TRAVELING WAVES FOR A LATTICE DYNAMICAL SYSTEM ARISING IN A DIFFUSIVE ENDEMIC MODEL

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Abstract. This paper is concerned with a lattice dynamical system modeling the evolution of susceptible and infective individuals at discrete niches. We prove the existence of traveling waves connecting the disease-free state to non-trivial leftover concentrations. We also characterize the minimal speed of traveling waves and we prove the non-existence of waves with smaller speeds.

1. Introduction

In this article, we consider the following lattice dynamical system (LDS)

$$\begin{cases} \dfrac{ds_n}{dt} &= (s_{n+1}+s_{n-1}-2s_n)+\mu-\mu\, s_n-\beta\, s_n\, i_n,\ n\in\mathbb{Z}, \\ \dfrac{di_n}{dt} &= d(i_{n+1}+i_{n-1}-2i_n)-\mu\, i_n+\beta\, s_n\, i_n-\gamma\, i_n,\ n\in\mathbb{Z}, \end{cases} \tag{1.1}$$

where $s_n=s_n(t)$, $i_n=i_n(t)$, $t\in\mathbb{R}$, and μ,β,γ are positive constants. Here $s_n(t)$ and $i_n(t)$ represent the population density of the susceptible individuals and the infective individuals at niches n at time t, 1 and d are the random migration coefficients for susceptible and infective population, respectively, and μ is regarded as the rate of the inflow of newborns into the susceptible population by assuming the total population of susceptible, infective and recovered individuals is normalized to be 1. The death rate of the susceptible population and the infective population are both assumed to be μ, β is the infective (transmission) coefficient and γ is the recovered/removed coefficient. Actually, as in [28], the equation for the recovered individuals $r_n(t)$ is given by

$$\frac{dr_n}{dt}=\gamma i_n-\mu r_n,$$

if there is no migration of the recovered individuals.

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For system (1.1), it easy to see there are two constant states $(1,0)$ and

$$(s^*, e^*) := \Big(\frac{1}{\sigma}, \frac{\mu}{\beta}\,(\sigma - 1)\Big), \tag{1.2}$$

where $\sigma := \beta/(\mu+\gamma)$. In this paper, we always assume that $\sigma > 1$, that is,

$$\beta > \mu + \gamma.$$

This means that, when the density of susceptible individuals is close to 1, the infective individuals have a positive per capita growth rate. Without migration, the steady state $(1,0)$ is dynamically unstable with respect to perturbations whose second component are positive, while the steady state (s^*, e^*) is dynamically stable. System (1.1) is therefore called monostable.

In this paper, we are interested in the existence of traveling wave solutions of (1.1). We first consider traveling waves which can be expressed as two bounded profiles of the continuous variable $n + ct$, namely

$$s_n(t) = \phi(n+ct) \ \text{ and } \ i_n(t) = \psi(n+ct) \tag{1.3}$$

for $n \in \mathbb{Z}$ and $t \in \mathbb{R}$, for some nonnegative bounded functions ϕ, ψ on $\mathbb{R}$ (the wave profiles) and some constant c (the wave speed). By setting $\xi = n + ct$ and substituting $(s_n(t), i_n(t)) = (\phi(\xi), \psi(\xi))$ into (1.1), we then obtain

$$\begin{cases} -c\,\phi'(\xi) + D[\phi](\xi) + \mu\,(1-\phi(\xi)) - \beta\,\phi(\xi)\,\psi(\xi) &= 0, \\ -c\,\psi'(\xi) + d\,D[\psi](\xi) - (\mu+\gamma)\,\psi(\xi) + \beta\,\phi(\xi)\,\psi(\xi) &= 0 \end{cases} \tag{1.4}$$

for all $\xi \in \mathbb{R}$, where

$$D[f](\xi) := f(\xi+1) + f(\xi-1) - 2f(\xi).$$

Furthermore, from the epidemic point of view, we are interested in traveling wave solutions connecting the trivial disease-free state $(1,0)$ as $\xi \to -\infty$ (ahead of the front) and non-trivial states as $\xi \to +\infty$.

Define the constant c^* by

$$c^* := \inf_{\lambda > 0} \frac{d\,(e^{\lambda} + e^{-\lambda} - 2) + \beta - \mu - \gamma}{\lambda}. \tag{1.5}$$

By the assumption $\beta > \mu + \gamma$ (i.e. $\sigma > 1$), we know that $c^* = c^*(d, \beta, \mu, \gamma)$ is a well-defined real number (and the infimum in (1.5) is a minimum) and $c^* > 0$.

Our first main result is the following theorem on the existence of traveling waves for (1.4) and the characterization of their minimal speed.

Theorem 1.1. *For any $c \geq c^*$, there exists a bounded classical solution (ϕ, ψ) of the system* (1.4) *such that*

$$0 < \phi < 1 \ \text{in } \mathbb{R}, \quad \psi > 0 \ \text{in } \mathbb{R} \tag{1.6}$$

and

$$\lim_{\xi \to -\infty} (\phi(\xi), \psi(\xi)) = (1, 0), \tag{1.7}$$

together with

$$0<\liminf_{\xi\to+\infty}\phi(\xi)\le s^*\le\limsup_{\xi\to+\infty}\phi(\xi)<1 \quad \text{and} \quad 0<\liminf_{\xi\to+\infty}\psi(\xi)\le e^*\le\limsup_{\xi\to+\infty}\psi(\xi)<+\infty, \tag{1.8}$$

where (s^*,e^*) *is given in* (1.2). *Furthermore, for any* $c<c^*$, *there is no classical solution* (ϕ,ψ) *of the system* (1.4) *satisfying* (1.6) *and* (1.7).

Behind the front, as $\xi\to+\infty$, the leftover concentrations of susceptible and infective individuals are non-trivial. It is still an open question to know whether the traveling wave solutions converge to the endemic state (s^*,e^*) as $\xi\to+\infty$, but Theorem 1.1 asserts that both susceptible and infective individuals coexist behind the front and that the endemic state (s^*,e^*) is the only possible constant leftover state. To show the convergence to the endemic state as $\xi\to+\infty$, the difficulties come from the fact that (1.4) is a system and is non-local (such issues also arise for equations with non-local nonlinear interaction, see e.g. [1, 2, 4, 18, 19, 23, 21, 40, 41]). We also point out that the random migration coefficient d for the infective individuals is any arbitrary positive real number and is therefore in general different from that for the susceptible individuals. Furthermore, we mention that, due to the transmission terms with opposite signs, the systems (1.1) and (1.4) are not monotone (neither cooperative nor competitive) and therefore do not satisfy the maximum principle. For the same reason, the question of the uniqueness, up to shifts in time, of the traveling wave profiles for a given speed $c\ge c^*$ is still open (by analogy with continuous thermal diffusion models [5, 36, 39], uniqueness or non-uniqueness of the profiles (ϕ,ψ) may depend on the sign of $d-1$). As for the monotonicity of the profiles for a given speed $c\ge c^*$, the numerical simulations presented in Section 7 show that the wave profiles ϕ and ψ are not always monotone.

Theorem 1.1 is concerned with traveling waves for the continuous problem (1.4). Let us now come back to the original discrete problem (1.1). For (1.1), besides (1.4), another natural definition of traveling waves consists of nonnegative and non-constant classical solutions $(s_n)_{n\in\mathbb{Z}}$ and $(i_n)_{n\in\mathbb{Z}}$ which are bounded (that is, each s_n and each i_n is of class $C^1(\mathbb{R})\cap L^\infty(\mathbb{R})$ and $\sup_{n\in\mathbb{Z}}\|s_n\|_{L^\infty(\mathbb{R})}+\sup_{n\in\mathbb{Z}}\|i_n\|_{L^\infty(\mathbb{R})}<+\infty$) and for which there is $c\in\mathbb{R}\setminus\{0\}$ such that

$$s_n\Big(t+\frac{1}{c}\Big)=s_{n+1}(t) \text{ and } i_n\Big(t+\frac{1}{c}\Big)=i_{n+1}(t) \text{ for all } n\in\mathbb{Z} \text{ and } t\in\mathbb{R}. \tag{1.9}$$

We also allow the possibility of bounded nonnegative and non-constant solutions $(i_n)_{n\in\mathbb{Z}}$ and $(s_n)_{n\in\mathbb{Z}}$ which are stationary (that is, independent of $t\in\mathbb{R}$). It immediately follows from Theorem 1.1 and (1.3) that, for any speed $c\ge c^*$, problem (1.1) admits traveling waves solutions in the sense of (1.9) connecting the trivial disease-free state $(1,0)$ as $t\to-\infty$ (that is, as $n\to-\infty$) and some non-trivial leftover concentrations. Furthermore, it turns out that, as for (1.4), problem (1.1) does not admit any such traveling wave solutions in the sense of (1.9) for any speed $c<c^*$ either. Namely, the following result holds.

Theorem 1.2. *For any* $c\ge c^*$, *there exists a bounded traveling wave of the system* (1.1) *in the sense of* (1.9), *such that*

$$0<s_n(t)<1 \quad \text{and} \quad i_n(t)>0 \quad \text{for all } n\in\mathbb{Z} \text{ and } t\in\mathbb{R}, \tag{1.10}$$

together with

$$\lim_{n\to-\infty}(s_n(t),i_n(t))=(1,0) \ \ \textit{locally uniformly in } t\in\mathbb{R}, \tag{1.11}$$

$$\lim_{t\to-\infty}(s_n(t),i_n(t))=(1,0) \ \ \textit{for all } n\in\mathbb{Z}, \tag{1.12}$$

and

$$\begin{cases} 0<\liminf\limits_{n\to+\infty} s_n(t)\le s^*\le \limsup\limits_{n\to+\infty} s_n(t)<1,\ \ 0<\liminf\limits_{n\to+\infty} i_n(t)\le e^*\le \limsup\limits_{n\to+\infty} i_n(t)<+\infty \\ \hfill \textit{for all } t\in\mathbb{R}, \\ 0<\liminf\limits_{t\to+\infty} s_n(t)\le s^*\le \limsup\limits_{t\to+\infty} s_n(t)<1,\ \ 0<\liminf\limits_{t\to+\infty} i_n(t)\le e^*\le \limsup\limits_{t\to+\infty} i_n(t)<+\infty \\ \hfill \textit{for all } n\in\mathbb{Z}. \end{cases} \tag{1.13}$$

Furthermore, for any $c<c^*$, there is no traveling wave of the system (1.1) *in the sense of* (1.9) *or stationary, and satisfying* (1.10) *and* (1.11)*.*

Theorem 1.2 implies in particular that the set of speeds of traveling waves for the original problem (1.1), in the sense of (1.9), is the same as for problem (1.4). However, as for (1.4), the characterization of the leftover concentrations as $n\to+\infty$ (or equivalently as $t\to+\infty$, since $c>0$) or the uniqueness of the profiles $s_n(t)$ and $i_n(t)$ up to shifts in time for a given speed $c\ge c^*$ are still open for (1.9).

Let us finally mention some references on related problems. Actually, there is a vast literature on the study of traveling wave solutions for lattice dynamical systems or discrete versions of continuous parabolic partial differential equations. For monostable equations or monostable monotone systems, we refer to e.g. [7, 8, 9, 16, 17, 22, 24, 26, 30, 32, 37, 38, 44, 47]. Waves for bistable lattice dynamical systems have been studied in e.g. [6, 10, 11, 12, 13, 14, 15, 25, 27, 29, 33, 34, 35, 45, 46].

Remark 1.3. Notice that the necessity condition $c\ge c^*$ holds in Theorem 1.1 (respectively in Theorem 1.2) for any traveling wave (ϕ,ψ) satisfying (1.4), (1.6) and (1.7) (respectively for any traveling wave of (1.1) satisfying (1.9), (1.10) and (1.11)). The limiting conditions (1.8) (respectively (1.13)) or the boundedness of ψ (respectively of the sequence $(i_n)_{n\in\mathbb{Z}}$ in $L^\infty(\mathbb{R})$) are not used here.

Outline of the paper. Sections 2 and 3 are devoted to the proof of the existence of a traveling wave in case $c>c^*$, with some preliminaries on lower and upper solutions in Section 2. Approximated solutions in bounded domains are constructed and the traveling wave solving (1.4) is obtained by passing to the limit in the whole real line. Some intricate issues are to show that the limiting ψ component is bounded and that the leftover concentrations are non-trivial. Section 4 is devoted to the proof of the existence of a traveling wave for the minimal speed c^*, by passing to the limit $c_k\to(c^*)^+$, after a suitable shift of the origin and after showing that the solutions with speed c_k are uniformly bounded. Section 5 is concerned with the proof of the necessity condition $c\ge c^*$ for any traveling wave satisfying (1.6) and (1.7), and Section 6 is devoted to the proof of Theorem 1.2 on the traveling waves for (1.1) in the sense of (1.9). Lastly, we present some numerical experiments in Section 7.

2. Preliminaries

In this section, we always assume that $c > c^*$. Then the equation

$$d\,(e^{\lambda} + e^{-\lambda} - 2) - c\,\lambda + \beta - \mu - \gamma = 0 \tag{2.1}$$

has two positive roots λ_1 and λ_2 with $0 < \lambda_1 < \lambda_2$. Notice that

$$d\,(e^{\lambda} + e^{-\lambda} - 2) - c\,\lambda + \beta - \mu - \gamma < 0$$

for all $\lambda \in (\lambda_1, \lambda_2)$.

2.1. **Upper and lower solutions.** First, we define the notion of upper solution $(\overline{\phi}, \overline{\psi})$ and lower solution $(\underline{\phi}, \underline{\psi})$ of (1.4) as follows.

Definition 2.1. *If $\overline{\phi}$, $\overline{\psi}$, $\underline{\phi}$, $\underline{\psi}$ are continuous in $\mathbb{R}$, of class C^1 on $\mathbb{R} \setminus \mathcal{F}$ for some finite set $\mathcal{F}$ and if they satisfy the following inequalities*

$$D[\overline{\phi}](\xi) - c\,\overline{\phi}'(\xi) + \mu\,(1 - \overline{\phi}(\xi)) - \beta\,\overline{\phi}(\xi)\,\underline{\psi}(\xi) \le 0, \tag{2.2}$$

$$D[\underline{\phi}](\xi) - c\,\underline{\phi}'(\xi) + \mu\,(1 - \underline{\phi}(\xi)) - \beta\,\underline{\phi}(\xi)\,\overline{\psi}(\xi) \ge 0, \tag{2.3}$$

$$d\,D[\overline{\psi}](\xi) - c\,\overline{\psi}'(\xi) - (\mu + \gamma)\,\overline{\psi}(\xi) + \beta\,\overline{\phi}(\xi)\,\overline{\psi}(\xi) \le 0, \tag{2.4}$$

$$d\,D[\underline{\psi}](\xi) - c\,\underline{\psi}'(\xi) - (\mu + \gamma)\,\underline{\psi}(\xi) + \beta\,\underline{\phi}(\xi)\,\underline{\psi}(\xi) \ge 0 \tag{2.5}$$

for all $\xi \in \mathbb{R}\setminus\mathcal{F}$, then the functions $(\overline{\phi}, \overline{\psi})$, $(\underline{\phi}, \underline{\psi})$ are called a pair of upper and lower solutions of (1.4).

Following [3, 20], we introduce

$$\overline{\phi}(\xi) = 1, \quad \overline{\psi}(\xi) = e^{\lambda_1 \xi}, \qquad \xi \in \mathbb{R}, \tag{2.6}$$

$$\underline{\phi}(\xi) = \begin{cases} 1 - \rho\, e^{\theta \xi}, & \xi \le \xi_1, \\ 0, & \xi \ge \xi_1, \end{cases} \tag{2.7}$$

$$\underline{\psi}(\xi) = \begin{cases} e^{\lambda_1 \xi} - q\, e^{\eta \lambda_1 \xi}, & \xi \le \xi_2, \\ 0, & \xi \ge \xi_2, \end{cases} \tag{2.8}$$

where

$$\xi_1 := -\frac{\ln \rho}{\theta} \quad \text{and} \quad \xi_2 := -\frac{\ln q}{(\eta - 1)\,\lambda_1}. \tag{2.9}$$

Here the constants θ, ρ, η and q are chosen *in sequence* such that the following assumptions (A1)-(A4) hold:

(A1) $\theta > 0$ is small enough such that $0 < \theta < \lambda_1$ and $e^{\theta} + e^{-\theta} - 2 - c\,\theta - \mu < 0$,

(A2) $\rho > \max\left\{1, \dfrac{\beta}{-(e^{\theta} + e^{-\theta} - 2 - c\,\theta - \mu)}\right\} \ge 1 > 0$,

(A3) $\eta \in (1, \min\{1 + \theta/\lambda_1, \lambda_2/\lambda_1\})$ such that

$$d\,(e^{\eta\lambda_1} + e^{-\eta\lambda_1} - 2) - c\,\eta\,\lambda_1 + \beta - \mu - \gamma < 0,$$

(A4) $q > \max\left\{e^{(1-\eta)\lambda_1 \xi_1}, \dfrac{\beta\,\rho}{-(d\,(e^{\eta\lambda_1} + e^{-\eta\lambda_1} - 2) - c\,\eta\,\lambda_1 + \beta - \mu - \gamma)}\right\} > 0$.

Note that we have

$$\xi_2 = -\frac{\ln q}{(\eta-1)\lambda_1} < \xi_1 = -\frac{\ln \rho}{\theta} < 0.$$

Also, it easy to see that

$$\max\{0, 1-\rho\, e^{\theta\xi}\} \le \underline{\phi}(\xi) \le \overline{\phi}(\xi) = 1, \quad \max\{0, e^{\lambda_1\xi} - q\, e^{\eta\lambda_1\xi}\} \le \underline{\psi}(\xi) \le \overline{\psi}(\xi) = e^{\lambda_1\xi}$$

for all $\xi \in \mathbb{R}$.

The next lemma gives the existence of a pair of upper and lower solutions.

Lemma 2.2. *The functions* $(\overline{\phi}, \overline{\psi})$ *and* $(\underline{\phi}, \underline{\psi})$ *defined by* (2.6)-(2.8) *are a pair of upper and lower solutions of* (1.4).

Proof. First, the functions $\overline{\phi}$ and $\overline{\psi}$ are of class $C^1(\mathbb{R})$ and the inequalities (2.2) and (2.4) hold on $\mathbb{R}$, since

$$\begin{cases} D[\overline{\phi}](\xi) - c\,\overline{\phi}'(\xi) + \mu\,(1-\overline{\phi}(\xi)) - \beta\,\overline{\phi}(\xi)\,\underline{\psi}(\xi) = -\beta\,\underline{\psi}(\xi) \le 0, \\ d\,D[\overline{\psi}](\xi) - c\,\overline{\psi}'(\xi) - (\mu+\gamma)\,\overline{\psi}(\xi) + \beta\,\overline{\phi}(\xi)\,\overline{\psi}(\xi) \\ \quad = e^{\lambda_1\xi}\left[d\,(e^{\lambda_1} + e^{-\lambda_1} - 2) - c\,\lambda_1 + \beta - \mu - \gamma\right] = 0 \end{cases}$$

for all $\xi \in \mathbb{R}$.

Next, the function $\underline{\phi}$ is continuous in $\mathbb{R}$ and of class $C^1(\mathbb{R}\backslash\{\xi_1\})$ and we would like to show that (2.3) holds for $\xi \ne \xi_1$. For $\xi > \xi_1$, this is trivial since $\underline{\phi}(\xi) = 0$. When $\xi < \xi_1\,(<0)$, we have $\underline{\phi}(\xi) = 1 - \rho e^{\theta\xi}$ and so

$$\begin{aligned} & D[\underline{\phi}](\xi) - c\,\underline{\phi}'(\xi) + \mu\,(1-\underline{\phi}(\xi)) - \beta\,\underline{\phi}(\xi)\,\overline{\psi}(\xi) \\ = \; & 1 - \rho\,e^{\theta(\xi+1)} + 1 - \rho\,e^{\theta(\xi-1)} - 2 + 2\,\rho\,e^{\theta\xi} + c\,\theta\,\rho\,e^{\theta\xi} + \mu\,\rho\,e^{\theta\xi} - \beta\,e^{\lambda_1\xi} + \beta\,\rho\,e^{(\theta+\lambda_1)\xi} \\ \ge \; & e^{\theta\xi}\left[-\rho\,(e^{\theta} + e^{\theta} - 2\,c\,\theta - \mu) - \beta\,e^{(\lambda_1-\theta)\xi}\right] \\ \ge \; & \beta\,e^{\theta\xi}\left[1 - e^{(\lambda_1-\theta)\xi}\right] \ge 0 \end{aligned}$$

by $\theta < \lambda_1$ and the choice of ρ.

Finally, the function $\underline{\psi}$ is continuous in $\mathbb{R}$ and of class $C^1(\mathbb{R}\backslash\{\xi_2\})$ and we claim that (2.5) holds for $\xi \ne \xi_2$. Clearly, (2.5) holds for $\xi > \xi_2$. For the case $\xi < \xi_2$, due to $\xi_2 < \xi_1 < 0$, we know that $\underline{\phi}(\xi) = 1 - \rho e^{\theta\xi}$ and $\underline{\psi}(\xi) = e^{\lambda_1\xi} - q e^{\eta\lambda_1\xi}$. Then we obtain

$$\begin{aligned} & d\,D[\underline{\psi}](\xi) - c\,\underline{\psi}'(\xi) - (\mu+\gamma)\,\underline{\psi}(\xi) + \beta\,\underline{\phi}(\xi)\,\underline{\psi}(\xi) \\ \ge \; & d\left[-q\,e^{\eta\lambda_1(\xi+1)} - q\,e^{\eta\lambda_1(\xi-1)} + 2\,q\,e^{\eta\lambda_1\xi}\right] + c\,\eta\,\lambda_1\,q\,e^{\eta\lambda_1\xi} - (\beta-\mu-\gamma)\,q\,e^{\eta\lambda_1\xi} - \beta\,\rho\,e^{(\theta+\lambda_1)\xi} \\ = \; & e^{\eta\lambda_1\xi}\Big\{-q\left[d\,(e^{\eta\lambda_1} + e^{-\eta\lambda_1} - 2) - c\,\eta\,\lambda_1 + \beta - \mu - \gamma\right] - \beta\,\rho\,e^{[\theta+(1-\eta)\lambda_1]\xi}\Big\} \\ \ge \; & \beta\,\rho\,e^{\eta\lambda_1\xi}\big(1 - e^{(\theta+(1-\eta)\lambda_1)\xi}\big) \ge 0 \end{aligned}$$

by the choices of η and q. Therefore, the proof of this lemma has been completed. □

2.2. **An auxiliary truncated problem.** Now, given $l > -\xi_2\,(>0)$, we consider the following truncated problem

$$
\begin{cases}
D[\phi] - c\,\phi' + \mu\,(1-\phi) - \beta\,\phi\,\psi = 0 & \text{in } [-l,l],\\
d\,D[\psi] - c\,\psi' - (\mu+\gamma)\,\psi + \beta\,\phi\,\psi = 0 & \text{in } [-l,l],\\
(\phi,\psi) = (\overline{\phi},\overline{\psi}) & \text{on } (-\infty,-l),\\
(\phi,\psi) = (\phi(l),\psi(l)) & \text{on } (l,+\infty),
\end{cases}
\tag{2.10}
$$

where

$$
\begin{cases}
\phi'(-l) := \lim\limits_{h\searrow 0} \dfrac{\phi(-l+h)-\phi(-l)}{h}, & \psi'(-l) := \lim\limits_{h\searrow 0} \dfrac{\psi(-l+h)-\psi(-l)}{h},\\
\phi'(l) := \lim\limits_{h\searrow 0} \dfrac{\phi(l)-\phi(l-h)}{h}, & \psi'(l) := \lim\limits_{h\searrow 0} \dfrac{\psi(l)-\psi(l-h)}{h}.
\end{cases}
$$

Next, we give some notations. Set $\mathcal{C}^l := C([-l,l]) \times C([-l,l])$ and

$$\mathcal{S}^l := \big\{(\phi,\psi) \in \mathcal{C}^l \mid \underline{\phi} \le \phi \le \overline{\phi},\ \underline{\psi} \le \psi \le \overline{\psi} \text{ in } [-l,l] \text{ and } (\phi,\psi)(-l) = (\overline{\phi},\overline{\psi})(-l)\big\}.$$

From the definition of $\overline{\phi}, \underline{\phi}, \overline{\psi}, \underline{\psi}$, we know that $0 \le \phi \le 1$ and $0 \le \psi \le e^{\lambda_1 l}$ in $[-l,l]$ for any $(\phi,\psi) \in \mathcal{S}^l$. Hence $\mathcal{S}^l$ is a nonempty bounded closed convex set in $(\mathcal{C}^l, \|\cdot\|)$, where $\|\cdot\|$ is the usual sup norm. For any $(\phi,\psi) \in \mathcal{S}^l$, we extend (ϕ,ψ) be continuous outside the interval $[-l,l]$ as in (2.10) and we introduce the continuous functions $H_1^l(\phi,\psi)$ and $H_2^l(\phi,\psi)$ defined in $\mathbb{R}$ by

$$
\begin{cases}
H_1^l(\phi,\psi)(\xi) &= \alpha\,\phi(\xi) + D[\phi](\xi) + \mu\,(1-\phi(\xi)) - \beta\,\phi(\xi)\,\psi(\xi),\\
H_2^l(\phi,\psi)(\xi) &= \alpha\,\psi(\xi) + d\,D[\psi](\xi) - (\mu+\gamma)\,\psi(\xi) + \beta\,\phi(\xi)\,\psi(\xi),
\end{cases}
$$

where $\alpha = \alpha^l$ is a positive constant such that

$$\alpha > \max\big\{2 + \mu + \beta\,e^{\lambda_1 l}, 2\,d + \mu + \gamma\big\}.$$

For $(\phi_i,\psi_i) \in \mathcal{S}^l$, $i = 1,2$, with $\phi_1 \le \phi_2$ and $\psi_1 \le \psi_2$ in $[-l,l]$, we have

$$H_1^l(\phi_1,\psi_2)(\xi) \le H_1^l(\phi_1,\psi_1)(\xi) \le H_1^l(\phi_2,\psi_1)(\xi) \ \text{ and } \ H_2^l(\phi_1,\psi_1)(\xi) \le H_2^l(\phi_2,\psi_2)(\xi) \tag{2.11}$$

for all $\xi \in [-l,l]$. Finally, we define the operator $F^l = (F_1^l, F_2^l)$ from $\mathcal{S}^l$ into $\mathcal{C}^l$ as follows

$$
\begin{cases}
F_1^l(\phi,\psi)(\xi) &= e^{\alpha(-l-\xi)/c}\,\overline{\phi}(-l) + \displaystyle\int_{-l}^{\xi} \frac{e^{\alpha(z-\xi)/c}}{c} H_1^l(\phi,\psi)(z)\,dz, \quad \xi \in [-l,l],\\
F_2^l(\phi,\psi)(\xi) &= e^{\alpha(-l-\xi)/c}\,\overline{\psi}(-l) + \displaystyle\int_{-l}^{\xi} \frac{e^{\alpha(z-\xi)/c}}{c} H_2^l(\phi,\psi)(z)\,dz, \quad \xi \in [-l,l].
\end{cases}
$$

Note that a fixed point (ϕ,ψ) of the operator F^l, extended outside the interval $[-l,l]$ as in (2.10), gives a solution of (2.10) which is continuous in $\mathbb{R}$ and of class $C^1(\mathbb{R}\backslash\{-l,l\})$.

To show the existence of such a fixed point, we apply Schauder's fixed point theorem in the next lemma.

Lemma 2.3. *Given $l > -\xi_2$, there exists a $C(\mathbb{R}) \times C(\mathbb{R})$ and $C^1(\mathbb{R}\backslash\{-l,l\}) \times C^1(\mathbb{R}\backslash\{-l,l\})$ solution (ϕ,ψ) of (2.10) such that*

$$0 \le \underline{\phi} \le \phi \le 1 \quad \text{and} \quad 0 \le \underline{\psi} \le \psi \le \overline{\psi} \quad \text{in } (-\infty,l]. \tag{2.12}$$

Proof. First, we claim that $F^l(\mathcal{S}^l) \subset \mathcal{S}^l$. By (2.11), for any $(\phi, \psi) \in \mathcal{S}^l$, we have

$$F_1^l(\underline{\phi}, \overline{\psi}) \le F_1^l(\phi, \psi) \le F_1^l(\overline{\phi}, \underline{\psi}) \ \text{ and } \ F_2^l(\underline{\phi}, \underline{\psi}) \le F_2^l(\phi, \psi) \le F_1^l(\overline{\phi}, \overline{\psi}) \ \text{ in } [-l, l].$$

By Lemma 2.2 and the definition of the upper and lower solutions, we also derive that

$$\underline{\phi} \le F_1^l(\underline{\phi}, \overline{\psi}), \quad F_1^l(\overline{\phi}, \underline{\psi}) \le \overline{\phi}, \quad \underline{\psi} \le F_2^l(\underline{\phi}, \underline{\psi}) \ \text{ and } \ F_2^l(\overline{\phi}, \overline{\psi}) \le \overline{\psi} \ \text{ in } [-l, l].$$

Hence $F^l(\mathcal{S}^l) \subset \mathcal{S}^l$.

By using the Arzela-Ascoli theorem, the operator $F^l : \mathcal{S}^l \to \mathcal{S}^l$ is completely continuous with respect to the sup norm. With the help of Schauder's fixed point theorem, we conclude that there exists a pair $(\phi, \psi) \in \mathcal{S}^l$ such that $(\phi, \psi) = F^l(\phi, \psi)$. Therefore, (ϕ, ψ), extended outside the interval $[-l, l]$ as in (2.10), solves (2.10) and satisfies the properties stated in Lemma 2.3. □

3. Existence of a traveling wave for $c > c^*$

3.1. Proof of Theorem 1.1 for $c > c^*$.

In this section, we show Theorem 1.1 for any fixed real number $c \in (c^*, +\infty)$. Namely, we show the existence of a bounded solution (ϕ, ψ) of (1.4) satisfying $0 < \phi < 1$ in $\mathbb{R}$, $\psi > 0$ in $\mathbb{R}$, and such that (1.7) and (1.8) hold.

First, we consider a positive increasing sequence $\{l_k\}_{k \in \mathbb{N}}$ such that $l_k \to \infty$ as $k \to \infty$, and $l_k > -\xi_2$ for all $k \in \mathbb{N}$, where $\xi_2 < 0$ is as in (2.9). By Lemma 2.3, for each $k \in \mathbb{N}$, there exists a $C(\mathbb{R}) \cap C^1(\mathbb{R} \backslash \{-l_k, l_k\})$ solution (ϕ_k, ψ_k) of (2.10) and (2.12) for $l = l_k$. For each $K \in \mathbb{N}$ such that $l_K \ge 2$, since $\overline{\psi}$ is bounded above in $[-l_K, l_K]$, it follows from (2.12) that the sequences

$$\{\phi_k\}_{k \ge K}, \quad \{\psi_k\}_{k \ge K}, \quad \{\phi_k \psi_k\}_{k \ge K}$$

are uniformly bounded on $[-l_K, l_K]$. Also, the sequences $\{\phi_k'\}_{k \ge K}$ and $\{\psi_k'\}_{k \ge K}$ are uniformly bounded in $[-l_K + 1, l_K - 1]$, due to (2.10) and (2.12). Since $\phi_k''(\xi)$ and $\psi_k''(\xi)$ can be expressed in terms of $\phi_k(\xi)$, $\psi_k(\xi)$, $\phi_k(\xi \pm 1)$, $\psi_k(\xi \pm 1)$, $\phi_k(\xi \pm 2)$, $\psi_k(\xi \pm 2)$, $\phi_k'(\xi)$ and $\psi_k'(\xi)$ in $[-l_K + 2, l_K - 2]$, one infers that the sequences $\{\phi_k''\}_{k \ge K}$ and $\{\psi_k''\}_{k \ge K}$ are uniformly bounded in $[-l_K + 2, l_K - 2]$. By using the Arzela-Ascoli theorem on $[-l_K + 2, l_K - 2]$ for every $K \in \mathbb{N}$ large enough, we obtain a subsequence $\{(\phi_{k_j}, \psi_{k_j})\}$ of $\{(\phi_k, \psi_k)\}$ through the diagonal process such that

$$\phi_{k_j} \to \phi, \ \psi_{k_j} \to \psi, \ \phi_{k_j}' \to \phi', \ \psi_{k_j}' \to \psi' \ \text{ as } j \to +\infty$$

uniformly in any compact subinterval of $\mathbb{R}$, for some functions $\phi \in C^1(\mathbb{R})$ and $\psi \in C^1(\mathbb{R})$. Then (ϕ, ψ) is a solution of the system (1.4) with

$$0 \le \underline{\phi} \le \phi \le 1 \ \text{ and } \ 0 \le \underline{\psi} \le \psi \le \overline{\psi} \ \text{ in } \mathbb{R}. \tag{3.1}$$

By the definitions of $\underline{\phi}$, $\overline{\psi}$ and $\underline{\psi}$, it easy to check that

$$(\phi, \psi)(-\infty) = (1, 0).$$

Notice also that, by differentiating the equations (1.4), one infers by induction that the functions ϕ and ψ are of class C^∞ in $\mathbb{R}$.

Lemma 3.1. *The functions ϕ and ψ are non-trivial, in the sense that*

$$0 < \phi < 1 \ \textit{ and } \ \psi > 0 \ \textit{ in } \mathbb{R}.$$

Proof. Firstly, owing to the definition of $\underline{\psi}$, we have $\psi > 0$ in $(-\infty, \xi_2)$. For contradiction, we assume that there exists a real number $\xi_0 \in [\xi_2, +\infty)$ such that $\psi(\xi_0) = 0$ and $\psi(\xi) > 0$ for all $\xi < \xi_0$. Since $\psi \geq 0$ in $\mathbb{R}$, we also have $\psi'(\xi_0) = 0$. From the second equation of (1.4), we get that $\psi(\xi_0 - 1) = \psi(\xi_0 + 1) = 0$, a contradiction to the definition of ξ_0.

Let us now show that $\phi > 0$ over $\mathbb{R}$. Indeed, if $\phi(\xi^*) = 0$ for some real number ξ^*, then

$$0 = -c\,\phi'(\xi^*) + D[\phi](\xi^*) + \mu\left(1 - \phi(\xi^*)\right) - \beta\,\phi(\xi^*)\,\psi(\xi^*) = -c\,\phi'(\xi^*) + D[\phi](\xi^*) + \mu > 0,$$

since $\phi'(\xi^*) = 0$, $D[\phi](\xi^*) \geq 0$ and $\mu > 0$. This contradiction leads to the inequality $\phi > 0$ in $\mathbb{R}$.

Similarly, we claim that $\phi < 1$ in $\mathbb{R}$ by a contradiction argument. If there exists a real number $\tilde{\xi}$ such that $\phi(\tilde{\xi}) = 1$, then

$$0 = -c\,\phi'(\tilde{\xi}) + D[\phi](\tilde{\xi}) + \mu\left(1 - \phi(\tilde{\xi})\right) - \beta\,\phi(\tilde{\xi})\,\psi(\tilde{\xi}) = -c\,\phi'(\tilde{\xi}) + D[\phi](\tilde{\xi}) - \beta\,\psi(\tilde{\xi}) < 0,$$

since $\phi'(\tilde{\xi}) = 0$, $D[\phi](\tilde{\xi}) \leq 0$ and $\psi(\tilde{\xi}) > 0$. This contradiction leads to the inequality $\phi < 1$ in $\mathbb{R}$. □

The next main step consists in showing that the function ψ is actually bounded. A first key-point is the following Harnack type property for equations of the type (1.4) satisfied by the second component ψ. We state this property in a more general framework.

Lemma 3.2. *Let M be a positive real number. Then there exists a constant $C = C(M) > 0$ such that, for any continuous functions a and b with $M^{-1} \leq a(\xi) \leq M$ and $b(\xi) \geq -M$ for all $\xi \in \mathbb{R}$ and for any positive $C^1(\mathbb{R})$ function u satisfying*

$$u'(\xi) \geq a(\xi)\,u(\xi+1) + b(\xi)\,u(\xi) \quad \text{for all } \xi \in \mathbb{R},$$

there holds

$$C^{-1} \leq \frac{u(\xi+1)}{u(\xi)} \leq C \quad \text{for all } \xi \in \mathbb{R}.$$

In order not to lengthen too much the main line of the proof of Theorem 1.1 with $c > c^*$, the proof of Lemma 3.2 is postponed in Section 3.2.

Coming back to our solutions (ϕ, ψ) of (1.4), since $c > 0$ and ϕ is nonnegative, it follows from Lemma 3.2 applied to the positive function $u = \psi$ solving $\psi'(\xi) \geq (d/c)\psi(\xi+1) - (2d/c + \mu/c + \gamma/c)\,\psi(\xi)$ that the functions $\xi \mapsto \psi(\xi \pm 1)/\psi(\xi)$ are bounded in $\mathbb{R}$. Hence, from the equation (1.4) itself and since ϕ is bounded, the function

$$\xi \mapsto \frac{\psi'(\xi)}{\psi(\xi)}$$

is therefore bounded too.

The following two lemmas deal with the behavior of ϕ and ψ at $+\infty$ if $\limsup_{\xi \to +\infty} \psi(\xi) = +\infty$. The first one says that ϕ is small when ψ is large. This property actually holds locally uniformly with respect to the speed c. It is stated in this more general framework since it will be used again in Section 4 to get the existence of a bounded solution (ϕ, ψ) of (1.4) with speed c^*.

Lemma 3.3. *Let $0 < \underline{c} \le \overline{c}$ be two given positive real numbers. Let $\{c_k\}$ be a sequence of real numbers in $[\underline{c}, \overline{c}]$ and let $\{(\phi_k, \psi_k)\}$ be a sequence of solutions of* (1.4) *with speed c_k and satisfying* (1.6)*. If $\{\xi_k\}$ is a sequence of real numbers such that $\psi_k(\xi_k) \to +\infty$ as $k \to +\infty$, then $\phi_k(\xi_k) \to 0$ as $k \to +\infty$.*

Since this lemma is concerned with general sequences of solutions with different speeds, and in order not to lengthen too much the main line of the proof of Theorem 1.1 with given speed $c > c^*$, the proof of Lemma 3.3 is postponed in Section 3.2.

Coming back to our solution (ϕ, ψ) of (1.4) satisfying (1.6) and (1.7), the following result shows the convergence of ψ to $+\infty$ at $+\infty$ if it were not bounded.

Lemma 3.4. *If* $\limsup_{\xi \to +\infty} \psi(\xi) = +\infty$*, then* $\lim_{\xi \to +\infty} \psi(\xi) = +\infty$.

Proof. Assume by way of contradiction that $\limsup_{\xi \to +\infty} \psi(\xi) = +\infty$ and $\liminf_{\xi \to +\infty} \psi(\xi) < +\infty$. Since ψ'/ψ is globally bounded, there are then $M \in \mathbb{R}$ and two sequences $\{\theta_k\}$ and $\{\xi_k\}$ converging to $+\infty$ and such that

$$\psi(\theta_k) \le M, \quad \theta_k < \xi_k - 1 < \xi_k < \xi_k + 1 < \theta_{k+1}, \quad \psi(\xi_k) = \max_{[\theta_k, \theta_{k+1}]} \psi \ \ \Big(= \max_{[\xi_k - 1, \xi_k + 1]} \psi\Big)$$

for all $k \in \mathbb{N}$ and $\lim_{k \to +\infty} \psi(\xi_k) = +\infty$. Therefore, $\psi'(\xi_k) = 0$ and $d\,D[\psi](\xi_k) \le 0$. Hence, by (1.4), one infers that $\big(\mu + \gamma - \beta\,\phi(\xi_k)\big)\,\psi(\xi_k) \le 0$ for all $k \in \mathbb{N}$. This is clearly impossible for large k since $\psi(\xi_k) > 0$, and $\phi(\xi_k) \to 0$ as $k \to +\infty$ by Lemma 3.3. The proof is thereby complete. ☐

To proceed further, we recall the following useful fundamental theory from [9] (or [7]) in dealing with the asymptotic tail behavior of wave profiles for a lattice dynamical system.

Proposition 3.5. [9] *Let $\varsigma > 0$ be a positive constant, let $B : \mathbb{R} \to \mathbb{R}$ be a continuous function having finite $B(\pm\infty) := \lim_{x \to \pm\infty} B(x)$ and let z be a continuous function such that*

$$\varsigma\, z(x) = e^{\int_x^{x+1} z(s)ds} + e^{\int_x^{x-1} z(s)ds} + B(x), \ \forall x \in \mathbb{R}. \tag{3.2}$$

Then z is uniformly continuous and bounded in $\mathbb{R}$. In addition, the limits $\omega^\pm = \lim_{x \to \pm\infty} z(x)$ exist and are real roots of the characteristic equations

$$\varsigma\, \omega = e^{\omega} + e^{-\omega} + B(\pm\infty).$$

With this result and the previous lemmas in hand, we can show that ψ is bounded in $\mathbb{R}$.

Lemma 3.6. *The function ψ is bounded.*

Proof. Assume not. Then $\limsup_{\xi \to +\infty} \psi(\xi) = +\infty$, since ψ is continuous, positive, and $\psi(-\infty) = 0$. Therefore, Lemmas 3.3 and 3.4 imply that $\psi(\xi) \to +\infty$ and $\phi(\xi) \to 0$ as $\xi \to +\infty$. From (1.4), the continuous function $z := \psi'/\psi$ satisfies

$$\frac{c}{d}\, z(x) = e^{\int_x^{x+1} z(s)ds} + e^{\int_x^{x-1} z(s)ds} - 2 - \frac{\mu + \gamma}{d} + \frac{\beta\,\phi(x)}{d}$$

for all $x \in \mathbb{R}$. Since ϕ has finite limits at $\pm\infty$ and $\phi(+\infty) = 0$, it then follows from Proposition 3.5 that, in particular, z has a finite limit ω at $+\infty$, with

$$d\left(e^{\omega} + e^{-\omega} - 2\right) = c\,\omega + \mu + \gamma. \tag{3.3}$$

Since μ and γ are positive, this equation has a negative and a positive root. The function $z = \psi'/\psi$ cannot converge to the negative root at $+\infty$, since $\psi(+\infty) = +\infty$. Therefore, ψ'/ψ converges at $+\infty$ to the positive root ω of (3.3). Remember now that $\lambda_1 < \lambda_2$ are the two positive roots of equation (2.1). Since $\beta > 0$, one infers immediately that $\lambda_1 < \lambda_2 < \omega$. But $\lim_{\xi\to+\infty} \psi'(\xi)/\psi(\xi) = \omega > 0$ yields $\ln\psi(\xi) \sim \omega\,\xi$ as $\xi \to +\infty$, while (3.1) implies that $\psi(\xi) \le \overline{\psi}(\xi) = e^{\lambda_1\xi}$ for all $\xi \in \mathbb{R}$. One gets a contradiction, since $\lambda_1 < \omega$. As a conclusion, the function ψ is bounded and the proof of Lemma 3.6 is complete. ☐

To complete the proof of Theorem 1.1 in case $c > c^*$, we show in the following lemmas that none of the components ϕ and ψ can be trivial at $+\infty$.

Lemma 3.7. *There holds* $\inf_{\mathbb{R}} \phi > 0$.

Proof. Remember that the C^∞ function ϕ satisfies $0 < \phi < 1$ in $\mathbb{R}$ and $\phi(-\infty) = 1$. Assume by contradiction that $\inf_{\mathbb{R}} \phi = 0$. Then there exists a sequence $\{\xi_k\}$ converging to $+\infty$ such that $\phi(\xi_k) \to 0$ as $k \to +\infty$. On the other hand, since both functions ϕ and ψ are bounded, the equations (1.4) guarantee that the functions ϕ and ψ have bounded derivatives at any order. Therefore, by the Arzela-Ascoli theorem, the functions $\xi \mapsto \phi(\xi + \xi_k)$ and $\xi \mapsto \psi(\xi + \xi_k)$ converge in $C^\infty_{loc}(\mathbb{R})$ as $k \to +\infty$, up to extraction of a subsequence, to some nonnegative C^∞ functions ϕ_∞ and ψ_∞. Furthermore,

$$-c\,\phi'_\infty + D[\phi_\infty] + \mu\,(1 - \phi_\infty) - \beta\,\phi_\infty\,\psi_\infty = 0 \tag{3.4}$$

in $\mathbb{R}$ and $\phi_\infty(0) = 0$. Since 0 is a global minimum of ϕ_∞, one has $\phi'_\infty(0) = 0$ and the above equality at 0 leads to a contradiction, since $\phi_\infty \ge 0$ and $\mu > 0$. Therefore, $\inf_{\mathbb{R}} \phi > 0$. ☐

To show that ψ cannot approach 0 at $+\infty$, even for a sequence, the key-step is the following lemma saying that ψ is increasing when it is small. The property actually holds locally uniformly with respect to the speed c and we state the lemma in this slightly more general framework, since it will be used as such in Section 4.

Lemma 3.8. *Let* $0 < \underline{c} \le \overline{c}$ *be two given positive real numbers. There is* $\varepsilon > 0$ *such that, for any* $\Gamma \in [\underline{c}, \overline{c}]$ *and for any solution* (Φ, Ψ) *of* (1.4) (*with speed* Γ *in place of* c) *satisfying* (1.6), *there holds*

$$\forall\,\xi \in \mathbb{R}, \quad \big(\Psi(\xi) \le \varepsilon\big) \Longrightarrow \big(\Psi'(\xi) > 0\big).$$

In order to conclude now the proof of Theorem 1.1 with $c > c^*$, the proof of Lemma 3.8 is postponed in Section 3.2. Coming back to our solution (ϕ, ψ), we immediately get from Lemma 3.8 and the positivity of ψ in $\mathbb{R}$ that

$$\liminf_{\xi\to+\infty} \psi(\xi) > 0. \tag{3.5}$$

We also claim that

$$\limsup_{\xi\to+\infty} \phi(\xi) < 1. \tag{3.6}$$

Indeed, otherwise, there exists a sequence of real numbers $\{\xi_k\}$ converging to $+\infty$ such that $\phi(\xi_k) \to 1$ as $k \to +\infty$. As in the proof of Lemma 3.7, up to extraction of a subsequence, the functions $\xi \mapsto \phi(\xi + \xi_k)$ and $\xi \mapsto \psi(\xi + \xi_k)$ converge as $k \to +\infty$ in $C^\infty_{loc}(\mathbb{R})$ to some nonnegative C^∞ functions ϕ_∞ and ψ_∞ solving (1.4). Furthermore, $0<\phi_\infty\le 1$ and $\psi_\infty>0$ in $\mathbb{R}$ from Lemma 3.7 and (3.5). Since $\phi_\infty(0)=1$, one has $\phi'_\infty(0)=0$. The equation (3.4) satisfied by ϕ_∞ at 0 leads to a contradiction, since $D[\phi_\infty](0)\le 0$ and $-\beta\,\phi_\infty(0)\,\psi_\infty(0)=-\beta\,\psi_\infty(0)<0$. Therefore, the claim (3.6) holds.

In order to complete the proof of (1.8), let us finally show that

$$\liminf_{\xi\to+\infty}\phi(\xi)\le s^*\le\limsup_{\xi\to+\infty}\phi(\xi)\ \text{ and }\ \liminf_{\xi\to+\infty}\psi(\xi)\le e^*\le\limsup_{\xi\to+\infty}\psi(\xi). \tag{3.7}$$

Call $\phi_-=\liminf_{\xi\to+\infty}\phi(\xi)$, $\phi_+=\limsup_{\xi\to+\infty}\phi(\xi)$, $\psi_-=\liminf_{\xi\to+\infty}\psi(\xi)$ and $\psi_+=\limsup_{\xi\to+\infty}\psi(\xi)$. One already knows from (3.5), (3.6) and Lemmas 3.6 and 3.7 that

$$0<\phi_-\le\phi_+<1\ \text{ and }\ 0<\psi_-\le\psi_+<+\infty.$$

Consider now a sequence $\{\xi_k\}$ converging to $+\infty$ such that $\psi(\xi_k)\to\psi_-$ as $k\to+\infty$. Up to extraction of a subsequence (as for instance in the proof of Lemma 3.7), the functions $\xi\mapsto\phi(\xi+\xi_k)$ and $\xi\mapsto\psi(\xi+\xi_k)$ converge in $C^\infty_{loc}(\mathbb{R})$ to some bounded functions $0<\phi_\infty<1$ and $\psi_\infty>0$ satisfying (1.4). Furthermore, $0<\psi_-=\psi_\infty(0)=\min_{\mathbb{R}}\psi_\infty$. Therefore, $\psi'_\infty(0)=0$ and $D[\psi_\infty](0)\ge 0$. Hence

$$-(\mu+\gamma)\,\psi_-+\beta\,\phi_\infty(0)\,\psi_-\le 0,$$

that is, $\beta\,\phi_\infty(0)\le\mu+\gamma$. This yields $\phi_-=\liminf_{\xi\to+\infty}\phi(\xi)\le(\mu+\gamma)/\beta=1/\sigma=s^*$. Similarly, it follows that $\phi_+=\limsup_{\xi\to+\infty}\phi(\xi)\ge s^*$. Consider also a sequence $\{\zeta_k\}$ converging to $+\infty$ such that $\phi(\xi_k)\to\phi_-$ as $k\to+\infty$. As above, up to extraction of a subsequence, the functions $\xi\mapsto\phi(\xi+\zeta_k)$ and $\xi\mapsto\psi(\xi+\zeta_k)$ converge in $C^\infty_{loc}(\mathbb{R})$ to some bounded functions $0<\Phi_\infty<1$ and $\Psi_\infty>0$ satisfying (1.4). Furthermore, $0<\phi_-=\Phi_\infty(0)=\min_{\mathbb{R}}\Phi_\infty$. Therefore, $\Phi'_\infty(0)=0$ and $D[\Phi_\infty](0)\ge 0$. Hence

$$\mu\,(1-\phi_-)-\beta\,\phi_-\,\Psi_\infty(0)\le 0.$$

Since $0<\phi_-\le s^*=1/\sigma$, one gets immediately that $\Psi_\infty(0)\ge(\mu/\beta)(\sigma-1)=e^*$, whence $\psi_+=\limsup_{\xi\to+\infty}\psi(\xi)\ge e^*$. Similarly, it follows that $\psi_-=\liminf_{\xi\to+\infty}\psi(\xi)\le e^*$.

As a conclusion, (1.8) is proved and the proof of Theorem 1.1 in case $c>c^*$ is thereby complete.

As explained after the statement of Theorem 1.1 in Section 1, the question of the existence of a limit of (ϕ,ψ) at $+\infty$ is unclear. However, we can say that the a priori existence of a limit of one of these two functions guarantees the convergence of both, and that the endemic state (s^*,e^*) defined in (1.2) is the only possible limit.

Lemma 3.9. *Let (ϕ,ψ) be a bounded classical solution of* (1.4) *satisfying* (1.6), (1.7) *and* (1.8), *with speed $c\ge c^*$. If $\phi(+\infty)$ or $\psi(+\infty)$ exists, then they both exist and*

$$(\phi(+\infty),\psi(+\infty))=(s^*,e^*).$$

Proof. Assume first that $l=\lim_{\xi\to+\infty}\phi(\xi)$ exists. Property (1.8) yields $0<l=s^*<1$. Consider now any sequence $\{\xi_k\}$ converging to $+\infty$. Up to extraction of a subsequence, the functions $\xi \mapsto \phi(\xi+\xi_k)$ and $\xi \mapsto \psi(\xi+\xi_k)$ converge in $C^\infty_{loc}(\mathbb{R})$ to some functions $\phi_\infty = l = s^*$ and ψ_∞ such that

$$\mu\,(1-s^*) - \beta\, s^*\, \psi_\infty(\xi) = 0 \quad \text{for all } \xi \in \mathbb{R}.$$

Therefore, the function ψ_∞ is identically equal to the constant $\mu(1-s^*)/(\beta s^*) = e^*$. Since the limit does not depend on the sequence $\{\xi_k\}$, one infers that $\lim_{\xi\to+\infty}\psi(\xi) = e^*$.

Conversely, if $L = \lim_{\xi\to+\infty}\psi(\xi)$ exists, property (1.8) yields $0 < L = e^*$. For any sequence $\{\xi_k\}$ converging to $+\infty$, the functions $\xi \mapsto \phi(\xi+\xi_k)$ and $\xi \mapsto \psi(\xi+\xi_k)$ converge in $C^\infty_{loc}(\mathbb{R})$, up to a subsequence, to some functions ϕ_∞ and $\psi_\infty = L = e^*$ such that

$$-(\mu+\gamma)\, e^* + \beta\, e^*\, \phi_\infty(\xi) = 0 \quad \text{for all } \xi \in \mathbb{R}.$$

Therefore, the function ϕ_∞ is identically equal to the constant $(\mu+\gamma)/\beta = s^*$. Since the limit does not depend on the sequence $\{\xi_k\}$, one infers that $\lim_{\xi\to+\infty}\phi(\xi) = s^*$.

Therefore, if the limit $l = \phi(+\infty)$ or the limit $L = \psi(+\infty)$ exists, then they both exist such that $(\phi(+\infty), \psi(+\infty)) = (s^*, e^*)$. □

Remark 3.10. The condition $c \geq c^*$ in Lemma 3.9 is not a restriction, since we shall prove in Section 5 that, for any solution (ϕ, ψ) of (1.4) satisfying (1.6) and (1.7) with speed c, there holds $c \geq c^*$.

3.2. **Proof of Lemmas 3.2, 3.3 and 3.8.** In this section, we prove some technical lemmas stated in Section 3.1.

Proof of Lemma 3.2. Although the idea of the proof is similar to the one given in [9], we provide the details here for completeness. Up to multiplication of u by a positive constant and up to a shift in space, one can assume without loss of generality that $u(0) = 1$ and it is sufficient to show that $u(\pm 1) \leq C = C(M)$. Firstly, since $u'(\xi) \geq -M\, u(\xi)$ for all $\xi \in \mathbb{R}$, the function $\xi \mapsto v(\xi) := u(\xi)\, e^{M\xi}$ is nondecreasing, hence

$$u(-1) \leq e^M\, u(0) = e^M.$$

Secondly, for all $\xi \in [0,1]$, one has

$$v'(\xi) = (u'(\xi) + M u(\xi))\, e^{M\xi} \geq a(\xi)\, u(\xi+1)\, e^{M\xi} \geq \frac{v(\xi+1)\, e^{-M}}{M} \geq \frac{v(1)\, e^{-M}}{M} = \frac{u(1)}{M}.$$

Hence, $v(\xi) \geq v(0) + u(1)\,\xi/M = 1 + u(1)\,\xi/M$ for all $\xi \in [0,1]$. In other words,

$$u(\xi) \geq \Big(1 + \frac{u(1)\,\xi}{M}\Big)\, e^{-M\xi} \quad \text{for all } \xi \in [0,1].$$

Finally, for all $\xi \in [-1/2, 0]$,

$$v'(\xi) \geq a(\xi) u(\xi+1)\, e^{M\xi} \geq \frac{e^{M\xi}}{M} \times \Big(1 + \frac{u(1)\,(\xi+1)}{M}\Big)\, e^{-M(\xi+1)} \geq \frac{e^{-M}}{M} \times \Big(1 + \frac{u(1)}{2M}\Big).$$

Therefore,

$$1 = v(0) \geq \underbrace{v(-1/2)}_{\geq 0} + \frac{e^{-M}}{2M} \times \Big(1 + \frac{u(1)}{2M}\Big) \geq \frac{e^{-M}}{2M} \times \Big(1 + \frac{u(1)}{2M}\Big).$$

Hence

$$u(1) \le 2\,M\left(2\,M\,e^{M}-1\right)$$

and the proof of Lemma 3.2 is thereby complete with $C(M)=\max\{e^{M},2M(2Me^{M}-1)\}$. □

Proof of Lemma 3.3. Let $0<\underline{c}\le\overline{c}$, $\{c_k\}$, $\{(\phi_k,\psi_k)\}$ and $\{\xi_k\}$ be as in the statement and assume by way of contradiction that there are $\varepsilon>0$ and a subsequence, still denoted with the same index k, such that $\psi_k(\xi_k)\to+\infty$ as $k\to+\infty$ and $\phi_k(\xi_k)\ge\varepsilon$ for all $k\in\mathbb{N}$. Since $0<\phi_k<1$ and $\psi_k>0$ in $\mathbb{R}$, the equation (1.4) for ϕ_k (with $c_k\in[\underline{c},\overline{c}]$) implies that $\phi_k'\le(2+\mu)/\underline{c}$ in $\mathbb{R}$. Hence

$$\phi_k(\xi)\ge\frac{\varepsilon}{2}\ \text{ for all } \xi\in[\xi_k-\delta,\xi_k] \text{ and for all } k\in\mathbb{N}, \tag{3.8}$$

where $\delta=\varepsilon\,\underline{c}/(4+2\mu)>0$. On the other hand, since

$$\psi_k'(\xi)\ge\frac{d}{\overline{c}}\,\psi_k(\xi+1)-\frac{2d+\mu+\gamma}{\underline{c}}\,\psi_k(\xi)\ \text{ for all } \xi\in\mathbb{R} \text{ and for all } k\in\mathbb{N},$$

Lemma 3.2 applied to the positive functions ψ_k implies that the functions $\xi\mapsto\psi_k(\xi\pm1)/\psi_k(\xi)$ are globally bounded independently of $k\in\mathbb{N}$. Hence, the functions ψ_k'/ψ_k are globally bounded in $\mathbb{R}$ independently of $k\in\mathbb{N}$. Therefore, the limit $\lim_{k\to+\infty}\psi_k(\xi_k)=+\infty$ implies that $0<M_k:=\min_{[\xi_k-\delta,\xi_k]}\psi_k\to+\infty$ as $k\to+\infty$. Now, equation (1.4) and the inequalities $0<\phi_k<1$ and (3.8) yield

$$\max_{[\xi_k-\delta,\xi_k]}\phi_k'\le\frac{2+\mu}{\underline{c}}-\frac{\beta\,\varepsilon\,M_k}{2\,\overline{c}}\to-\infty\ \text{ as } k\to+\infty.$$

This contradicts the global boundedness of the functions ϕ_k. The proof of Lemma 3.3 is thereby complete. □

Proof of Lemma 3.8. Assume by way of contradiction that there is no such ε. Then there exist a sequence of real numbers $\{c_k\}$ in $[\underline{c},\overline{c}]$, a sequence of solutions $\{(\phi_k,\psi_k)\}$ of (1.4) with speed $c=c_k$ and $0<\phi_k<1$, $\psi_k>0$ in $\mathbb{R}$, and a sequence of real numbers $\{\xi_k\}$ such that

$$\psi_k(\xi_k)\to0 \text{ as } k\to+\infty\ \text{ and }\ \psi_k'(\xi_k)\le0 \text{ for all } k\in\mathbb{N}. \tag{3.9}$$

Up to a shift of the origin, one can assume without loss of generality that

$$\xi_k=0 \tag{3.10}$$

for all $k\in\mathbb{N}$. Up to extraction of a subsequence, one can also assume that $c_k\to c_\infty\in[\underline{c},\overline{c}]$ as $k\to+\infty$.

Notice first that Lemma 3.2 and the equations (1.4) satisfied by (ϕ_k,ψ_k) with $c_k\in[\underline{c},\overline{c}]\subset(0,+\infty)$ imply that the sequence $\{\psi_k'/\psi_k\}$ is bounded in $L^\infty(\mathbb{R})$, that is, there is $C>0$ such that $|\psi_k'(\xi)|\le C\,\psi_k(\xi)$ for all $k\in\mathbb{N}$ and $\xi\in\mathbb{R}$. Since $\psi_k(0)\to0^+$ as $k\to+\infty$, it follows that

$$\psi_k\to0 \text{ locally uniformly in } \mathbb{R} \text{ as } k\to+\infty.$$

As a consequence, there also holds that $\psi_k'\to0$ locally uniformly in $\mathbb{R}$ as $k\to+\infty$.

Furthermore, by differentiating the equation (1.4) satisfied by ϕ_k, one gets that the functions ϕ_k' and ϕ_k'' are locally bounded (and the functions ϕ_k are globally bounded). Therefore, the functions ϕ_k converge in $C^1_{loc}(\mathbb{R})$, up to extraction of a subsequence, to a function $0 \le \phi_\infty \le 1$ solving (1.4) with speed c_∞ and with $\psi = 0$, that is,

$$c_\infty \, \phi'_\infty = D[\phi_\infty] + \mu\,(1-\phi_\infty) \ \text{ in } \mathbb{R}. \tag{3.11}$$

Call $\alpha = \inf_{\mathbb{R}} \phi_\infty$ and let $\{\zeta_m\}$ be sequence of real numbers such that $\phi_\infty(\zeta_m) \to \alpha$ as $m \to +\infty$. Up to extraction of a subsequence, the functions $\xi \mapsto \phi_\infty(\xi + \zeta_m)$ converge as $m \to +\infty$ in $C^\infty_{loc}(\mathbb{R})$ to a function Φ_∞ solving $c_\infty \Phi'_\infty = D[\Phi_\infty] + \mu\,(1-\Phi_\infty)$ in $\mathbb{R}$, $\alpha \le \Phi_\infty \le 1$ in $\mathbb{R}$ and $\Phi_\infty(0) = \alpha$. Consequently, $\Phi'_\infty(0) = 0$ and $D[\Phi_\infty](0) \ge 0$, whence $\mu\,(1-\alpha) = \mu\,(1-\Phi_\infty(0)) \le 0$. Thus, $\alpha \ge 1$. Since $\alpha = \inf_{\mathbb{R}} \phi_\infty$ and $\phi_\infty \le 1$ in $\mathbb{R}$, one concludes that

$$\phi_\infty = 1 \ \text{ in } \mathbb{R}.$$

Now set

$$\Psi_k(\xi) = \frac{\psi_k(\xi)}{\psi_k(0)}$$

for $k \in \mathbb{N}$ and $\xi \in \mathbb{R}$. Since the sequence $\{\psi'_k/\psi_k\}$ is bounded in $L^\infty(\mathbb{R})$, the positive functions Ψ_k are locally bounded, in the sense that $\sup_{k\in\mathbb{N},\,|\xi|\le R} \Psi_k(\xi) < +\infty$ for all $R > 0$. Therefore, the functions

$$\Psi'_k(\xi) = \frac{\psi'_k(\xi)}{\psi_k(0)} = \frac{\psi'_k(\xi)}{\psi_k(\xi)} \times \Psi_k(\xi)$$

are locally bounded too. Since each Ψ_k satisfies

$$-c_k\, \Psi''_k(\xi) + d\, D[\Psi'_k](\xi) - (\mu+\gamma)\, \Psi'_k(\xi) + \beta\, \phi'_k(\xi)\, \Psi_k(\xi) + \beta\, \phi_k(\xi)\, \Psi'_k(\xi) = 0$$

in $\mathbb{R}$ and the sequence $\{\phi_k\}$ is bounded in $C^1_{loc}(\mathbb{R})$, one infers that the functions Ψ''_k are locally bounded too. By the Arzela-Ascoli theorem, it follows that, up to extraction of a subsequence, the positive functions Ψ_k converge in $C^1_{loc}(\mathbb{R})$ to a nonnegative solution Ψ_∞ of

$$c_\infty\, \Psi'_\infty = d\, D[\Psi_\infty] + (\beta - \mu - \gamma)\, \Psi_\infty \ \text{ in } \mathbb{R}, \tag{3.12}$$

where one used the fact that $\phi_k(\xi) \to \phi_\infty(\xi) = 1$ as $k \to +\infty$ for all $\xi \in \mathbb{R}$. Furthermore, we claim that $\Psi_\infty > 0$ in $\mathbb{R}$. Otherwise, there is $\xi_0 \in \mathbb{R}$ such that $\Psi_\infty(\xi_0) = 0$, and $\Psi'_\infty(\xi_0) = 0$. It follows from (3.12) applied at ξ_0 that $\Psi_\infty(\xi_0+1) = \Psi_\infty(\xi_0-1) = 0$, and then $\Psi_\infty(\xi_0+m) = 0$ for all $m \in \mathbb{Z}$ by immediate induction. Since $c_\infty \Psi'_\infty \ge (\beta-\mu-\gamma-2)\, \Psi_\infty$ in $\mathbb{R}$, the nonnegative function $\xi \mapsto \Psi_\infty(\xi)\, e^{-(\beta-\mu-\gamma-2)\xi/c_\infty}$ is nondecreasing. Since it vanishes at ξ_0+m for all $m \in \mathbb{Z}$, one concludes that it is identically equal to 0, whence $\Psi_\infty = 0$ in $\mathbb{R}$. This contradicts the fact that $\Psi_\infty(0) = 1$. Therefore,

$$\Psi_\infty(\xi) > 0$$

for all $\xi \in \mathbb{R}$.

The continuous function $z := \Psi'_\infty/\Psi_\infty$ obeys

$$\frac{c_\infty}{d}\, z(\xi) = e^{\int_\xi^{\xi+1} z(s)ds} + e^{\int_\xi^{\xi-1} z(s)ds} - 2 + \frac{\beta-\mu-\gamma}{d} \ \text{ in } \mathbb{R}. \tag{3.13}$$

Therefore, by Proposition 3.5, $z(\xi) = \Psi'_\infty(\xi)/\Psi_\infty(\xi)$ has finite limits $\omega_\pm$ as $\xi \to \pm\infty$, which are roots of the characteristic equation

$$c_\infty\, \omega_\pm = d\,(e^{\omega_\pm} + e^{-\omega_\pm} - 2) + \beta - \mu - \gamma.$$

Since $c_\infty \geq \underline{c} > 0$ and $\beta > \mu + \gamma$, the roots of the previous equation are necessarily positive. In particular, Ψ'_∞ is positive at $\pm\infty$. Furthermore, by differentiating (3.13), one gets that

$$c_\infty \, z'(\xi) = d\,(z(\xi+1) - z(\xi))\,\frac{\Psi_\infty(\xi+1)}{\Psi_\infty(\xi)} + d\,(z(\xi-1) - z(\xi))\,\frac{\Psi_\infty(\xi-1)}{\Psi_\infty(\xi)} \ \text{ in } \mathbb{R}. \tag{3.14}$$

Therefore, if z has a minimum $\underline{\xi}$ in $\mathbb{R}$, then $z'(\underline{\xi}) = 0$ and $z(\underline{\xi}+1) = z(\underline{\xi}-1) = z(\underline{\xi})$, whence $z(\underline{\xi}+m) = z(\underline{\xi})$ for all $m \in \mathbb{Z}$ by immediate induction. As a consequence,

$$\inf_{\mathbb{R}} z \geq \min\{z(-\infty), z(+\infty)\} > 0.$$

Finally, $\Psi'_\infty > 0$ in $\mathbb{R}$, hence $0 < \Psi'_\infty(0) = \lim_{k\to+\infty} \Psi'_k(0) = \lim_{k\to+\infty} \psi'_k(0)/\psi_k(0)$ and $\psi'_k(0) > 0$ for all k large enough. This contradicts the fact that $\psi'_k(0) \leq 0$ for all $k \in \mathbb{N}$ (remember (3.9) and (3.10)).

As a conclusion, there is $\varepsilon > 0$ such that $\psi'(\xi) > 0$ for any $\xi \in \mathbb{R}$ with $\psi(\xi) \leq \varepsilon$ for any solution (ϕ, ψ) of (1.4) with $c \in [\underline{c}, \overline{c}]$, $0 < \phi < 1$ and $\psi > 0$ in $\mathbb{R}$. The proof of Lemma 3.8 in thereby complete. □

4. The case $c = c^*$

This section is devoted to the proof of the existence of a traveling wave (ϕ, ψ) of (1.4) satisfying (1.6), (1.7) and (1.8) with speed $c = c^*$. To do so, we consider a sequence $\{c_k\}$ of real numbers such that $c_k \in (c^*, c^*+1]$ for each $k \in \mathbb{N}$, and

$$c_k \to c^* \ \text{ as } k \to +\infty.$$

For each $k \in \mathbb{N}$, Section 3 provides the existence of a traveling wave (ϕ_k, ψ_k) of (1.4) (with speed c_k) satisfying (1.6), (1.7) and (1.8). The natural strategy is to pass to the limit as $k \to +\infty$, in order to get the existence of a traveling wave with the limiting speed c^*. To achieve this goal, we need some a priori bounds for the functions ψ_k in order to get a non-trivial solution at the limit. We also point out that the inequalities (3.1) satisfied by the approximated waves (ϕ_k, ψ_k) do not carry over at the limit $c_k \to c^*$ (since the coefficients in the definitions of the lower solutions depend on c_k and degenerate at the limit $c_k \to c^*$). Therefore, we will have to suitably shift and renormalize the approximated waves (ϕ_k, ψ_k) before passing to the limit as $k \to +\infty$.

The first a priori bound asserts that the functions ψ_k do not converge to 0 uniformly as $k \to +\infty$.

Lemma 4.1. *There holds* $\liminf_{k\to+\infty} \|\psi_k\|_{L^\infty(\mathbb{R})} > 0$.

Proof. Assume that the conclusion does not hold. Then, up to extraction of a subsequence, one can assume without loss of generality that $\|\psi_k\|_{L^\infty(\mathbb{R})} \to 0$ as $k \to +\infty$. Since $c_k \in [c^*, c^*+1] \subset (0, +\infty)$ for each $k \in \mathbb{N}$, Lemma 3.8 implies that $\psi'_k > 0$ in $\mathbb{R}$ for all k large enough. Since each ψ_k is bounded, it follows that the limit $\psi_k(+\infty)$ exists in $\mathbb{R}$, for all k large enough. Since each (ϕ_k, ψ_k) satisfies the assumptions of Lemma 3.9, one then infers in particular that, for all k large enough,

$$\psi_k(+\infty) = e^* = \frac{\mu}{\beta}\,(\sigma - 1) > 0.$$

This contradicts the fact that $\lim_{k\to+\infty}\|\psi_k\|_{L^\infty(\mathbb{R})}=0$. Thus, the conclusion of Lemma 4.1 holds. □

The second key-point is the boundedness of the sequence $\{\psi_k\}$ in $L^\infty(\mathbb{R})$.

Lemma 4.2. *There holds* $\limsup_{k\to+\infty}\|\psi_k\|_{L^\infty(\mathbb{R})}<+\infty$.

Proof. Assume that the conclusion does not hold. Then, up to extraction of a subsequence, one has $\|\psi_k\|_{L^\infty(\mathbb{R})}\to+\infty$ as $k\to+\infty$. For each $k\in\mathbb{N}$, since the function ψ_k is bounded and positive in $\mathbb{R}$, there is then $\xi_k\in\mathbb{R}$ such that

$$\psi_k(\xi_k)\ge\Big(1-\frac{1}{k+1}\Big)\,\|\psi_k\|_{L^\infty(\mathbb{R})}. \tag{4.1}$$

In particular, $\psi_k(\xi_k)\to+\infty$ as $k\to+\infty$. Furthermore, one has

$$\psi_k'(\xi)\ge\frac{d}{c^*}\,\psi_k(\xi+1)-\frac{2d+\mu+\gamma}{c^*}\,\psi_k(\xi)\ \text{ in }\mathbb{R}$$

for all $k\in\mathbb{N}$. Since each ψ_k is positive, it follows from Lemma 3.2 that the functions $\xi\mapsto\psi_k(\xi\pm1)/\psi_k(\xi)$ are globally bounded in $\mathbb{R}$ independently of $k\in\mathbb{N}$, and so are the functions $\xi\mapsto\psi_k'(\xi)/\psi_k(\xi)$, from the equation (1.4) satisfied with speed $c_k\in(c^*,c^*+1]$ (remember also that $0<\phi_k<1$ in $\mathbb{R}$). As a consequence,

$$\psi_k(\xi+\xi_k)\underset{k\to+\infty}{\longrightarrow}+\infty\ \text{ locally uniformly in }\xi\in\mathbb{R}.$$

Lemma 3.3 then implies that

$$\Phi_k(\xi):=\phi_k(\xi+\xi_k)\to0$$

as $k\to+\infty$ locally uniformly in $\xi\in\mathbb{R}$.

From the boundedness of the sequence $\{\psi_k'/\psi_k\}$ in $L^\infty(\mathbb{R})$, one also infers that the functions

$$\xi\mapsto\Psi_k(\xi)=\frac{\psi_k(\xi+\xi_k)}{\psi_k(\xi_k)}$$

are locally bounded independently of k (in the sense that $\sup_{k\in\mathbb{N}}\|\Psi_k\|_{L^\infty(K)}<+\infty$ for any compact set $K\subset\mathbb{R}$). Each function Ψ_k obeys

$$c_k\,\Psi_k'=d\,D[\Psi_k]-(\mu+\gamma)\,\Psi_k+\beta\,\Phi_k\,\Psi_k\ \text{ in }\mathbb{R},$$

whence the functions Ψ_k' are locally bounded too. From the Arzela-Ascoli theorem, the positive functions Ψ_k converge locally uniformly in $\mathbb{R}$, up to extraction of a subsequence, to a continuous nonnegative function Ψ_∞. Furthermore, from the above equation and the fact that $\Phi_k\to0$ as $k\to+\infty$ locally uniformly in $\mathbb{R}$ (together with $c_k\to c^*>0$), the functions Ψ_k' converge locally uniformly in $\mathbb{R}$ too. Therefore, the functions Ψ_k converge in $C^1_{loc}(\mathbb{R})$ to Ψ_∞ and the function Ψ_∞ satisfies

$$c^*\,\Psi_\infty'=d\,D[\Psi_\infty]-(\mu+\gamma)\,\Psi_\infty\ \text{ in }\mathbb{R}. \tag{4.2}$$

Notice that this function Ψ_∞ is thus automatically of class $C^\infty(\mathbb{R})$. Furthermore, Ψ_∞ is nonnegative and $\Psi_\infty(0)=\lim_{k\to+\infty}\Psi_k(0)=1$. As in the proof of Lemma 3.8 for the solution of (3.12), one then infers that Ψ_∞ is positive in $\mathbb{R}$.

Finally, for every $\xi\in\mathbb{R}$, there holds $\psi_k(\xi+\xi_k)\le\|\psi_k\|_{L^\infty(\mathbb{R})}\le(1+1/k)\,\psi_k(\xi_k)$ from (4.1). In other words, $\Psi_k(\xi)\le1+1/k$ for every $\xi\in\mathbb{R}$ and $k\in\mathbb{N}$ with $k\ge1$, whence $\Psi_\infty(\xi)\le1$

for every $\xi \in \mathbb{R}$. Therefore, since $\Psi_\infty(0) = 1$, 0 is a global maximum of the function Ψ_∞, and $\Psi'_\infty(0) = 0$, $D[\Psi_\infty](0) \le 0$. The equation (4.2) evaluated at 0 leads to a contradiction, since μ and γ are positive. The proof of Lemma 4.2 is thereby complete. □

End of the proof of Theorem 1.1 *in case* $c = c^*$. First of all, Lemma 3.8 applied with $\underline{c} = c^* > 0$ and $\overline{c} = c^* + 1$ yields the existence of $\varepsilon > 0$ such that $\Psi'(\xi) > 0$ for every $\xi \in \mathbb{R}$ with $\Psi(\xi) \le \varepsilon$, and for every solution (Φ, Ψ) of (1.4) and (1.6) with speed $c \in [c^*, c^* + 1]$. Without loss of generality, one can assume that

$$0 < \varepsilon \le e^* = \frac{\mu}{\beta}(\sigma - 1). \tag{4.3}$$

Coming back to our solutions (ϕ_k, ψ_k) of (1.4) (with speed c_k) satisfying (1.6), (1.7) and (1.8), it follows from Lemma 4.1 and the positivity of each ψ_k that one can also assume without loss of generality that

$$0 < \varepsilon < \inf_{k \in \mathbb{N}} \|\psi_k\|_{L^\infty(\mathbb{R})}.$$

Therefore, for each $k \in \mathbb{N}$, since $\psi_k(-\infty) = 0$ and $\psi_k > 0$, there is $\xi_k \in \mathbb{R}$ such that

$$\psi_k(\xi_k) = \varepsilon.$$

Shift the origin at ξ_k and denote

$$\tilde{\phi}_k(\xi) = \phi_k(\xi + \xi_k) \;\text{ and }\; \tilde{\psi}_k(\xi) = \psi_k(\xi + \xi_k).$$

From Lemma 4.2, the sequence $\{\tilde{\psi}_k\}$ is bounded in $L^\infty(\mathbb{R})$. Remember also that $0 < \tilde{\phi}_k < 1$ in $\mathbb{R}$ and $c_k \to c^* > 0$ as $k \to +\infty$. Therefore, up to extraction of a subsequence, the functions $\tilde{\phi}_k$ and $\tilde{\psi}_k$ converge in $C^\infty_{loc}(\mathbb{R})$ to some bounded $C^\infty(\mathbb{R})$ functions ϕ and ψ solving (1.4) with speed c^*. Furthermore, $0 \le \phi \le 1$ and $\psi \ge 0$ in $\mathbb{R}$, while

$$\psi(0) = \varepsilon > 0.$$

In order to complete the proof of Theorem 1.1 in case $c = c^*$, one shall show that the pair (ϕ, ψ) is non-trivial and satisfies the desired limiting conditions at $\pm\infty$, that is, the conditions (1.6), (1.7) and (1.8) hold.

Let us first show that

$$\psi > 0 \;\text{ in } \mathbb{R}.$$

Indeed, if there is $\xi^* \in \mathbb{R}$ such that $\psi(\xi^*) = 0$, then $\psi'(\xi^*) = 0$ and equation (1.4) at ξ^* yields $\psi(\xi^* \pm 1) = 0$, whence $\psi(\xi^* + m) = 0$ for all $m \in \mathbb{Z}$ by immediate induction. But $c^*\psi' \ge -(2d + \mu + \gamma)\,\psi$ in $\mathbb{R}$, whence the function $\xi \mapsto \psi(\xi)\,e^{(2d+\mu+\gamma)\xi/c^*}$ is nondecreasing. Since $\psi \ge 0$ in $\mathbb{R}$ and $\psi(\xi^* + m) = 0$ for all $m \in \mathbb{Z}$, one infers that $\psi = 0$ in $\mathbb{R}$, which is impossible since $\psi(0) = \varepsilon > 0$. Thus, $\psi > 0$ in $\mathbb{R}$. Once the positivity of ψ is known, it follows as in the proof of Lemma 3.1 that

$$0 < \phi < 1 \;\text{ in } \mathbb{R}.$$

In other words, the pair (ϕ, ψ) fulfills (1.6).

Let us then show that the pair (ϕ, ψ) satisfies the limiting conditions (1.7) at $-\infty$. Since the pair (ϕ, ψ) solves (1.4) and (1.6) with speed c^*, the choice of $\varepsilon > 0$ above and the property $\psi(0) = \varepsilon$ imply that $\psi' > 0$ in $(-\infty, 0]$. In particular, the limit $L = \lim_{\xi \to -\infty} \psi(\xi)$ exists, and $L \in [0, \varepsilon)$. If $L > 0$, then the same arguments as in the proof of Lemma 3.9 imply that $\phi(-\infty)$ exists and $\phi(-\infty) = (\mu + \gamma)/\beta = 1/\sigma \in (0, 1)$. The same arguments also yield $L = \psi(-\infty) = \mu(1 - \phi(-\infty))/(\beta\phi(-\infty)) = (\mu/\beta)(\sigma - 1) = e^*$. Hence, $e^* = L < \varepsilon$, contradicting (4.3). Therefore,

$$L = \psi(-\infty) = 0.$$

Furthermore, for any sequence $\{\tilde{\xi}_k\}$ converging to $-\infty$, the functions $\xi \mapsto \phi(\xi + \tilde{\xi}_k)$ and $\xi \mapsto \psi(\xi + \tilde{\xi}_k)$ converge in $C^\infty_{loc}(\mathbb{R})$, up to extraction of a subsequence, to a pair $(\phi_\infty, 0)$, for some function $0 \le \phi_\infty \le 1$ solving (3.11) with speed $c_\infty = c^*$. It follows as in the proof of Lemma 3.8 that $\phi_\infty = 1$ in $\mathbb{R}$. Since the limit does not depend on the choice the sequence $\{\tilde{\xi}_k\}$, one gets that the limit $\lim_{\xi \to -\infty} \phi(\xi)$ exists, and

$$\phi(-\infty) = 1.$$

In other words, the pair (ϕ, ψ) satisfies (1.7).

Let us finally show that the non-triviality conditions (1.8) hold at $+\infty$. Firstly, as in the proof of Lemma 3.7, there holds $\inf_{\mathbb{R}} \phi > 0$. Secondly, Lemma 3.8 and (1.6) imply at once that $\liminf_{\xi \to +\infty} \psi(\xi) > 0$. Thirdly, one concludes that $\limsup_{\xi \to +\infty} \phi(\xi) < 1$ as in the proof of (3.6) and that (3.7) holds as in the case $c > c^*$. The solution (ϕ, ψ) thus fulfills all desired properties and the proof of Theorem 1.1 in case $c = c^*$ is thereby complete. □

5. Non-existence of traveling waves for $c < c^*$

In this section, (ϕ, ψ) denotes a classical solution of (1.4) satisfying (1.6) and (1.7), with a speed $c \in \mathbb{R}$. By classical, we mean that ϕ and ψ are of class $C^1(\mathbb{R})$ (and then of class $C^\infty(\mathbb{R})$) if $c \ne 0$, and that ϕ and ψ are continuous if $c = 0$. We shall prove that, necessarily, $c \ge c^*$. To do so, we consider separately the cases $c > 0$, $c < 0$ and $c = 0$.

First case: $c > 0$. Since the positive function ψ satisfies

$$\psi'(\xi) \ge \frac{d}{c}\,\psi(\xi + 1) - \frac{2d + \mu + \gamma}{c}\,\psi(\xi)$$

for all $\xi \in \mathbb{R}$, Lemma 3.2 implies that the functions $\xi \mapsto \psi(\xi \pm 1)/\psi(\xi)$ are bounded, and then so is the function $\xi \mapsto \psi'(\xi)/\psi(\xi)$. Consider now any sequence $\{\xi_k\}$ converging to $-\infty$. The positive functions

$$\xi \mapsto \psi_k(\xi) := \frac{\psi(\xi + \xi_k)}{\psi(\xi_k)}$$

are locally bounded and they satisfy

$$c\,\psi_k' = d\,D[\psi_k] - (\mu + \gamma)\,\psi_k + \beta\,\phi(\cdot + \xi_k)\,\psi_k \quad \text{in } \mathbb{R}.$$

Therefore, the functions ψ_k' are locally bounded too (remember that $\phi(-\infty) = 1$). From the Arzela-Ascoli theorem, up to extraction of a subsequence, the functions ψ_k converge locally uniformly (and then in $C^1_{loc}(\mathbb{R})$ from the above equation) to a function ψ_∞ solving

$$c\,\psi_\infty' = d\,D[\psi_\infty] + (\beta - \mu - \gamma)\,\psi_\infty \quad \text{in } \mathbb{R}. \tag{5.1}$$

Furthermore, $\psi_\infty \geq 0$ in $\mathbb{R}$ and $\psi_\infty(0) = 1$. As in the proof of Lemma 3.8 for the function Ψ_∞ solving (3.12), it follows that $\psi_\infty > 0$ in $\mathbb{R}$. Now, the function $z = \psi'_\infty/\psi_\infty$ solves

$$\frac{c}{d}\,z(x) = e^{\int_x^{x+1} z(s)ds} + e^{\int_x^{x-1} z(s)ds} - 2 + \frac{\beta-\mu-\gamma}{d} \ \text{ in } \mathbb{R}. \tag{5.2}$$

Proposition 3.5 implies that the limits $z(\pm\infty)$ exist in $\mathbb{R}$ and are roots ω of the equation

$$c\,\omega = d\,(e^{\omega} + e^{-\omega} - 2) + \beta - \mu - \gamma.$$

Since $c > 0$ and $\beta > \mu + \gamma$, the roots must be positive and $c \geq c^*$ by definition of c^* in (1.5). The proof of the necessity condition is thereby complete in the case $c > 0$.

Second case: $c < 0$. Denote $\Phi(\xi) = \phi(-\xi)$ and $\Psi(\xi) = \psi(-\xi)$. The functions Φ and Ψ satisfy (1.4) and (1.6) with speed $|c| > 0$, together with the limiting conditions $(\Phi(+\infty), \Psi(+\infty)) = (1, 0)$. Furthermore, since the positive function Ψ satisfies

$$\Psi'(\xi) \geq \frac{d}{|c|}\,\Psi(\xi+1) - \frac{2d+\mu+\gamma}{|c|}\,\Psi(\xi)$$

and (1.4) with speed $|c|$, it follows as in the above case $c > 0$ that the function Ψ'/Ψ is bounded. Since $\Psi > 0$ in $\mathbb{R}$ and $\Psi(+\infty) = 0$, one can consider a sequence $\{\xi_k\}$ converging to $+\infty$ such that

$$\Psi'(\xi_k) \leq 0 \ \text{ for all } k \in \mathbb{N}.$$

As above, up to extraction of a subsequence, the functions $\xi \mapsto \Psi_k(\xi) := \Psi(\xi + \xi_k)/\Psi(\xi_k)$ converge in $C^1_{loc}(\mathbb{R})$ to a positive solution Ψ_∞ of (5.1) with $|c|$ instead of c, and such that $\Psi_\infty(0) = 1$. Furthermore, here, $\Psi'_\infty(0) \leq 0$. The function $Z := \Psi'_\infty/\Psi_\infty$ satisfies (5.2) with $|c|$ instead of c and it follows from Proposition 3.5 that the limits $Z(\pm\infty)$ exist in $\mathbb{R}$ and are roots Ω of the equation

$$|c|\,\Omega = d\,(e^{\Omega} + e^{-\Omega} - 2) + \beta - \mu - \gamma.$$

Since $\beta > \mu+\gamma$, the roots are positive (and $|c| \geq c^*$). In particular, Z is positive at $\pm\infty$. But $Z(0) = \Psi'_\infty(0)/\Psi_\infty(0) = \Psi'_\infty(0) \leq 0$. Hence, the continuous function Z has a minimum Ξ in $\mathbb{R}$, that is $Z(\Xi) \leq Z(\xi)$ for all $\xi \in \mathbb{R}$. By differentiating the equation satisfied by Z, one gets as in (3.14) that

$$|c|\,Z'(\xi) = d\,(Z(\xi+1) - Z(\xi))\,\frac{\Psi_\infty(\xi+1)}{\Psi_\infty(\xi)} + d\,(Z(\xi-1) - Z(\xi))\,\frac{\Psi_\infty(\xi-1)}{\Psi_\infty(\xi)} \ \text{ in } \mathbb{R}.$$

Hence, $Z(\Xi \pm 1) = Z(\Xi)$, and $Z(\Xi + m) = Z(\Xi) = \min_{\mathbb{R}} Z$ for all $m \in \mathbb{Z}$ by immediate induction. Therefore, $Z(\pm\infty) = \min_{\mathbb{R}} Z \leq Z(0) \leq 0$, a contradiction with the positivity of $Z(\pm\infty)$. As a consequence, the case $c < 0$ is ruled out.

Third case: $c = 0$. Here, the function ψ satisfies $d\,D[\psi] + (\beta\,\phi - \mu - \gamma)\,\psi = 0$ in $\mathbb{R}$. Since $d > 0$, $\beta > \mu + \gamma$, $\phi(-\infty) = 1$ and $\psi > 0$ in $\mathbb{R}$, it follows that there exists $\xi_0 \in \mathbb{R}$ such that $D[\psi](\xi) < 0$ for all $\xi \leq \xi_0$. Denote

$$\theta(\xi) = \psi(\xi) - \psi(\xi+1).$$

The condition $D[\psi] < 0$ in $(-\infty, \xi_0]$ means that $\theta(\xi-1) < \theta(\xi)$ for all $\xi \leq \xi_0$. Furthermore, since $\psi > 0$ in $\mathbb{R}$ and $\psi(-\infty) = 0$, there is $\xi_1 \leq \xi_0$ such that $\theta(\xi_1) < 0$. Since $\theta(\xi_1 - m) < \theta(\xi_1)$

for all $m \in \mathbb{N}$ with $m \geq 1$, one infers that

$$\psi(\xi_1 - m) - \psi(\xi_1) = \sum_{j=1}^{m} \theta(\xi_1 - j) < m\,\theta(\xi_1)$$

for all $m \in \mathbb{N}$ with $m \geq 1$. Thus, $\psi(\xi_1 - m) < \psi(\xi_1) + m\,\theta(\xi_1) \to -\infty$ as $m \to +\infty$ since $\theta(\xi_1) < 0$. This contradicts the positivity of ψ. As a consequence, the case $c = 0$ is ruled out too and the proof of Theorem 1.1 is thereby complete. □

Remark 5.1. We give here another proof of the positivity of c when ψ is bounded (cf. [26]). Since $\phi(-\infty) = 1$ and $\beta > \mu + \gamma$, there is a sufficiently large K such that

$$\beta\,\phi(\xi) - \mu - \gamma > \frac{\beta - \mu - \gamma}{2} > 0 \quad \text{for } \xi \in (-\infty, -K).$$

Integrating the second equation of (1.4) from $-\infty$ to $\xi < -K$, using $\psi(-\infty) = 0$ and the positivity and boundedness of ψ, we obtain

$$\begin{aligned} c\,\psi(\xi) &= d\left\{\int_{\xi}^{\xi+1} \psi(s)ds - \int_{\xi-1}^{\xi} \psi(s)ds\right\} + \int_{-\infty}^{\xi} [\beta\,\phi(s) - \mu - \gamma]\,\psi(s)\,ds \\ &\geq -d\,\{\sup_{s\in\mathbb{R}} \psi(s)\} + \frac{\beta - \mu - \gamma}{2} \int_{-\infty}^{\xi} \psi(s)\,ds \end{aligned}$$

for all $\xi < -K$. It follows that the integral

$$R(\xi) := \int_{-\infty}^{\xi} \psi(s)\,ds$$

is well-defined for all $\xi < -K$ (and then for all $\xi \in \mathbb{R}$ by continuity of ψ). Integrating the second equation of (1.4) twice, we obtain

$$c\,R(x) = d\left\{\int_{x}^{x+1} R(\xi)\,d\xi - \int_{x-1}^{x} R(\xi)\,d\xi\right\} + \int_{-\infty}^{x}\int_{-\infty}^{\xi} [\beta\,\phi(s) - \mu - \gamma]\,\psi(s)\,ds\,d\xi$$

for all $x \in \mathbb{R}$. Since $R(\xi)$ is strictly increasing, we conclude that $c > 0$.

6. Proof of Theorem 1.2

First of all, for any speed $c \geq c^*$, the bounded classical solution (ϕ, ψ) of the system (1.4) given in Theorem 1.1 and satisfying (1.6), (1.7) and (1.8), gives rise to a traveling wave $(s_n(t), i_n(t))_{n\in\mathbb{Z},\,t\in\mathbb{R}}$ of (1.1) in the sense of (1.9) by setting $s_n(t) = \phi(n + ct)$ and $i_n(t) = \psi(n + ct)$ for all $n \in \mathbb{Z}$ and $t \in \mathbb{R}$. Furthermore, properties (1.10), (1.11), (1.12) and (1.13) immediately follow.

Consider now any traveling wave $(s_n(t), i_n(t))_{n\in\mathbb{Z},\,t\in\mathbb{R}}$ of (1.1) in the sense of (1.9) with a speed $c \in \mathbb{R} \setminus \{0\}$, and satisfying (1.10) and (1.11). Set

$$S_n(\xi) = s_n\Big(\frac{\xi}{c}\Big) \ \text{ and } \ I_n(\xi) = i_n\Big(\frac{\xi}{c}\Big)$$

for $n \in \mathbb{Z}$ and $\xi \in \mathbb{R}$. It then follows from (1.9) that, whatever the sign of c may be, each pair of functions (S_n, I_n) is a classical solution of

$$\begin{cases} c\,S_n'(\xi) &= \big(S_n(\xi+1)+S_n(\xi-1)-2\,S_n(\xi)\big)+\mu\,(1-S_n(\xi))-\beta\,S_n(\xi)\,I_n(\xi),\\ c\,I_n'(\xi) &= d\,\big(I_n(\xi+1)+I_n(\xi-1)-2\,I_n(\xi)\big)-\mu\,I_n(\xi)+\beta\,S_n(\xi)\,I_n(\xi)-\gamma\,I_n(\xi)\end{cases}$$

for all $\xi \in \mathbb{R}$. In other words, each pair (S_n, I_n) is a solution of (1.4). Furthermore, by (1.9), (1.10) and (1.11), each function S_n satisfies $0 < S_n < 1$ in $\mathbb{R}$, each I_n is positive in $\mathbb{R}$ and $(S_n(-\infty), I_n(-\infty)) = (1, 0)$, whatever the sign of c may be. The same arguments as in Section 5 for the functions (ϕ, ψ) therefore apply, both for $c > 0$ and $c < 0$. Hence, the case $c < 0$ is ruled out, and c is positive and satisfies $c \geq c^*$.

Let us finally assume by contradiction that there is a stationary wave $(s_n, i_n)_{n\in\mathbb{Z}}$ of (1.1) (the left-hand sides are thus equal to 0) satisfying (1.10) and (1.11). Since $\beta > \mu + \gamma$ and since $i_n > 0$ for all $n \in \mathbb{Z}$ and $s_n \to 1$ as $n \to -\infty$, it follows that there is $n_0 \in \mathbb{Z}$ such that $i_{n+1} + i_{n-1} - 2\,i_n < 0$ for all $n \leq n_0$. In other words, by setting

$$j_n = i_n - i_{n+1},$$

one has $j_{n-1} < j_n$ for all $n \leq n_0$. Since $i_{n_0} > 0$ and $i_n \to 0$ as $n \to -\infty$, there is $N \leq n_0$ such that $j_N < 0$. As $j_{N-p} < j_N$ for all $p \in \mathbb{N}\backslash\{0\}$, one infers that

$$i_{N-m} - i_N = j_{N-m} + \cdots + j_{N-1} < m\,j_N$$

for all $m \in \mathbb{N}\backslash\{0\}$. The left-hand side of the above inequality converges to $-i_N$ as $m \to +\infty$, while the right-hand side converges to $-\infty$ (since $j_N < 0$). One has then reached a contradiction and the existence of stationary waves of (1.1) satisfying (1.10) and (1.11) is therefore ruled out. The proof of Theorem 1.2 is thereby complete. □

7. Numerical experiments

This section is devoted to the numerical experiments in order to understand the dynamics of the system (1.1). We use MATLAB to run some simple numerical experiments.

First, we observe that the endemic state (s^*, e^*) can be either a stable spiral point or stable node of the kinetic system (i.e., (1.1) without the discrete diffusion terms), depending on the parameters (μ, γ, β). Indeed, let

$$\beta_\pm := 2\frac{(\gamma+\mu)^2}{\mu}\left(1 \pm \sqrt{\gamma/[\gamma+\mu]}\right).$$

Then (s^*, e^*) is stable spiral point if and only if $\beta \in (\beta_-, \beta_+) \cap (\gamma + \mu, \infty)$; while it is a stable node if and only if $\beta \in \{(0, \beta_-] \cup [\beta_+, \infty)\} \cap (\gamma + \mu, \infty)$. For convenience, we try $\gamma = 1$ and $\mu = 3$ so that $\beta_- = 16/3$ and $\beta_+ = 16$. In our numerical experiments, we set $d = 1$ and choose $\beta = 5, 6, 10, 15, 20$, so that both cases are considered.

We consider the truncated problem of (1.1) for the set $n \in \{0, \cdots, N+1\}$ so that the number of equations for s_n (and i_n) is $N+2$. The left-hand boundary condition is set to be the Dirichlet boundary condition, i.e., $(s_0(t), i_0(t)) = (1, 0)$; and the right-hand boundary condition is chosen to be the zero Neumann boundary condition, i.e., $(s_{N+1}(t), i_{N+1}(t)) = (s_N(t), i_N(t))$.

β=5, μ=3, γ=1 (minimal speed=2.073)

β=5, μ=3, γ=1, $s_n(t)$=0.95		
n	arrival time (t)	speed
1005	99.617	2.061005771
1007	98.6466	2.054231717
1009	97.673	2.057613169
1011	96.701	2.053915276
1013	95.72725	2.065582236
1015	94.759	2.05338809
1017	93.785	2.056026728
1019	92.81225	2.06291903
1021	91.84275	2.053915276
1023	90.869	2.059732235
1025	89.898	

β=6, μ=3, γ=1 (minimal speed=3.017)

β=6, μ=3, γ=1, $s_n(t)$=0.95		
n	arrival time (t)	speed
907	99.845	3.006012024
910	98.847	3.006012024
913	97.849	3.006012024
916	96.851	3.006012024
919	95.853	3.006012024
922	94.855	3.003003003
925	93.856	3.006012024
928	92.858	3.006012024
931	91.86	3.003003003
934	90.861	3.006012024
937	89.863	

β=10, μ=3, γ=1 (minimal speed= 5.672)

β=10, μ=3, γ=1, $s_n(t)$=0.95		
n	arrival time (t)	speed
640	99.91167	5.66390394
646	98.85233	5.663957407
652	97.793	5.663957407
658	96.73367	5.66390394
664	95.67433	5.663957407
670	94.615	5.663957407
676	93.55567	5.66390394
682	92.49633	5.663957407
688	91.437	5.663957407
694	90.37767	5.6621401
700	89.318	

β=15, μ=3, γ=1 (minimal speed= 8.231)

β=15, μ=3, γ=1, $s_n(t)$=0.95		
n	arrival time (t)	speed
383	99.941	8.22622108
391	98.9685	8.22622108
399	97.996	8.221993834
407	97.023	8.221993834
415	96.05	8.22622108
423	95.0775	8.22622108
431	94.105	8.221993834
439	93.132	8.221993834
447	92.159	8.221993834
455	91.186	8.22622108
463	90.2135	

β=20, μ=3, γ=1(minimal speed=10.460)

β=20, μ=3, γ=1, $s_n(t)$=0.95		
n	arrival time (t)	speed
160	99.97662	10.45226971
170	99.01989	10.45292525
180	98.06322	10.45303452
190	97.10656	10.45281599
200	96.14988	10.45183273
210	95.19311	10.45292525
220	94.23644	10.45183273
230	93.27967	10.45292525
240	92.323	10.45172349
250	91.36622	10.45303452
260	90.40956	

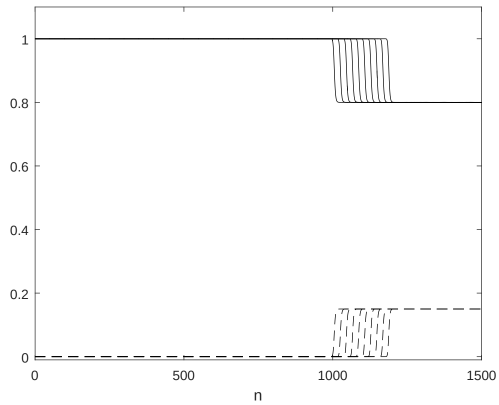
TABLE 1. The tables for computed speeds for different β.

We look for left-moving waves. Therefore, for the initial condition, we choose $s_n(0) = 1$ for all n, $i_n(0) = 0$ for $n = 1, 2, \cdots, 4N/5$, and $i_n(0) \in (0, 1)$ randomly for $n = 4N/5+1, \cdots, N$. Here we run the program with $N = 1500$ on the time interval $[0, 100]$ with time step $\Delta t = 0.001$.

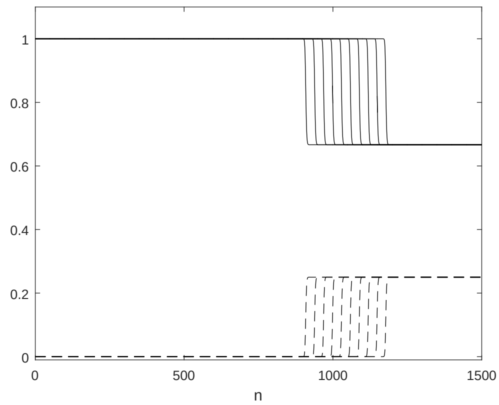
For the *wave speed selection problem*, we observe from our numerical experiments that initially compact perturbations converge to traveling waves with approximately the minimal speed defined in (1.5). See Figure 1, where we plot the wave profiles at $t = 10k$, $k = 1, 2, \cdots, 10$. The wave propagates from the right to the left as time increases.

To compute the approximated wave speed, we choose the front position to be $s_n(t) = 0.95$ for $t \in (99, 100)$ and compute the approximated speed c_i by $(n_{i+1}-n_i)/(t_i-t_{i+1})$ for $i = 1, \cdots, 10$, where n_i is the position and t_i is the arrival time. Here we first choose n_1 to be the position such that $s_{n_1}(t_1) = 0.95$ for $t_1 \approx 100$. Then set $n_{i+1} = n_i + m$ for $i \geq 1$ and find the corresponding arrival time t_{i+1} (here m is chosen to be the nearest integer to c^*, but other choices of this integer m would give similar results for the computed ratio $(n_{i+1} - n_i)/(t_i - t_{i+1})$). The computed speeds are presented in Table 1.

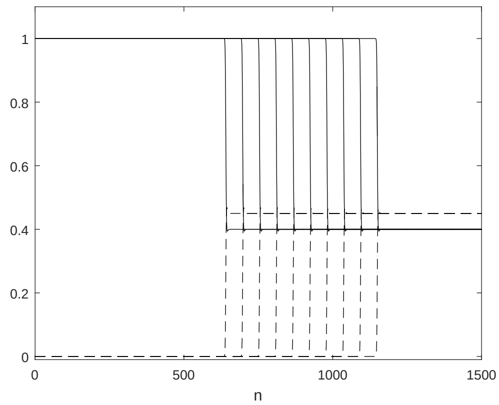
Moreover, from our numerical experiments, it indicates that the leftover state is always the endemic state. Finally, for the monotonicity of wave profiles, we have observed numerically the traveling waves are non-monotone for $\beta \in \{10, 15, 20\}$. It seems that the ones for $\beta \in \{5, 6\}$ are monotone. This indeed is a very interesting open question. On the other hand, unfortunately



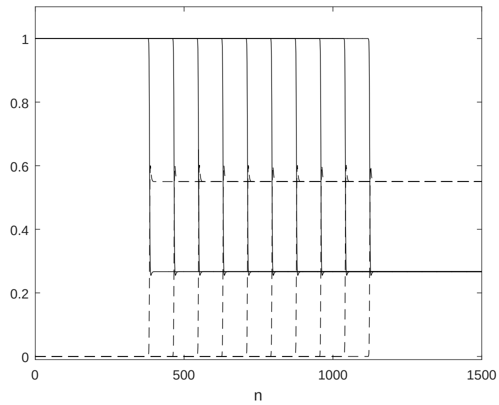
(a) $\beta = 5$



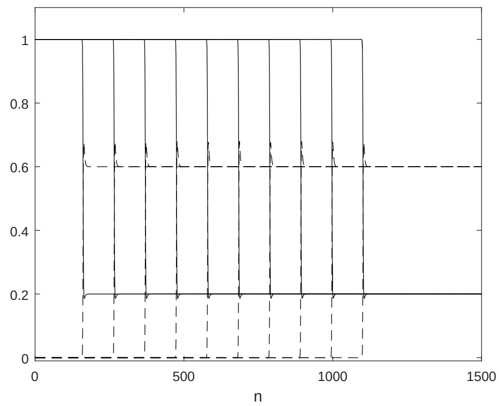
(b) $\beta = 6$



(c) $\beta = 10$



(d) $\beta = 15$



(e) $\beta = 20$

FIGURE 1. Plot of wave profiles at 10 different times for each β: solid curves for $\{s_n\}$ and dashed curves for $\{i_n\}$ (waves propagate leftwards).

we were unable to observe whether the right-hand wave tails are non-monotone (even with very high precision), in particular, when $\beta = 6, 10, 15 \in (\beta_-, \beta_+)$.

References

[1] M. Alfaro, J. Coville, *Rapid traveling waves in the nonlocal Fisher equation connect two unstable states*, Appl. Math. Letters 25 (2012), 2095–2099.

[2] N. Apreutesei, N. Bessonov, V. Volpert, V. Vougalter, *Spatial structures and generalized travelling waves for an integro-differential equation*, Disc. Cont. Dyn. Syst. B 13 (2010), 537–557.

[3] H. Berestycki, F. Hamel, A. Kiselev, L. Ryzhik, *Quenching and propagation in KPP reaction-diffusion equations with a heat loss*, Arch. Ration. Mech. Anal. 178 (2005), 57–80.

[4] H. Berestycki, G. Nadin, B. Perthame, L. Ryzhik, *The non-local Fisher-KPP equation: traveling waves and steady states*, Nonlinearity 22 (2009), 2813–2844.

[5] A. Bonnet, *Non-uniqueness for flame propagation when the Lewis number is less than 1*, Europ. J. Appl. Math. 6 (1995), 287–306.

[6] J.W. Cahn, J. Mallet-Paret, E.S. van Vleck, *Travelling wave solutions for systems of ODE's on a two-dimensional spatial lattice*, SIAM J. Appl. Math. 59 (1999), 455–493.

[7] X. Chen, S.-C. Fu, J.-S. Guo, *Uniqueness and asymptotics of traveling waves of monostable dynamics on lattices*, SIAM J. Math. Anal. 38 (2006), 233–258.

[8] X. Chen, J.-S. Guo, *Existence and asymptotic stability of traveling waves of discrete quasilinear monostable equations*, J. Diff. Equations 184 (2002), 549–569.

[9] X. Chen, J.-S. Guo, *Uniqueness and existence of traveling waves for discrete quasilinear monostable dynamics*, Math. Ann. 326 (2003), 123–146.

[10] X. Chen, J.-S. Guo, C.-C. Wu, *Traveling waves in discrete periodic media for bistable dynamics*, Arch. Ration. Mech. Anal. 189 (2008), 189–236.

[11] S.N. Chow, J. Mallet-Paret, W. Shen, *Travelling waves in lattice dynamical systems*, J. Diff. Equations 149 (1998), 249–291.

[12] R. Coutinho, B. Fernandez, *Extended symbolic dynamics in bistable CML: existence and stability of fronts*, Physica D 108 (1997), 60–80.

[13] R. Coutinho, B. Fernandez, *Fronts and interfaces in bistable extended mappings*, Nonlinearity 11 (1998), 1407–1433.

[14] R. Coutinho, B. Fernandez, *Fronts in extended systems of bistable maps coupled via convolutions*, Nonlinearity 17 (2004), 23–47.

[15] R. Coutinho, B. Fernandez, *Fronts between periodic patterns for bistable recursions on lattices*, J. Dyn. Diff. Equations 25 (2013), 17–31.

[16] J. Fang, J. Wei, X.-Q. Zhao, *Spreading speeds and travelling waves for non-monotone time-delayed lattice equations*, Proc. Royal Soc. A 466 (2010), 1919–1934.

[17] J. Fang, J. Wei, X.-Q. Zhao, *Uniqueness of traveling waves for nonlocal lattice equations*, Proc. Amer. Math. Soc. 139 (2011), 1361–1373.

[18] J. Fang, X.-Q. Zhao, *Monotone wavefronts of the nonlocal Fisher-KPP equation*, Nonlinearity 24 (2011), 3043–3054.

[19] G. Faye, M. Holzer, *Modulated traveling fronts for a nonlocal Fisher-KPP equation: a dynamical systems approach*, J. Diff. Equations 258 (2015), 2257–2289.

[20] S.-C. Fu, J.-S. Guo, C.-C. Wu, *Traveling wave solutions for a discrete diffusive epidemic model*, J. Nonlinear Convex Anal. (to appear).

[21] S. Genieys, V. Volpert, P. Auger, *Pattern and waves for a model in population dynamics with nonlocal consumption of resources*, Math. Model. Nat. Phenom. 1 (2006), 63–80.

[22] J.-S. Guo, F. Hamel, *Front propagation for discrete periodic monostable equations*, Math. Ann. 335 (2006), 489–525.

[23] J.-S. Guo, Y.-C. Lin, *Traveling wave solution for a lattice dynamical system with convolution type nonlinearity*, Disc. Cont. Dyn. Syst. 32 (2012), 101–124.

[24] J.-S. Guo, C.-H. Wu, *Existence and uniqueness of traveling waves for a monostable 2-D lattice dynamical system*, Osaka J. Math. 5 (2008), 327–346.

[25] J.-S. Guo, C.-H. Wu, *Wave propagation for a two-component lattice dynamical system arising in strong competition models*, J. Diff. Equations 250 (2011), 3504–3533.

[26] J.-S. Guo, C.-H. Wu, *Traveling wave front for a two-component lattice dynamical system arising in competition models*, J. Diff. Equations 252 (2012), 4357–4391.

[27] D. Hankerson, B. Zinner, *Wavefronts for a cooperative tridiagonal system of differential equations*, J. Dyn. Diff. Equations 5 (1993), 359–373.

[28] H.W. Hethcote, *The mathematics of infectious diseases*, SIAM Review 42 (2000), 599–653.

[29] A. Hoffman, H.J. Hupkes, E.S. Van Vleck, *Multi-dimensional stability of waves travelling through rectangular lattices in rational directions*, Trans. Amer. Math. Soc. 367 (2015), 8757–8808.

[30] A. Hoffman, B. Kennedy, *Existence and uniqueness of traveling waves in a class of unidirectional differential equations*, Disc. Cont. Dyn. Syst. 30 (2011), 137–167.

[31] M. Holzer, A. Scheel, *A slow pushed front in a LotkaVolterra competition model*, Nonlinearity 25 (2012), 2151–2179.

[32] W. Hudson, B. Zinner, *Existence of traveling waves for a generalized discrete Fisher's equation*, Comm. Appl. Nonlinear Anal. 1 (1994), 23–46.

[33] J.P. Keener, *Propagation and its failure in coupled systems of discrete excitable cells*, SIAM J. Appl. Math. 47 (1987), 556–572.

[34] J.P. Keener, *Homogenization and propagation in the bistable equation*, Physica D 136 (2000), 1–17.

[35] S.-G. Liao, C.-C. Wu, *Propagation failure in discrete periodic media*, J. Diff. Equ. Appl. 19 (2013), 1268–1275.

[36] E. Logak, *Mathematical analysis of a condensed phase combustion model without ignition temperature*, Nonlinear Anal. Theory Meth. Appl. 28 (1997), 1–38.

[37] S. Ma, P. Weng, X. Zou, *Asymptotic speed of propagation and traveling wavefronts in a non-local delayed lattice differential equation*, Nonlinear Anal. Theory Meth. Appl. 65 (2006), 1858–1890.

[38] S. Ma, X. Zou, *Existence, uniqueness and stability of travelling waves in a discrete reactiondiffusion monostable equation with delay*, J. Diff. Equations 217 (2005), 54–87.

[39] M. Marion, *Qualitative properties of a nonlinear system for laminar flames without ignition temperature*, Nonlinear Anal. Theory Meth. Appl. 9 (1985), 1269–1292.

[40] G. Nadin, B. Perthame, M. Tang, *Can a traveling wave connect two unstable states? The case of the nonlocal Fisher equation*, C. R. Acad. Sci. Paris I 349 (2010), 553–557.

[41] G. Nadin, L. Rossi, L. Ryzhik, B. Perthame, *Wave-like solutions for nonlocal reaction-diffusion equations: a toy model*, Math. Model. Nat. Phenom. 8 (2013), 33–41.

[42] W. van Saarloos, *Front propagation into unstable states: marginal stability as a dynamical mechanism for velocity selection*, Phys. Rev. A 37 (1988), 211–229.

[43] W. van Saarloos, *Front propagation into unstable states. II. Linear versus nonlinear marginal stability and rate of convergence*, Phys. Rev. A 39 (1989), 6367–6390.

[44] Z. Xu, J. Ma, *Monotonicity, asymptotics and uniqueness of travelling wave solution of a non-local delayed lattice dynamical system*, Disc. Cont. Dyn. Syst. 35 (2015), 5107–5131.

[45] B. Zinner, *Stability of traveling wavefronts for the discrete Nagumo equation*, SIAM J. Math. Anal. 22 (1991), 1016–1020.

[46] B. Zinner, *Existence of traveling wavefront solutions for the discrete Nagumo equation*, J. Diff. Equations 96 (1992), 1–27.

[47] B. Zinner, G. Harris, W. Hudson, *Traveling wavefronts for the discrete Fisher's equation*, J. Diff. Equations 105 (1993), 46–62.

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