EXISTENCE AND UNIQUENESS OF RIGIDLY ROTATING SPIRAL WAVES BY A WAVE FRONT INTERACTION MODEL

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ABSTRACT. We study the spiral wave in an unbounded excitable medium from the wave front interaction model derived by Zykov **in 2009**. This model consists of two systems of ordinary differential equations that describe the wave front and wave back, respectively. First, we derive some properties of the back by the shooting argument and the comparison principle. Next we show the global existence of the solution of the back. Then we study its asymptotic behavior at infinity. Finally, we prove the uniqueness of the solution.

Keywords: spiral wave, angular speed, front, back

1. INTRODUCTION

Wave patterns in excitable media have been studied in many fields in physics, chemistry, biology and so on. One of the typical wave patterns is a spiral pattern, which is observed in Belousov-Zhabotinsky reaction [7], cyclic-AMP signaling in social amoeba colonies of Dictyostelium discoideum [8], etc. It is also known that the spiral waves is one of causes of a ventricular fibrillation [9]. Therefore, to understand the mechanism of the appearance of spiral waves is very important. For more details on the mathematics aspect and physical background of spiral waves, we refer to Fife [3], Tyson and Keener [14], Meron [10], Fiedler and Scheel [1], Mikhailov [13] and so on.

Many researchers studied the spiral wave as a thickless curve in the plane **(see, e.g.,** [2, 4]**)**, though most of experiments exhibit thick spiral waves. Under the assumption that the tip is rotating along a circle, namely, the front is perpendicular to the core circle, they derived some information of spiral waves, such as the behavior of the wave and the multiple existence. To study the motion of the tip and the core of the spiral wave, we need more information of the spiral wave, especially near its tip. Therefore, it is meaningful to study it as a thick region to derive the more information of the spiral wave. Zykov studied the spiral wave rotating along a core circle by free boundary approach in [16, 17]. It shows the selection mechanism that uniquely determines the shape and the rotation frequency of spiral waves in an unbounded excitable medium. **More precisely, given an admissible rotating**

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frequency ω, we can choose the constant b related to the excitability so that the spiral wave in an unbounded excitable medium exists. For example, when we consider the photosensitive Belousov-Zhabotinsky reaction as in [11, 17], we can choose the parameter b such that the spiral wave pattern appears by changing the light intensity. In this work, we will show this result by mathematical analysis. The method used in [17] is also applicable to other kinds of wave patterns, [11, 12, 15] for a wave segment in the plane and [16] for a rotating wave pattern in a disk.

In this paper, we give a mathematical proof of the existence and the uniqueness of spiral waves which consist of the front and the back by using the system proposed by Zykov [17]. This system is the so-called *wave front interaction model*. For the mathematical studies of two related wave front interaction models, we refer the reader to some recent works on the propagating wave segment in the plane [5] and the rotating wave pattern in a disk [6]. We shall describe this model in the successive section.

Comparing to the works ([2, 4]) that regard the spiral wave as a thickless curve, we need to treat not only a 3-component ordinary differential system describing the front part, but also a 3-component ordinary differential system for the back. Moreover, the system for the back involves a function related to the front. Motivated by [4], the system for the front can be easily solved by a change of variables which reduces the 3-component system into a 2-component autonomous system. However, a similar change of variables for the system of the back gives us a 2-component non-autonomous system, since the reduced system includes the angular distance between the front and the back. The usual phase plane analysis is no longer applicable. To overcome this difficulty, a delicate analysis is carried out for the reduced system (see (4.7) below). For this, we need to derive some useful properties of the back and the angular distance between the front and the back. Indeed, this is based on comparing with a special solution constructed from the solution describing the front. Hence we are able to show the global existence of back with positive angular distance using the continuity argument and the dependency of the parameter. Using the properties of the angular distance, we can also show the asymptotic behavior and uniqueness of the back. This result implies that the size of the core is determined only by the front. We emphasize that the front is tangential to the core circle at the tip, because the spiral wave is rotating along the core circle.

We organize this paper as follows. First, in Section 2, we describe the wave front interaction model and state the main result of this paper. In Section 3, we derive several key properties of the back and study the angular distance between front and back. Finally, we show the existence and uniqueness of the back for a spiral wave in Section 4.

2. The problem setting and main result

In this section, we first describe the problem setting by the *wave front interaction model* derived by Zykov [17].

A planar curve can be described by the Euclidean coordinates (x, y) and the angle θ of the normal vector (right-hand to the tangent) measuring from the positive x-axis. Then we have

$$\frac{dx}{ds} = -\sin\theta, \quad \frac{dy}{ds} = \cos\theta, \quad \frac{d\theta}{ds} = \kappa,$$

where s is the arc length parameter and κ is the (signed) curvature. Using the polar coordinates (r, γ) with the relation $(x, y) = (r\cos\gamma, r\sin\gamma)$, we have

$$\frac{dr}{ds} = \sin\varphi, \quad \frac{d\gamma}{ds} = \frac{1}{r}\cos\varphi, \quad \frac{d\theta}{ds} = \kappa, \tag{2.1}$$

where $\varphi := \gamma - \theta$.

In this paper, we consider a spiral wave pattern in the plane which is rotating counterclockwise with a constant positive angular speed ω. Here a wave pattern corresponds to the excited region in the media. Let the origin be the center of the rotation. Then we have the relations

$$r(s,t) = r(s), \quad \gamma(s,t) = \gamma(s) + \omega t, \quad \theta(s,t) = \theta(s) + \omega t. \tag{2.2}$$

Since the normal velocity V can be computed by

$$V = \frac{dx}{dt}\cos\theta + \frac{dy}{dt}\sin\theta = \frac{dr}{dt}\cos(\gamma - \theta) - r\frac{d\gamma}{dt}\sin(\gamma - \theta),$$

it follows from (2.2) that

$$V = -\omega r \sin\varphi. \tag{2.3}$$

As in [6], we let the *tip* (or, phase change point) to be the unique point on the boundary of the excited region with zero normal velocity. Moreover, the radius function has its minimum at the tip. Then we define the *front* to be the wave boundary before the tip and the *back* to be the one after the tip. Furthermore, we measure the arc length s from the tip *forward* on the back and *backward* on the front so that $s > 0$ on the back and $s < 0$ on the front. For clarity, the functions of the front curve and the back curve are denoted by

$$(x_+, y_+, r_+, \gamma_+, \theta_+, \varphi_+, V_+), \quad (x_-, y_-, r_-, \gamma_-, \theta_-, \varphi_-, V_-),$$

respectively. Note that by our choice the normal vector is always pointed outward to the excited region.

For the front, as in [17], we choose the normalized interface equation by the following linear eikonal equation

$$V_+ = 1 - \kappa_+. \tag{2.4}$$

Using (2.1) and (2.3), we obtain

$$\begin{cases} \dfrac{dr_+}{ds} &= \sin\varphi_+, \\ \dfrac{d\varphi_+}{ds} &= \dfrac{\cos\varphi_+}{r_+} - 1 - \omega r_+ \sin\varphi_+. \end{cases} \tag{2.5}$$

We take the tip to be $(r_+, \gamma_+)|_{s=0} = (R_0, 3\pi/2)$ for some $R_0 > 0$. Also, we choose $\theta_+(0) = \pi/2$. Recall that the arc length is measured *backward* so that $s < 0$ for the front. Therefore, (2.5) is equipped with the terminal condition

$$r_+|_{s=0} = R_0, \quad \varphi_+|_{s=0} = \pi. \tag{2.6}$$

We denote the solution of (2.5) with (2.6) by $(r_+(s;\omega), \varphi_+(s;\omega))$, or simply $(r_+(s), \varphi_+(s))$.

Using the change of variables

$$X_+(s) := \omega r_+(s)\cos\varphi_+(s), \quad Y_+(s) := 1 + \omega r_+(s)\sin\varphi_+(s),$$

we end up with

$$\begin{cases} \dfrac{dX_+}{ds} &= Y_+(Y_+ - 1), \quad \dfrac{dY_+}{ds} = \omega - X_+Y_+, \quad s < 0, \\ X_+(0) &= -\omega R_0, \quad Y_+(0) = 1. \end{cases} \tag{2.7}$$

Note that Y_+ is the curvature function.

We look for solutions such that the radius function is monotone in s. Since $\varphi_+(0;\omega) = \pi$, $\theta_+(s;\omega) < \pi/2$, and $\gamma_+(s;\omega) > 3\pi/2$ for $0 < -s \ll 1$, we see from (2.5) that $\varphi_+ \in [\pi, 2\pi]$ for the front. This gives us the condition that $Y_+ < 1$ for the front, except when $\varphi_+ = \pi$ or 2π.

The following proposition for the existence of the front was shown in [4, Lemma 2.1 and Theorem 1].

Proposition 2.1 ([4, Lemma 2.1 and Theorem 1]). *There exists a positive constant ω^* such that for each $\omega \in (0, \omega^*)$ there is a unique positive constant $R_*(\omega)$ such that a unique solution (X_+, Y_+) of (2.7) for $R_0 = R_*(\omega)$ with $0 < Y_+ < 1$ exists for all $s < 0$. Moreover, $X'_+(s) < 0$, $Y'_+(s) > 0$ for all $s < 0$, and the properties*

$$\lim_{s\to-\infty} X_+(s) = \infty, \quad \lim_{s\to-\infty} Y_+(s) = 0 \tag{2.8}$$

hold for this solution.

Indeed, the solution obtained in Proposition 2.1 will be used for the front of a spiral wave. Note that this solution curve is of positive curvature. Hence for each $\omega \in (0, \omega^*)$ there is a unique solution (r_+, φ_+) of (2.5) and (2.6) with $\varphi_+ \in [\pi, 2\pi]$ defined for all $s \le 0$ such that $R_0 = R_*(\omega)$ and

$$\lim_{s\to-\infty} r_+(s) = \infty, \quad \lim_{s\to-\infty} \varphi_+(s) = 2\pi, \quad \lim_{s\to-\infty} \frac{d\theta_+}{ds}(s) = 0, \tag{2.9}$$

by using (2.8). Moreover, since $\varphi_+(s) \in (\pi, 2\pi)$ for all $s \in (-\infty, 0)$, we can invert $r_+(s)$ and define $\Gamma_+(r) := \gamma_+(s(r))$ for all $r > R_0$.

Next, we consider the back of a spiral wave. The back of a spiral wave is influenced by the front through the inhibitor of the excitable medium. **If the activator and the inhibitor are close to constants in the excited region, the density of the inhibitor v is expected to satisfy the equation**

$$\frac{\partial v}{\partial t} = \widehat{b} \tag{2.10}$$

for some constant $\widehat{b}$. The continuity condition at the front also implies that $v = v_+$ on the front where v_+ is the density of the inhibitor on the front. Now we focus on the spiral wave pattern rotating along the core with a constant angular speed ω. Hence, we can rewrite (2.10) as

$$-\omega\frac{\partial v}{\partial \gamma} = \widehat{b}.$$

Using the condition at the front we have

$$v_- = v_+ + \frac{\widehat{b}}{\omega}(\Gamma_+(r_-) - \gamma_-),$$

where v_+ (resp. v_-) is the inhibitor on the front (resp. back). The speed of the planar wave of the activator is determined by the density of the inhibitor. As in [14], under the normalization, we simply assume that the wave velocity c_- of the back is

$$c_- = 1 - b(\Gamma_+(r) - \Gamma_-(r)). \tag{2.11}$$

Here b is equal to the quantity $\widetilde{B}/\omega$ in [17], where $\widetilde{B}$ is a dimensionless constant and it is actually related to the excitability of the medium. Using the normalized eikonal equation, we obtain that the interface equation for the back is given by

$$V_- = 1 - \kappa_- - b(\Gamma_+(r_-) - \gamma_-). \tag{2.12}$$

It is also noted that the patterns in the experiments of photosensitive Belousov-Zhabotinsky reaction become different if we change the light intensity. Combining (2.12) with (2.1) and (2.3), the equations for the back can be written as

$$\begin{cases} \dfrac{dr_-}{ds} &= \sin\varphi_-, \\ \dfrac{d\gamma_-}{ds} &= \dfrac{\cos\varphi_-}{r_-}, \\ \dfrac{d\varphi_-}{ds} &= \dfrac{\cos\varphi_-}{r_-} - 1 - \omega r_- \sin\varphi_- + b(\Gamma_+(r_-) - \gamma_-). \end{cases} \tag{2.13}$$

Here we have the arc length $s \geq 0$ and the initial condition is given by

$$r_-|_{s=0} = R_0, \quad \gamma_-|_{s=0} = \frac{3}{2}\pi, \quad \varphi_-|_{s=0} = \pi. \tag{2.14}$$

We denote the solution of (2.13) with (2.14) by $(r_-(s;b), \gamma_-(s;b), \varphi_-(s;b))$.

Now we state the following main theorem of this paper.

Theorem 2.2. *For any $\omega \in (0, \omega^*)$, there exist a unique positive constant $R_0 := R_*(\omega)$ defined as in Proposition 2.1 and a unique positive constant b_* such that the corresponding solution $(r_-, \gamma_-, \varphi_-)(s; b_*)$ of (2.13) with (2.14) is defined for all $s \geq 0$. Moreover, it satisfies the asymptotic condition*

$$\lim_{s\to\infty} r_-(s) = \infty, \quad \lim_{s\to\infty} V_-(s) = -1, \quad \lim_{s\to\infty} \frac{d\theta_-}{ds}(s) = 0. \tag{2.15}$$

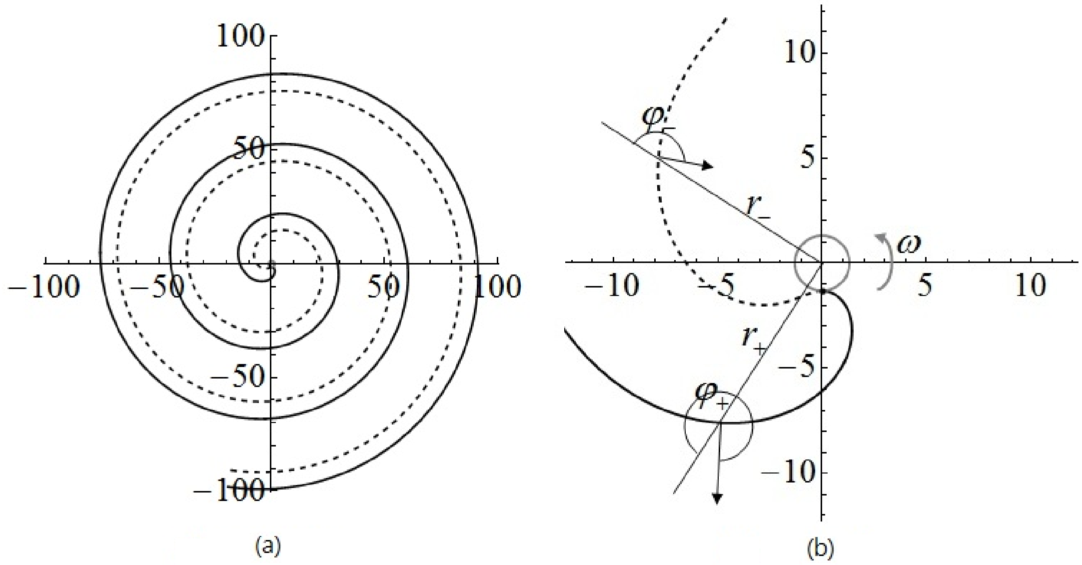

FIGURE 1. Numerical solutions of (2.5) (solid curve) and (2.13) (dotted curve) when $\omega = 0.2$, $R_0 \approx 1.305$ and $b = b_* \approx 1.312$:**(a) whole plane (b) near the tip**

In [17], the author shows the selection mechanism about the rotating spiral wave in an unbounded excitable medium. Proposition 2.1 and Theorem 2.2 give a mathematical proof to this result. Therefore, given any $\omega \in (0, \omega^*)$, the unbounded rotating spiral wave pattern only appears by choosing the specific core radius $R_0 = R_*(\omega)$ and excitability $b = b_*$.

The numerical solution of Proposition 2.1 and Theorem 2.2 for $\omega = 0.2$ is given in Figure 1. The solid curve corresponds to the front of the chemical waves and the dotted one does to the back. **The excited region and the resting one are separated by the front and the back.** We can observe the topological changes when the parameter b varies (see Figure 2). From this numerical result, it suggests that the existence of the back can be shown by the continuity argument. Also we can observe that the angular distance is always positive and tends to some constant when $b = b_*$. This means that there is no intersection between front and back.

Remark 2.1. (*Wavelength*) **As $r_- \to \infty$, we can derive the asymptotic wavelength for the back. From the conclusion in Proposition2.1 and Theorem2.2, we can easily to obtain that**

$$\begin{aligned} \frac{dr_+}{d\gamma_+} &= \frac{r_+ \sin\phi_+}{\cos\phi_+} \to -\frac{1}{\omega} \quad \text{as} \quad r_+ \to \infty, \\ \frac{dr_-}{d\gamma_-} &= \frac{r_- \sin\phi_-}{\cos\phi_-} \to -\frac{1}{\omega} \quad \text{as} \quad r_- \to \infty. \end{aligned}$$

These results suggest that the asymptotic wavelength for the front and the back when r_+ and r_- tends to infinity are equal to $2\pi/\omega$.

Note that the second condition in (2.15) is **natural**, since $V_+(s) \to 1$ as $s \to -\infty$.

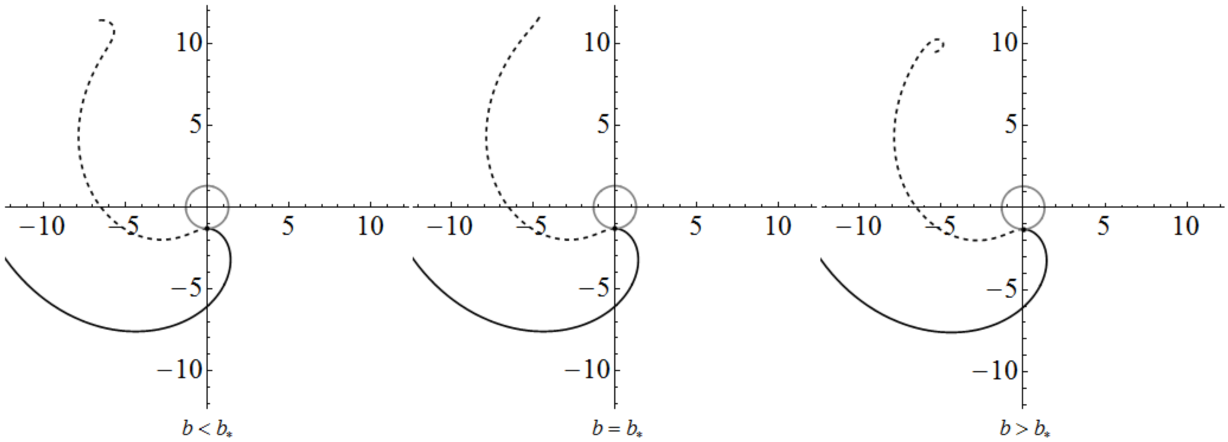

FIGURE 2. Dependence of solutions of (2.13) on b

3. SOME PROPERTIES OF THE BACK

In this section, we shall study the properties of the solution $(r_-, \gamma_-, \varphi_-)$ of (2.13) and (2.14) for any given $b > 0$. For the simplicity of the notation, we shall drop the subscript minus sign from now on. As the front, we want to find the radius function for the back to be monotone in s. So we require that $\varphi \in [0, \pi]$ for the back, due to

$$\varphi(0; b) = \pi,\ \ \theta(s; b) > \pi/2,\ \ \gamma(s; b) < 3\pi/2 \text{ for } 0 < s \ll 1$$

and (2.13). First, we recall a result from [6].

Lemma 3.1 ([6, Lemma 4.2])**.** *Let $b > 0$. Then the following statements hold:*

(i) *If $\varphi(s_*) = 0$ for some $s_* > 0$ and $0 < \varphi(s) < \pi$ for $0 < s < s_*$, then $\varphi(s) < 0$ for $s > s_*$ with $s - s_*$ small.*
(ii) *If $\varphi(s_*) = \pi$ for some $s_* > 0$ and $0 < \varphi(s) < \pi$ for $0 < s < s_*$, then $\varphi(s) > \pi$ for $s > s_*$ with $s - s_*$ small.*

Now we consider the following open strip domain

$$Q := (R_0, \infty) \times \mathbb{R} \times (0, \pi)$$

for a given $\omega \in (0, \omega^*)$. For each $b > 0$, we define the *exit-length* $S = S(b)$ and the *exit-point* $(R_e, \gamma_e, \varphi_e)(b)$ as follows:

(i) if there is a positive number $\overline{s}$ such that the orbit stays in Q for $0 < s < \overline{s}$, $\varphi(\tau) > \pi$ for some τ close to $\overline{s}$ with $\tau > \overline{s}$, then $S = S(b) = \overline{s}$ and $(R_e, \gamma_e, \varphi_e)(b) = (r(S), \gamma(S), \pi)$;
(ii) if there is a positive number $\underline{s}$ such that the orbit stays in Q for $0 < s < \underline{s}$, $\varphi(\tau) < 0$ for some τ close to $\underline{s}$ with $\tau > \underline{s}$, then $S = S(b) = \underline{s}$ and $(R_e, \gamma_e, \varphi_e)(b) = (r(S), \gamma(S), 0)$;
(iii) set $S(b) = \infty$ and $R_e(b) = r(\infty; b)$, if $\varphi(s) \in (0, \pi)$ for all $s > 0$.

We remark that since r is increasing in s as long as the orbit stays in Q, the orbit never touches the plane $r = R_0$. Moreover, by Lemma 3.1, the definitions of exit-length and exit-point are well-defined. Note that $R_e(b) < \infty$ if $S(b) < \infty$.

Lemma 3.2. *If $S(b) = \infty$, then $R_e(b) = \infty$.*

Proof. Assume for contradiction that $R_e(b) < \infty$. Note that $r(s)$ is monotone increasing in s, $0 < \varphi < \pi$ for $0 < s < \infty$ and $R_e(b) = r(\infty; b)$. With this $R_e(b)$ as R_D in [6, Lemma 4.3], we have that $|\gamma(s)| \le \gamma^*(b)$ for $0 < s < \infty$, where

$$\gamma^*(b) := \frac{1}{b}\Big(\frac{1}{R_0} + R_e(b)\Big) + \max\Big\{3, \max_{R_0 \le r \le R_e(b)} \Gamma_+(r)\Big\} + \frac{\pi}{R_0}.$$

Suppose that the limit $\lim_{s\to\infty} \gamma(s)$ exists (which is finite). Then we can find a sequence $\{s_n\}$ tending to infinity such that $(r'(s_n), \gamma'(s_n)) \to (0, 0)$ as $n \to \infty$. By (2.13), we have

$$\lim_{n\to\infty} \sin\varphi(s_n) = \lim_{n\to\infty} \frac{\cos\varphi(s_n)}{r(s_n)} = 0,$$

which is a contradiction.

On the other hand, suppose that $\gamma(s)$ is oscillatory. Then, by the same argument as in the proof of the last part of [6, Lemma 4.3], we also reach a contradiction. Thus we conclude that $R_e(b) = \infty$. $\square$

We can regard the orbit as a function of r instead of s. Namely, let $\Phi(r) := \varphi(s(r))$ and $\Gamma(r) := \gamma(s(r))$. Then (Φ, Γ) is a solution of the system

$$\begin{cases} \dfrac{d\Gamma}{dr} = \dfrac{\cos\Phi}{r\sin\Phi}, \\[2mm] \dfrac{d\Phi}{dr} = \dfrac{\cos\Phi}{r\sin\Phi} - \dfrac{1}{\sin\Phi} - \omega r + \dfrac{b(\Gamma_+(r) - \Gamma)}{\sin\Phi}. \end{cases} \tag{3.1}$$

We now recall a comparison principle from [6] as follows.

Lemma 3.3 ([6, Lemma 4.8])**.** *Let $(\Gamma_j(r), \Phi_j(r)) := (\Gamma(r; b_j), \Phi(r; b_j))$ be the solution of (3.1) with $\Gamma_j(R_0) = 3\pi/2$ and $\Phi_j(R_0) = \pi$ defined on $[R_0, r(S(b_j))]$, $j = 1, 2$. If $0 < b_1 < b_2$, then*

$$\Gamma_1(r) > \Gamma_2(r), \quad \Phi_1(r) < \Phi_2(r)$$

on $(R_0, \min\{r(S(b_1)), r(S(b_2))\})$ as long as $\Gamma_+(r) > \Gamma_1(r)$.

Let $(r_+, \gamma_+, \varphi_+)(s)$ be the front solution defined in Proposition 2.1. Since the trajectory of the solution (X_+, Y_+) passes through the point $(0, Y_0)$ for some $Y_0 \in (0, 1)$, there is a positive constant s_1 such that $\varphi_+(-s_1) = 3\pi/2$. In the sequel, we define $R_1 = r_+(-s_1)$. Note that R_1 satisfies that

$$\omega R_1 < 1. \tag{3.2}$$

This radius R_1 shall play the role as the disk radius R_D in [6] in the sequel.

The following simple observation gives a clue for our shooting argument to obtain the existence of the solution of the back.

Proposition 3.4. *Given $b>0$. Set*

$$(\hat{r}(s),\hat{\gamma}(s),\hat{\varphi}(s)) := \Big(r_+(-s),\gamma_+(-s)-\frac{2}{b},\varphi_+(-s)-\pi\Big),\quad s\geq 0.$$

Then $(\hat{r},\hat{\gamma},\hat{\varphi})$ becomes a solution of the system (2.13) defined for all $s\geq 0$ and so

$$(\hat{\Gamma},\hat{\Phi}) := \Big(\Gamma_+(r)-\frac{2}{b},\Phi_+(r)-\pi\Big)$$

is also a solution of (3.1).

Proof. It is easy to see that

$$\begin{aligned}
\frac{d\hat{r}(s)}{ds} &= \frac{dr_+(-s)}{ds} = -\sin(\varphi_+(-s)) = \sin(\varphi_+(-s)-\pi) = \sin\hat{\varphi}(s),\\
\frac{d\hat{\gamma}(s)}{ds} &= \frac{d(\gamma_+(-s)-2/b)}{ds} = \frac{-\cos(\varphi_+(-s))}{r_+(-s)} = \frac{\cos(\varphi_+(-s)-\pi)}{r_+(-s)} = \frac{\cos(\hat{\varphi}(s))}{\hat{r}(s)},\\
\frac{d\hat{\varphi}(s)}{ds} &= \frac{d(\varphi_+(-s)-\pi)}{ds} = -\Big(\frac{\cos(\varphi_+(-s))}{r_+(-s)} - 1 - \omega r_+(-s)\sin(\varphi_+(-s))\Big)\\
&= \frac{\cos(\varphi_+(-s)-\pi)}{r_+(-s)} - 1 - \omega r_+(-s)\sin(\varphi_+(-s)-\pi) + 2\\
&= \frac{\cos(\hat{\varphi}(s))}{\hat{r}(s)} - 1 - \omega\hat{r}(s)\sin(\hat{\varphi}(s)) + b(\Gamma_+(\hat{r})-\hat{\gamma}(s)).
\end{aligned}$$

Hence we know that $(\hat{r},\hat{\gamma},\hat{\varphi})$ is a solution of (2.13) on $[0,\infty)$ and the proposition is proved. □

The following lemma is useful.

Lemma 3.5. *For each $b>0$, the solution $(r(s;b),\gamma(s;b),\varphi(s;b))$ of (2.13) and (2.14) satisfies one of the following:*

(i) $\Phi_+(r)-\pi<\Phi(r)<\pi$ *for any* $r\in(R_0,R_e(b))$.
(ii) *There exists a $R_p=R_p(b)\in(R_0,R_e(b))$ such that $\Phi_+(r)-\pi<\Phi(r)<\pi$ for any $r\in(R_0,R_p)$ and $\Phi(R_p)=\Phi_+(R_p)-\pi$. Moreover, we have*

$$\Gamma_+(r)-\Gamma(r)<\frac{2}{b}\quad \textit{for } R_0<r\leq R_p.$$

Proof. Recall $\Phi_+(r)\in(\pi,2\pi)$ for $r>R_0$ and $\Phi(r)\in(0,\pi)$ for all $r\in(R_0,R_e)$. Since

$$\frac{d(\Gamma_+-\Gamma)}{dr} = \frac{\cos\Phi_+}{r\sin\Phi_+} - \frac{\cos\Phi}{r\sin\Phi} = \frac{\sin(\Phi-\Phi_+)}{r\sin\Phi_+\sin\Phi}, \tag{3.3}$$

the function $(\Gamma_+-\Gamma)(r)$ is increasing in r as long as $0<(\Phi_+-\Phi)(r)<\pi$. This is the case whenever $\Phi_+(r)-\pi<\Phi(r)<\pi$. Using $(\Gamma_+-\Gamma)(R_0)=0$ and [6, Lemma 4.1], we have $(\Gamma_+-\Gamma)(r)>0$ for $r>R_0$ and sufficiently close to R_0.

Assume that $0 < \Phi_+(r) - \Phi(r) < \pi$ for any $r \in (R_0, R_p)$ and $(\Phi_+ - \Phi)(R_p) = \pi$ for some $R_p \in (R_0, R_e)$. Then

$$\frac{d(\Phi_+ - \pi - \Phi)}{dr}(R_p) \geq 0.$$

We compute

$$\begin{aligned}\frac{d(\Phi_+ - \pi - \Phi)}{dr} &= \frac{1}{\sin\Phi_+}\Big(\frac{\cos\Phi_+}{r} - 1\Big) - \frac{1}{\sin\Phi}\Big(\frac{\cos\Phi}{r} - 1 + b(\Gamma_+ - \Gamma)\Big) \\ &= \frac{\sin(\Phi - \Phi_+)}{r\sin\Phi_+\sin\Phi} - \frac{1}{\sin\Phi_+} + \frac{1}{\sin\Phi} - \frac{b(\Gamma_+ - \Gamma)}{\sin\Phi}.\end{aligned}$$

If $(\Gamma_+ - \Gamma)(R_p) > 2/b$, then we have

$$\frac{d(\Phi_+ - \pi - \Phi)}{dr}(R_p) < 0,$$

a contradiction. Hence we have $(\Gamma_+ - \Gamma)(R_p) \leq 2/b$.

On the other hand, if $(\Gamma_+ - \Gamma)(R_p) = 2/b$, then, by the uniqueness of solution of (3.1) with the help of Proposition 3.4, we have $(\Gamma_+ - \Gamma)(r) \equiv 2/b$ and $(\Phi_+ - \Phi)(r) \equiv \pi$, a contradiction. Hence we must have $(\Gamma_+ - \Gamma)(R_p) < 2/b$. It follows from (3.3) that

$$\Gamma_+(r) - \Gamma(r) < \frac{2}{b} \quad \text{for } R_0 < r \leq R_p.$$

This completes the proof of the lemma. $\square$

Hereafter we shall define the set

$$B := \left\{ b > 0 \,\middle|\, \begin{array}{l} \text{There exist a constant } R_p = R_p(b) \in (R_0, R_e(b)) \text{ such that} \\ \Phi(R_p; b) = \Phi_+(R_p) - \pi \text{ and} \\ \Phi_+(r) - \pi < \Phi(r; b) < \pi \text{ for any } r \in (R_0, R_p) \end{array} \right\}.$$

Then we have the following property for any $b \in B$.

Lemma 3.6. *For any $b \in B$, the corresponding solution of* (2.13) *and* (2.14) *satisfies*

$$(\Gamma_+ - \Gamma)(r) < \frac{2}{b}, \quad 0 < \Phi(r) < \Phi_+(r) - \pi$$

for all $r \in (R_p, R_e(b))$.

Proof. For any r with $\Phi(r) = \Phi_+(r) - \pi$, we compute that

$$\begin{aligned}\frac{d(\Gamma_+ - \Gamma)}{dr}(r) &= 0, \\ \frac{d(\Phi_+ - \Phi)}{dr}(r) &= \frac{1}{\sin\Phi(r)}[2 - b(\Gamma_+ - \Gamma)(r)], \qquad (3.4) \\ \frac{d^2(\Gamma_+ - \Gamma)}{dr^2}(r) &= \frac{1}{r\sin^3\Phi(r)}[b(\Gamma_+ - \Gamma)(r) - 2].\end{aligned}$$

From these equalities and Lemma 3.5, it follows that

$$\frac{d(\Phi_+ - \Phi)}{dr}(R_p) > 0, \quad \frac{d^2(\Gamma_+ - \Gamma)}{dr^2}(R_p) < 0.$$

Hence we have

$$(\Phi_+ - \Phi)(r) > \pi, \quad (\Gamma_+ - \Gamma)(r) < \frac{2}{b} \;\text{ for }\; 0 < r - R_p \ll 1.$$

By (3.3), we know $(\Gamma_+ - \Gamma)(r) < 2/b$ as long as $(\Phi_+ - \Phi)(r) > \pi$. If there exists a smallest $R_p' > R_p$ such that $(\Phi_+ - \Phi)(R_p') = \pi$, then we have

$$\frac{d(\Phi_+ - \Phi)}{dr}(R_p') \leq 0.$$

However, we also have

$$(\Gamma_+ - \Gamma)(R_p') < \frac{2}{b},$$

since $(\Gamma_+ - \Gamma)(r)$ is nonincreasing on $[R_p, R_p']$. This contradicts (3.4). Therefore, the lemma follows. □

Lemma 3.6 means that the projection of the trajectory (r, Γ, Φ) onto (r, Φ)-plane stays in the region $\{(r, \Phi) \in (R_0, \infty) \times (0, \pi) \mid r > R_0,\ 0 < \Phi < \Phi_+(r) - \pi\}$ as long as it hits the boundary $\Phi = \Phi_+(r) - \pi$. Moreover, we have $\Gamma_+(r) - \Gamma(r; b) < 2/b$ for all $r \in (R_0, R_e(b))$, if $b \in B$.

Remark 3.1. If $\Phi_+(r) - \pi < \Phi(r; b) < \pi$ (resp. $0 < \Phi(r; b) < \Phi_+(r) - \pi < \Phi(r; b)$), then $\Gamma_+(r) - \Gamma(r; b)$ is monotone increasing (resp. decreasing) in r. Moreover, we always have the property that $\Gamma(r; b) < \Gamma_+(r)$ for $r \in (R_0, R_p(b))$, due to the fact that $\Gamma(r; b) < 3\pi/2 < \Gamma_+(r)$ when $0 < r - R_0 \ll 1$ and the above monotonicity property.

We have the following characterization for the set B.

Proposition 3.7. *The set B is a nonempty bounded open set. In fact, there is a finite positive constant b_* such that $B = (0, b_*)$.*

Proof. Recall (3.2). Then, by [6, Lemma 4.9], there is a positive constant $b^* = b^*(\omega)$ such that $S(b) = \bar{s}$ for all $b \geq b^*$. This implies that $b \notin B$ for all $b \geq b^*$, by Lemma 3.6. Hence B is bounded. Moreover, by [6, Theorem 4.10], there is a positive constant b_0 such that the corresponding solution (Γ_0, Φ_0) of (3.1) satisfying $\Phi_0(R_1) = \pi/2$ and

$$\Phi_+(r) - \pi < \Phi_0(r) < \pi,\ \Gamma_0(r) < \Gamma_+(r) \quad \text{for} \quad R_0 < r < R_1.$$

Note that $\Phi_+(R_1) = 3\pi/2$. This implies that $b_0 \in B$ and so $B \neq \emptyset$.

Next, we claim that

(i) If $b_1 \in B$, then $(0, b_1] \subset B$;
(ii) If $b \in B$, then there is a positive constant δ such that $b + \delta \in B$.

Then the proposition follows from the above two properties by setting $b_* := \sup B$.

To show (i), we take any b which is smaller than b_1. It follows from Lemma 3.3 that

$$\Gamma(r; b_1) < \Gamma(r; b), \quad \Phi(r; b) < \Phi(r; b_1), \quad R_0 < r < \min\{R_e(b), R_e(b_1)\}.$$

Since

$$\Phi(R_p(b_1); b) < \Phi(R_p(b_1); b_1) = \Phi_+(R_p(b_1)) - \pi,$$

there is $r \in (R_0, R_p(b_1))$ such that $\Phi(r; b) = \Phi_+(r) - \pi$. Hence $b \in B$. The first statement is shown.

For (ii), suppose for contradiction that there is a $b_1 \in B$ such that $b \notin B$ for any $b > b_1$. Lemma 3.5 and (3.4) imply that

$$\Gamma_+(R_p(b_1)) - \Gamma(R_p(b_1); b_1) < \frac{2}{b_1}, \quad \frac{d(\Phi_+(R_p(b_1)) - \pi - \Phi(R_p(b_1); b_1))}{dr} > 0.$$

Then there is a positive constant $R > R_p(b_1)$ such that $0 < \Phi(R; b_1) < \Phi_+(R) - \pi$. If b is close to b_1, then $\Phi(R; b)$ is close to $\Phi(R; b_1)$ by the continuity of the parameter b. Then $b \in B$ for all $b > b_1$ with $b - b_1 \leq \delta$ for some constant $\delta > 0$. Therefore, the statement (ii) holds true and the proposition is proved. □

Remark 3.2. Note that for $b = 0$ the quantity $R_p(0)$ is well-defined. Indeed, as in the proof of [6, Lemma 4.4], the corresponding solution Φ (to $b = 0$) satisfies

$$\frac{d(\cos\Phi - \cos\Phi_+)}{dr} = -\frac{\cos\Phi - \cos\Phi_+}{r} + \omega r\left(\sqrt{1-\cos^2\Phi} + \sqrt{1-\cos^2\Phi_+}\right)$$

for $r > R_0$ with the initial condition

$$\cos\Phi_+(R_0) = \cos\Phi(R_0) = -1.$$

Since $\cos\Phi(r), \cos\Phi_+(r) > -1$ for $r > R_0$ with $r - R_0$ small, we see that

$$(\cos\Phi - \cos\Phi_+)(r) > 0$$

for $r > R_0$. Hence $S(0) \neq \bar{s}$. On the other hand, if $\Phi_+(r) - \pi < \Phi(r) < \pi$ for all $r > R_0$, then

$$\cos\Phi_+(r) < \cos\Phi(r) < \cos(\Phi_+(r) - \pi) = -\cos\Phi_+(r)$$

for all $r > R_0$. This contradicts with the fact that the range of $\Phi_+(r)$ is equal to $[\pi, 2\pi]$. Therefore, $R_p(0)$ is a well-defined finite positive number.

4. Proof of Theorem 2.2

For any $b > 0$, the local existence of the solution of (2.13) and (2.14) has been studied by [6, Lemma 3.1]. Now, we shall show the global existence of the solution of the back that satisfies (2.15) for a certain $b > 0$.

Recall the constant $b_* = \sup B$ defined in Proposition 3.7.

Proposition 4.1. *The solution $(r, \gamma, \varphi)(s; b_*)$ of (2.13) and (2.14) exists globally for all $s \geq 0$ such that $0 < \varphi(s; b_*) < \pi$ for all $s > 0$. Moreover,*

$$R_e(b_*) = \infty, \quad \varphi_e(b_*) = \pi, \quad 0 < \Gamma_+(r(s; b_*)) - \gamma(s; b_*) < \frac{2}{b_*} \text{ for } s > 0. \tag{4.1}$$

Proof. First, by the definition of b_*, we can take a sequence $\{b_n\}$ such that $b_n \in B$ and $b_n \nearrow b_*$. Since $b_* \notin B$, we have

$$\Phi_+(r) - \pi < \Phi(r; b_*) < \pi \text{ for all } r \in (R_0, R_e(b_*)). \tag{4.2}$$

If $R_e(b_*) < \infty$, then Lemma 3.2 implies that $S(b_*) < \infty$ and so, by (4.2), $\varphi(S(b_*); b_*) = \pi$. By Lemma 3.1 and the continuous dependence on b, there is a $n \gg 1$ such that $S(b_n) = \bar{s}$. This contradicts to $b_n \in B$. Therefore, we have $R_e(b_*) = \infty$ and so $S(b_*) = \infty$. Since $\Phi_+(\infty) = 2\pi$, it follows from (4.2) that $\varphi_e(b_*) = \varphi(\infty; b_*) = \pi$.

Finally, the last inequality in (4.1) follows from Lemma 3.5 (ii). This completes the proof of this proposition. □

To complete the proof of the existence part in Theorem 2.2, it remains to show that the solution given in Proposition 4.1 satisfies (2.15). For this, we set

$$k = k(r; b_*) := b_*(\Gamma_+ - \Gamma)(r).$$

Since $\Gamma_+(r) - \Gamma(r; b_*)$ is monotone increasing in r, the limit

$$k_\infty := \lim_{r\to\infty} k(r; b_*) = \lim_{s\to\infty} k(r(s; b_*); b_*)$$

exists and we have $k(r; b_*) < k_\infty$ for all $r \geq R_0$. Note that $k_\infty \leq 2$ by (4.1).

By the fact $\varphi(\infty; b_*) = \pi$, we can choose a sequence $\{s_n\}$ tending to infinity such that $\lim_{n\to\infty} \varphi'(s_n; b_*) = 0$. Define $r_n = r(s_n; b_*)$. Then we have

$$\lim_{n\to\infty} \omega r_n \sin\Phi(r_n; b_*) = \lim_{n\to\infty} \omega r(s_n; b_*) \sin\varphi(s_n; b_*) = k_\infty - 1, \tag{4.3}$$

by using the equation of φ in (2.13). Using (3.3), (2.8) and (4.3), we compute that

$$0 = \lim_{n\to\infty} \frac{d(\Gamma_+ - \Gamma)(r_n)}{dr} = \lim_{n\to\infty} \left(\frac{\cos\Phi_+(r_n)}{r_n \sin\Phi_+(r_n)} - \frac{\cos\Phi(r_n)}{r_n \sin\Phi(r_n)} \right) = -\omega + \frac{\omega}{k_\infty - 1}.$$

Thus we conclude that $k_\infty = 2$, i.e.,

$$\lim_{s\to\infty} \{b_*[\Gamma_+(r(s; b_*)) - \gamma(s; b_*)]\} = 2. \tag{4.4}$$

We next claim that

$$\lim_{s\to\infty} \omega r(s; b_*) \sin\varphi(s; b_*) = 1. \tag{4.5}$$

For this, we define

$$X = X(s) := \omega r(s; b_*) \cos\varphi(s; b_*), \quad Y = Y(s) := 1 + \omega r(s; b_*) \sin\varphi(s; b_*). \tag{4.6}$$

Then (X, Y) satisfies

$$\begin{cases} \dfrac{dX}{ds} &= (Y - 1)[Y - k(r(s; b_*); b_*)], \\ \dfrac{dY}{ds} &= \omega - X[Y - k(r(s; b_*); b_*)]. \end{cases} \tag{4.7}$$

Proposition 4.1 implies that $X(s) < 0, Y(s) > 1$ for all $s \geq s_0$ for some $s_0 \gg 1$ and $X(s) \to -\infty$ as $s \to \infty$.

Suppose that there is a positive constant $s_1 \geq s_0$ such that

$$Y(s_1) < k(r(s_1; b_*); b_*) + \frac{\omega}{X(s_1)}.$$

Then

$$\frac{dY}{ds}(s_1) < 0.$$

Moreover, we compute that

$$\frac{d}{ds}\Big(k(r(s; b_*)); b_*) + \frac{\omega}{X(s)}\Big) = \frac{d}{ds}k(r(s; b_*)); b_*) - \frac{\omega}{X^2(s)}\frac{dX(s)}{ds} > 0$$

at $s = s_1$. Therefore, by a contradiction argument and using (4.7), we deduce that

$$Y(s) < k(r(s; b_*); b_*) + \frac{\omega}{X(s)} \text{ for all } s \geq s_1.$$

Notice that $1 < Y(s) < k(r(s; b_*)); b_*)$ for all $s \geq s_1$. Hence it follows from (4.7) that the functions $X(s)$ and $Y(s) - [k(r(s; b_*)); b_*) + \omega/X(s)]$ are strictly decreasing for $s \geq s_1$.

Now, we choose a constant $s_2 > s_1$ and a sufficiently small constant ϵ such that

$$Y(s) < k(r(s; b_*); b_*) + \frac{\omega}{X(s)} - \epsilon \text{ for all } s \geq s_2.$$

Then we derive that

$$\frac{dY}{ds}(s) < \epsilon X(s) < \epsilon X(s_2) < 0 \text{ for all } s > s_2.$$

Letting $s \to \infty$, we obtain that $Y(s) \to -\infty$, a contradiction. Thus we conclude that

$$Y(s) \geq k(r(s; b_*); b_*) + \frac{\omega}{X(s)} \text{ for all } s \geq s_0. \tag{4.8}$$

On the other hand, using $\pi/2 < \Phi_+(r) - \pi < \Phi(r; b_*) < \pi$ for all $r \gg 1$, we have

$$Y(s) < 1 - \omega r(s) \sin \Phi_+(r(s)) \text{ for all } s \gg 1. \tag{4.9}$$

Letting $s \to \infty$ in (4.8) and (4.9), we obtain that $\lim_{s\to\infty} Y(s; b_*) = 2$. Thus (4.5) follows.

Using (4.4), (4.5) and (2.13), we obtain that

$$\lim_{s\to\infty} \frac{d\theta}{ds}(s) = 0.$$

Since $\kappa = d\theta/ds$, it follows from (2.12) that

$$\lim_{s\to\infty} V(s) = -1.$$

Hence (2.15) follows and this completes the proof of the existence part of Theorem 2.2.

By (2.11)**, we immediately see that the asymptotic wave speed of the back is**

$$\lim_{r\to\infty} c_- = \lim_{r\to\infty} \{1 - b[\Gamma_+(r) - \Gamma_-(r)]\} = 1 - k_\infty = -1.$$

We now turn to show the uniqueness of the back.

Lemma 4.2. *For any $b > b_*$, $R_e(b) < \infty$.*

Proof. Assume for contradiction that $R_e(b) = \infty$ for some $b > b_*$. Note that $S(b) = \infty$. Recall from (4.1) that $\Gamma(r; b_*) < \Gamma_+(r)$ for all $r > R_0$. It follows from Lemma 3.3 that

$$\Gamma(r; b_*) > \Gamma(r; b), \quad \Phi(r; b_*) < \Phi(r; b) \quad \text{for all } r \in (R_0, \infty). \tag{4.10}$$

We claim that $b \in B$. Otherwise, we have $\Phi_+(r) - \pi < \Phi(r; b) < \pi$ for all $r > R_0$. Hence we have $\Phi(\infty; b) = \pi$, since $\Phi_+(\infty) = 2\pi$.

First, we define $\eta_1(r) = \cos\Phi(r; b)$ and $\eta_2(r) = \cos\Phi(r; b_*)$. Then

$$\lim_{r\to\infty} (\eta_1 - \eta_2)(r) = 0, \tag{4.11}$$

since $\Phi(\infty; b) = \Phi(\infty; b_*) = \pi$. Moreover, due to (4.10) we have

$$-1 < \eta_1(r) < \eta_2(r) < 0 \tag{4.12}$$

for all $r \geq \tilde{R}$ for some large $\tilde{R}$.

Next, we compute

$$\begin{aligned} \frac{d(\eta_1 - \eta_2)}{dr} &= -\frac{\eta_1 - \eta_2}{r} + \omega r \left(\sqrt{1 - \eta_1^2} - \sqrt{1 - \eta_2^2} \right) \\ &\quad - b[\Gamma_+(r) - \Gamma(r; b)] + b_*[\Gamma_+(r) - \Gamma(r; b_*)]. \end{aligned}$$

From (4.10), we have

$$b[\Gamma_+(r) - \Gamma(r; b)] > b[\Gamma_+(r) - \Gamma(r; b_*)].$$

Then it follows from $k_\infty(b_*) = 2$ that

$$\liminf_{r\to\infty} \{b[\Gamma_+(r) - \Gamma(r; b)]\} \geq \frac{2b}{b_*} > 2,$$

due to $b > b_*$. Combining this with (4.11) and (4.12), by choosing a larger $\widetilde{R}$ (if it is necessary), we have

$$\frac{d(\eta_1 - \eta_2)}{dr} < 0 \tag{4.13}$$

for all $r \geq \widetilde{R}$. Then from (4.11) it follows that $(\eta_1 - \eta_2)(\widetilde{R}) > 0$. This contradicts (4.12). Therefore, we conclude that $b \in B$, i.e., $R_p(b) < \infty$.

Now, from the proof of Proposition 3.7, we have $b_* \in (0, b) \subset B$. This contradicts the definition of b_*. Thereby the proof is complete. □

To complete the proof of the uniqueness, we consider the case $0 < b < b_*$. In this case, $R_p(b)$ exists by Proposition 3.7 and then there is a positive constant $s_p = s_p(b)$ such that $r(s_p; b) = R_p(b)$. Let (X, Y) be defined by (4.6) and (4.7) with b_* replacing by b. Note that $k(s; b)$ is decreasing in s for $s > s_p$ and $Y(s; b) > 1$ for $s > 0$.

Lemma 4.3. *For any $b \in (0, b_*)$ with $R_e(b) = \infty$ and any $s_- \in (s_p(b), \infty)$ satisfying $X(s_-) \leq 0$, either $\varphi(\infty; b) = \pi$, or there is a $s_+ = s_+(s_-)$ such that $s_+ > s_-$ and $X(s_+) > 0$.*

Proof. We assume that $X(s) \le 0$ for all $s \ge s_-$. Note that $\varphi(s) \in [\pi/2, \pi]$ for all $s \ge s_-$ and so

$$\varphi(s;b) = \pi - \sin^{-1}\Big[\frac{Y(s)-1}{\omega r(s)}\Big]. \tag{4.14}$$

We divide our discussion into two cases.

Case (i). $Y(s_0) > k(s_0;b)$ for some $s_0 \ge s_-$. Note that X and Y are increasing whenever $X \le 0$ and $Y > k$. Hence

$$0 \ge X(s) > X(s_0), \quad Y(s) > Y(s_0) > k(s_0;b) > k(s;b) \quad \text{for all } \ s > s_0.$$

Then

$$\frac{dY}{dX} = \frac{\omega - X(Y-k)}{(Y-1)(Y-k)} \le M \quad \text{for all } \ s > s_0 + 1$$

where

$$M := \frac{\omega}{[Y(s_0)-1][Y(s_0)-k(s_0+1)]} + \frac{-X(s_0)}{Y(s_0)-1} > 0.$$

Thus it follows that Y is bounded, since

$$1 < Y(s) \le Y(s_0+1) + M[X(s) - X(s_0+1)] \le Y(s_0+1) - MX(s_0+1)$$

for all $s > s_0 + 1$. Using (4.14), we see that $\varphi(\infty;b) = \pi$.

Case (ii). $Y(s) \le k(s_-;b)$ for all $s \ge s_-$. In this case, it is easy to see from (4.14) that $\varphi(\infty;b) = \pi$.

Therefore, if $\varphi(s;b)$ does not converge to π as $s \to \infty$, then there is $s_+ > s_-$ such that $X(s_+) > 0$. This proves the lemma. $\square$

Lemma 4.4. *For $0 < b < b_*$, $S_e(b) = \underline{s} < \infty$.*

Proof. Suppose that there is a $b \in (0, b_*)$ satisfying $R_e(b) = \infty$. Let $R_e = R_e(b)$. By Remark 3.1 and Lemmas 3.5 and 3.6, we have

$$k_\infty(b) < k_M := \sup_{R_0 < r < \infty} k(r,b) < 2. \tag{4.15}$$

Moreover, by Lemma 3.6 and Proposition 3.7, we recall that $Y(s;b) > 1$ for $s > 0$,

$$r(s_p;b) = R_p(b), \quad k(s_p;b) = k_M,$$
$$0 < \varphi(s) < \Phi_+(r(s)) - \pi \text{ for all } s \ge s_p$$

and $k(s;b)$ is decreasing in s for $s > s_p$. Note that $S(b) = \infty$.

First, we consider the case $\varphi(\infty;b) = \pi$. Define $\xi_1(r) = \cos\Phi(r;b)$ and $\xi_2(r) = \cos\Phi(r;b_*)$. Then $\lim_{r\to\infty}(\xi_1 - \xi_2)(r) = 0$. Note that $\Phi(r;b) < \Phi(r;b_*) < \pi$ by Lemma 3.3. Since $\Phi(\infty;b) = \pi$ and $k_\infty(b_*) = 2$, we can choose a sufficiently large constant $\widetilde{R}$ such that

$$-1 < \xi_2(r) < \xi_1(r) < 0 \tag{4.16}$$

and, due to (4.15),

$$\begin{aligned}\frac{d(\xi_1-\xi_2)}{dr} &= -\frac{\xi_1-\xi_2}{r}+\omega r\left(\sqrt{1-\xi_1^2}-\sqrt{1-\xi_2^2}\right)\\ &\quad -b[\Gamma_+(r)-\Gamma(r;b)]+b_*[\Gamma_+(r)-\Gamma(r;b_*)]>0\end{aligned}$$

for all $r\geq\widetilde{R}$. Then $(\xi_1-\xi_2)(r)<0$ for all $r>\widetilde{R}$, since $(\xi_1-\xi_2)(\infty)=0$. This contradicts (4.16). Hence either $R_e<\infty$ or $\varphi(s;b)$ does not converge to π as $s\to\infty$. The former case implies that $S(b)=\underline{s}$, due to $b\in B$, Lemma 3.2 and Lemma 3.6.

Now we assume that $\varphi(s;b)$ does not converge to π as $s\to\infty$. We claim that there is a $\bar{s}\geq s_p$ such that

$$X(\bar{s})>0,\quad Y(\bar{s})\geq k(\bar{s};b). \tag{4.17}$$

To see this, we consider the sign of $X(s_p)$. If $X(s_p)\leq 0$, then, by Lemma 4.3, there exists a $s_+\geq s_p$ such that $X(s)>0$. Hence there is a $\bar{s}\in[s_p,s_+]$ satisfying

$$X(\bar{s})>0,\quad \frac{dX}{ds}(\bar{s})\geq 0. \tag{4.18}$$

Recalling (4.7) and $Y>1$, (4.17) follows. Consider the case where $X(s_p)>0$. If there is a $s_-\geq s_p$ such that $X(s_-)\leq 0$, then we have $X(s_+)>0$ for some $s_+>s_-$, by Lemma 4.3. Similarly, we have a $\bar{s}\in[s_p,s_+]$ satisfying (4.18) and then (4.17) holds. Thus we only need to consider the case where $X(s)>0$ for all $s\geq s_p$. For this case, suppose for contradiction that $Y(s)<k(s;b)$ for all $s\geq s_p$. Then $dY/ds\geq\omega$ for all $s\geq s_p$. Hence there is a $s_+\in[s_p,s_p+(k(s_p;b)-Y(s_p))/\omega]$ such that $Y(s_+)=k(s_+;b)$, a contradiction. Therefore, our claim (4.17) has been proved.

Next, we will prove $X(s)>0$ for all $s\geq\bar{s}$, where $\bar{s}$ is a point such that (4.17) holds.

For this, we first consider the case when $Y(\bar{s})<k(\bar{s})+\omega/X(\bar{s})$. We claim that $Y(s)>k(s)$ for all $s>\bar{s}$. Note that

$$Y'(s):=\frac{dY}{ds}(s)>0\quad\text{whenever } Y(s)<k(s)+\omega/X(s)\text{ and } X(s)>0. \tag{4.19}$$

Then, by the fact that k is decreasing in (s_p,∞), $(Y-k)'(\bar{s})>0$. Hence $Y>k$ for $s>\bar{s}$ with $s-\bar{s}\ll 1$. Suppose that there exists $s_1>\bar{s}$ such that $Y(s_1)=k(s_1)$. Without loss of generality, we may take s_1, the smallest one, such that $Y>k$ in $(\bar{s},s_1)$ and $Y(s_1)=k(s_1)$. Then $(Y-k)'(s_1)\leq 0$ which implies that $Y'(s_1)<0$. On the other hand, by (4.7), we have $X>0$ in $[\bar{s},s_1]$. In particular, $X(s_1)>0$ and $Y(s_1)<k(s_1)+\omega/X(s_1)$. It follows from (4.19) that $Y'(s_1)>0$, a contradiction. Therefore, we have proved that $Y>k$ in $(\bar{s},\infty)$. Then, by (4.7), X is increasing in $(\bar{s},\infty)$ and so $X(s)>0$ for all $s\geq\bar{s}$.

Suppose next that $Y(\bar{s})\geq k(\bar{s})+\omega/X(\bar{s})$. Since

$$\frac{dX}{ds}(s)>0,\quad \frac{d}{ds}\left(Y-k-\frac{\omega}{X}\right)(s)>0$$

whenever $Y\geq k+\omega/X$, $X>0$ and $s\geq s_p$ (in which k is non-increasing), the trajectory satisfies conditions $X>0$ and $Y>k+\omega/X$ for $s>\bar{s}$.

Therefore, we have proved that $X(s)>0$ for all $s\geq\bar{s}$.

Now we consider the case where $X(s_1) > 0$ and $Y(s_1) \geq k(s_1) + \omega/X(s_1)$ for some $s_1 > \bar{s}$. From the argument above, the trajectory satisfies $X > 0$ and $Y > k + \omega/X$ for $s > s_1$. It follows from (4.7) that $Y(s)$ is decreasing for $s > s_1$. Then $\varphi(s)$ is decreasing for $s > s_1$, since

$$\frac{d\varphi}{ds} = \frac{\frac{dY}{ds} - \frac{dr}{ds}\omega \sin\varphi}{X} = \frac{\frac{dY}{ds} - \omega \sin^2\varphi}{X} < 0.$$

Hence the limit $Y_\infty := \lim_{s\to\infty} Y(s)$ exists with $Y_\infty \in [1, Y(s_1)]$ which also implies that $\varphi(\infty; b) = 0$. Since there is a sequence $\{r_n\}$ tending to infinity such that

$$\begin{aligned} 0 &= \lim_{n\to\infty} \frac{d(\Gamma_+ - \Gamma)(r_n)}{dr} = \lim_{n\to\infty} \left(\frac{\cos\Phi_+(r_n)}{r_n \sin\Phi_+(r_n)} - \frac{\cos\Phi(r_n)}{r_n \sin\Phi(r_n)} \right) \\ &= -\omega - \frac{\omega}{Y_\infty - 1} < 0, \end{aligned}$$

we have a contradiction.

Finally, it remains to exclude the case that the trajectory satisfies conditions $X > 0$ and $Y < k + \omega/X$ for $s > \bar{s}$. For this, we suppose that $X(s) > 0$ and $Y(s) < k(s) + \omega/X(s)$ for all $s > \bar{s}$. Under this assumption, we see that $Y(s) - k(s)$ is increasing for all $s > \bar{s}$. Then we obtain that $k(s) < Y(s) < k(s) + \omega/X(s)$ for all $s \geq \bar{s}$. Hence $X(s)$ is increasing for $s \geq \bar{s}$ and this implies that $Y_\infty \in (1, k_\infty + \omega/X(\bar{s})]$. Using the fact that $\varphi(s) \in (0, \pi/2)$ for all $s \geq \bar{s}$, we may re-write

$$\varphi(s) = \sin^{-1}\left[\frac{Y(s) - 1}{\omega r(s)}\right]$$

and derive that $\varphi(\infty; b) = 0$. Then we reach a contradiction as before. Therefore, we conclude that $R_e(b) < \infty$. This completes the proof of this lemma. □

The uniqueness part of Theorem 2.2 immediately follows from Lemmas 4.2 and 4.4. Thus we have completed the proof of Theorem 2.2.

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