ENTIRE SOLUTIONS OF REACTION-DIFFUSION EQUATIONS WITH BALANCED BISTABLE NONLINEARITIES

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Abstract. This paper deals with entire solutions of a bistable reaction-diffusion equation for which the speed of the traveling wave connecting two constant stable equilibria is zero. Entire solutions which behave as two traveling fronts approaching, with super-slow speeds, from opposite directions and annihilating in a finite time are constructed by using a quasi-invariant manifold approach. Such solutions are shown to be unique up to space and time translations.

1. Introduction

In this paper, we study the following reaction-diffusion equation in one space dimension

$$u_t = u_{xx} - f(u), \tag{1.1}$$

where $f(0) = f(1) = 0$. For a cubic nonlinearity $f(u) = u(1-u)(a-u)$, this equation is called the Allen-Cahn equation ($a = 1/2$) originally used for phase transitions [1]. It is also called the Nagumo equation for propagations of nerve excitations. In various biological models, (1.1) is often written in the form

$$v_t = v_{xx} + g(v), \quad g(v) = f(1-v),$$

where $v = 1-u$ stands for population density. It becomes the classical KPP model [23] when the logistic growth $g(v) = v(1-v)$, that is, $f(u) = g(1-u) = u(1-u)$, is used.

In these models, it is assumed that $u \equiv 0$ and $u \equiv 1$ are two steady spatially homogeneous states of (1.1). We are interested in solutions representing the interaction of these two states. One such interaction can be described by a traveling wave which is a solution of the form $u(x,t) = Q(\xi)$, where $\xi = x - ct$. For the existence, uniqueness, and the stability of traveling wave solutions, we refer the readers to [2, 3, 4, 5, 9, 11, 12, 15, 14, 22, 23, 25], and the references cited therein.

From a dynamical point of view, the behavior of solutions are captured by attractors or omega limit sets. These sets are invariant under the flow governed by the reaction diffusion equation; in particular, they consists of **entire solutions**, defined here as solutions that exist for all $(x,t) \in \mathbb{R}^2$. Traveling waves are typical example of entire solutions. There are

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many other types of entire solutions. One that attracts recent interest is an entire solution describing two traveling fronts approaching from opposite directions and annihilating in a finite time; see [20, 21, 16, 26, 19] and references therein. We call such a solution a **2-front** entire solution. In the constructions of 2-front entire solutions in [20, 21, 16, 26, 19], it is mostly assumed that the state whose occupying spatial region will be wiped out in finite time is either unstable or lesser stable than the other eventual dominating state; that is, the wave speed c of the traveling wave solutions $u(x,t) = Q(x - ct)$ is non-zero.

In this paper, we consider the **bistable** case in which the corresponding ordinary differential equation $U_t = -f(U)$ has two stable equilibria 0 and 1. Especially, we shall consider the **balanced bistable** case, that is, the two stable constant states $u \equiv 1$ and $u \equiv 0$ have the same strength in the sense that the traveling wave connecting the two states has zero speed. In this case, the competition between these two states will mostly depend on the geometry of the region they occupy in general n space dimension.

In the one space dimension, the following **meta-stability** plays the role. Pick n different points $z_1, \cdots, z_n$ on $\mathbb{R}$, arranged from left to right. Let $u_0(\cdot)$ be a concatenation of n segments of traveling wave profile $Q((-1)^i(x - z_i))$ in $(\frac{1}{2}(z_{i-1} + z_i), \frac{1}{2}(z_i + z_{i+1}))$, where $z_0 = -\infty, z_{n+1} = \infty$. If $d = \min_i\{z_{i+1} - z_i\}$ is large, then u_0 is almost an equilibrium, called a quasi-equilibrium or a **meta-stable state**. The evolution of meta-stable states is super-slow and has been extensively investigated by Carr-Pego [6, 7], Fusco [17], Fusco-Hale [18] where an ode system describing the dynamics of $z_1(t), \cdots, z_n(t)$ was derived and rigorously verified. It is shown that the speed $\dot{z}_i(t)$ is proportional to $e^{-\alpha(z_{i+1}-z_i)} - e^{-\beta(z_i-z_{i-1})}$ where either (α, β) or (β, α) equals $(\sqrt{f'(1)}, \sqrt{f'(0)})$.

In [10, 24], the terminology "kink" was used instead of "front" here. In [10], among other things, Eckmann-Rougemont presented a description of the annihilation (collapse) of two nearby fronts. Rougemont [24] considered, in particular, the dynamics of four fronts. After the first annihilation of two middle fronts, the "bump" vanishes sufficiently fast so that one sees again two slowly moving fronts. Then the remaining two fronts shall again be annihilated after some time. This annihilation process can be applied to multi-front solutions. See also the recent paper by Chen [8], where generation, propagation, and annihilation (not necessarily pairwise) of traveling fronts are considered. While initial value problems are considered in [10, 24, 8], here we focus on entire solutions, those that are initiated from $t = -\infty$.

In this paper, we shall construct a 2-front entire solution which behaves like two traveling fronts approaching from opposite directions. Thus it is natural to adopt the following "**initial**

condition" at $t=-\infty$: (see Fig. 1)

$$(1.2)\quad \lim_{t\to-\infty}\ \inf_{p>q}\Big\{\sup_{x>(p+q)/2}\Big|u(x,t)-Q(p-x)\Big|+\sup_{x<(p+q)/2}\Big|u(x,t)-Q(x-q)\Big|\Big\}=0.$$

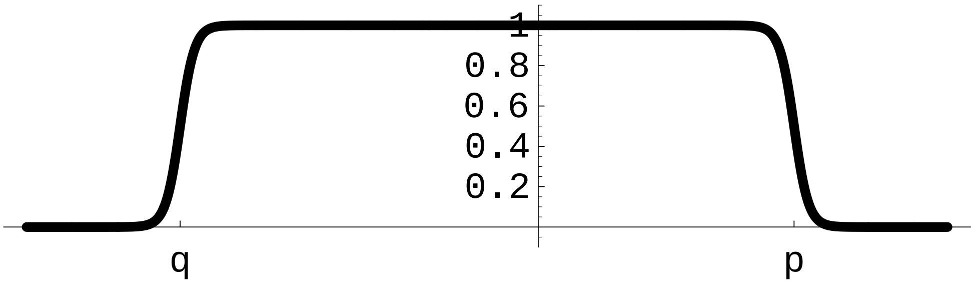

FIGURE 1. The profile of the entire solution at $-t$ for large $t>0$.

As mentioned, the existence, as well as partial uniqueness, of entire solutions of (1.1), (1.2) for the case $c\neq 0$ has been studied in [20, 21, 16, 26, 19]. Since we shall consider the case $c=0$, $f(s)=F'(s)$ becomes the derivative of a double–equal–well potential satisfying

$$(1.3)\qquad F\in C^4(\mathbb{R}),\ F''(0)>0,\ F''(1)>0,\ F(0)=F(1)=0<F(s)\ \forall\, s\neq 0,1.$$

In this case, (1.1) admits a monotonic standing wave: $u(x,t)=Q(x)$ where Q is the solution of

$$\ddot{Q}(z)=f(Q(z))\quad \forall\, z\in\mathbb{R},\quad Q(-\infty)=0,\ Q(\infty)=1.$$

The first integral $\dot{Q}=\sqrt{2F(Q)}$ provides a special solution, for $\mu:=\sqrt{f'(1)}$,

$$\mu z=\int_1^{Q(z)}\Big(\frac{\mu}{\sqrt{2F(s)}}-\frac{1}{(1-s)}\Big)ds-\ln[1-Q(z)]\quad \forall z\in\mathbb{R}.$$

In the sequel, Q always refers to this particular solution. It has the expansion

$$Q(z)=1-e^{-\mu z}+\tfrac{f''(1)}{6f'(1)}\,e^{-2\mu z}+O(1)e^{-3\mu z}\quad\text{as}\quad z\to\infty.$$

We shall prove the following existence and uniqueness theorem.

Theorem 1.1. *Assume (1.3). Then (1.1), (1.2) admits a solution. In addition, the solution is unique up to space and time translations; namely, if u_1 and u_2 are solutions of (1.1), (1.2), then there exist constants ξ,η such that*

$$(1.4)\qquad u_1(x,t)=u_2(x+\xi,t+\eta)\qquad \forall (x,t)\in\mathbb{R}^2.$$

Furthermore, the solution satisfies

$$\lim_{t\to\infty}\|u(\cdot,t)\|_{L^\infty(\mathbb{R})}=0,$$
$$\lim_{|x|\to\infty}u(x,t)=0\quad\forall t\in\mathbb{R},$$
$$\lim_{t\to-\infty}\sup_{y\in\mathbb{R}}\Big|u(y+z^*,t)-Q\Big(y+\tfrac{1}{2\mu}\ln|2\alpha\mu t|\Big)Q\Big(\tfrac{1}{2\mu}\ln|2\alpha\mu t|-y\Big)\Big|=0$$

for some $z^*\in\mathbb{R}$, where

$$\mu:=\sqrt{f'(1)},\qquad \alpha=\frac{2f'(1)}{\int_0^1\sqrt{2F(s)}ds}. \tag{1.5}$$

The "initial" condition (1.2) can also be replaced by the following:

There exist constants $L>0$ and $T<0$, and functions $p(\cdot)$ and $q(\cdot)$ such that for all $t\le T$,

$$\begin{cases} u(x,t)\le\alpha_0 & \forall\, x\in(-\infty,q(t)]\cup[p(t),\infty),\\ u(x,t)\ge\beta_0 & \forall\, x\in[q(t)+L,p(t)-L],\end{cases} \tag{1.6}$$

where α_0, β_0 are constants satisfying

$$f>0\quad in\ \ (0,\alpha_0],\qquad f<0\quad in\ [\beta_0,1). \tag{1.7}$$

In many applications, (1.1) takes the form

$$\varepsilon^2U^\varepsilon_\tau(y,\tau)=\varepsilon^2U^\varepsilon_{yy}-f(U^\varepsilon),\quad y\in\mathbb{R},\ \tau>0,$$

where ε is a small positive constant (cf. [1]). It relates to (1.1) by $u(x,t)=U^\varepsilon(\varepsilon x,\varepsilon^2t)$. Now consider a bounded initial data $U^\varepsilon(\cdot,0)$ on $\mathbb{R}$ that satisfies

$$U^\varepsilon(y,0)\ge\beta_0\qquad\forall\, y\in[-l,l],$$
$$U^\varepsilon(y,0)\le\alpha_0\qquad\forall\, y\in[-3l-\delta,-l-\sqrt{\varepsilon}]\cup[l+\sqrt{\varepsilon},3l+\delta],$$

where δ and l are fixed positive constants independent of ε. It is shown in [8] that the solution can be visualized as in Figure 1 and expressed as

$$U^\varepsilon(y,\tau)\sim Q\Big(\frac{y-q^\varepsilon(\tau)}{\varepsilon}\Big)Q\Big(\frac{p^\varepsilon(\tau)-y}{\varepsilon}\Big)\qquad\forall\, y\in[-3l,3l],\ \tau\in[\sqrt{\varepsilon},T^\varepsilon],$$

where $p^\varepsilon(\tau)$ and $q^\varepsilon(\tau)$, with $q^\varepsilon(\tau)<p^\varepsilon(\tau)$ and referred to as the positions of **interfaces**, are functions from $\tau\in[\sqrt{\varepsilon},T^\varepsilon]$ to $[-l-\sqrt{\varepsilon},l+\sqrt{\varepsilon}]$ and T^ε is the first time such that $p^\varepsilon(T^\varepsilon)-q^\varepsilon(T^\varepsilon)=\sqrt{\varepsilon}$. In their pioneer work, Carr and Pego [6, 7] derived and verified an ode system for the positions of interfaces, which in the current case implies that the distance $p^\varepsilon-q^\varepsilon\sim 2p^\varepsilon$ will decrease in an exponentially slow rate; see also [17, 18, 10, 24, 8]. A detailed calculation in [8] shows that any position of interface outside the interval $[-3l-\delta,3l+\delta]$ will not move into the interval $[-3l,3l]$ in the $[0,3T^\varepsilon]$ time interval. Note that at $\tau=T^\varepsilon$, the two positions p^ε and q^ε of the two interfaces are $\sqrt{\varepsilon}$ away, and are expected to approach

closer and closer and eventually annihilate each other. Clearly, using the standing wave Q alone one cannot expect to obtain an asymptotic expansion valid all the way up to the total annihilation of the two interfaces, i.e., to the disappearance of the phase region $\{y \in [-3l, 3l] \mid U^\varepsilon(y,\tau) \geq \beta_0\}$. Using our entire solution u to (1.1)-(1.2) and the analysis presented in [8], one can show that the solution has the asymptotic expansion, as $\varepsilon \searrow 0$,

$$U^\varepsilon(y,\tau) \sim u\Big(\frac{y-\hat{z}^\varepsilon}{\varepsilon}, \frac{\tau - \hat{T}^\varepsilon}{\varepsilon^2}\Big) \quad \forall\, y \in [-3l, 3l],\ \ \tau \in [\sqrt{\varepsilon}, 2\hat{T}^\varepsilon]$$

where $\hat{z}^\varepsilon = O(\sqrt{\varepsilon})$ is the "center" of the phase region and $\hat{T}^\varepsilon = [1+o(1)]T^\varepsilon$ is the first time at which the local maximum of $U^\varepsilon(\cdot,\tau)$ in $[-3l,3l]$ is equal $u(0,0)$, which can be normalized to be, say, α_0. Notice that at $\tau = 2\hat{T}^\varepsilon$, $U^\varepsilon(y,\tau)$ is exponentially small for all $y \in [-3l, 3l]$.

From the above discussion, we see that while a traveling wave describes the motion of a single front (between two different phase regions), an entire solution established here describes the detailed interactive behavior of two fronts when they approaching each other and annihilate after they are sufficiently close. In this sense, *the entire solutions, together with traveling waves, can be regarded as characteristic solutions for the non-linear dynamics* $u_t = u_{xx} - f(u)$. *They form fundamental building blocks for matched asymptotic expansions.*

Remark 1.2. *The initial and boundary conditions for entire solutions modeling the annihilation of wave fronts are*

$$\liminf_{t\to-\infty}\ \max_{x\in\mathbb{R}} u(x,t) \geq \beta_0, \qquad \limsup_{|x|\to\infty}\ u(x,t) \leq \alpha_0 \quad \forall\, t \in \mathbb{R}. \tag{1.8}$$

We believe that these two conditions are not sufficient for uniqueness, since the work of Chen in [8] *indicates the existence of entire solutions simulating the annihilation of multiple* (≥ 3) *fronts at the same time and the same spot. We leave it here as an open problem to characterize all entire solutions of (1.1) that satisfy (1.8).*

Remark 1.3. *In* [13]*, it is shown that the solution with (1.6) converges to* 0 *as* $t \to \infty$*. In* [26]*, Yagisita has proved the existence of 2-front entire solutions under certain conditions including our case* $c = 0$*. However, no details of the properties of these solutions are given.*

We note that the existence of a 2-front entire solution can also be established as follows. For each positive integer n, let $u_n(x,t)$ be the solution to (1.1) with initial value

$$u_n(x,0) = 1 \quad \text{when} \quad |x| < n, \qquad u_n(x,0) = 0 \quad \text{when} \quad |x| \geq n.$$

Let $t_n > 0$ be a time satisfying $u_n(0,t_n) = 0.5$. Using well-known results of [14, 10, 24], it is not very difficult to show that a subsequence of $\{u_n(x, t_n + t)\}_{n=1}^\infty$ approaches a limit. The well-known results on meta-stable motion of interface in any of the papers [6, 17, 18, 10, 24, 8] tell us that the limit is an entire solution that we want and has the asymptotic profile that we described.

However, to show the uniqueness of the entire solution, it seems to us that the available results on the meta-stable motion of interface (e.g. [6, 7, 17, 18, 10, 24, 8]) cannot be directly applied; new results are needed to be established. Indeed, most of our effort in this paper is devoted to this analysis. It turns out that we have to employ the full geometric method (cf. [6, 8]) and the method of global analysis [14] instead of applying the existing well-known results derived by these methods.

This paper is organized as follows. In §2, we prove the existence by using a quasi-invariant manifold, that is, a set of meta-stable states or approximate equilibria of (1.1). In §3, we define the quasi-invariant manifold. After showing that 2-front entire solutions stay very close to the spatial translations of the quasi-invariant manifold in §4, we prove the uniqueness of the entire solutions of (1.1), (1.2) in §5, using a geometric theory [7, 17, 18, 8]. Finally, in §6, we extend the uniqueness result to those entire solutions of (1.1) that satisfy (1.6) by using the methods of Fife-McLeod [14] and the method of Chen [8].

2. Existence

For existence, we need to consider only solutions that are even in x, namely, solutions of

$$\begin{cases} u_t = u_{xx} - f(u), & x > 0, t \in \mathbb{R}, \\ u_x(0,t) = 0, & t \in \mathbb{R}, \\ u(x,t) = Q(p(t) - x) + o(1) & \text{as } \ t \to -\infty, \ x \geq 0. \end{cases} \tag{2.1}$$

Here $p(t) > 0$ for all $t < 0$ is to be determined.

For this, we do the following steps.

1. Construct a pair $(c(p), \Phi(x,p))$, for $p \geq 0$ and $x \in [0,\infty)$, such that

$$\begin{aligned} \Phi_{xx} - f(\Phi) &= c(p)\Phi_p + O(1)|c(p)|^{2+\varepsilon}, \quad (\varepsilon > 0) \\ \Phi_x(0,p) &= 0, \\ \Phi(x,p) &= Q(p+x)Q(p-x) + o(1), \\ \Phi_p(x,p) &= \dot{Q}(p-x) + \dot{Q}(p+x) + o(1), \\ \Phi_x(x,p) &= \dot{Q}(p+x) - \dot{Q}(p-x) + o(1) \end{aligned} \tag{2.2}$$

where subscripts denote partial derivatives, and

$$\lim_{p\to\infty} \|o(1)\|_{C^0([0,\infty))} = 0.$$

Here and in the sequel, $O(1)$ is a quantity that is bounded by a constant independent of x and p:

$$\sup_{p\geq 0, x\geq 0} \Big|O(1)\Big| < \infty.$$

As we shall see later, for μ, α defined in (1.5),

$$c(p) = -\alpha e^{-2\mu p} + O(1) p e^{-4\mu p}. \tag{2.3}$$

The construction will be presented in the next section, with a precise statement given in Theorem 3.4.

2. Next we construct sub-super solutions. Note that $f'(0) > 0$ and $f'(1) > 0$ imply the existence of a positive constant A such that for all $z \in \mathbb{R}$,

$$A\dot{Q}(z) + f'(Q(z)) = A\sqrt{2F(Q)} + f'(Q) > 2\kappa, \quad \kappa := \tfrac{1}{4}\min\{f'(0), f'(1)\} > 0.$$

Hence, for every p large enough,

$$A\Phi_p + f'(\Phi) \geq \kappa \quad \forall x \geq 0.$$

A supersolution $\bar{u}(x,t)$, for $t \leq 0, x \in [0,\infty)$, can be constructed by setting

$$\bar{u}(x,t) := [\Phi(x,p) + B(p)]\Big|_{p=\bar{p}(t)}, \qquad B(p) := e^{-(4+\varepsilon)\mu p}$$

where $\bar{p}$ is the solution of the ordinary differential equation

$$\begin{cases} \dot{\bar{p}} = c(\bar{p}) + AB(\bar{p}) & \forall\, t < 0, \\ \bar{p}(0) = b, \end{cases} \tag{2.4}$$

where b is large enough such that $0 < AB(p) < -c(p)$ for all $p \geq b$. Note that $\dot{\bar{p}} < 0$ for all $t \leq 0$. Also, $|c|^{2+\varepsilon} = O(1)Be^{-\varepsilon\mu p}$.

Simple calculation gives

$$\begin{aligned} & \bar{u}_t - \bar{u}_{xx} + f(\bar{u}) \\ = \;& (\Phi_p + B_p)\dot{\bar{p}} - \Phi_{xx} + f(\Phi) + f'(\Phi)B + O(1)B^2 \\ = \;& (\dot{\bar{p}} - c(\bar{p}))\Phi_p + f'(\phi)B + O(1)(|c|^{2+\varepsilon} + B^2) + B_p\dot{p} \\ = \;& B\Big\{A\Phi_p + f'(\Phi)\Big\} + O(1)(|c|^{2+\varepsilon} + B^2 + |c|B) \\ \geq \;& \kappa B + O(1)(B^2 + Be^{-\varepsilon\mu p} + Be^{-2\mu p}) > 0 \quad \forall t \leq 0, x \geq 0 \end{aligned}$$

provided that we take b large enough.

The subsolution can be constructed by setting

$$\underline{u}(x,t) = [\Phi(x,p) - B(p)]\Big|_{p=\underline{p}(t)}, \qquad B(p) = e^{-(4+\varepsilon)\mu p}$$

where $\underline{p}$ is the solution of the ordinary differential equation

$$\begin{cases} \dot{\underline{p}} = c(\underline{p}) - AB(\underline{p}) & \forall\, t < 0 \\ \underline{p}(0) = a, \end{cases} \tag{2.5}$$

where a is sufficiently large.

3. To construct 2-front entire solutions of (1.1), we want to choose a and b such that the supersolution is bigger than the subsolution, and as $t \to -\infty$, the supersolution approaches the subsolution.

For this, we first claim that $\bar{u}(\cdot,t) \geq \underline{u}(\cdot,t)$ as long as $\bar{p}(t) \geq \underline{p}(t) \gg 1$.

Indeed, from the construction of Φ in the next section, there holds

$$\Phi_p(x,p) \geq \dot{Q}(p-x)\Big\{\tfrac{1}{2} - e^{-\mu p}(1+|x-p|^2)\Big\}.$$

Hence, $\Phi_p > 0$ when $x \leq e^{\mu p/3}$. When $x > e^{\mu p/3}$, we can use $0 < \dot{Q}(p-x) = O(1)e^{-\sqrt{f'(0)}(x-p)}$ to conclude that

$$\Phi_p \geq -e^{-2(3+2\varepsilon)\mu p}$$

for all p large enough. It then follows that when $p_2 > p_1 \gg 1$,

$$\Phi(x,p_2) - \Phi(x,p_1) = \int_{p_1}^{p_2} \Phi_p(x,s)ds \geq -\int_{p_1}^{p_2} e^{-2(3+2\varepsilon)\mu s}\, ds \geq -B(p_1).$$

Thus,

$$\underline{u}(x,t) \leq \bar{u}(x,t) \quad \text{provided that} \quad 1 \ll \underline{p}(t) \leq \bar{p}(t).$$

Now we show that for arbitrary large a, there is a unique $b > a$ such that solutions of (2.4) and (2.5) satisfy

$$\bar{p}(t) > \underline{p}(t) \quad \forall\, t \leq 0, \qquad \lim_{t\to-\infty} (\bar{p}(t) - \underline{p}(t)) = 0. \tag{2.6}$$

In fact, the solutions, for $t \leq 0$, can be written as

$$|t| = \int_b^{\bar{p}(t)} \frac{dp}{|c(p)| - AB(p)} = \int_a^{\underline{p}(t)} \frac{dp}{|c(p)| + AB(p)}.$$

Let $b > a \gg 1$ to be determined. Then

$$\begin{aligned}
\int_{\underline{p}(t)}^{\bar{p}(t)} \frac{dp}{|c(p)| - AB(p)} &= \int_a^b \frac{dp}{|c(p)| + AB(p)} + \int_b^{\underline{p}(t)} \Big(\frac{1}{|c(p)| + AB(p)} - \frac{1}{|c(p)| - AB(p)}\Big)dp \\
&= \int_a^b \frac{dp}{|c(p)| + AB(p)} - \int_b^{\underline{p}(t)} \frac{2AB(p)\, dp}{c^2(p) - A^2B^2(p)}.
\end{aligned}$$

It then follows that if

$$\int_a^b \frac{dp}{|c(p)| + AB(p)} = \int_b^\infty \frac{2AB(p)\, dp}{c^2(p) - A^2B^2(p)}, \tag{2.7}$$

then (2.6) holds. Since $c = (-\alpha + o(1))e^{-2\mu p}$ and $B = e^{-(4+\varepsilon)\mu p}$, the above improper integral is convergent. Hence, for each $a \gg 1$, there is a unique b such that (2.7) holds.

Once we have the sub-super solution pair, the existence of an even entire solution $u(x,t)$ of (1.1) with the property that

$$\bar{u}(x,t) > u(x,t) > \underline{u}(x,t) \quad \forall x \geq 0, t \leq 0 \tag{2.8}$$

can be obtained by taking the limit of the family $\{u_n(x,t)\}_{n=1}^{\infty}$ where u_n is the even solution of (1.1) on $\mathbb{R}\times[-n,\infty)$ with initial value at $t=-n$ given by $u_n(\cdot,-n)=\bar{u}(\cdot,-n)$. Indeed, since $\bar{u}$ is a supersolution, $u_n(\cdot,-n+1)\le\bar{u}(\cdot,-n+1)$ so that by comparison, $u_n\le u_{n-1}$ on $\mathbb{R}\times[-n+1,\infty)$. As $u_n\ge\underline{u}$, $\lim_{n\to\infty}u_n=:u$ exists and is an entire solution of (1.1) satisfying (2.8). As $\lim_{t\to-\infty}\bar{p}(t)=\infty$, such solution also satisfies (1.2).

Remark 2.1. *Solutions of (1.1) and (1.2) can also be obtained as follows:*

For each positive integer n, let $w_n(y,\tau)$ be the solution of

$$w_{n\tau}-w_{nyy}+f(w_n)=0 \text{ in } \mathbb{R}\times(0,\infty),\quad w_n(y,0)=Q(n+y)Q(n-y)\ \forall\, y\in\mathbb{R}.$$

Then for each $\tau>0$, $w_n(\cdot,\tau)$ is even and strictly decreasing in $[0,\infty)$. It is easy to show that if $Q^2(n)>\alpha_0$, there exists a unique time $T(n)>0$ such that

$$w_n(0,T(n))=\alpha_0.$$

With quite amount of work, one can show that

$$n=\frac{1}{2\mu}\ln(2\mu T(n))+o(1).$$

In addition, for each $(x,t)\in\mathbb{R}^2$, the limit

$$u(x,t):=\lim_{n\to\infty}w_n(x,T(n)+t)$$

exists and is the unique solution of (1.1), (1.2) satisfying the normalized condition

$$u(0,0)=\alpha_0=\max_{x\in\mathbb{R}}u(x,0).$$

We shall not provide the details here.

In the next section, we shall construct $(c(p),\Phi(x,p))$. We shall call

$$\{\Phi(\cdot,p)\mid p\ge 0\}$$

a **quasi-invariant manifold** under the dynamics $u_t=\mathbf{A}u:=u_{xx}-f(u)$.

3. The Quasi-Invariant Manifold

The function $\Phi(\cdot,p)$ and constant $c(p)$ are constructed via an iteration procedure. For our purposes, we use two iterations, so it can be written as

$$\Phi=\Phi_0(x,p)+\Phi_1(x,p)+\Phi_2(x,p),\quad c(p)=0+c_1(p)+c_2(p). \tag{3.1}$$

Here Φ_0 is a certain approximation to the equation

$$\Phi_{0xx}-f(\Phi_0)\approx 0,\qquad \Phi_0(\pm p,p)\approx Q(0).$$

The term Φ_1 is added to make the approximation better. Since there is no smooth solution to the exact equation $\Phi_{xx} - f(\Phi) = 0$ with two fronts, a forcing term is needed. We choose a forcing term of the form $c\Phi_{0p}$. Using a variation of constant formula and ignoring higher order terms, we obtain Φ_1 and c_1. Similarly, we obtain Φ_2 and c_2 for a second order approximation.

3.1. **Preparation.** For each $p \geq 0$, set

$$\phi_1(x,p) := Q(p-x), \quad \phi_2(x,p) := Q(p+x). \tag{3.2}$$

Note that $\phi_1(x,p) = \phi_2(-x,p)$;

$$\phi_{1p} = -\phi_{1x} = \sqrt{2F(\phi_1)} > 0, \quad \phi_{2p} = \phi_{2x} = \sqrt{2F(\phi_2)} > 0;$$

$$\text{for all } x > 0: \quad 1 - \phi_2 = O(1)\phi_{2x}, \qquad \phi_{2x} = \mu e^{-\mu(p+x)}[1 + O(1)e^{-\mu(p+x)}]; \tag{3.3}$$

$$\text{when } x \in [0,p]: \quad 1 - \phi_1 = O(1)\phi_{1x}, \quad \phi_{1x} = -\mu e^{-\mu(p-x)}[1 + O(1)e^{-\mu(p-x)}], \tag{3.4}$$
$$\phi_{1x}\phi_{2x} = -\mu^2 e^{-2\mu p}\Big[1 + O(1)e^{-\mu(p-x)}\Big];$$

$$\text{when } x \geq p: \quad \phi_1 = O(1)\phi_{1x}, \qquad \phi_{1x} = O(1)e^{-\sqrt{f'(0)}(x-p)}, \tag{3.5}$$
$$\phi_{1x}\phi_{2x} = O(1)e^{-2\mu p}|\phi_{1x}|^{1+\gamma}, \quad \gamma := \sqrt{f'(1)/f'(0)}.$$

Also, for all integers $k \geq 0, l \geq 0$ with $1 \leq k + l$ (≤ 5),

$$\frac{\partial^{k+l}}{\partial p^k \partial x^l}\phi_1 = (-1)^k \frac{\partial^{k+l}}{\partial x^{k+l}}\phi_1 = O(1)\phi_{1x}, \quad \frac{\partial^{k+l}}{\partial p^k \partial x^l}\phi_2 = \frac{\partial^{k+l}}{\partial x^{k+l}}\phi_2 = O(1)\phi_{2x}. \tag{3.6}$$

3.2. **The zeroth order Φ_0.** Although $\phi_1 = Q(p-x)$ is a good approximation for Φ, it is not even. A better choice would be $\phi_1\phi_2$. Note that $\phi_1\phi_2$ is even and

$$\phi_1\phi_2 - \phi_1 = \phi_1(1-\phi_2) = O(1)e^{-\mu(p+x)}.$$

An investigation for the residue $(\phi_1\phi_2)_{xx} - f(\phi_1\phi_2)$ indicates that the residue in the interval $[-p,p]$ is nearly a constant. As the function $\phi_{1x}\phi_{2x}$ is also nearly a constant in $[-p,p]$, we can add a multiple of $\phi_{1x}\phi_{2x}$ to $\phi_1\phi_2$ to make the residue small in $[-p,p]$. Hence, we set

$$\begin{aligned} \Phi_0 &:= \Phi_0(x,p) := \phi_1\phi_2 + k\phi_{1x}\phi_{2x}, \qquad k := \frac{f'(1) + f''(1)}{f'(1)^2}, \\ R_0 &:= \Phi_{0xx} - f(\Phi_0). \end{aligned}$$

Note that the term $k\phi_{1x}\phi_{2x}$ in the definition of Φ_0 is introduced to obtain the extra factor $(1-\phi_1)$ in the estimate of R_0, R_{0p}, and R_{0pp} as in the following lemma. This factor is very useful in controlling the size of Λ defined in (3.9) below.

Lemma 3.1. *For each $p \geq 0$, $\Phi_0(\cdot,p)$ is an even function and for all $x \geq 0$,*

$$\Big|R_0\Big| + \Big|R_{0p}\Big| + \Big|R_{0pp}\Big| = O(1)(1-\phi_1)\phi_{1x}\phi_{2x}. \tag{3.7}$$

Proof. Using $\phi_{ixx} = f(\phi_i)$, $\phi_{ixxx} = f'(\phi_i)\phi_{ix}$, and $\phi_{1x}\phi_{2x} = -\sqrt{4F(\phi_1)F(\phi_2)}$, we find

$$R_0 \quad = \quad W(\phi_1(x,p), \phi_2(x,p)),$$

where

$$\begin{aligned} W(s_1, s_2) \quad := \quad & s_1 f(s_2) + s_2 f(s_1) + 2k f(s_1) f(s_2) - f(s_1 s_2 - k\sqrt{4F(s_1)F(s_2)}\ ) \\ & -\sqrt{4F(s_1)F(s_2)}\Big(2 + kf'(s_1) + kf'(s_2)\Big). \end{aligned}$$

By the assumption (1.3), the function $s \to \sqrt{2F(s)}$ is a $C^3([0,1])$ function, so W is also $C^3([0,1]^2)$ function. One observes that

$$0 = W(0,s) = W(1,s) = W(s,0) = W(s,1) \quad \forall\ s \in [0,1].$$

Hence, for $m = 0$ or 1,

$$W(\phi_1, \phi_2) = \int_1^{\phi_2} \int_m^{\phi_1} \frac{\partial^2}{\partial s_1 \partial s_2} W(s_1, s_2)\, ds_1\, ds_2. \tag{3.8}$$

First we consider the case $x \geq p$. We take $m = 0$ in (3.8) to obtain

$$R_0 \quad = \quad W(\phi_1, \phi_2) = O(1)\phi_1(1 - \phi_2) = O(1)\phi_{1x}\phi_{2x}.$$

Also, using (3.6),(3.3), and (3.5), we obtain

$$\begin{aligned} R_{0p} \quad &= \quad -W_{s_1}\phi_{1x} + W_{s_2}\phi_{2x} = O(1)\phi_{1x}\phi_{2x}, \\ R_{0pp} \quad &= \quad W_{s_1}\phi_{1xx} + W_{s_2}\phi_{2xx} + W_{s_1 s_1}\phi_{1x}^2 + 2W_{s_1 s_2}\phi_{1x}\phi_{2x} + W_{s_2 s_2}\phi_{2x}^2 = O(1)\phi_{1x}\phi_{2x}. \end{aligned}$$

Hence, when $x \geq p$, (3.7) holds, since $1 - \phi_1 \geq 1 - Q(0) > 0$.

Next, we consider the case when $x \in [0,p]$. In this case, $1 - \phi_2 \leq 1 - \phi_1$. Also, $(1 - \phi_1) = O(1)\phi_{1x}$. Direct calculation shows that $W_{s_1 s_2}(1,1) = 0$ with our particular choice of k. Hence, taking $m = 1$ in (3.8) we obtain

$$\begin{aligned} R_0 \quad &= \quad W(\phi_1, \phi_2) = (1 - \phi_1)(1 - \phi_2)[O(1)(1 - \phi_1) + O(1)(1 - \phi_2)] \\ &= \quad O(1)(1 - \phi_1)^2(1 - \phi_2) = O(1)(1 - \phi_1)\phi_{1x}\phi_{2x}. \end{aligned}$$

Similar to the previous case, one also obtains the required estimate for R_{0p} and R_{0pp}. This completes the proof. □

Notice that there exists a constant $p_0 > 0$ such that for all $p \geq p_0$ and $x \in \mathbb{R}$,

$$\begin{aligned} \Phi_{0p} \quad &= \quad \dot{Q}(p - x)\Big\{Q(p + x) - k\ddot{Q}(p + x)\Big\} + \dot{Q}(p + x)\Big\{Q(p - x) - k\ddot{Q}(p - x)\Big\} \\ &> \quad \tfrac{1}{2}\Big\{\dot{Q}(p - x) + \dot{Q}(p + x)\Big\} > 0. \end{aligned}$$

Also

$$(\Phi_{0p})_{xx} - f'(\Phi_0)\Phi_{0p} = R_{0p} = \Lambda(x,p)\Phi_{0p},$$

where

$$\Lambda(x,p) := \frac{R_{0p}}{\Phi_{0p}} = O(1)(1-\phi_1)\phi_{2x} = O(1)e^{-2\mu p}, \quad \Lambda_p = O(1)e^{-2\mu p}. \tag{3.9}$$

Furthermore, for $m = 0, 1, 2$,

$$\text{when } x \in [0,p]: \qquad \int_0^x (p-y)^m \Phi_{0p}^2(y,p)dy = O(1)[1+(p-x)^m]\Phi_{0p}^2(x,p),$$

$$\text{when } x \geq p: \qquad \int_x^\infty (y-p)^m \Phi_{0p}^2(y,p)dy = O(1)[1+(x-p)^m]\Phi_{0p}^2(x,p).$$

3.3. **The first order Φ_1.** We now define, for $p \geq p_0$ and $x \geq 0$,

$$\begin{aligned} c_1(p) &:= \int_0^\infty R_0 \Phi_{0p}\, dx \Big/ \int_0^\infty \Phi_{0p}^2 dx, \\ \Phi_1(x,p) &:= \Phi_{0p} \int_p^x \frac{dy}{\Phi_{0p}^2(y,p)} \int_0^y (c_1\Phi_{0p} - R_0)\Phi_{0p}, \\ R_1(x,p) &:= (\Phi_0 + \Phi_1)_{xx} - f(\Phi_0 + \Phi_1) - c_1\,(\Phi_0+\Phi_1)_p. \end{aligned}$$

Here c_1 and Φ_1 are derived from the construction of solutions for

$$[\Phi_0 + \Phi_1]_{xx} - [f(\Phi_0) + f'(\Phi_0)\Phi_1] \approx c_1 \Phi_{0p},$$

via a variation of constant formula.

Direct calculation shows that

$$\Phi_{1x}(0,p) = 0 = \Phi_1(p,p),$$

$$\Phi_{1xx} - f'(\Phi_0)\Phi_1 = \Lambda \Phi_1 + c_1 \Phi_{0p} - R_0,$$

$$R_1 = f(\Phi_0) + f'(\Phi_0)\Phi_1 - f(\Phi_0 + \Phi_1) + \Lambda\Phi_1 - c_1 \Phi_{1p}. \tag{3.10}$$

Here $\Phi_1(p,p) = 0$ is chosen so that the fronts are located at $x = \pm p$.

Lemma 3.2. *For all $p \geq p_0$ and $x \geq 0$,*

$$c_1(p) = -\alpha e^{-2\mu p} + O(1) p e^{-4\mu p},$$

$$\Big|c_{1p}\Big| + \Big|c_{1pp}\Big| = O(1)e^{-2\mu p},$$

$$\Big|\Phi_1\Big| + \Big|\Phi_{1p}\Big| + \Big|\Phi_{1pp}\Big| = O(1)e^{-2\mu p}[1+|x-p|]\Phi_{0p},$$

$$\Big|R_1\Big| + \Big|R_{1p}\Big| = O(1)e^{-4\mu p}[1+|x-p|]\Phi_{0p}.$$

Proof. We divide the proof into the following steps.

1. Since $1 - \phi_2 + |\phi_{2xx}| = O(1)\phi_{2x}$,

$$\Phi_{0p} = -\phi_{1x}\phi_2 + \phi_1\phi_{2x} - k\phi_{1xx}\phi_{2x} + k\phi_{1x}\phi_{2xx} = -\phi_{1x} + O(1)\phi_{2x}.$$

It follows that

$$\begin{aligned}\int_0^\infty \Phi_{0p}^2 dx &= \int_0^\infty \{\phi_{1x}^2 + O(1)\phi_{1x}\phi_{2x} + O(1)\phi_{2x}^2\} dx \\ &= \int_0^\infty \dot{Q}(p-x)^2\, dx + O(1)pe^{-2\mu p} \\ &= \int_0^1 \sqrt{2F(s)}\, ds + O(1)pe^{-2\mu p}.\end{aligned}$$

2. Observe that $\Phi_{0p} + \Phi_{0x} = 2\phi_1\phi_{2x} + 2k\phi_{1x}\phi_{2xx}$. It follows that

$$\begin{aligned}\int_0^\infty R_0\Phi_{0p} dx &= -\int_0^\infty R_0\Phi_{0x} dx + 2\int_0^\infty R_0(\phi_1\phi_{2x} + k\phi_{1x}\phi_{2xx}) dx \\ &= -\int_0^\infty \Big\{\Phi_{0xx} - f(\Phi_0)\Big\}\Phi_{0x} dx + \int_0^\infty O(1)((1-\phi_1)\phi_1 + |\phi_{1x}|)|\phi_{1x}\phi_{2x}^2 dx \\ &= -F(\Phi_0(0,p)) + O(1)pe^{-4\mu p} \\ &= -\frac{f'(1)}{2}[\Phi_0(0,p) - 1]^2 + \frac{f''(1)}{6}[1 - \Phi_0(0,p)]^3 + O(1)pe^{-4\mu p}.\end{aligned}$$

Using the expansion of Q and the definition of k, we have

$$\Phi_0(0,p) = Q(p)^2 - k\dot{Q}(p)^2 = 1 - 2e^{-\mu p} - \tfrac{2f''(1)}{3f'(1)} e^{-2\mu p} + O(e^{-3\mu p}).$$

It then follows that

$$\int_0^\infty R_0\Phi_{0p} dx = -2f'(1)e^{-2\mu p} + O(1)pe^{-4\mu p}.$$

The required estimate for $c_1(p)$ then follows from its definition.

3. Using the estimate for R_{0p}, we have

$$\frac{d}{dp}\int_0^\infty R_0\Phi_{0p} dx = \int_0^\infty (R_{0p}\Phi_{0p} - R_0\Phi_{0pp})\, dx = \int_0^\infty O(1)e^{-2\mu p}\phi_{1x}^2 = O(1)e^{-2\mu p}.$$

It then follows that

$$c_{1p} = \frac{\frac{d}{dp}\int_0^\infty R_0\Phi_{0p} dx \;\; - c_1 \frac{d}{dp}\int_0^\infty \Phi_{0p}^2}{\int_0^\infty \Phi_{0p}^2} = O(1)e^{-2\mu p}.$$

Similarly, $c_{1pp} = O(1)e^{-2\mu p}$.

4. Now we estimate Φ_1. First consider $x \in [0,p]$. Note that $R_0 = O(1)(1-\phi_1)\phi_{1x}\phi_{2x} = O(1)e^{-2\mu p}\phi_{1x} = O(1)e^{-2\mu p}\Phi_{0p}$. It follows that for $y \in [0,p]$,

$$\int_0^y (R_0 - c_1\Phi_{0p})\Phi_{0p} = O(1)e^{-2\mu p}\int_0^y \Phi_{0p}^2 = O(1)e^{-2\mu p}\Phi_{0p}^2(y,p).$$

Hence, from the definition of Φ_1,

$$\Phi_1(x,p) = O(1)e^{-2\mu p}(p-x)\Phi_{0p}.$$

Next consider the case when $x \geq p$. From the definition of c_1,

$$\begin{aligned}\int_0^x (c_1\Phi_{0p} - R_0)\Phi_{0p} &= -\int_x^\infty (c_1\Phi_{0p} - R_0)\Phi_{0p}\\ &= O(1)e^{-2\mu p}\int_x^\infty \Phi_{0p}^2 = O(1)e^{-2\mu p}\Phi_{0p}^2(x,p).\end{aligned}$$

It then follows that, when $x \geq p$, $\Phi_1 = O(1)(x-p)\Phi_{0p}$. The estimate for Φ_1 thus follows.

5. Differentiation gives

$$\begin{aligned}\Phi_{1p} &= \Phi_{0pp}\int_p^x \frac{dy}{\Phi_{0p}^2}\int_0^y (c_1\Phi_{0p} - R_0)\Phi_{0p} - \frac{\Phi_{0p}(x,p)}{\Phi_{0p}^2(p,p)}\int_0^p (c_1\Phi_{0p} - R_0)\Phi_{0p}\\ &\quad -2\Phi_{0p}\int_p^x \frac{\Phi_{0pp}dy}{\Phi_{0p}^3}\int_0^y (c_1\Phi_{0p} - R_0)\Phi_{0p} + \Phi_{0p}\int_p^x \frac{dy}{\Phi_{0p}^2}\frac{\partial}{\partial p}\int_0^y (c_1\Phi_{0p} - R_0)\Phi_{0p}.\end{aligned}$$

Using $\Phi_{0pp} = O(1)\Phi_{0p}$ and following the same technique as that for the estimate for Φ_1 in the previous step and the fact that

$$\frac{\partial}{\partial p}\int_0^\infty (c_1\Phi_{0p} - R_0)\Phi_{0p} = 0$$

we can show that $\Phi_{1p} = O(1)e^{-2\mu p}(1+|x-p|)\Phi_{0p}$. Similarly, $\Phi_{1pp} = O(1)e^{-2\mu p}(1+|x-p|)\Phi_{0p}$.

6. Finally, from (3.10) and the fact that $|x-p|\Phi_{1p} = O(1)$, we see that

$$R_1 = O(1)\Phi_1^2 + \Lambda\Phi_1 - c_0\Phi_{1p} = O(1)e^{-4\mu p}(1+|x-p|)\Phi_{0p}.$$

Similarly, we can show that $|R_{1p}| = O(1)e^{-4\mu p}(1+|x-p|)\Phi_{0p}$. This completes the proof. □

3.4. **The second order Φ_2.** Analogous to the construction of (Φ_1, c_1), we define (Φ_2, c_2) by

$$\begin{aligned}c_2(p) &:= \int_0^\infty R_1\Phi_{0p}\,dx \Big/ \int_0^\infty \Phi_{0p}^2 dx,\\ \Phi_2(x,p) &:= \Phi_{0p}\int_p^x \frac{dy}{\Phi_{0p}^2(y,p)}\int_0^y \Big(c_2\Phi_{0p} - R_1\Big)\Phi_{0p},\\ \Phi(x,p) &:= \Phi_0 + \Phi_1 + \Phi_2, \quad c(p) := c_1 + c_2,\\ R(x,p) &:= \Phi_{xx} - f(\Phi) - c\,\Phi_p.\end{aligned}$$

Here, c_2 and Φ_2 are constructed by a variation of constant formula for the problem

$$[\Phi_0 + \Phi_1 + \Phi_2]_{xx} - [f(\Phi_0 + \Phi_1) + f'(\Phi_0)\Phi_2] \approx (c_1 + c_2)(\Phi_{0p} + \Phi_{1p}).$$

Direct calculation shows that

$$\Phi_{2x}(0,p) = 0 = \Phi_2(p,p),$$

$$\Phi_{2xx} - f'(\Phi_0)\Phi_2 = \Lambda\Phi_2 + \Phi_{0p}c_2 - R_1,$$

$$R = f(\Phi_0 + \Phi_1) + f'(\Phi_0)\Phi_2 - f(\Phi_0 + \Phi_1 + \Phi_2) + \Lambda\Phi_2 - (c_1 + c_2)\,\Phi_{2p} - c_2\Phi_{1p}. \tag{3.11}$$

Lemma 3.3. *For all $p \ge p_0$ and $x \ge 0$,*

$$\Big|c_2\Big| + \Big|c_{2p}\Big| = O(1)e^{-4\mu p},$$
$$\Big|\Phi_2\Big| + \Big|\Phi_{2p}\Big| = O(1)(1+|p-x|^2)e^{-4\mu p}\phi_{0p},$$
$$\Big|R\Big| = O(1)e^{-6\mu p}(1+|x-p|^2)\Phi_{0p} = O(1)e^{-6\mu p}.$$

Proof. First, the estimate for R_1 and R_{1p} implies that $|c_2| + |c_{2p}| = O(1)e^{-4\mu p}$.

Next, same as before, when $y \in [0,p]$,

$$\begin{aligned}\int_0^y \Big(c_2\Phi_{0p} - R_1\Big)\Phi_{0p}(z,p)\,dz &= O(1)e^{-4\mu p}\int_0^y (1+|z-p|)\Phi_{0p}^2\\ &= O(1)e^{-4\mu p}(1+|x-p|)\Phi_{0p}^2.\end{aligned}$$

It then follows that when $x \in [0,p]$, $\Phi_2(x,p) = O(1)e^{-4\mu p}(1+|x-p|^2)\Phi_{0p}$.

Similarly, when $y > p$,

$$\int_0^y \Big(c_2\Phi_{0p} - R_1\Big)\Phi_{1p}(z,p)\,dz = -\int_y^\infty \Big(c_2\Phi_{0p} - R_1\Big)\Phi_{0p}\,dz$$
$$= O(1)e^{-4\mu p}\int_y^\infty (1+|x-p|)\Phi_{0p}^2 = O(1)e^{-4\mu p}(1+|x-p|)\Phi_{0p}^2.$$

Hence $\Phi_2 = O(1)(1+|x-p|^2)e^{-4\mu p}\Phi_{0p}$. Upon differentiation, we also obtain

$$\Phi_{2p} = O(1)e^{-4\mu p}(1+|x-p|^2)\Phi_{0x}.$$

Finally, from (3.11) one obtains that

$$\begin{aligned}R &= (f'(\Phi_0+\Phi_1) - f'(\Phi_0))\Phi_2 + O(1)\Phi_2^2 + \Lambda\Phi_2 - (c_1+c_2)\Phi_{2p} - c_2\Phi_{1p}\\ &= O(1)e^{-6\mu p}(1+|x-p|^2)\Phi_{1p}.\end{aligned}$$

This completes the proof. □

We can now summarize the properties of Φ. We extend our function Φ evenly over $x \in (-\infty, 0]$. Also, we can extend it smoothly to $p \in [0, p_0]$.

Theorem 3.4. *There exists $(c(p), \Phi(y,p))$ for $p \ge 0$ and $y \in \mathbb{R}$ such that $\Phi(\cdot,p)$ is even, and*

$$\begin{aligned}\Phi_{yy} - f(\Phi) &= c\Phi_p + O(1)e^{-6\mu p},\\ c &= -\alpha\, e^{-2\mu p} + O(1)pe^{-4\mu p},\\ \Phi(y,p) &= \Phi_{00}(y,p)\Big\{1 + O(1)e^{-2\mu p}[1+|y-p|^2]\Big\},\\ \Phi_p(y,p) &= \Phi_{00p}(y,p)\Big\{1 + O(1)e^{-2\mu p}(1+|y-p|^2)\Big\},\\ \Phi_y(y,p) &= \Phi_{00y}(y,p) + O(1)e^{-2\mu p}[1+|y-p|^2]\Phi_{00p},\end{aligned} \tag{3.12}$$

where

$$\Phi_{00}(y,p) := Q(p-y)Q(p+y),$$
$$\sup_{p\geq 0, y\in\mathbb{R}} \Big|O(1)\Big| =: M < \infty.$$

3.5. **An Eigenvalue Estimate.** For later applications, we consider the linear operator

$$\mathcal{L}\phi := \phi_{yy} - f'(\Phi(y,p))\phi \tag{3.13}$$

where p is any large enough constant.

For convenience, we use the notation

$$\langle \phi, \psi\rangle = \int_{\mathbb{R}} \phi(y)\psi(y)dy, \qquad \|\phi\| = \sqrt{\langle \phi, \phi\rangle}.$$

Also, we use the notation $\phi \perp \psi$ when $\langle \phi, \psi\rangle = 0$.

Since $\Phi(\cdot, p)$ is an even function, $\Phi_p(\cdot, p)$ is even and $\Phi_y(\cdot, p)$ is odd. Hence,

$$\langle \Phi_y, \Phi_p\rangle = 0, \qquad \forall p \geq 0.$$

The following theorem shows that the self-adjoint operator $\mathcal{L}$ has two eigenvalues of order $e^{-2\mu p}$, and all the remaining eigenvalues are strictly negative.

Theorem 3.5. *Let $\mathcal{L}$ be defined as in (3.13). Then for all $p \geq 0$,*

$$\begin{aligned} \mathcal{L}\Phi_y &= O(1)e^{-2\mu p}, \\ \mathcal{L}\Phi_p &= O(1)e^{-2\mu p}. \end{aligned} \tag{3.14}$$

In addition, there exist positive constants ν, p_0 such that for all $p \geq p_0$,

$$\langle \mathcal{L}\phi, \phi\rangle \leq -3\nu\Big(\|\phi\|^2 + \|\phi_y\|^2\Big) \qquad \forall \phi \in H^2(\mathbb{R}), \phi\perp \Phi_y, \phi \perp \Phi_p. \tag{3.15}$$

Proof. The estimates for $\mathcal{L}\Phi_y$ and $\mathcal{L}\Phi_p$ follows by differentiating the equation (3.12) with respect to y and p respectively. To prove (3.15), consider the symmetric bilinear form

$$\mathcal{L}(\phi,\psi) = (\mathcal{L}\phi, \psi) = -\int_{\mathbb{R}} \Big\{\phi_y\psi_y + f'(\Phi)\phi\psi\Big\}.$$

We need only show that $\mathcal{L}(\phi, \phi) \leq -3\nu$ for any ϕ satisfying

$$\|\phi\|^2 + \|\phi_y\|^2 = 1, \quad \phi \perp \Phi_y, \quad \phi\perp\Phi_p. \tag{3.16}$$

We divide the proof into several steps.

1. Denote

$$Q_\pm = Q(p\pm y), \qquad \mathcal{L}_0^\pm \phi := \phi_{yy} - f'(Q_\pm)\phi.$$

Since $\mathcal{L}_0^\pm \dot{Q}_\pm = 0$, $(0, \dot{Q}_\pm)$ is an eigenpair of the self-adjoint operator $\mathcal{L}_0^\pm$. In addition, all the spectrum in $(-\min\{f'(0), f'(1)\}, \infty)$ are eigenvalues. Since $\dot{Q}_\pm > 0$, 0 is the principal eigenvalue of $\mathcal{L}_0^\pm$ and the next eigenvalue, if it exists, is strictly negative. Hence, there exists a positive constant $\nu \le \frac{1}{4}\min\{1, f'(1)\}$ such that

$$\langle \mathcal{L}_0^\pm \phi, \phi) \le -4\nu(\|\phi\|^2 + \|\phi_y\|^2) \qquad \forall \phi \perp Q_\pm.$$

2. Let R be a large constant to be determined. Define a cut-off function ζ by

$$\zeta(y) := \begin{cases} 1 & \text{if } \ y > R, \\ 0 & \text{if } \ y < -R, \\ \frac{1}{2}(1 + \sin\frac{\pi y}{R}) & \text{if } \ |x| \le R. \end{cases}$$

Let $\phi \in H^2(\mathbb{R})$ be any function satisfying (3.16). We can decompose it as

$$\phi = \phi_1 + \phi_2, \qquad \phi_1 := \zeta\phi, \quad \phi_2 := (1-\zeta)\phi_2.$$

Then

$$\mathcal{L}(\phi,\phi) = \mathcal{L}(\phi_1,\phi_1) + \mathcal{L}(\phi_2,\phi_2) + 2\mathcal{L}(\phi_1,\phi_2).$$

3. Easy calculation gives

$$\begin{aligned} \mathcal{L}(\phi_1,\phi_2) &= -\int_{\mathbb{R}} \Big\{\zeta(1-\zeta)\phi_y^2 + \zeta(1-\zeta)\phi^2 f'(\Phi) + (\zeta - 1/2)\zeta_{yy}\phi^2\Big\} \\ &\le \frac{\pi^2}{4R^2} - \int_{\mathbb{R}} \zeta(1-\zeta)(\phi_y^2 + \hat{\nu}\phi^2) \end{aligned}$$

where

$$\hat{\nu} = \min_{y\in[-R,R]} f'(\Phi(y,p)) = f'(1) + O(1)e^{-\mu(p-R)}.$$

4. Note that

$$\Phi_p - \Phi_y = 2\dot{Q}_- + O(1)e^{-2\mu p}[1 + (|p-|y|)^2](\dot{Q}_+ + \dot{Q}_-),$$
$$\langle \phi_2, \dot{Q}_-\rangle = \int_{-\infty}^R (1-\zeta)\phi\dot{Q}(p-y) \le \Big(\int_{p-R}^\infty \dot{Q}^2(z)\Big)^{1/2} = O(1)e^{-\mu(p-R)}.$$

It follows that

$$0 = \langle \phi, \Phi_p - \Phi_y\rangle = \langle \phi, \dot{Q}_-\rangle + O(1)e^{-2\mu p} = \langle \phi_1, \dot{Q}_-\rangle + O(1)e^{-\mu(p-R)}.$$

Writing

$$a_1 := \langle \phi_1, \dot{Q}_-\rangle/\|\dot{Q}\|^2 = O(1)e^{-\mu(p-R)}, \qquad \phi_1 = a_1\dot{Q}_- + \phi_1^\perp, \quad \phi_1^\perp \perp \dot{Q}_-.$$

Then from Step 1,

$$\begin{aligned} \mathcal{L}_0^-(\phi_1,\phi_1) &= \mathcal{L}_0^-(\phi_1^\perp,\phi_1^\perp) \\ &\le -4\nu(\|\phi_1^\perp\|^2 + \|\phi_{1y}^\perp\|^2) \\ &= -4\nu(\|\phi_1\|^2 + \|\phi_{1y}\|^2) + O(1)e^{-\mu(p-R)}. \end{aligned}$$

5. Notice that

$$\mathcal{L}(\phi_1,\phi_1) - \mathcal{L}_0^-(\phi_1,\phi_1) = \int_{-R}^\infty \Big\{f'(Q_-) - f'(\Phi)\Big\}\phi_1^2 = O(1)e^{-\mu(p-R)}.$$

It then follows that

$$\mathcal{L}(\phi_1,\phi_1) \le -4\nu(\|\phi_1\|^2 + \|\phi_{1y}\|^2) + O(1)e^{-\mu(p-R)}.$$

Similarly, we can show that

$$\mathcal{L}(\phi_2,\phi_2) \le -4\nu(\|\phi_1\|^2 + \|\phi_{1y}\|^2) + O(1)e^{-\mu(p-R)}).$$

Combining all the estimates, we then obtain

$$\begin{aligned}
\mathcal{L}(\phi,\phi) &\le -4\nu\{\|\phi_1\|^2 + \|\phi_2\|^2 + \|\phi_{1y}\|^2 + \|\phi_{2y}\|^2\} - 2\int_{\mathbb{R}} \zeta(1-\zeta)(\phi^2 + f'(1)\phi_y^2) \\
&\quad + \frac{\pi^2}{2R^2} + O(1)e^{-\mu(p-R)} \\
&\le -4\nu(\|\phi\|^2 + \|\phi_y\|^2) + \frac{\pi^2}{R^2} + O(1)e^{-\mu(p-R)} \\
&= -4\nu + \frac{\pi^2}{R^2} + O(1)e^{-\mu(p-R)}
\end{aligned}$$

since $4\nu \le f'(1)$. Thus, first taking $R = \pi/\sqrt{\nu/2}$ and then taking $p_0 \gg 1$ such that

$$O(1)e^{-\mu(p_0-R)} \le \nu/2,$$

We then obtain, when $p \ge p_0$,

$$\mathcal{L}(\phi,\phi) \le -3\nu = -3\nu(\|\phi\|^2 + \|\phi_y\|^2).$$

This completes the proof. $\square$

4. Properties of Entire Solutions

In this section, we always assume that u is an entire solution of (1.1), (1.2). We shall establish some basic properties of u. For convenience, we denote by α_0, β_0 the constants in (1.7).

4.1. **Some L^∞ Estimates.**

Lemma 4.1. *Suppose u is a solution of (1.1) and (1.2). Then*

$$0 < u < 1 \quad \forall (x,t) \in \mathbb{R}^2.$$

Proof. Set

$$M(t) = \sup_{x\in\mathbb{R}} u(x,t) \quad \forall\, t \in \mathbb{R}. \tag{4.1}$$

Then, by a comparison principle, $M(t) \le V(M(\tau); t-\tau)$ for every $t \ge \tau$, where $V(a;t)$ is the unique solution of

$$\dot V = -f(V), \quad V(a;0) = a.$$

From (1.2), $\limsup_{\tau\to-\infty} M(\tau) \le 1$, so that for every $t \in \mathbb{R}$,

$$M(t) \le \lim_{\tau\to-\infty} V(M(\tau); t-\tau) \le 1,$$

since $f'(1) > 0 = f(1)$. Since $u \not\equiv 1$, a strong maximum principle then gives us $M(t) < 1$ for all $t \in \mathbb{R}$.

Similarly, we can show that $u > 0$ in $\mathbb{R} \times \mathbb{R}$. The lemma follows. □

Lemma 4.2. *Let $M(t)$ be defined as in (4.1). Then*

$$\liminf_{t\to-\infty} M(t) > \alpha_0. \tag{4.2}$$

Proof. Let $\hat{\alpha}_0$ be a constant such that $\hat{\alpha}_0 > \alpha_0$ and $f > 0$ in $(0, \hat{\alpha}_0]$. If the assertion is not true, then there exists a sequence $\{t_j\}_{j=1}^{\infty}$ such that $\lim_{j\to\infty} t_j = -\infty$ and $M(t_j) \le \hat{\alpha}_0$ for all j. By comparison, for all $t > t_j$, $M(t) \le V(\hat{\alpha}_0; t - t_j)$, so that $M(t) \le \lim_{j\to\infty} V(\hat{\alpha}_0; t-t_j) = V(\hat{\alpha}_0; \infty) = 0$, a contradiction. Thus, (4.2) holds. □

4.2. **The Traveling Fronts.** From (4.2) and (1.2), we can define, for all $t \ll -1$,

$$l(t) = \min\{x \mid u(x,t) = \alpha_0\}, \qquad r(t) = \max\{x \mid u(x,t) = \alpha_0\}$$
$$m(t) = \tfrac{1}{2}[r(t) + l(t)], \qquad s(t) = \tfrac{1}{2}[r(t) - l(t)].$$

Lemma 4.3. *Assume that u is a solution of (1.1) and (1.2). Then*

$$\lim_{t\to-\infty} s(t) = \infty. \tag{4.3}$$

Consequently,

$$\lim_{t\to-\infty} \Big\{ \Big\| u_t(\cdot,t) \Big\|_{C^0(\mathbb{R})} + \Big\| u(\cdot,t) - Q(m_0 + r(t) - \cdot) Q(m_0 + \cdot - l(t)) \Big\|_{C^2(\mathbb{R})} \Big\} = 0$$

where m_0 is the constant such that $Q(m_0) = \alpha_0$.

Proof. For each $L > 0$, set

$$g(L; y) := \begin{cases} 1 & \text{when } \ y \in [-L, L], \\ \alpha_0 & \text{when } \ |y| > L. \end{cases}$$

Denote by $W(g; y, \tau)$ the solution of

$$\begin{aligned} &w_\tau = w_{yy} - f(w) && \text{in } \ \mathbb{R} \times (0, \infty), \\ &w(\cdot, 0) = g(\cdot) && \text{on } \ \mathbb{R} \times \{0\}. \end{aligned} \tag{4.4}$$

Then by a classical result of Fife-McLeod [14], there exists a constant $K^*(L)$ such that

$$0 \le W(g; y, \tau) \le \alpha_0 \qquad \forall \tau \ge K^*(L), y \in \mathbb{R}.$$

Since for each $\tau \ll -1$, $u(\cdot, \tau) \le g(s(\tau); \cdot - m(\tau))$, by comparison,

$$u(x,t) \le \alpha_0 \quad \forall\, x \in \mathbb{R}, t \ge \tau + K^*(s(\tau)).$$

From (4.2), we obtain (4.3).

The assertion of the consequence part follows from (1.2) and the continuous dependence of parabolic equation with respect to initial data. □

Corollary 4.4. *There exists a constant $T_0 \le 0$ such that $r(t), l(t) \in C^1((-\infty, T_0])$ and*

$$\lim_{t\to-\infty} \Big\{ |r'(t)| + |l'(t)| \Big\} = 0. \tag{4.5}$$

In addition, for all $t \le T_0$,

$$\begin{aligned} u(x,t) &< \alpha_0 \qquad \forall x \in (-\infty, l(t)) \cup (r(t), \infty), \\ u(x,t) &> \alpha_0 \qquad \forall x \in (l(t), r(t)). \end{aligned} \tag{4.6}$$

Furthermore, there exists a positive constant $L_0 > 0$ such that for all $t_1 < t_2 \le 0$,

$$l(t_2) \ge l(t_1) - L_0, \qquad r(t_2) \le r(t_1) + L_0. \tag{4.7}$$

Proof. From the previous lemma,

$$u_x(r(t), t) = -\dot{Q}(m_0) + o(1), \qquad u_x(l(t), t) = \dot{Q}(m_0) + o(1)$$

where $o(1) \to 0$ as $t \to -\infty$. Thus, by the Implicit Function Theorem, for all large negative t,

$$r'(t) = -\frac{u_t(r(t),t)}{u_x(r(t),t)}, \quad l'(t) = -\frac{u_t(l(t),t)}{u_x(l(t),t)}.$$

The limits in (4.5) thus follows. Also, $r, l \in C^1((-\infty, T_0])$ for some $T_0 < 0$, and (4.6) holds.

To prove the last assertion, consider the function $W(g_1; y, \tau)$ where

$$g_1(y) = \begin{cases} 1 & \text{if } \ y > 0, \\ \alpha_0 & \text{if } \ y \le 0. \end{cases}$$

By a result of [14], there exist constants $\xi_1 \in \mathbb{R}$, $K > 0$ and $\sigma > 0$, such that

$$\|W(g_1; \cdot, \tau) - Q(\cdot + \xi_1)\|_{L^\infty(\mathbb{R})} \le K e^{-\sigma\tau}.$$

It then follows that there exists a constant $L_0 \ge 0$ such that

$$W(g_1; -L_0, \tau) \le \alpha_0 \quad \forall \tau \ge 0.$$

Since for any $t_1 < 0$ and $\tau \ge 0$, $y \in \mathbb{R}$,

$$u(l(t_1) + y, t_1 + \tau) \le W(g; y, \tau),$$

it follows that for all $t_2 \in (t_1, 0)$, $l(t_2) \ge l(t_1) - L_0$. Similarly, $r(t_2) \le r(t_1) + L_0$. □

4.3. **Exponential Tails.**

Lemma 4.5. *There exist constants $T_1 \le 0$, $K > 0$ and $\varepsilon > 0$ such that for all $t \le T_1$,*

$$0 < u(x,t) \le \alpha_0 e^{-\varepsilon \min\{|x-r(t)|,|x-l(t)|\}} \qquad \forall x \in (-\infty, l(t)] \cup [r(t), \infty), \tag{4.8}$$

$$0 < 1 - u(x,t) \le K e^{-\varepsilon \min\{|x-r(t)|,|x-l(t)|\}} \qquad \forall\, x \in [l(t), r(t)]. \tag{4.9}$$

Consequently,

$$\lim_{t\to-\infty} \Big\| u(\cdot,t) - Q(m_0 + r(t) - \cdot)Q(m_0 + \cdot - l(t)) \Big\|_{L^2(\mathbb{R})\cap L^\infty(\mathbb{R})} = 0. \tag{4.10}$$

Proof. Let $t_0 \le T_0$ be any fixed constant. Without loss generality, we assume that

$$r'(t) \ge -1 \qquad \forall t \le T_0.$$

Then $r(t) \le r(t_0) + (t_0 - t)$ for all $t < t_0$. It then follows that

$$u(x,t) \le \alpha_0 \qquad \text{in } \ \Omega := \{(x,t) \mid t \le t_0, x \ge r(t_0) + t_0 - t\}.$$

Let $\varepsilon > 0$ be a constant to be determined. For any sufficiently small positive δ, consider the function

$$w(x,t) := \alpha_0 e^{-\varepsilon[x - r(t_0) + t - t_0]} + \delta.$$

We calculate, for all $(x,t) \in \Omega$ and small positive δ,

$$\begin{aligned} f(w) &\ge \eta_0 w, \qquad \eta_0 := \tfrac{1}{2} \sup_{s\in(0,\alpha_0]} \frac{f(s)}{s}, \\ w_t - w_{xx} + f(w) &\ge \alpha_0 e^{-\varepsilon[x-r(t_0)+t-t_0]}\{-\varepsilon - \varepsilon^2 + \eta_0\} > 0 \qquad \text{in } \ \Omega \end{aligned}$$

provided that $\varepsilon + \varepsilon^2 \le \eta_0$.

Now we compare u with w in Ω. On the lateral boundary of Ω, $x = r(t_0) + t_0 - t$, we have

$$w = \alpha_0 + \delta \ge \alpha_0 \ge u.$$

We compare "initial" value at $t = T \ll -1$. Since there exists a positive constant $\hat\varepsilon$ such that

$$Q(m_0 + z) \le \alpha_0 e^{-\hat\varepsilon z} \qquad \forall z \ge 0,$$

taking $\varepsilon \in (0, \hat\varepsilon]$ we have, for all $T \ll -1$ and $x \ge r(T)$,

$$\begin{aligned} u(x,T) &= Q(m_0 + x - r(T)) + o(1) \le \alpha_0 e^{-\hat\varepsilon[x-r(T)]} + o(1) \\ &\le \alpha e^{-\varepsilon[x-r(t_0)+T-t_0]} + o(1) \le w(x,T). \end{aligned}$$

Hence, by comparison,

$$u(x,t) \le w(x,t) \qquad \forall x > r(t_0) + t_0 - t, t \in [T, t_0].$$

First sending $T \to -\infty$ and then $\delta \to 0$ we then obtain

$$u(x,t_0) \le \alpha_0 e^{-\varepsilon[x-r(t_0)]} \quad \forall\, x \ge r(t_0).$$

In a similar manner, we can show that

$$u(x,t_0) \le \alpha_0 e^{-\varepsilon|x-l(t_0)|} \quad \forall\, x \le l(t_0).$$

The estimate (4.8) thus follows.

We continue to prove (4.9). From the previous Lemma, we have for all $t \le t_0$,

$$r(t) \ge r(t_0) - L_0, \qquad l(t) \le l(t_0) + L_0.$$

Hence

$$u(x,t) \ge \alpha_0 \qquad \text{on } \ [l(t_0)+L_0, r(t_0)-L_0] \times (-\infty, t_0].$$

Also, from (1.2) and the fact that $\lim_{t\to-\infty} |l(t)-r(t)| = \infty$ we have

$$\lim_{t\to-\infty} \|u(\cdot,t) - \min\{Q(m_0 + r(t) - \cdot), Q(m_0 + \cdot - l(t))\}\|_{L^\infty(\mathbb{R})} = 0.$$

It then follows from a comparison that

$$u(x,t) \ge \hat{Q}(x - \tfrac{1}{2}[r(t_0)+l(t_0)]) \quad \forall x \in [r(t_0)+L_0, r(t_0)-L_0], t \le t_0,$$

where $\hat{Q}$ is an even function and satisfies

$$\begin{array}{l} \ddot{\hat{Q}} = f(\hat{Q}) \quad \text{in } \ [0,\ell], \qquad (\ell := \tfrac{1}{2}(r(t_0) - l(t_0)) - L_0,) \\ \hat{Q}(\ell) = \alpha_0, \quad \dot{\hat{Q}}(0) = 0, \quad \hat{Q}(0) > \beta_0, \quad \dot{\hat{Q}} < 0 \text{ in } \ (0,\ell). \end{array}$$

When ℓ is large enough, such $\hat{Q}$ exists and is unique. In addition,

$$\hat{Q}(z) \le \min\{Q(m_0 + \ell - z), Q(m_0 + \ell + z)\} \quad \forall z \in [-\ell, \ell].$$

In terms of a first integral, one can show that there exists a positive constant ε that is independent of $\ell \gg 1$ such that

$$0 < 1 - \hat{Q}(z) \le (1-\alpha_0) e^{-\varepsilon(\ell - z)} \quad \forall z \in [0,\ell].$$

This implies that

$$0 < 1 - u(x,t_0) \le (1-\alpha_0) e^{\varepsilon L_0} e^{-\varepsilon|x-r(t_0)|} \quad \forall x \in [\tfrac{1}{2}[r(t_0)+l(t_0)], r(t_0)).$$

Hence (4.9) follows, since u is even in x. This completes the proof. □

4.4. **Projection Onto the Quasi-Invariant Manifold.** We define

$$\Psi(x,z,p) := \Phi(x-z,p) \qquad \forall x \in \mathbb{R}, z \in \mathbb{R}, p \ge 0.$$

We define quasi-invariant manifold by

$$\mathcal{M} := \{\Psi(\cdot, z, p) \mid z \in \mathbb{R}, p > p_0\} \subset L^2(\mathbb{R}),$$

where p_0 is a large positive constant.

Lemma 4.6. *There exists a constant $T_2 < 0$ with the property that for each $t \le T_2$ there exist unique $z = z(t) \in \mathbb{R}$ and $p = p(t) \ge p_0 + 1$ such that*

$$u(x,t) \;=\; \Psi(x,z,p) + \phi(x,t) \quad \forall x \in \mathbb{R}, \tag{4.11}$$

where

$$\|\phi\| = \operatorname{dist}(u(\cdot,t),\mathcal{M}) := \min_{\psi\in\mathcal{M}} \Big\| u(\cdot,t) - \psi \Big\|.$$

In addition, (z,p) satisfies the orthogonality condition, for $\Psi = \Psi(x,z,p)$,

$$\langle \Psi - u, \Psi_z\rangle = 0, \qquad \langle \Psi - u, \Psi_p\rangle = 0. \tag{4.12}$$

Furthermore, $z(t), p(t), \phi$ are smooth functions.

Proof. Set

$$\hat z = \tfrac12[r(t) + l(t)], \quad \hat p = \tfrac12[r(t) - l(t)] + m_0.$$

From (4.10), we see that $\|\Psi(\cdot,z,p) - u(\cdot,t)\|$ is small if and only if (z,p) is close to $(\hat z,\hat p)$. That is, the distance from $u(\cdot,t)$ to $\mathcal{M}$ can be attained only at those $\Psi(\cdot,z,p)$ for which (z,p) is close to $(\hat z,\hat p)$. Hence we need only to solve the algebraic system (4.12) of two equations and two unknowns (z,p) near $(\hat z,\hat p)$.

When $(z,p) = (\hat z,\hat p)$, both $\langle \Psi - u, \Psi_z\rangle$ and $\langle \Psi - u, \Psi_p\rangle$ are small quantities. Also, using $\langle \Psi_z, \Psi_p\rangle = -\langle \Phi_y, \Phi_p\rangle = 0$ we can calculate the Jacobi matrix associated with the algebraic system (4.12) to be

$$J(z,p) := \begin{pmatrix} \|\Psi_z\|^2 & 0 \\ 0 & \|\Psi_p\|^2 \end{pmatrix} + \begin{pmatrix} \langle \Psi - u, \Psi_{zz}\rangle & \langle \Psi - u, \Psi_{zp}\rangle \\ \langle \Psi - u, \Psi_{pz}\rangle & \langle \Psi - u, \Psi_{pp}\rangle \end{pmatrix}. \tag{4.13}$$

Observe that

$$\|\Psi_z\|^2, \|\Psi_p\|^2 = 2\int_{\mathbb{R}} \dot Q^2(x)\,dx + O(1)e^{-2\mu p}.$$

Thus, for all (z,p) close to $(\hat z,\hat p)$, the Jacobi matrix J is close to the identity matrix multiplied by the constant $2\int_{\mathbb{R}} \dot Q^2(x)dx$. The Implicit Function Theorem then implies that for every small A and B, there are unique (z,p) near $(\hat z,\hat p)$ such that $A = \langle \Psi - u, \Psi_z\rangle$, $B = \langle \Psi - u, \Psi_p\rangle$. In particular, setting $(A,B) = (0,0)$ we obtain the assertion of the Lemma. □

4.5. **Super-Slow Interfacial Motion.** Now we study the dynamics of $z(t), p(t)$. We can write (1.1) as

$$\dot p \Psi_p + \dot z \Psi_y + \phi_t = (\Psi + \phi)_{xx} - f(\Psi + \phi) \quad \forall\, x \in \mathbb{R}, t \le T_2, \tag{4.14}$$

where

$$\phi = \phi(x,t), \qquad \Psi = \Psi(x,z,p) = \Phi(x - z, p), \quad z = z(t), \quad p = p(t).$$

Recall from (4.12) that $\langle \phi, \Psi_z\rangle = \langle \phi, \Psi_p\rangle = 0$ for all $t \le T_2$. For $j = z$ and p, taking the inner product of (4.14) with Ψ_j and using

$$\langle \phi_t, \Psi_j\rangle = \langle \phi, \Psi_j\rangle_t - \langle \phi, \Psi_{zj}\rangle \dot z - \langle \phi, \Psi_{pj}\rangle \dot p = -\langle \phi, \Psi_{zj}\rangle \dot z - \langle \phi, \Psi_{pj}\rangle \dot p,$$

we then obtain

$$J\begin{pmatrix}\dot z\\ \dot p\end{pmatrix}=\begin{pmatrix}R^z\\ R^p\end{pmatrix},$$

where $J=J(\phi,z,p)$ is as in (4.13) and for $j=z,p$,

$$\begin{aligned}R^j &= \langle(\Psi+\phi)_{xx}-f(\Psi+\phi),\Psi_j\rangle\\ &= \langle\Psi_{xx}-f(\Psi),\Psi_j\rangle+\langle\phi_{xx}-f'(\Psi)\phi,\Psi_j\rangle+\langle O(1)\phi^2,\Psi_j\rangle.\end{aligned}$$

We calculate each of the three terms as follows.

Denote

$$\begin{aligned}\mathcal{E}(p) &= \int_{\mathbb{R}}\Big\{\tfrac12\Phi_y^2(y,p)+F(\Phi(y,p))\Big\}dy\\ &= \int_{\mathbb{R}}\Big\{\tfrac12\Psi_x^2(x,z,p)+F(\Psi(x,z,p))\Big\}dx.\end{aligned}$$

We obtain

$$\langle\Psi_{xx}-f(\Psi),\Psi_z\rangle=-\frac{\partial}{\partial z}\mathcal{E}(p)=0,$$
$$\langle\Psi_{xx}-f(\Psi),\Psi_p\rangle=-\frac{\partial}{\partial p}\mathcal{E}(p)=-\mathcal{E}_p.$$

Also, we calculate, for $j=z$ or p,

$$\langle\phi_{xx}-f'(\Psi)\phi,\Psi_j\rangle=\langle\mathcal{L}\Psi_j,\phi\rangle=O(1)e^{-2\mu p}\|\phi\|$$

by (3.14).

Hence we have

$$\begin{aligned}R^z &= O(1)e^{-2\mu p}\|\phi\|+O(1)\|\phi\|^2,\\ R^p &= -\mathcal{E}_p+O(1)e^{-2\mu p}\|\phi\|+O(1)\|\phi\|^2.\end{aligned}$$

It then follows that

$$\begin{pmatrix}\dot z\\ \dot p\end{pmatrix} = J^{-1}\begin{pmatrix}R^z\\ R^p\end{pmatrix}=\begin{pmatrix}0\\ C(p)\end{pmatrix}+O(1)e^{-2\mu p}\|\phi\|+O(1)\|\phi\|^2, \tag{4.15}$$

where

$$C(p)=-\frac{\mathcal{E}_p}{\|\Phi_p\|^2}=c(p)+O(e^{-6\mu p})=-\alpha e^{-2\mu p}+O(1)pe^{-4\mu p}. \tag{4.16}$$

Finally, we estimate ϕ. Taking the inner product of (4.14) with ϕ and using the orthogonality, we obtain

$$\begin{aligned}\frac12\frac{d}{dt}\|\phi\|^2 &= \langle\phi_t,\phi\rangle\\ &= \langle\Psi_{xx}-f(\Psi),\phi\rangle+\langle\mathcal{L}\phi,\phi\rangle+\langle O(1)\phi^2,\phi\rangle.\end{aligned}$$

Using $\phi\perp\Psi_p$ and

$$\Psi_{xx}-f(\Psi)=\Phi_{xx}-f(\Phi)=c\Psi_p+O(1)e^{-6\mu p}(1+[x-|p|]^2)\Psi_p,$$

we obtain

$$\langle \Psi_{xx} - f(\Psi), \phi \rangle = O(1)e^{-6\mu p}\|\phi\|.$$

Also, using the eigenvalue estimate (3.15), since $\phi \perp \Phi_x, \phi \perp \Phi_p$, it follows that

$$\langle \mathcal{L}\phi, \phi \rangle \leq -3\nu\|\phi\|^2.$$

Furthermore, we have

$$\langle O(1)\phi^2, \phi \rangle = O(1)\|\phi\|_{L^\infty(\mathbb{R})}\|\phi\|^2 = o(1)\|\phi\|^2.$$

Hence, we obtain, for all $t \leq T_3 \ll -1$,

$$\frac{1}{2}\frac{d}{dt}\|\phi\|^2 \leq -2\nu\|\phi\|^2 + M_1 e^{-6\mu p}\|\phi\|.$$

This is equivalent to

$$\frac{d}{dt}\|\phi\| \leq -2\nu\|\phi\| + M_1 e^{-6\mu p}. \tag{4.17}$$

For every $t_0 < T_3$, we claim that

$$\|\phi(t)\| \leq \|\phi(t_0)\|e^{-2\nu(t-t_0)} + \frac{M_1}{\nu}e^{-6\mu p(t)} \qquad \forall t \in [t_0, T_3]. \tag{4.18}$$

Suppose it is not true. Then there exists a $t^* \in (t_0, T_3)$ such that

$$\|\phi(t^*)\| = \|\phi(t_0)\|e^{-2\nu(t^*-t_0)} + \frac{M_1}{\nu}e^{-6\mu p(t^*)}$$

and

$$\begin{aligned}
\frac{d}{dt}\|\phi\|\,\Big|_{t=t^*} &\geq \frac{d}{dt}\Big\{\|\phi(t_0)\|e^{-2\nu(t-t_0)} + \frac{M_1}{\nu}e^{-6\mu p(t)}\Big\}\Big|_{t=t^*} \\
&= -2\nu\|\phi(t_0)\|e^{-2\nu(t^*-t_0)} - \frac{6M_1\mu}{\nu}\dot{p}(t^*)e^{-6\mu p(t^*)} \\
&= -2\nu\|\phi(t^*)\| + e^{-6\mu p}M_1\left\{2 - \frac{6\mu}{\nu}\dot{p}(t^*)\right\} \\
&\geq -2\nu\|\phi(t^*)\| + \tfrac{3}{2}M_1 e^{-6\mu p}
\end{aligned}$$

since $\dot{p} = o(1)$. But this contradicts (4.17).

Therefore, (4.18) holds. Sending $t_0 \to -\infty$ and using the boundedness of $\|\phi(t_0)\|$ we then obtain

$$\|\phi(t)\| \leq \frac{M_1}{\nu}e^{-6\mu p} \quad \forall\, t \leq T_3 \ll 1.$$

Substituting this estimate into (4.15) we then obtain the following.

Theorem 4.7. *Assume (1.3) and let u be a solution of (1.1), (1.2). Then there exists a large negative constant T_3, unique functions $z(t), p(t)$ defined on $(-\infty, T_3]$ such that for all $t \le T_3$,*

$$
\begin{aligned}
&u(x,t) = \Phi(x - z(t), p(t)) + \phi(x,t), \qquad (4.19)\\
&\langle \phi, \Phi_x \rangle = \langle \phi, \Phi_p \rangle = 0,\\
&\|\phi(\cdot, t)\| = O(1)e^{-6\mu p},\\
&\dot z(t) = O(1)e^{-8\mu p},\\
&0 > \dot p(t) = C(p) + O(1)e^{-8\mu p} = -\alpha e^{-2\mu p} + O(1)pe^{-4\mu p}.
\end{aligned}
$$

Consequently,

$$\lim_{t\to-\infty} \Big\{ p(t) - \frac{1}{2\mu}\ln(2\alpha\mu|t|) \Big\} = 0, \tag{4.20}$$

$$z(t) = z(-\infty) + \frac{O(1)}{|t|^3}, \tag{4.21}$$

where $z(-\infty)$ is a finite number.

Proof. It remains to show the consequence part.

Integrating $\dot p = -\alpha e^{-2\mu p} + O(1)pe^{-4\mu p}$ we obtain, for any $t < T_3$,

$$
\begin{aligned}
\alpha(T_3 - t) &= \int_{p(T_3)}^{p(t)} \frac{ds}{e^{-2\mu s}[1 + O(1)se^{-2\mu s}]}\\
&= \int_{p(T_3)}^{p(t)} \Big\{ e^{2\mu s} + O(1)s + O(1)s^2 e^{-2\mu s} \Big\}\, ds\\
&= \frac{1}{2\mu} e^{2\mu p(t)} + O(1)p(t)^2 + O(1).
\end{aligned}
$$

It then follows that for all $t \ll -1$, $p(t) = O(1)\ln|t|$ and

$$
\begin{aligned}
p(t) &= \frac{1}{2\mu}\ln\Big\{ 2\alpha\mu|t| + O(1)p^2(t) \Big\}\\
&= \frac{1}{2\mu}\ln(2\alpha\mu|t|) + O(1)\frac{p^2(t)}{|t|}\\
&= \frac{1}{2\mu}\ln(2\alpha\mu|t|) + O(1)\frac{\ln^2|t|}{|t|}.
\end{aligned}
$$

The estimate (4.20) thus follows.

This estimate implies that

$$\dot z = O(1)e^{-8\mu p} = O(1)t^{-4}.$$

After integration, we obtain (4.21). □

We remark that the uniqueness theorem show that u has to be even above the line $x = z(-\infty)$, so that $z(t)$ is indeed a constant function.

4.6. **A Change of Coordinates.** Since $\dot{p} < 0$ for all $t \ll -1$, we can use p to replace the variable t. Hence, we make a change of variable, for all $t \ll -1$:

$$(t, x, u) \longrightarrow (p, y, \phi, a, b)$$

via

$$\begin{aligned} &p = p(t), \\ &y = x - z(t), \\ &u(x,t) = \Phi(y,p) + \phi(y,p), \\ &a(p) = \dot{p}(t), \\ &b(p) = -\dot{z}(t). \end{aligned} \tag{4.22}$$

Then (1.1) can be written as, for the unknowns $(\phi(y,p), a(p), b(p))$,

$$\begin{aligned} &\langle \phi(\cdot,p), \Phi_y(\cdot,p)\rangle = 0, && p > p_0, \\ &\langle \phi(\cdot,p), \Phi_p(\cdot,p)\rangle = 0, && p > p_0, \\ &a(\Phi_p + \phi_p) + b(\Phi_y + \phi_y) = (\Phi + \phi)_{yy} - f(\Phi + \phi), && p > p_0, y \in \mathbb{R}. \end{aligned} \tag{4.23}$$

We remark that ϕ here is slightly different from that ϕ in (4.19).

5. Uniqueness

Suppose u_1 and u_2 are two solutions of (1.1), (1.2). Using the transformation (4.22) we denote the corresponding solutions of (4.23) by (ϕ_1, a_1, b_1) and (ϕ_2, a_2, b_2). We denote

$$\psi(y,p) = \phi_1(y,p) - \phi_2(y,p), \qquad a(p) = a_1(p) - a_2(p), \qquad b(p) = b_1(p) - b_2(p).$$

Also, we denote the corresponding functions in Theorem 4.7 by

$$z_1(t), p_1(t), z_2(t), p_2(t).$$

Taking the differences of the corresponding equations satisfied by (ϕ_1, a_1, b_1) and (ϕ_2, a_2, b_2), we obtain

$$\langle \psi, \Phi_y\rangle = \langle \psi, \Phi_p\rangle = 0, \tag{5.1}$$

$$a_1\psi_p + b_1\psi_y + a(\Phi_p + \phi_{2p}) + b(\Phi_y + \phi_{2y}) = \psi_{yy} - f'(\Phi + \phi_2 + \theta\psi)\psi, \tag{5.2}$$

where $\theta = \theta(y,p) \in (0,1)$.

5.1. **Estimation of a and b.** Taking the inner product of (5.2) with Φ_p and Φ_y respectively, we obtain

$$\begin{pmatrix} \langle \Phi_p + \phi_{2p}, \Phi_p \rangle & \langle \phi_{2y}, \Phi_p \rangle \\ \langle \phi_{2p}, \Phi_y \rangle & \langle \Phi_y + \phi_{2y}, \Phi_y \rangle \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} r^p + \rho^p \\ r^y + \rho^y \end{pmatrix}$$

where for $j = p$ and $j = y$,

$$\begin{aligned} r^j &= \langle \psi_{yy} - f'(\Phi + \phi_2 + \theta\psi)\psi, \Phi_j \rangle, \\ \rho^j &= -\langle a_1 \psi_p + b_1 \psi_y, \Phi_j \rangle. \end{aligned}$$

To estimate ρ^j, we notice that, for $i = p$ or y,

$$\langle \psi_i, \Phi_j \rangle = \langle \psi, \Phi_j \rangle_i - \langle \psi, \Phi_{ij} \rangle = -\langle \psi, \Phi_{ij} \rangle = O(1)\|\psi\|.$$

Since $|a_1| + |b_1| = O(1)e^{-2\mu p}$, we then obtain

$$\rho^j = O(1)e^{-2\mu p}\|\psi\|.$$

To estimate r^j, we use

$$f'(\Phi + \phi_2 + \theta\psi) = f'(\Phi) + O(1)(|\phi_1| + |\phi_2|) = f'(\Phi) + O(1)e^{-2\mu p}$$

to obtain

$$\begin{aligned} r^j &= \langle \psi_{yy} - f'(\Phi)\psi, \Phi_j \rangle + O(1)e^{-2\mu p}\|\psi\| \\ &= \langle \psi, (\Phi_{yy} - f(\Phi))_j \rangle + O(1)e^{-2\mu p}\|\psi\| \\ &= O(1)e^{-2\mu p}\|\psi\|. \end{aligned}$$

It then follows that

$$|a| + |b| = O(1)e^{-2\mu p}\|\psi\| \qquad \forall p > p_0. \tag{5.3}$$

5.2. **Estimate of ψ.** Taking the inner product of (5.2) with ψ we obtain

$$\begin{aligned} & a_1\langle \psi_p, \psi \rangle + b_1\langle \psi_y, \psi \rangle + a\langle \phi_{2p}, \psi \rangle + b\langle \phi_{2y}, \psi \rangle \\ = \ & \langle \psi_{yy} - f'(\Phi + \phi_2 + \theta\psi)\psi, \psi \rangle \\ \le \ & -2\nu\|\psi\|^2. \end{aligned}$$

Note that

$$\langle \psi_y, \psi \rangle = \int_{\mathbb{R}} \psi_y \psi \, dy = \tfrac{1}{2}\psi^2 \Big|_{-\infty}^{\infty} = 0.$$

Using the estimate of a and b in (5.3) we then obtain

$$\frac{a_1}{2}\frac{d}{dp}\|\psi\|^2 \ \le \ \|\psi\|^2(-2\nu + O(1)e^{-2\mu p}) \le -\nu\|\psi\|^2. \tag{5.4}$$

Since $a_1 = \dot{p}_1(t)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\psi(\cdot, p_1(t))\|^2 \leq -\nu\|\psi(\cdot, p_1(t))\|^2.$$

An integration then gives, for any $\tau < t \ll -1$,

$$\|\psi(\cdot, p_1(t)\| \leq e^{-\nu(t-\tau)}\|\psi(\cdot, p_1(\tau)\|.$$

Since $\|\psi(\cdot, p)\|$ is uniformly bounded for all $p > p_0$, sending $\tau \to -\infty$ we then obtain

$$\|\psi(\cdot, p_1(t))\| = 0 \qquad \forall t \ll -1.$$

Thus $\phi_1(y, p) = \phi_2(y, p)$. We may suppose that $p_1(t_1) = p_2(t_2)$ for some $t_1, t_2 \in \mathbb{R}$. Then

$$u_1(x + z_1(t_1), t_1) = u_2(x + z_2(t_2), t_2) \quad \forall x \in \mathbb{R}.$$

Consequently, (1.4) holds with $\xi = z_2(t_2) - z_1(t_1)$ and $\eta = t_2 - t_1$.

This completes the proof of Theorem 1.1.

6. Another Initial Condition

In this section, we provide another condition that can be used to replace (1.2).

6.1. **An L^∞ Bound.** We shall assume that either $0 \leq u \leq 1$ on $\mathbb{R}^2$ is known, or conditions on $f|_{(-\infty,0]\cup[1,\infty)}$ is made such that we can derive such an estimate. Here we provide such a condition:

$$|f| > 0 \text{ in } (-\infty, 0) \cup (1, \infty), \qquad \int_{|s|>2} \frac{ds}{|f(s)|} < \infty. \tag{6.1}$$

Lemma 6.1. *Assume (1.3) and (6.1). Let $u \in C(\mathbb{R}; L^\infty(\mathbb{R}))$ be a non-constant solution of (1.1). Then*

$$0 < u < 1 \quad \forall\,(x, t) \in \mathbb{R}^2.$$

Proof. Define $M(t)$ as in (4.1). Then for any $\tau < t$, by comparison $M(t) \leq V(M(\tau); t-\tau)$, which is equivalent to $M(\tau) \geq V(M(t); \tau - t)$.

Now suppose $M(t_0) > 1$ for some $t_0 \in \mathbb{R}$. Then

$$M(\tau) \geq V(M(t_0); \tau - t_0) = \infty \quad \text{at} \quad \tau := t_0 - \int_{M(t_0)}^{\infty} \frac{ds}{f(s)},$$

a contradiction. Hence, we must have $M(t) \leq 1$ for all $t \in \mathbb{R}$. As u is non-constant, we must have $u < 1$ in $\mathbb{R}$. Similarly, one can show that $u > 0$ in $\mathbb{R}$. ☐

6.2. **A Replacement.**

Theorem 6.2. *Suppose that u is an entire solution of (1.1) that satisfies (1.6) and $0 < u < 1$ on $\mathbb{R}^2$. Then u also satisfies (1.2).*

Proof. Following the same proof as that for (4.3), we can show that

$$\lim_{t\to-\infty}\{p(t)-q(t)\}=\infty. \tag{6.2}$$

To continue, we need two auxiliary results. Let L be as in (1.6). Define

$$X(L):=\Big\{g\in C(\mathbb{R}\to[0,1])\;\Big|\; g\le\alpha_0 \;\text{ in }\; (-\infty,0], \quad g>\beta_0 \;\text{ in }\; [L,\infty)\Big\}.$$

The following Lemma can be proven by the method used by Fife-McLeod [14].

Lemma 6.3. *There exist positive constant σ and $K(L)$ (σ is independent of L) such that if $g\in X(L)$, the solution $W(g;y,\tau)$ to (4.4) satisfies*

$$|W(g;y,\tau)-Q(y-\xi)|\le K(L)e^{-\sigma\tau}\quad \forall y\in\mathbb{R},\tau\ge 0,$$

where $\xi=\xi(g)$ is some constant satisfying $|\xi|\le K(L)$.

Lemma 6.4. *For every $\varepsilon, T>0$, there exists $\ell(\varepsilon,L)>0$ such that if g_1,g_2 are two functions satisfying*

$$0\le g_1,g_2\le 1 \;\text{ in }\; \mathbb{R}, \qquad g_1=g_2 \;\text{ in }\; (-\infty,0].$$

Then

$$\Big|W(g_1;y,\tau)-W(g_2;y,\tau)\Big|\le\varepsilon\quad\forall\tau\in[0,T], y\in(-\infty,-\ell).$$

The assertion follows from a continuous dependence argument and is omitted; cf. [9].

We now continue the proof of Theorem 6.2.

Let $\varepsilon>0$ be arbitrary. Define T such that

$$K(L)e^{-\sigma T}=\varepsilon.$$

From (6.2), there exists $t_0<0$ such that for all $t\le t_0$, the quantity $m(t):=p(t)-q(t)$ is large enough such that

$$m(t)\ge 4\ell(\varepsilon,L)+2L,\qquad Q(m(t)/2-K(L))\ge 1-\varepsilon.$$

Fix any $t\le t_0$. Define $t_1=t-T$ and

$$g(x)=\begin{cases}1 & \text{if } \; x>p(t_1)-L,\\ u(x,t_1) & \text{if } \; x\le p(t_1)-L.\end{cases}$$

Then $g(q(t_1)+\cdot)\in X(L)$. It then follows that for some $\xi\in[-K(L),K(L)]$,

$$|W(g;x,T)-Q(x-q(t_1)-\xi)|\le K(L)e^{-\sigma T}=\varepsilon.$$

Also, by the second lemma,

$$|W(g;x,T) - u(x,t_1+T)| \le \varepsilon \quad \forall x \le p(t_1) - L - \ell.$$

Since $t_1 + T = t$ and $p(t_1) - L - \ell \ge \frac{1}{2}(p(t_1) + q(t_1))$, we then obtain

$$|u(x,t) - Q(x - q(t_1) - \xi)| \le 2\varepsilon \qquad \forall\, x \le \tfrac{1}{2}(p(t_1) + q(t_1)).$$

Similarly, we can show that, for some $|\eta| \le K(L)$,

$$|u(x,t) - Q(p(t_1) + \eta - x)| \le 2\varepsilon \qquad \forall\, x \ge \tfrac{1}{2}(p(t_1) + q(t_1)).$$

As $|\xi|, |\eta| \le K(L)$, we have

$$\begin{aligned} 1 > Q(p(t_1) + \eta - x) \ge 1 - \varepsilon \quad \forall\, x \le \tfrac{1}{2}(p(t_1) + q(t_1)),\\ 1 > Q(x - q(t_1) - \eta) \ge 1 - \varepsilon \quad \forall\, x \ge \tfrac{1}{2}(p(t_1) + q(t_1)). \end{aligned}$$

We then obtain

$$\Big| u(x,t) - Q(p(t_1) + \eta - x) Q(x - q(t_1) - \xi) \Big| \le 3\varepsilon \quad \forall x \in \mathbb{R}.$$

Thus,

$$\sup_{t<t_0} \inf_{p>q} \| u(\cdot,t) - Q(p - \cdot) Q(\cdot - q) \|_{L^\infty(\mathbb{R})} \le 3\varepsilon.$$

Hence u satisfies (1.2). □

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