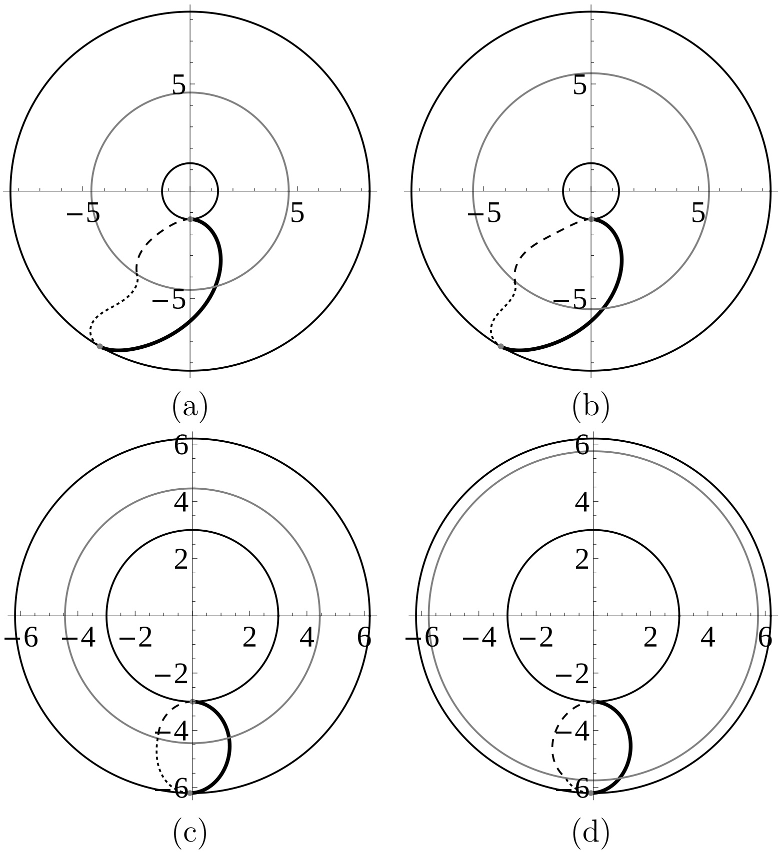

FIGURE 3. Solid circles: $r = R_1$ (inner); $r = R_2$ (outer); dotted circle: $r = \widetilde{R}$. Examples of point-tipped rotating spot for (a) $(\omega, R_1, b_1, b_2, \widetilde{R}) = (0.2, 1.305, 1.105, 4.295, 4.6)$, (b) $(\omega, R_1, b_1, b_2, \widetilde{R}) = (0.2, 1.305, 0.5, 4.04, 3)$, (c) $(\omega, R_1, b_1, b_2, \widetilde{R}) = (0.1, 3, 1.1, 2, 4.5)$, (d) $(\omega, R_1, b_1, b_2, \widetilde{R}) = (0.1, 3, 1.4, 1.7, 5.75)$.

(i) if there is a positive number $\hat{s}_1$ such that the orbit stays in Q_1 for $0 < s_1 < \hat{s}_1$ and $r_1(\hat{s}_1) = R_2$, then $S_1 = S_1(b_1) = \hat{s}_1$ and $(r_1^e, \gamma_1^e, \varphi_1^e)(b_1) = (R_2, \gamma_1(S_1), \varphi_1(S_1))$;

(ii) if there is a positive number $\overline{s}_1$ such that the orbit stays in Q_1 for $0 < s_1 < \overline{s}_1$, $\varphi_1(\tau) > \pi$ for any τ close to $\overline{s}_1$ with $\tau > \overline{s}_1$, then $S_1 = S_1(b_1) = \overline{s}_1$ and $(r_1^e, \gamma_1^e, \varphi_1^e)(b_1) = (r_1(S_1), \gamma_1(S_1), \pi)$;

(iii) if there is a positive number $\underline{s}_1$ such that the orbit stays in Q_1 for $0 < s_1 < \underline{s}_1$, $\varphi_1(\tau) < 0$ for any τ close to $\underline{s}_1$ with $\tau > \underline{s}_1$, then $S_1 = S_1(b_1) = \underline{s}_1$ and $(r_1^e, \gamma_1^e, \varphi_1^e)(b_1) = (r_1(S_1), \gamma_1(S_1), 0)$.

Note that the solution $(r_1, \gamma_1, \varphi_1)(s; b_1)$ is continuous in b_1 for $b_1 \geq 0$.

Since $r_1(s_1)$ is increasing in s_1 when the trajectory $(r_1, \gamma_1, \varphi_1)$ stays in the region Q_1, we consider $\Phi_1(r) := \varphi_1(s(r))$ and $\Gamma_1(r) := \gamma_1(s(r))$. Then (Φ_1, Γ_1) is a solution of the system

$$\begin{cases} \dfrac{d\Gamma_1}{dr} = \dfrac{\cos\Phi_1}{r\sin\Phi_1}, \\ \dfrac{d\Phi_1}{dr} = \dfrac{\cos\Phi_1}{r\sin\Phi_1} - \dfrac{1}{\sin\Phi_1} - \omega r + \dfrac{b_1(\Gamma_+(r) - \Gamma_1)}{\sin\Phi_1}. \end{cases} \tag{3.1}$$

Moreover, we have the following monotonicity property.

Lemma 3.2 ([5, Lemma 4.8]). *Let $(\Gamma_{1,j}(r), \Phi_{1,j}(r)) := (\Gamma_1(r; b_{1,j}), \Phi_1(r; b_{1,j}))$ be the solution of (3.1) with $\Gamma_{1,j}(R_1) = 3\pi/2$ and $\Phi_{1,j}(R_1) = \pi$ defined on $[R_1, r_1(S_1(b_{1,j}))]$, $j = 1, 2$. If $0 < b_{1,1} < b_{1,2}$, then*

$$\Gamma_{1,1}(r) > \Gamma_{1,2}(r), \quad \Phi_{1,1}(r) < \Phi_{1,2}(r)$$

on $(R_1, \min\{r_1(S(b_{1,1})), r_1(S(b_{1,2}))\}]$ as long as $\Gamma_+(r) > \Gamma_{1,1}(r)$.

For the solution $(r_2, \gamma_2, \varphi_2)(s_2; b_2)$ of (2.11)-(2.12) for a given $b_2 \geq 0$, we have the following property similar to Lemma 3.1.

Lemma 3.3. *Let $b_2 > 0$. Then the following statements hold:*

(i) *If $\varphi_2(s_*) = 2\pi$ for some $s_* < 0$ and $2\pi < \varphi_2(s) < 3\pi$ for $s_* < s < 0$, then $\varphi_2(s) < 2\pi$ for $s < s_*$ with $s_* - s$ small.*
(ii) *If $\varphi_2(s_*) = 3\pi$ for some $s_* < 0$ and $2\pi < \varphi_2(s) < 3\pi$ for $s_* < s < 0$, then $\varphi_2(s) > 3\pi$ for $s < s_*$ with $s_* - s$ small.*

Let $Q_2 := (R_1, R_2) \times \mathbb{R} \times (2\pi, 3\pi)$. Again, by Lemma 3.3, the following *exit-length* $S_2 = S_2(b_2)$ and the *exit-point* $(r_2^e, \gamma_2^e, \varphi_2^e)(b_2)$ are well-defined.

(i) if there is a positive number $\hat{s}_2$ such that the orbit stays in Q_2 for $-\hat{s}_2 < s_2 < 0$ and $r_2(-\hat{s}_2) = R_1$, then $S_2 = S_2(b_2) = \hat{s}_2$ and $(r_2^e, \gamma_2^e, \varphi_2^e)(b_2) = (R_1, \gamma_2(-S_2), \varphi_2(-S_2))$;
(ii) if there is a positive number $\overline{s}_2$ such that the orbit stays in Q_2 for $-\overline{s}_2 < s_2 < 0$, $\varphi_2(-\tau) < 2\pi$ for any τ close to $\overline{s}_2$ with $\tau > \overline{s}_2$, then $S_2 = S_2(b_2) = \overline{s}_2$ and $(r_2^e, \gamma_2^e, \varphi_2^e)(b_2) = (r_2(-S_2), \gamma_2(-S_2), 2\pi)$;
(iii) if there is a positive number $\underline{s}_2$ such that the orbit stays in Q_2 for $-\underline{s}_2 < s_2 < 0$, $\varphi_2(-\tau) > 3\pi$ for any τ close to $\underline{s}_2$ with $\tau > \underline{s}_2$, then $S_2 = S_2(b_2) = \underline{s}_2$ and $(r_2^e, \gamma_2^e, \varphi_2^e)(b_2) = (r_2(-S_2), \gamma_2(-S_2), 3\pi)$.

Moreover, the solution $(r_2, \gamma_2, \varphi_2)(s; b_2)$ is continuous in b_2 for $b_2 \geq 0$.

Also, we can regard the orbit as a function of r instead of s. Namely, let $\Phi_2(r) := \varphi_2(s(r))$ and $\Gamma_2(r) := \gamma_2(s(r))$. Then (Φ_2, Γ_2) is a solution of the system

$$\begin{cases} \dfrac{d\Gamma_2}{dr} = \dfrac{\cos\Phi_2}{r\sin\Phi_2}, \\ \dfrac{d\Phi_2}{dr} = \dfrac{\cos\Phi_2}{r\sin\Phi_2} - \dfrac{1}{\sin\Phi_2} - \omega r + \dfrac{b_2(\Gamma_+(r) - \Gamma_2)}{\sin\Phi_2}. \end{cases} \tag{3.2}$$

Furthermore, we have the following monotonicity property as [5, Lemma 4.8].

Lemma 3.4. *Let* $(\Gamma_{2,j}(r), \Phi_{2,j}(r)) := (\Gamma_2(r; b_{2,j}), \Phi_2(r; b_{2,j}))$ *be the solution of* (3.2) *with* $\Gamma_{2,j}(R_2) = \gamma^*$ *and* $\Phi_{2,j}(R_2) = 2\pi$ *defined on* $[r_2(-S_2(b_{2,j})), R_2]$, $j = 1, 2$. *If* $0 < b_{2,1} < b_{2,2}$, *then*

$$\Gamma_{2,1}(r) > \Gamma_{2,2}(r), \quad \Phi_{2,1}(r) > \Phi_{2,2}(r)$$

on $[\max\{r_2(-S(b_{2,1})), r_2(-S(b_{2,2}))\}, R_2)$ *as long as* $\Gamma_+(r) > \Gamma_{2,1}(r)$.

4. Proof of the main theorem

This section is devoted to the proof of Theorem 1. For the reader's convenience, we provide a rough guideline of the proof here. First, we determine the constant R_* in Theorem 1 to guarantee no intersection of the extended trajectories of the front from two tips. This is shown in Lemma 2.2. Next, for shooting we need to find two solution trajectories of the back meeting together smoothly. For this, using the ideas from [1], we first define two sets (B_1 and B_2 as below) such that the corresponding solution trajectories of the supremum of these two sets have "convex-intersection" in the sense (4.3) stated in Proposition 4.4 (see Figure 4). This also ensures that the set B (defined below) is not empty. However, the connection of these trajectories is not smooth. In order to obtain a smooth solution of the back, we then prepare Lemma 4.6 which plays a key role in the proof of Theorem 1. Namely, the existence of trajectories with "concave-intersection" implies that of a smooth solution. Since it is difficult to show the existence of such trajectories directly, we introduce the set B which consists of trajectories with "convex-intersection". Finally, we derive some properties of B in Proposition 4.7, from which Theorem 1 can be readily proved.

Recall from Lemma 2.2 that there is no $r \in [R_1, r_1^e(0)] \cap [r_2^e(0), R_2]$ such that $\Gamma_1(r; 0) = \Gamma_2(r; 0)$ when $R_1 \in [R_*, R^*)$, since $r_1^e(0) = R_0$ and $r_2^e(0) = R_3$.

For a fix $\omega \in (0, \omega_*)$ and $R_1 \in [R_*(\omega), R^*(\omega))$, we define the following sets:

$$B_1 := \left\{ b_1 \geq 0 \;\middle|\; \begin{array}{l} \text{There exists a constant } R_p(b_1) \in (R_1, r_1^e(b_1)) \text{ such that} \\ \Phi_1(R_p(b_1); b_1) = \Phi_+(R_p(b_1)) - \pi \text{ and} \\ \Phi_+(r) - \pi < \Phi_1(r; b_1) < \pi \text{ for any } r \in (R_1, R_p(b_1)) \end{array} \right\},$$

$$B_2 := \left\{ b_2 \geq 0 \;\middle|\; \begin{array}{l} \text{There exists a constant } R_p(b_2) \in (r_2^e(b_2), R_2) \text{ such that} \\ \Phi_2(R_p(b_2); b_2) = \Phi_+(R_p(b_2)) + \pi \text{ and} \\ \Phi_+(r) + \pi > \Phi_2(r; b_2) > 2\pi \text{ for any } r \in (R_p(b_2), R_2) \end{array} \right\},$$

where $\Phi_+(r) := \varphi_+(s(r))$.

By a similar argument as in [1, Lemma 3.5, Proposition 3.7], we obtain the following properties for the sets B_1 and B_2.

Proposition 4.1. *If* $b_1 \in B_1$, *then we have*

$$\frac{d(\Gamma_+ - \Gamma_1)}{dr}(r) > 0 \quad \textit{for any } r \in (R_1, R_p(b_1))$$

and

$$\frac{d(\Gamma_+ - \Gamma_1)}{dr}(r) \le 0, \quad (\Phi_+ - \Phi_1)(r) > \pi \quad \text{for any } r \in [R_p(b_1), r_1^e(b_1)].$$

If $b_2 \in B_2$, then we have

$$\frac{d(\Gamma_+ - \Gamma_2)}{dr}(r) < 0 \quad \text{for any } r \in (R_p(b_2), R_2)$$

and

$$\frac{d(\Gamma_+ - \Gamma_2)}{dr}(r) \ge 0, \quad (\Phi_2 - \Phi_+)(r) > \pi \quad \text{for any } r \in [r_2^e(b_2), R_p(b_2)].$$

Proposition 4.2. *There are positive constants $\overline{b_1}$ and $\overline{b_2}$ such that $B_1 = [0, \overline{b_1})$ and $B_2 = [0, \overline{b_2})$.*

By [1, Proposition 4.1], we also have

$$r_1^e(\overline{b_1}) = R_2, \quad r_2^e(\overline{b_2}) = R_1. \tag{4.1}$$

Moreover, we have the next proposition.

Proposition 4.3. *The following hold:*

(i) *For any $b_1 \in B_1$, we have either $r_1^e(b_1) < R_2$ with $\varphi_1^e(b_1) = 0$ or $r_1^e(b_1) = R_2$ with $0 \le \varphi_1^e(b_1) < \pi$.*

(ii) *For any $b_2 \in B_2$, we have either $r_2^e(b_2) > R_1$ with $\varphi_2^e(b_2) = 3\pi$ or $r_2^e(b_2) = R_1$ with $2\pi < \varphi_2^e(b_2) \le 3\pi$.*

Proof. For the case (i), since $b_1 \in B_1$, we have $\Phi_+(r_1^e(b_1)) - \Phi_1(r_1^e(b_1); b_1) > \pi$. If $r_1^e(b_1) = R_2$, we have $\varphi_1^e(b_1) < \Phi_+(R_2) - \pi = \pi$. Otherwise, by the definition of $r_1^e(b_1)$, we know that $r_1^e(b_1) < R_2$ with $\varphi_1^e(b_1) = 0$. For the case (ii), $\Phi_2(r_2^e(b_2); b_2) - \Phi_+(r_2^e(b_2)) > \pi$ holds for $b_2 \in B_2$. Thus, $\Phi_2(r_2^e(b_2); b_2) > \Phi_+(r_2^e(b_2)) + \pi = 2\pi$ if $r_2^e(b_2) = R_1$. If $r_2^e(b_2) > R_1$, it follows from the definition of $r_2^e(b_2)$ that $\varphi_2^e(b_2) = 3\pi$. □

Recall that $\overline{b_1} = \sup B_1$, $\overline{b_2} = \sup B_2$. Now, we consider the corresponding solutions of (3.1) and (3.2), which are denoted by

$$(\overline{\Gamma_1}(r), \overline{\Phi_1}(r)) := (\Gamma_1(r; \overline{b_1}), \Phi_1(r; \overline{b_1})), \quad (\overline{\Gamma_2}(r), \overline{\Phi_2}(r)) := (\Gamma_2(r; \overline{b_2}), \Phi_2(r; \overline{b_2})),$$

respectively. Then, by (4.1), we have

$$0 < (\Phi_+ - \overline{\Phi_1})(r) < \pi, \quad 0 < (\overline{\Phi_2} - \Phi_+)(r) < \pi \quad \text{for all } r \in (R_1, R_2) \tag{4.2}$$

and

$$(\Phi_+ - \overline{\Phi_1})(R_2) = \pi, \quad (\overline{\Phi_2} - \Phi_+)(R_1) = \pi.$$

Moreover, we obtain the next proposition.

Proposition 4.4. *There exists a positive constant $\overline{R} \in (R_1, R_2)$ such that*

$$\overline{\Gamma_1}(\overline{R}) = \overline{\Gamma_2}(\overline{R}) \quad \text{and} \quad \overline{\Phi_2}(\overline{R}) - \overline{\Phi_1}(\overline{R}) < 2\pi \tag{4.3}$$

Moreover, we have $\overline{\Gamma_1}(r) > \overline{\Gamma_2}(r)$ for $r \in [R_1, \overline{R})$ and $\overline{\Gamma_1}(r) < \overline{\Gamma_2}(r)$ for $r \in (\overline{R}, R_2]$.

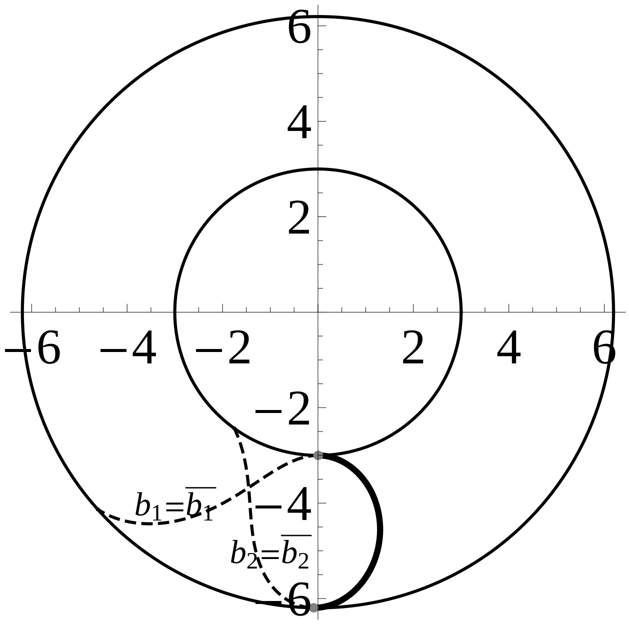

FIGURE 4. The solutions of (3.1)-(3.2) for $b_1 = \overline{b_1}$ and $b_2 = \overline{b_2}$, respectively, with $\omega = 0.1$ and $R_1 = 3$.

Proof. From (4.2), it is easy to see that $(\Gamma_+ - \overline{\Gamma_1})(r) > 0$ for all $r \in (R_1, R_2]$ and $(\Gamma_+ - \overline{\Gamma_2})(r) > 0$ for all $r \in [R_1, R_2)$. Then we obtain that

$$\overline{\Gamma_2}(R_1) < \Gamma_+(R_1) = \overline{\Gamma_1}(R_1) \quad \text{and} \quad \overline{\Gamma_2}(R_2) = \Gamma_+(R_2) > \overline{\Gamma_1}(R_2).$$

Therefore, there is a positive constant $\overline{R} \in (R_1, R_2)$ such that $\overline{\Gamma_1}(\overline{R}) = \overline{\Gamma_2}(\overline{R})$. By (4.2), we also derive that $\overline{\Phi_2}(\overline{R}) - \overline{\Phi_1}(\overline{R}) < 2\pi$.

Since $\Phi_+(r) - \overline{\Phi_1}(r) < \pi$ for all $r \in [R_1, R_2)$ and $\overline{\Phi_2}(r) - \Phi_+(r) < \pi$ for any $r \in (R_1, R_2]$, $\Gamma_+(r) - \overline{\Gamma_1}(r)$ is increasing on $[R_1, R_2)$ and $\Gamma_+(r) - \overline{\Gamma_2}(r)$ is decreasing on $(R_1, R_2]$. Hence

$$\begin{aligned}\overline{\Gamma_1}(r) - \overline{\Gamma_2}(r) &= [\overline{\Gamma_1}(r) - \Gamma_+(r)] + [\Gamma_+(r) - \overline{\Gamma_2}(r)] \\ &> [\overline{\Gamma_1}(\overline{R}) - \Gamma_+(\overline{R})] + [\Gamma_+(\overline{R}) - \overline{\Gamma_2}(\overline{R})] = 0\end{aligned}$$

for $r \in [R_1, \overline{R})$ and

$$\begin{aligned}\overline{\Gamma_1}(r) - \overline{\Gamma_2}(r) &= [\overline{\Gamma_1}(r) - \Gamma_+(r)] + [\Gamma_+(r) - \overline{\Gamma_2}(r)] \\ &< [\overline{\Gamma_1}(\overline{R}) - \Gamma_+(\overline{R})] + [\Gamma_+(\overline{R}) - \overline{\Gamma_2}(\overline{R})] = 0\end{aligned}$$

for $r \in (\overline{R}, R_2]$. The proof of the proposition is completed. □

Motivated by the proof of [5, Lemma 4.9], we obtain the following result.

Lemma 4.5. *The following hold:*

(i) *For any given $R \in (R_1, R_2)$, there exists a unique $b_1^u(R) > \overline{b_1}$ such that $\Phi_1(R; b_1^u(R)) = \pi$.*

(ii) *For any given* $R \in (R_1, R_2)$*, there exists a unique* $b_2^u(R) > \overline{b_2}$ *such that* $\Phi_2(R; b_2^u(R)) = 2\pi$*.*

Proof. Since the proofs of statements (i) and (ii) are similar, we only give the proof of the statement (i).

Let $R \in (R_1, R_2)$ be fixed. By Lemma 3.2, Proposition 4.3 (i) and the definition of $\overline{b_1}$, if there is a $b_1^u(R)$ such that the corresponding solution of (3.1) satisfies $\Phi_1(R; b_1^u(R)) = \pi$ for $R \in (R_1, R_2)$, then $b_1^u(R) > \overline{b_1}$ and it is unique.

Next, for this given $R \in (R_1, R_2)$, we let

$$b_1^*(R) := \max\left\{\overline{b_1}, \frac{\dfrac{4(\pi-\varphi_0)}{R} + \dfrac{1}{R_1} + 1 + \omega R}{\Gamma_+(R/2) - \gamma_0}\right\},$$

where

$$\varphi_0 := \Phi_1\left(\frac{R}{2}; \overline{b_1}\right), \quad \gamma_0 := \Gamma_1\left(\frac{R}{2}; \overline{b_1}\right).$$

Then we shall claim that

$$r_1^e(b_1) < R \quad \text{and} \quad \varphi_1^e(b_1) = \pi \quad \text{for} \ \ b_1 \geq b_1^*(R). \tag{4.4}$$

For this, by the first inequality in (4.2), we have $\Gamma_+(r) - \Gamma_1(r; \overline{b_1}) > 0$ for $r \in (R_1, R_2]$. Then we apply Lemma 3.2 and derive that

$$\Gamma_1(r; \overline{b_1}) \geq \Gamma_1(r; b_1) \quad \text{and} \quad \Phi_1(r; \overline{b_1}) \leq \Phi(r; b_1) \tag{4.5}$$

for any $b_1 \geq \overline{b_1}$ and $r \in [R_1, \min\{R_2, r_1^e(b_1)\}]$.

Now, we divide our discussion into two cases: $r_1^e(b_1) \leq R/2$ and $r_1^e(b_1) > R/2$. If $r_1^e(b_1) \leq R/2 < R < R_2$, then by (4.5) we have

$$\Phi_1(r_1^e(b_1); b_1) \geq \Phi_1(r_1^e(b_1); \overline{b_1}) > 0$$

which implies that $\Phi_1(r_1^e(b_1); b_1) = \pi$ by the definition of $r_1^e(b_1)$. Hence (4.4) holds. On the other hand, if $r_1^e(b_1) > R/2$, then by (4.5) we have

$$\Gamma_+(r) - \Gamma_1(r; b_1) > \Gamma_+(R/2) - \Gamma_1(R/2; b_1) \geq \Gamma_+(R/2) - \gamma_0$$

for $r > R/2$. Moreover, we compute that

$$\begin{aligned}
\frac{d\Phi_1}{dr} &= \frac{1}{\sin\Phi_1}\Big[\frac{\cos\Phi_1}{r} - 1 - r\omega\sin\Phi_1 + b_1(\Gamma_+(r) - \Gamma_1(r))\Big] \\
&\geq \frac{1}{\sin\Phi_1}\Big[\frac{\cos\Phi_1}{r} - 1 - r\omega\sin\Phi_1 + \frac{\frac{4(\pi-\varphi_0)}{R} + \frac{1}{R_1} + 1 + \omega R}{\Gamma_+(R/2) - \gamma_0}(\Gamma_+(r) - \Gamma_1(r))\Big] \\
&\geq \frac{1}{\sin\Phi_1}\Big[\frac{4(\pi-\varphi_0)}{R}\Big] \geq \frac{4(\pi-\varphi_0)}{R} > 0
\end{aligned}$$

for $b_1 \geq b_1^*(R)$ and $R/2 < r < R$. Hence $\Phi_1(r; b_1)$ reaches to π for some $r \in (R/2, 3R/4)$, which implies that $r_1^e(b_1) < 3R/4 < R$. Hence (4.4) holds.

Finally, the statement (i) follows from the continuous dependence on b_1, (4.4) and $(r_1^e, \varphi_1^e)(\overline{b_1}) = (R_2, \pi)$. Therefore, the proof of this lemma has been completed. $\square$

Now we give a key lemma of this paper as below.

Lemma 4.6. *Suppose that there exist $\widetilde{b_1} \geq 0$, $\widetilde{b_2} \geq 0$ and $\widetilde{R} \in (R_1, R_2)$ such that*

$$\Gamma_1(\widetilde{R}; \widetilde{b_1}) = \Gamma_2(\widetilde{R}; \widetilde{b_2}), \quad \Phi_2(\widetilde{R}; \widetilde{b_2}) - \Phi_1(\widetilde{R}; \widetilde{b_1}) \geq 2\pi, \tag{4.6}$$

Then a point-tipped rotating spot exists.

Proof. It is clear that a point-tipped rotating spot exists when $\Phi_2(\widetilde{R}; \widetilde{b_2}) - \Phi_1(\widetilde{R}; \widetilde{b_1}) = 2\pi$ holds in (4.6) (in this case, $\tilde{R}$ can be in $[R_1, R_2]$). Hence we may assume without loss of generality that $\Phi_2(\widetilde{R}; \widetilde{b_2}) - \Phi_1(\widetilde{R}; \widetilde{b_1}) > 2\pi$ in (4.6) for some $\widetilde{R} \in (R_1, R_2)$. Then we have either

$$\Phi_+(\widetilde{R}) - \Phi_1(\widetilde{R}; \widetilde{b_1}) > \pi \tag{4.7}$$

or

$$\Phi_2(\widetilde{R}; \widetilde{b_2}) - \Phi_+(\widetilde{R}) > \pi. \tag{4.8}$$

Suppose that (4.7) holds. Then $\widetilde{b_1}$ belongs to B_1, which implies that $\widetilde{b_1} < \overline{b_1}$ and $R_p(\widetilde{b_1}) < \widetilde{R}$. Lemma 3.2 implies that $\Gamma_1(r; \widetilde{b_1}) > \Gamma_1(r; \overline{b_1})$ for $r \in (R_1, R_p(\widetilde{b_1})]$.

By Proposition 4.1 and $\overline{b_1} \notin B_1$, we have $\Phi_+(r) - \Phi_1(r; \widetilde{b_1}) > \pi$ and $\Phi_+(r) - \Phi_1(r; \overline{b_1}) < \pi$ for $r \in (R_p(\widetilde{b_1}), r_1^e(\widetilde{b_1})]$. This implies that

$$\begin{aligned}
\Gamma_+(r) - \Gamma_1(r; \overline{b_1}) &> \Gamma_+(R_p(\widetilde{b_1})) - \Gamma_1(R_p(\widetilde{b_1}); \overline{b_1}) \\
&> \Gamma_+(R_p(\widetilde{b_1})) - \Gamma_1(R_p(\widetilde{b_1}); \widetilde{b_1}) > \Gamma_+(r) - \Gamma_1(r; \widetilde{b_1})
\end{aligned}$$

for $r \in (R_p(\widetilde{b_1}), r_1^e(\widetilde{b_1}))$. Therefore, we obtain that

$$\Gamma_1(r; \widetilde{b_1}) > \Gamma_1(r; \overline{b_1}) \quad \text{for any } r \in (R_1, r_1^e(\widetilde{b_1})). \tag{4.9}$$

Now, we show that a point-tipped rotating spot exists under the assumption of the lemma.

Case 1. $\widetilde{R} = \overline{R}$, where $\overline{R}$ defined as in Proposition 4.4. We have $\widetilde{b_2} < \overline{b_2}$ by Lemma 3.4 and

$$\Gamma_2(\overline{R}; \overline{b_2}) = \Gamma_1(\overline{R}; \overline{b_1}) < \Gamma_1(\widetilde{R}; \widetilde{b_1}) = \Gamma_2(\widetilde{R}; \widetilde{b_2}).$$

Hence it follows from the continuous dependence on both b_1 and b_2, Lemmas 3.2 and 3.4 that there exist $b_1 \in (\widetilde{b_1}, \overline{b_1})$ and $b_2 \in (\widetilde{b_2}, \overline{b_2})$ such that

$$\Gamma_1(\widetilde{R}; b_1) = \Gamma_2(\widetilde{R}; b_2), \quad \Phi_2(\widetilde{R}; b_2) - \Phi_1(\widetilde{R}; b_1) = 2\pi. \tag{4.10}$$

Indeed, for a given $b_1 \in (\widetilde{b_1}, \overline{b_1})$, it follows from Lemma 3.2 that

$$\Gamma_1(\widetilde{R}; \overline{b_1}) < \Gamma_1(\widetilde{R}; b_1) < \Gamma_1(\widetilde{R}; \widetilde{b_1}).$$

Also, by Lemma 3.4, we have

$$\Gamma_1(\widetilde{R};\overline{b_1}) = \Gamma_2(\widetilde{R};\overline{b_2}) < \Gamma_2(\widetilde{R};b_2) < \Gamma_2(\widetilde{R};\widetilde{b_2}) = \Gamma_1(\widetilde{R};\widetilde{b_1})$$

for $b_2 \in (\widetilde{b_2},\overline{b_2})$. The continuous dependence on b_2 implies that there exists a unique $b_2 = b_2(b_1)$ such that $\Gamma_1(\widetilde{R};b_1) = \Gamma_2(\widetilde{R};b_2)$. Note that $b_2(\overline{b_1}) = \overline{b_2}$ and $b_2(\widetilde{b_1}) = \widetilde{b_2}$. It follows from

$$\Phi_2(\widetilde{R};b_2(\widetilde{b_1})) - \Phi_1(\widetilde{R};\widetilde{b_1}) > 2\pi, \quad \Phi_2(\widetilde{R};b_2(\overline{b_1})) - \Phi_1(\widetilde{R};\overline{b_1}) < 2\pi,$$

and the continuous dependence on b_1 that (4.10) holds for some $b_1 \in (\widetilde{b_1},\overline{b_1})$ and $b_2 = b_2(b_1) \in (\widetilde{b_2},\overline{b_2})$. Hence a point-tipped rotating spot exists.

Case 2. $\widetilde{R} < \overline{R}$. In this case, it suffices to show that there exist b_1 and b_2 such that

$$\Gamma_1(\widetilde{R};b_1) = \Gamma_2(\widetilde{R};b_2) \quad \text{and} \quad \Phi_2(\widetilde{R};b_2) - \Phi_1(\widetilde{R};b_1) \leq 2\pi. \tag{4.11}$$

Then we are done, if $\Phi_2(\widetilde{R};b_2) - \Phi_1(\widetilde{R};b_1) = 2\pi$. Otherwise, it reduces to Case 1.

For this, we first claim that $\widetilde{b_2} < \overline{b_2}$. By the contradiction argument, we assume that $\widetilde{b_2} \geq \overline{b_2}$. Since $\Phi_2(\widetilde{R};\widetilde{b_2}) - \Phi_1(\widetilde{R};\widetilde{b_1}) > 2\pi$, we have $d(\Gamma_1 - \Gamma_2)(\widetilde{R})/dr > 0$. This implies that $\Gamma_1(\widetilde{R}-\eta;\widetilde{b_1}) < \Gamma_2(\widetilde{R}-\eta;\widetilde{b_2})$ for $\eta > 0$ sufficiently small. By Lemmas 3.2 and 3.4 with $\widetilde{b_1} < \overline{b_1}$ and $\widetilde{b_2} \geq \overline{b_2}$, we obtain that

$$\Gamma_1(\widetilde{R}-\eta;\overline{b_1}) < \Gamma_1(\widetilde{R}-\eta;\widetilde{b_1}) < \Gamma_2(\widetilde{R}-\eta;\widetilde{b_2}) \leq \Gamma_2(\widetilde{R}-\eta;\overline{b_2}).$$

Then it follows from Proposition 4.4 that $\widetilde{R} > \widetilde{R}-\eta > \overline{R}$, a contradiction. Hence $\widetilde{b_2} < \overline{b_2}$, and we have $\Gamma_2(r;\widetilde{b_2}) > \Gamma_2(r;\overline{b_2})$ for any $r \in [r_2^e(\widetilde{b_2}), R_2)$, by the similar argument as that for (4.9).

By using Proposition 4.4, we know that

$$\Gamma_2(\widetilde{R};\overline{b_2}) < \Gamma_1(\widetilde{R};\overline{b_1}) < \Gamma_1(\widetilde{R};\widetilde{b_1}) = \Gamma_2(\widetilde{R};\widetilde{b_2}). \tag{4.12}$$

By the inequality (4.12) and the continuous dependence on b_2, there is a unique $b_2' \in (\widetilde{b_2},\overline{b_2})$ such that $\Gamma_1(\widetilde{R};\overline{b_1}) = \Gamma_2(\widetilde{R};b_2')$. If $\Phi_2(\widetilde{R};b_2') - \Phi_1(\widetilde{R};\overline{b_1}) \leq 2\pi$, then we are done.

Otherwise, we assume that $\Phi_2(\widetilde{R};b_2') - \Phi_1(\widetilde{R};\overline{b_1}) > 2\pi$. By Lemma 4.5 (i), there exist a $b_1^u(\widetilde{R}) > \overline{b_1}$ such that $\Phi_1(\widetilde{R};b_1^u(\widetilde{R})) = \pi$. If $\Gamma_1(\widetilde{R};b_1^u(\widetilde{R})) \leq \Gamma_2(\widetilde{R};\overline{b_2})$, then there exist a $b_1^* \in (\overline{b_1}, b_1^u(\widetilde{R})]$ such that

$$\Gamma_1(\widetilde{R};b_1^*) = \Gamma_2(\widetilde{R};\overline{b_2})$$

by the continuous dependence on b_1, Lemma 3.2 and $\Gamma_2(\widetilde{R};\overline{b_2}) < \Gamma_1(\widetilde{R};\overline{b_1})$. Since $b_1^* > \overline{b_1}$, we know that

$$\Phi_+(\widetilde{R}) - \Phi_1(\widetilde{R};b_1^*) < \Phi_+(\widetilde{R}) - \Phi_1(\widetilde{R};\overline{b_1}) < \pi$$

Also, we have $\Phi_2(\widetilde{R};\overline{b_2}) - \Phi_+(\widetilde{R}) < \pi$. Hence we obtain that $\Phi_2(\widetilde{R};\overline{b_2}) - \Phi_1(\widetilde{R};b_1^*) < 2\pi$.

On the other hand, when $\Gamma_1(\widetilde{R};b_1^u(\widetilde{R})) > \Gamma_2(\widetilde{R};\overline{b_2})$, by the continuous dependence on b_2 and the inequality $\Gamma_1(\widetilde{R};b_1^u(\widetilde{R})) < \Gamma_1(\widetilde{R};\overline{b_1}) = \Gamma_2(\widetilde{R};b_2')$, there exists a $b_2^* \in (b_2',\overline{b_2})$ such that

$$\Gamma_1(\widetilde{R};b_1^u(\widetilde{R})) = \Gamma_2(\widetilde{R};b_2^*).$$

Moreover, we obtain that

$$\Phi_2(\widetilde{R};b_2^*)-\Phi_1(\widetilde{R};b_1^u(\widetilde{R}))=\Phi_2(\widetilde{R};b_2^*)-\pi\le 2\pi$$

which implies that (4.11) holds.

Case 3. $\widetilde{R}>\overline{R}$. We divide this case into two subcases: $\widetilde{b_2}<\overline{b_2}$ and $\widetilde{b_2}\ge\overline{b_2}$.

For the case $\widetilde{b_2}<\overline{b_2}$, by Proposition 4.4 and Lemma 3.4, we know that

$$\Gamma_1(\widetilde{R};\overline{b_1})<\Gamma_2(\widetilde{R};\overline{b_2})<\Gamma_2(\widetilde{R};\widetilde{b_2})=\Gamma_1(\widetilde{R};\widetilde{b_1}). \tag{4.13}$$

Then there is a unique $b_1'\in(\widetilde{b_1},\overline{b_1})$ such that

$$\Gamma_1(\widetilde{R};b_1')=\Gamma_2(\widetilde{R};\overline{b_2}),$$

by (4.13) and the continuous dependence on b_1. If $\Phi_2(\widetilde{R};\overline{b_2})-\Phi_1(\widetilde{R};b_1')\le 2\pi$, then we are done. Otherwise, we suppose that $\Phi_2(\widetilde{R};\overline{b_2})-\Phi_1(\widetilde{R};b_1')>2\pi$. By Lemma 4.5 (ii), there is a $b_2^u(\widetilde{R})$ such that $\Phi_2(\widetilde{R};b_2^u(\widetilde{R}))=2\pi$. Using the similar argument as that for the case $\widetilde{R}<\overline{R}$, there exist a $b_2^*\in(\overline{b_2},b_2^u(\widetilde{R}))$ such that (4.11) holds, if $\Gamma_2(\widetilde{R};b_2^u(\widetilde{R}))\le\Gamma_1(\widetilde{R};\overline{b_1})$, and there exists a $b_1^*\in(b_1',\overline{b_1})$ such that (4.11) holds, if $\Gamma_2(\widetilde{R};b_2^u(\widetilde{R}))>\Gamma_1(\widetilde{R};\overline{b_1})$.

For the other case $\widetilde{b_2}\ge\overline{b_2}$, it is easy to see that $\Phi_2(\widetilde{R};\widetilde{b_2})>2\pi+\Phi_1(\widetilde{R};\widetilde{b_1})\ge 2\pi$. By Lemma 4.5 (ii) and Lemma 3.4, there exists a $b_2^u(\widetilde{R})>\widetilde{b_2}$ with $\Phi_2(\widetilde{R};b_2^u(\widetilde{R}))=2\pi$. Then (4.11) holds by the similar argument as the case $\widetilde{R}>\overline{R}$ and $\widetilde{b_2}<\overline{b_2}$.

Therefore, the existence of a point-tipped rotating spot is proved, if (4.7) holds. Similar argument can be applied, if (4.8) holds. Therefore, the proof of this lemma is completed. □

In the sequel, for the notational convenience, we shall denote

$$\Delta\Gamma(r;b_1,b_2):=\Gamma_2(r;b_2)-\Gamma_1(r;b_1),\quad \Delta\Phi(r;b_1,b_2):=\Phi_2(r;b_2)-\Phi_1(r;b_1).$$

Now we define the set B as follows.

$$\begin{aligned} B\ :=\ &\{(b_1,b_2)\in[0,\infty)\times[0,\infty)\mid \text{there exists a unique } R_I(b_1,b_2)\in(R_1,R_2)\\ &\text{such that }\ \Delta\Gamma(R_I;b_1,b_2)=0,\ \Delta\Phi(R_I;b_1,b_2)<2\pi\}.\end{aligned}$$

Note that $(b_1,b_2)\in B$ implies that $(R_1,r_1^e(b_1)]\cap[r_2^e(b_2),R_2)\neq\emptyset$ implicitly. Also, B is nonempty, since $(\overline{b_1},\overline{b_2})\in B$ by Proposition 4.4.

We give some properties of the set B.

Proposition 4.7. *Let $(b_1,b_2)\in B$ and let $R_I=R_I(b_1,b_2)$. Then we have*

$$\Gamma_2(R_I;b_2)=\Gamma_1(R_I;b_1)<\Gamma_+(R_I). \tag{4.14}$$

Moreover, if $b_1>0$ (resp. $b_2>0$), then for $\varepsilon>0$ small enough there exists $\delta>0$ such that

$$\Gamma_1(R_I+\delta;b_1-\varepsilon)=\Gamma_2(R_I+\delta;b_2), \tag{4.15}$$

(resp. $\Gamma_1(R_I-\delta;b_1)=\Gamma_2(R_I-\delta;b_2-\varepsilon)$).

Proof. First, we give the proof of (4.14). For contradiction, we assume that

$$\Gamma_2(R_I; b_2) = \Gamma_1(R_I; b_1) \geq \Gamma_+(R_I). \tag{4.16}$$

Then we claim that $b_1 \in B_1$ and $b_2 \in B_2$. If $b_1 \notin B_1$, we have $\Gamma_1(r) < \Gamma_+(r)$ for all $r \in (R_1, r_1^e(b_1)]$. This contradicts (4.16). On the other hand, we also get a contradiction by $\Gamma_2(r) < \Gamma_+(r)$ for all $r \in [r_2^e(b_2), R_2)$ if $b_2 \notin B_2$. Hence $b_1 \in B_1$ and $b_2 \in B_2$.

Moreover, we have $\Gamma_+(r) > \Gamma_1(r; b_1)$ for any $r \in (R_1, R_p(b_1))$ and $\Gamma_+(r) > \Gamma_2(r; b_2)$ for any $r \in (R_p(b_2), R_2)$ which implies that $R_p(b_1) < R_I < R_p(b_2)$. Hence we obtain that

$$\begin{aligned}
& \Phi_2(R_I; b_2) - \Phi_1(R_I; b_1) \\
= \; & [\Phi_2(R_I; b_2) - \Phi_+(R_I)] + [\Phi_+(R_I) - \Phi_1(R_I; b_1)] \\
> \; & [\Phi_2(R_p(b_2); b_2) - \Phi_+(R_p(b_2))] + [\Phi_+(R_p(b_1)) - \Phi_1(R_p(b_1); b_1)] = 2\pi.
\end{aligned}$$

which contradicts $(b_1, b_2) \in B$. Thus (4.14) is proved.

Next, we prove the second statement. Given a pair $(b_1, b_2) \in B$ with $b_1 > 0$. By (4.14), applying Lemma 3.2 and the continuous dependence on b_1, we derive that

$$\Gamma_1(R_I; b_1 - \varepsilon) > \Gamma_1(R_I; b_1) = \Gamma_2(R_I; b_2) \tag{4.17}$$

with $\varepsilon > 0$ sufficiently small. On the other hand, at $R = R_I$ we have

$$\frac{d(\Gamma_2(R; b_2) - \Gamma_1(R; b_1))}{dr} = \frac{-\sin(\Phi_2(R; b_2) - \Phi_1(R; b_1))}{R \sin \Phi_1(R; b_2) \sin \Phi_2(R; b_2)} > 0 \tag{4.18}$$

by using $\Phi_1 \in (0, \pi)$, $\Phi_2 \in (2\pi, 3\pi)$ and $\pi < \Phi_2(R_I; b_2) - \Phi_1(R_I; b_1) < 2\pi$. Hence

$$\Gamma_1(R_I + \eta; b_1) < \Gamma_2(R_I + \eta; b_2)$$

with $\eta > 0$ sufficiently small. By the continuous dependence on b_1, we also have

$$\Gamma_1(R_I + \eta; b_1 - \varepsilon) < \Gamma_2(R_I + \eta; b_2) \tag{4.19}$$

with $\varepsilon > 0$ sufficiently small. Thus, by (4.17) and (4.19), there exists a $\delta \in (0, \eta)$ such that (4.15) holds. The case for b_2 can be treated similarly. Therefore, the proposition is proved. □

Proof of Theorem 1. We divide our discussion into three cases: (i) $\overline{R} \in [R_0, R_3]$, (ii) $\overline{R} < R_0$ and (iii) $\overline{R} > R_3$.

Case (i). $\overline{R} \in [R_0, R_3]$. In this case, we define $\underline{b_1} = \inf\{b_1 > 0 \mid (b_1, \overline{b_2}) \in B\}$. Then it follows from the continuous dependence on the parameters that there exists a $\hat{R} \in [R_1, R_2]$ such that $\Delta\Gamma(\hat{R}; \underline{b_1}, \overline{b_2}) = 0$ and $\Delta\Phi(\hat{R}; \underline{b_1}, \overline{b_2}) \leq 2\pi$. Due to Proposition 4.4, we have

$$\Gamma_2(R_1; \overline{b_2}) < \Gamma_1(R_1; \overline{b_1}) = \Gamma_1(R_1; \underline{b_1}).$$

Hence $\hat{R} \in (R_1, R_2]$. If $\Delta\Phi(\hat{R}; \underline{b_1}, \overline{b_2}) = 2\pi$, then we have found a solution. Hence we may assume that $\Delta\Phi(\hat{R}; \underline{b_1}, \overline{b_2}) < 2\pi$.

Since $\overline{R} \geq R_0$, by Proposition 4.4 and Lemma 3.2, we have

$$\Gamma_2(r; \overline{b_2}) \leq \Gamma_1(r; \overline{b_1}) < \Gamma_1(r; 0) \quad \text{for} \quad r \in (R_1, R_0].$$

Thus $\underline{b_1} > 0$. We claim that $(\underline{b_1}, \overline{b_2}) \notin B$. Otherwise, if $(\underline{b_1}, \overline{b_2}) \in B$, then $\hat{R} \in (R_1, R_2)$ and by Proposition 4.7 there exist $\varepsilon, \delta > 0$ such that $\Delta\Gamma(\hat{R}+\delta; \underline{b_1}-\varepsilon, \overline{b_2}) = 0$. By choosing ε, δ small enough, we also have $\hat{R}+\delta \in (R_1, R_2)$ and $\Delta\Phi(\hat{R}+\delta; \underline{b_1}-\varepsilon, \overline{b_2}) < 2\pi$. This implies that $(\underline{b_1}-\varepsilon, \overline{b_2}) \in B$, a contradiction to the definition of $\underline{b_1}$. Hence $(\underline{b_1}, \overline{b_2}) \notin B$.

Recall from Lemma 4.6 that it suffices to consider the case $\hat{R} = R_2$.

Suppose that $\hat{R} = R_2$. Then $\hat{R}$ is the unique intersection point. Otherwise, let R_s be another intersection point such that $R_s \neq R_2$ and so $R_s \in (R_1, R_2)$. We may assume without loss of generality that there is no intersection point in (R_s, R_2). If $\Delta\Phi(R_s; \underline{b_1}, \overline{b_2}) \geq 2\pi$, then we are done. Otherwise, if $\Delta\Phi(R_s; \underline{b_1}, \overline{b_2}) < 2\pi$, then $\Delta\Gamma(R_s+\eta; \underline{b_1}, \overline{b_2}) > 0$ for $0 < \eta \ll 1$ due to (4.18). Hence $\Delta\Gamma(r; \underline{b_1}, \overline{b_2}) > 0$ for all $r \in (R_s, R_2)$. Recall that $\Delta\Gamma(R_2; \underline{b_1}, \overline{b_2}) = 0$. By Lemma 3.2, $\Phi_1(R_2; \underline{b_1}) < \Phi_1(R_2; \overline{b_1}) = \pi$. Hence $\Delta\Phi(R_2; \underline{b_1}, \overline{b_2}) \in (\pi, 2\pi)$. By the continuity of $\Delta\Phi$, we also have $\Delta\Phi(r; \underline{b_1}, \overline{b_2}) \in (\pi, 2\pi)$ for r with $0 < R_2 - r \ll 1$. Using $\Phi_1 \in (0, \pi)$ and $\Phi_2 \in (2\pi, 3\pi)$ in (R_1, R_2), integrating (4.18) over $(R_2 - \varepsilon, R_2)$ with $0 < \varepsilon \ll 1$ we obtain that $\Delta\Gamma(R_2; \underline{b_1}, \overline{b_2}) > 0$, a contradiction. Therefore, $\hat{R}$ is the unique intersection point and we conclude that

$$\Gamma_1(r; \underline{b_1}) > \Gamma_2(r; \overline{b_2}) \quad \text{for all } r \in [R_1, R_2). \tag{4.20}$$

Moreover, $\underline{b_1} < \overline{b_1}$ and, by comparison, $\Gamma_1(r; \underline{b_1}) > \Gamma_1(r; \overline{b_1})$ for all $r \in (R_1, R_2]$.

Now, we consider the trajectory for $b_2 = 0$. Note that $\Phi_2(R_2; 0) = 2\pi$. If $\Phi_1(R_2; \underline{b_1}) = 0$, then we are done. Otherwise, $\Phi_1(R_2; \underline{b_1}) > 0$ and so $\Delta\Phi(R_2; \underline{b_1}, 0) < 2\pi$. Hence we have $\Gamma_2(r; 0) < \Gamma_1(r; \underline{b_1})$ for r with $0 < R_2 - r \ll 1$. Suppose that there is the second intersection point, say at R_s. Similar argument as before, we then have $\Delta\Phi(R_s; \underline{b_1}, 0) \geq 2\pi$. Since $R_s \in (R_1, R_2)$, Lemma 4.6 implies the existence of a point-tipped rotating spot.

It remains to consider the case that $\Gamma_2(r; 0) < \Gamma_1(r; \underline{b_1})$ for all $r \in [R_3, R_2)$. Since $\overline{R} \leq R_3$, we have $\Gamma_1(R_3; \overline{b_1}) \leq \Gamma_2(R_3; \overline{b_2}) < \Gamma_2(R_3; 0)$, by Proposition 4.4 and Lemma 3.2. Hence we obtain that $\Gamma_1(R_3; \overline{b_1}) < \Gamma_2(R_3; 0) < \Gamma_1(R_3; \underline{b_1})$. Then by the continuous dependence on b_1 we can find a $b_1 \in (\underline{b_1}, \overline{b_1})$ such that $\Gamma_1(R_3; b_1) = \Gamma_2(R_3; 0)$. Note that $\Phi_2(R_3; 0) = 3\pi$. Since $\Phi_1 \in (0, \pi)$, we have $\Delta\Phi(R_3; b_1, \overline{b_2}) \geq 2\pi$. Then the existence of a point-tipped rotating spot again follows from Lemma 4.6.

Case (ii). $\overline{R} < R_0$. In this case, we also define $\underline{b_1} = \inf\{b_1 > 0 \mid (b_1, \overline{b_2}) \in B\}$. if $\underline{b_1} > 0$, then the proof of Theorem 1 can be completed by the argument similar to **Case (i)**. Thus, we only consider the case that $\underline{b_1} = 0$.

Suppose that $\underline{b_1} = 0$. Then there is at least one $\hat{R} \in (R_1, R_0]$ such that $\Delta\Gamma(\hat{R}; 0, \overline{b_2}) = 0$. If $\hat{R} = R_0$, then we have $\Delta\Phi(R_0; 0, \overline{b_2}) \geq 2\pi$, since $\Phi_1(R_0; 0) = 0$. Hence we are done. Also, if we have $\hat{R} \in (R_1, R_0)$ and there is the second intersection point, then a point-tipped rotating spot exists.

We consider the remaining case that $\hat{R} \in (R_1, R_0)$, $\Delta\Phi(\hat{R}; 0, \overline{b_2}) < 2\pi$, $\Gamma_1(r; 0) > \Gamma_2(r; \overline{b_2})$ for all $r \in (R_1, \hat{R})$, and $\Gamma_1(r; 0) \in (\Gamma_1(r; \overline{b_1}), \Gamma_2(r; \overline{b_2}))$ for all $r \in (\hat{R}, R_0]$. Assume that there exist $b_1 > 0$ and $r \in (R_1, R_e(b_1))$ such that $\Delta\Gamma(r; b_1, \overline{b_2}) \le 0$, then $\Delta\Gamma(\breve{R}; b_1, \overline{b_2}) = 0$ for some $\breve{R} \in (R_1, r] \subset (R_1, R_2)$. These conditions imply that

$$\frac{d\Delta\Gamma(\breve{R}; b_1, \overline{b_2})}{dr} \le 0.$$

Using $\Phi_1(\breve{R}; b_1) \in (0, \pi)$, $\Phi_2(\breve{R}; \overline{b_2}) \in (2\pi, 3\pi)$ and $\Delta\Phi(\breve{R}; b_1, \overline{b_2}) \in (\pi, 3\pi)$ together with

$$\frac{d\Delta\Gamma(\breve{R}; b_1, \overline{b_2})}{dr} = \frac{-\sin\Delta\Phi(\breve{R}; b_1, \overline{b_2})}{\breve{R}\sin\Phi_2(\breve{R}; \overline{b_2})\sin\Phi_1(\breve{R}; b_1)} \le 0,$$

we see that $\Delta\Phi(\breve{R}; b_1, \overline{b_2}) \ge 2\pi$. This shows that (4.6) holds with $\widetilde{b_1} = b_1$ and $\widetilde{b_2} = \overline{b_2}$. Hence a point-tipped rotating spot can be derived. Otherwise, we have $\Delta\Gamma(r; b_1, \overline{b_2}) > 0$ for $r \in (R_1, R_e(b_1))$ for each $b_1 \ge 0$. By applying the continuous dependence on b_1 in the (r, ϕ)-phase plane, there exists a $b_1 \in (0, \overline{b_1})$ such that the corresponding solution of (2.9)-(2.10) satisfies

$$r_1(\xi_1; b_1) = R_2, \quad \varphi_1(\xi_1; b_1) = 0.$$

Note that we have $\gamma_1(\xi_1; b_1) \le \gamma^*$. This produces a point-tipped rotating spot, if $\gamma_1(\xi_1; b_1) = \gamma^*$. When $\gamma_1(\xi_1; b_1) < \gamma^*$, it gives an arc-tipped rotating spot.

Case (iii) $\overline{R} > R_3$. For this case, we define $\underline{b_2} = \inf\{b_2 > 0 \mid (\overline{b_1}, b_2) \in B\}$. Then Theorem 1 can be proved by a similar argument as **Case (ii)**. In conclusion, we have completed the proof of Theorem 1. ☐

Acknowledgments. The authors would like to thank the Mathematics Division of NCTS (Taipei Office) for the support of the third author's visit to Taiwan for which this paper was inspired. We also thank the anonymous referees for their valuable comments and suggestions.

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