MOTION BY CURVATURE OF PLANAR CURVES WITH END POINTS MOVING FREELY ON A LINE

XINFU CHEN AND JONG-SHENQ GUO

Abstract. This paper deals with the motion by curvature of planar curves having end points moving freely along a line with fixed contact angles to this line. We first prove the existence and uniqueness of self-similar shrinking solution. Then we show that the curve shrinks to a point in a self-similar manner, if initially the curve is a graph.

Keywords: Motion by curvature, contact angle, triple junction, self-similar solution

1. Introduction

In this paper, we study the following problem on the evolution of planar curves.

Problem (P): Given an initial curve $\Gamma(0)$, find a family of curves $\{\Gamma(t)\}_{0<t<T}$ that lie on the upper-half plane, have end points on the x-axis with contact angle ψ_- on the left and ψ_+ on the right, and evolve according to the motion by curvature; see Figure 1 (a).

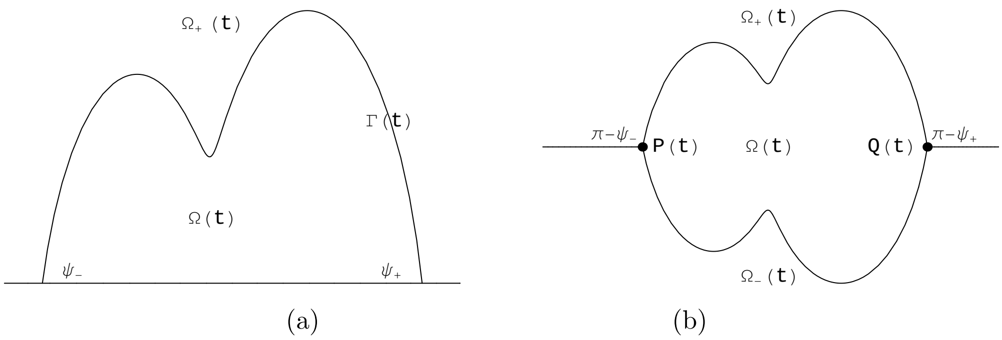

Figure 1. Figure (b) is a schematic snap shot of a diminishing grain domain $\Omega(t)$ surrounded by two other grain domains $\Omega_+(t)$ and $\Omega_-(t)$; the dots $P(t)$ and $Q(t)$ are the so-called triple junctions of three grain domains. When $\Omega(t)$ is symmetric about the x-axis, figure (a), modelled by problem (P), is the upper-half part of figure (b)

One motivation of our investigation of problem **(P)** originates from the study of evolution of grain domains in polycrystals. Here by a grain it refers to a periodic lattice structure of composite particles of a crystal; see Angenent and Gurtin [8, 29], Herring [30, 31], Mullins [40, 41, 42], Sutton

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and Baluffi [47], Woodruff [48], as well as Kobayashi, Warren, and Carter [36, 37, 38] for more physical background. In such a sense, all grains are physically and chemically identical, except their orientations. A grain boundary is the intersection of two grain domains at which orientations of different lattices do not match. Similarly, a triple junction is the meeting place of three grain domains. It is commonly believed that at a triple junction, the intersection angles are fixed, a principal quite often called the Herring condition [30, 31] (under such a principal, we indeed should have $\psi_+ = \psi_-$). Grain boundaries are often modelled by the (mean) curvature flows; see the theoretical and laboratorial studies of the group of Adams, Ta'asan, Kinderlehrer, Livshits, Manolache, Mason, Wu, Mullins, Rother, Rollett and Saylor [2, 3, 34], and also mathematical oriented studies of Bronsard and Retich [12], Kinderleherer and Liu [33], Mantegazza, Novaga and Tortorelli [39].

It is observed that the evolution of grain boundaries makes a network of grains topologically simpler and simpler. This is achieved by diminishing of grains; creation of grains is very rare, except at very early stage of the formation of polycrystals. Here in this paper we consider a mostly observed scenario depicted in Figure 1 (b). Due to the mathematical challenge, here we shall focus only on a situation where $\Omega(t)$ is symmetric about the x-axis. Then the evolution of the grain boundary between $\Omega(t)$ and $\Omega_+(t)$ is described in the Problem **(P)**.

When no triple junctions are involved, mathematically one studies the curvature flow of a simple closed curve (the boundary of a bounded smooth domain). A fundamental result in this direction is that of Grayson [27] who proved that the curvature evolution of a simple smooth curve remains simple and smooth until it shrinks to a single point; in addition, in its final stage the curve, after an appropriate magnification, becomes closer and closer to a circle. Here we shall prove a similar result: $\Omega(t)$ shrinks to a single point in an asymptotically self-similar manner.

In the literature, there have been many studies on the (mean) curvature flow of non-simple curves (or hypersurfaces in higher spatial dimension), notably the work of Brakke [10], Evans and Spruck [20, 21, 22, 23], Chen, Giga and Goto [16]. In these studies, either there is non-uniqueness, such as the varifold solution [10, 32], or there is uniqueness, such as the viscosity solution [20, 16], but the uniqueness is obtained in a sense by taking the union of all Brakke's varifold solutions [32]. There is also an approach by regarding the curvature flow as the limit of a scalar Allen-Cahn equation [4, 11, 14, 18, 25, 24, 32, 43, 44]; however, the scalar Allen-Cahn equation [4] can model only two grains.

Thus, in the study of (mean) curvature flow, the existence theory established in [10, 20, 16, 32, 46] on the one hand are beautiful and complete in modelling two phase problems such as the phase transition between liquid and solid; on the other hand, the uniqueness for multiple ($\geqslant 3$) phase problems has to be reconsidered. For evolution of grains in polycrystals, for example, one has to take into account conditions at triple junctions [12, 39, 33, 44]. Indeed, this is another motivation of this paper. In addition to an earlier work [15], we intend to address relevant problems in resolving non-uniqueness in the classical curvature flow. For the existence theory, we also refer the reader to the nice book of Giga [26] for the level-set approach to surface evolution equations.

This paper is organized as follows. In the next section, we first provide, for reader's convenience, three formulations for the curvature flow that is relevant to our problem **(P)**. Then we recall some classical well known local existence and uniqueness results for solution of **(P)** (cf. [12, 13, 17, 39]). Moreover, some geometric properties are given. In §3, we show that there exists a unique self-similar solution, following the discussion of our earlier paper [15] and also Abresch and Langer [1];

as a byproduct we supply an analytic proof for the monotonicity of a period function originally proven by Abresch and Langer [1] with the help of numerical verifications of certain quantities. The main analysis is in §4, in which we show that $\Gamma(t)$ shrinks to a point in a self-similar manner. Due to technical difficulties, we assume that $\psi_\pm \in (0, \pi/2)$ and initially $\Gamma(0)$ is a graph $y = u^0(x)$. We expect the same conclusion holds for a generic simple initial curve and positive $\psi_\pm$ satisfying $\psi_+ + \psi_- < \pi$. We leave this important extension as an open problem.

After submitting this paper, we learned from the referee that there is a similar work done independently by a group led by Oliver Schnürer [45]. In this work, the authors considered the case of convex curve. Our work is for general curve. We do not assume the convexity of the initial curve. Also, our proof of the main theorems are totally different from the ones given in [45]. We are grateful to the referee for informing us the relevant work [45]. We also thank Bellettini and Novoga for sending us their work [9] for the nonconvex case with $\psi_\pm = \pi/3$ after this paper was accepted.

2. Preliminaries

In this section, we shall supply three PDE formulations for our problem **(P)** and study their well-posedness. Moreover, some geometric properties of the solution shall be given.

First, we can regard $\Gamma(0)$ as the union of the positions of a collection of particles, each particle being designated by a real number $z \in [0, 1]$; that is,

$$\begin{cases} \Gamma(0) = \{(x^0(z), y^0(z)) \mid 0 \leqslant z \leqslant 1\}, \quad |x^0_z(z)| + |y^0_z(z)| > 0 \quad \forall\, z \in [0,1], \\ y^0(0) = y^0(1) = 0, \quad x^0_z(0) = y^0_z(0)\cot\psi_-, \qquad x^0_z(1) = -y^0_z(1)\cot\psi_+. \end{cases} \tag{2.1}$$

Each particle moves and $\Gamma(t)$ is the union of the positions of all the particles at time t. Hence,

$$\Gamma(t) := \{X(z,t) \mid z \in [0,1]\}, \qquad X(z,t) := (x(z,t), y(z,t)).$$

Here $X(z,t)$ is the position of the particle named z. At $X(z,t)$, the unit tangent $\mathbf{t}$, unit normal $\mathbf{n}$, normal velocity V and curvature κ of $\Gamma(t)$ are given by

$$\mathbf{t} = \frac{X_z}{|X_z|} = \frac{(x_z, y_z)}{\sqrt{x_z^2 + y_z^2}}, \qquad \mathbf{n} = \frac{(y_z, -x_z)}{\sqrt{x_z^2 + y_z^2}}, \qquad V = X_t \cdot \mathbf{n}, \qquad \kappa = \frac{X_{zz}}{|X_z|^2} \cdot \mathbf{n}.$$

The motion by curvature equation $V = \kappa$ is equivalent to

$$\left\{X_t - \frac{X_{zz}}{|X_z|^2}\right\} \cdot \mathbf{n} = 0. \tag{2.2}$$

It is easy to see that, up to a reprameterization, (2.2) takes the special form $X_t = X_{zz}/|X_z|^2$. Hence, problem (**P**) can be formulated as follows: Find $X = (x, y)$ such that

$$\begin{cases} x_t = \dfrac{x_{zz}}{x_z^2 + y_z^2}, \qquad y_t = \dfrac{y_{zz}}{x_z^2 + y_z^2}, & z \in (0,1), t \in (0,T), \\ y(0,t) = 0, \quad y(1,t) = 0, & t \in [0,T), \\ x_z(0,t) = y_z(0,t)\cot\psi_-, & t \in [0,T), \\ x_z(1,t) = -y_z(1,t)\cot\psi_+, & t \in [0,T), \\ x(\cdot,0) = x^0(z), \quad y(\cdot,0) = y^0(z), & z \in [0,1]. \end{cases} \tag{2.3}$$

The well-posedness of the problem (2.3) is stated as follows.

Theorem 1. *Let $\psi_+, \psi_- \in (0,\pi)$ and assume that $\Gamma(0) \in C^{1+\alpha}$ for some $\alpha \in (0,1)$, i.e., (2.1) holds for some $X^0 = (x^0, y^0) \in C^{1+\alpha}([0,1])$. Then there exists a positive T such that (2.3) admits a unique solution $(x,y) \in C^\infty([0,1]\times(0,T)) \cap C^{1+\alpha,(1+\alpha)/2}([0,1]\times[0,T))$ and T is the time of blow-up of curvature, i.e., $\lim_{t\nearrow T}\|\kappa\|_{L^\infty(\Gamma(t))} = \infty$.*

Since the proof of this theorem is rather standard, we safely omit its details here. For example, one can use a fixed point theorem and a bootstrap argument to derive the existence and uniqueness of solution to (2.3). Indeed, a much more general construction has been given by Daskalopoulos and Hamilton [17] to prove the short time existence for the porous medium equation in arbitrary dimensions. Note that the norm

$$\|\kappa\|_{L^\infty(\Gamma(t))} := \max_{z\in[0,1]} \frac{|x_{zz}(z,t)y_z(z,t) - y_{zz}(z,t)x_z(z,t)|}{(x_z^2(z,t)+y_z^2(z,t))^{3/2}}$$

is intrinsic and it does not depend on any parameterization of $\Gamma(t)$.

Next, when $\psi_\pm \in (0,\pi/2)$ and $\Gamma(0)$ is a graph $y = u^0(x)$, $x \in [l^0_-, l^0_+]$, one can expect that $\Gamma(t)$ is also a graph given by $y = u(x,t)$, $x \in [l_-(t), l_+(t)]$. At $(x, u(x,t))$, the relevant geometrical quantities of $\Gamma(t)$ are given by

$$\mathbf{t} = \frac{(1,u_x)}{\sqrt{1+u_x^2}}, \quad \mathbf{n} = \frac{(u_x,-1)}{\sqrt{1+u_x^2}}, \quad V = -\frac{u_t}{\sqrt{1+u_x^2}}, \quad \kappa = -\frac{(\arctan u_x)_x}{\sqrt{1+u_x^2}}.$$

Thus problem **(P)** is equivalent to find unknowns u and $\{l_\pm(t)\}$ such that

$$\begin{cases} u_t = (\arctan u_x)_x, & x \in (l_-(t), l_+(t)), \quad t \in (0,T), \\ u(l_\pm(t),t) = 0, & t\in[0,T), \\ u_x(l_\pm(t),t) = \mp\tan\psi_\pm, & t \in [0,T), \\ u(x,0) = u^0(x), & x \in [l_-(0), l_+(0)] := [l^0_-, L^0_+]. \end{cases} \tag{2.4}$$

Comparing with (2.3), this formulation is simpler in the sense that it is a scalar equation. However, the disadvantage is that (2.4) is a free boundary problem since a priori $\{l_\pm(t)\}_{0<t<T}$ are unknown. The well-posedness of (2.4) was established by Chang, Guo and Kohsaka [13] in a much more general setting, using a semigroup theory.

The third formulation is related to the polar coordinates. Fix a reference point $x_0 + 0\mathbf{i} \in \mathbb{C} = \mathbb{R}^2$ and suppose that $\{\Gamma(t)\}$ can be expressed as

$$\Gamma(t) = \{x_0 + R(\varsigma,t)e^{\mathbf{i}\varsigma} \mid 0 \leqslant \varsigma \leqslant \pi\}.$$

At point $X(\varsigma,t) := x_0 + R(\varsigma,t)e^{\mathbf{i}\varsigma}$,

$$X_\varsigma(\varsigma,t) = (R_\varsigma + \mathbf{i}R)\, e^{\mathbf{i}\varsigma} = \sqrt{R^2+R_\varsigma^2}\, e^{\mathbf{i}(\varsigma+\phi)}, \quad \phi := \operatorname{arccot}\frac{R_\varsigma}{R} \in (0,\pi).$$

Retaining the earlier convention of clockwise rotation as positive direction, we have

$$\mathbf{t} = -e^{\mathbf{i}(\varsigma+\phi)}, \quad \mathbf{n} = -\mathbf{i}\mathbf{t}, \quad V = -R_t\sin\phi, \quad \kappa = \frac{(\varsigma+\phi)_\varsigma}{\sqrt{R^2+R_\varsigma^2}} = \frac{R^2+2R_\varsigma^2 - RR_{\varsigma\varsigma}}{(R^2+R_\zeta^2)^{3/2}}.$$

Then problem (**P**) can be expressed as

$$(2.5)\quad \begin{cases} R_t = \dfrac{RR_{\varsigma\varsigma} - 2R_\varsigma^2 - R^2}{R(R^2 + R_\varsigma^2)}, & \varsigma \in (0,\pi), t \in (0,T), \\ R_\varsigma(0,t) = -R(0,t)\cot\psi_+, \quad R_\varsigma(\pi,t) = R(\pi,t)\cot\psi_-, & t \in [0,T), \\ R(\varsigma,0) = R^0(\varsigma), & \varsigma \in [0,\pi]. \end{cases}$$

This is a scalar quasilinear parabolic PDE whose short time existence follows from a standard theory [35]. For maximal existence (i.e. to time where $\Gamma(t)$ shrinks to a single point), one needs to shift the origin from time to time.

If two variables (R,ϕ) are used, then the problem can be written as

$$(2.6)\quad \begin{cases} RR_t = -1 - \phi_\zeta, \quad R_\varsigma = R\cot\phi, & \varsigma \in (0,\pi), t \in (0,T), \\ \phi(0,t) = \pi - \psi_+, \quad \phi(\pi,t) = \psi_-, & t \in [0,T), \\ R(\varsigma,0) = R^0(\varsigma), & \varsigma \in [0,\pi]. \end{cases}$$

In the sequel, we denote by $X(z,t) = (x(z,t), y(z,t))$, $(z,t) \in [0,1]\times[0,T)$, the unique maximal solution to (2.3). It is easy to establish the following geometric properties of $\Gamma(t) = \{X(z,t) \mid z \in [0,1]\}$. Since the proof is very standard, we leave the verification to the reader.

Theorem 2. *Assume that* $\psi_\pm > 0$, $\psi_+ + \psi_- \leqslant \pi$, *and* $\Gamma(0)$ *is a simple curve whose interior lies in the upper-half plane. Then for each* $t \in (0,T)$, $x(0,t) < x(1,t)$ *and* $\Gamma(t)$ *is also a simple curve whose interior lies in the upper-half plane. In addition, the area* $A(t)$ *of the region bounded by* $\Gamma(t)$ *and the* x*-axis is given by*

$$(2.7)\quad A(t) = A(0) - [\psi_- + \psi_+]t \quad \forall t \in [0,T), \qquad T \leqslant T_{\max} := \frac{A(0)}{\psi_+ + \psi_-}.$$

Furthermore, if denote by $L(t)$ *the arclength of* $\Gamma(t)$, $\ell(t) = x(1,t) - x(0,t)$ *the distance of the end points, and* $c(t) = \frac{1}{2}[x(0,t) + x(1,t)]$ *the "center", then*

$$(2.8)\quad \frac{d}{dt}\left(L(t) - \frac{\cos\psi_- + \cos\psi_+}{2}\ell(t) + (\cos\psi_- - \cos\psi_+)c(t)\right) + \int_{\Gamma(t)} \kappa^2 ds = 0,$$

where ds *is the arclength element.*

3. Self-Similar Solutions

In this section, we study a class of special solutions that dominate the asymptotic behavior, as $t \nearrow T$, of solutions to problem **(P)**. These are self-similar solutions.

3.1. Main Result.

Here by **self-similar** it means a solution $\{\Gamma(t)\}_{0<t<T}$ that satisfies, for some fixed shape Γ_0,

$$\Gamma(t) = \sqrt{2(T-t)}\,\Gamma_0 := \{\sqrt{2(T-t)}(x,y) \mid (x,y) \in \Gamma_0\} \qquad \forall t \in [0,T).$$

Since one can show that Γ_0 has to be convex, any of the systems that we derived in §2 can be used. Here we follow our earlier paper [15] using polar coordinates. Hence, we seek solutions to (2.6) in the form

$$R(\varsigma,t) = \sqrt{2(T-t)\rho(\varsigma)}, \quad \phi(\varsigma,t) = \psi(\varsigma) \qquad \forall \varsigma \in [0,\pi], t \in [0,T).$$

This is equivalent to solve the ODE system

$$\rho' = 2\rho \cot \psi, \qquad \rho > 0, \qquad \psi' = \rho - 1 \qquad \text{in } [0, \pi], \tag{3.1}$$

subject to the boundary conditions

$$\psi(0) = \pi - \psi_+, \qquad \psi(\pi) = \psi_-. \tag{3.2}$$

This section is devoted to prove the following result.

Theorem 3. *Assume that* $\psi_\pm \in (0, \pi/2]$. *Then* (3.1)–(3.2) *admits a unique solution. Consequently, problem* **(P)** *admits a unique self-similar solution.*

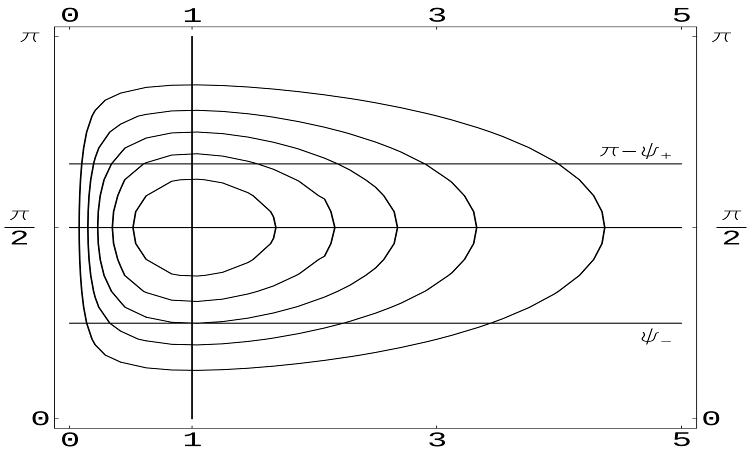

FIGURE 2. Trajectories to (3.1) on the ρ–ψ phase plane.

3.2. **Solution Trajectories.**

To solve the problem, we investigate trajectories of the solutions to (3.1). There is a first integral obtained from $[\ln \sin^2 \psi + \ln \rho - \rho]' = 0$. Hence, a generic trajectory to (3.1) is given by, for some constant $c \in [1, \infty)$,

$$e^{\rho - 1} = c\,\rho\, \sin^2 \psi. \tag{3.3}$$

Note that the function $\rho \to e^{\rho-1}/\rho$ is decreasing in $(0, 1]$ and increasing in $[1, \infty)$, with its unique minimum value 1 attained at $\rho = 1$. When $c = 1$, the solution is given by $(\rho, \psi) \equiv (1, \pi/2)$, which corresponds to the unit circle. A few other sample trajectories are depicted in Figure 2, where counterclockwise rotation is the positive direction.

In the sequel, we denote by $\gamma(c)$ the trajectory given in (3.3) and by G_1 and G_2 the two inverses of $\rho \to e^{\rho-1}/\rho$:

$$s = \frac{e^{\rho-1}}{\rho} \qquad \Longleftrightarrow \qquad \rho = \begin{cases} G_1(s) & \text{if } \rho \leqslant 1, \\ G_2(s) & \text{if } \rho \geqslant 1. \end{cases}$$

Every trajectory $\gamma(c)$ is periodic. Its leftmost, rightmost, top, and bottom points are, respectively,

$$\Big(G_1(c), \frac{\pi}{2}\Big), \quad \Big(G_2(c), \frac{\pi}{2}\Big), \quad \Big(1, \pi - \arcsin\frac{1}{\sqrt{c}}\Big), \quad \Big(1, \arcsin\frac{1}{\sqrt{c}}\Big).$$

It is shown in [1] and also in the next subsection that the period of $\gamma(c)$ is strictly bigger than π so a trajectory γ to (3.1)–(3.2) is only a part of $\gamma(c)$. Since $\gamma \subset \gamma(c)$ has to intersect the horizontal lines $\psi = \pi - \psi_+$ and $\psi = \psi_-$, we must have

$$c \geqslant c_* := \max\Big\{\frac{1}{\sin^2\psi_+}, \frac{1}{\sin^2\psi_-}\Big\}.$$

We denote by $A_1(c), A_2(c)$ the intersections of $\gamma(c)$ with $\psi = \pi - \psi_+$ and by $B_1(c), B_2(c)$ that with $\psi = \psi_-$:

$$A_1(c) = \Big(G_1(c\sin^2\psi_+), \pi - \psi_+\Big), \qquad A_2(c) = \Big(G_2(c\sin^2\psi_+), \pi - \psi_+\Big),$$
$$B_1(c) = \Big(G_1(c\sin^2\psi_-), \psi_-\Big), \qquad B_2(c) = \Big(G_2(c\sin^2\psi_-), \psi_-\Big).$$

Assume that $\psi_+ + \psi_- \leqslant \pi$ so $\pi - \psi_+ \geqslant \psi_-$. Then, a solution trajectory γ to (3.1)-(3.2) can have only four possible choices,

(1) $A_1 \neq A_2$ and γ is the part of $\gamma(c)$ from A_2 to B_2 (counterclockwise);
(2) γ is the par of $\gamma(c)$ from A_1 to B_2 (counterclockwise);
(3) γ is the par of $\gamma(c)$ from A_2 to B_1 (counterclockwise);
(4) γ is the part of $\gamma(c)$ from A_1 to B_1 (counterclockwise).

Clearly, any of these trajectories gives a solution to (3.1)–(3.2) if and only if the "time" spent on the trajectory is exactly π. To calculate the time spent on these trajectories, we use the following

$$d\varsigma = \frac{d\psi}{\rho - 1} = \frac{d\rho}{2\rho\cot\psi}, \qquad \frac{1}{\sin^2\psi} = c\rho e^{1-\rho}, \qquad \cot\psi = \pm\sqrt{c\rho e^{1-\rho} - 1}. \tag{3.4}$$

Also, we introduce the following functions. For every $\varphi \in (0, \pi/2]$ and $c \geqslant 1/\sin^2\varphi$,

$$\ell_1(c,\varphi) := \int_\varphi^{\pi/2} \frac{d\phi}{1 - G_1(c\sin^2\phi)}, \qquad \ell_2(c,\varphi) := \int_{G_1(c)}^{G_2(c\sin^2\varphi)} \frac{ds}{2s\sqrt{cse^{1-s} - 1}}. \tag{3.5}$$

Note that $\ell_1(c,\varphi)$ is the time spent on $\gamma(c)$ from the leftmost point $(G_1(c), \pi/2)$ to $(G_1(c\sin^2\varphi), \varphi)$ (the first intersection of $\gamma(c)$ with the line $\psi = \varphi$), and $\ell_2(c,\varphi)$ is the time spent on $\gamma(c)$ from the left most point $(G_1(c), \pi/2)$ to $(G_2(c\sin^2\varphi), \varphi)$ (the second intersection of $\gamma(c)$ with the line $\psi = \varphi$). In particular, by symmetry, the period $\omega(c)$ of $\gamma(c)$ is given by

$$\omega(c) := 2\ell_2(c, \pi/2) = \int_{G_1(c)}^{G_2(c)} \frac{d\rho}{\rho\sqrt{c\,\rho\, e^{1-\rho} - 1}}. \tag{3.6}$$

Also, denote by $\omega_1(c)$ the "time" spent on $\gamma(c)$ from the leftmost point $(G_1(c), \pi/2)$ to the bottom point $(1, \arcsin[1/\sqrt{c}])$. Then

$$\omega_1(c) := \ell_1(c, \arcsin[1/\sqrt{c}]) = \ell_2(c, \arcsin[1/\sqrt{c}]) = \int_{G_1(c)}^{1} \frac{d\rho}{2\rho\sqrt{c\,\rho\, e^{1-\rho} - 1}}. \tag{3.7}$$

Furthermore, when $\pi - \psi_+ \geqslant \pi/2 \geqslant \psi_-$, by symmetry $\ell_i(\pi - \psi_+) = \ell_i(c, \psi_+)$, so the time spent on the part of the trajectory described in cases (1)–(4) are respectively the following:

$$h_I(c) := \ell_2(c, \psi_+) + \ell_2(c, \psi_-),$$
$$h_{II}(c) := \ell_1(c, \psi_+) + \ell_2(c, \psi_-),$$
$$h_{III}(c) := \ell_2(c, \psi_+) + \ell_1(c, \psi_-),$$
$$h_{IV}(c) := \ell_1(c, \psi_+) + \ell_1(c, \psi_-).$$

Hence, solving (3.1)–(3.2) is equivalent to find $c \geqslant c_*$ such that one of the following holds:

$$h_I(c) = \pi, \qquad h_{II}(c) = \pi, \qquad h_{III}(c) = \pi, \qquad h_{IV}(c) = \pi.$$

A direct investigation for these functions from their integral formulations seems vary hard. Here we shall utilize the differential equations that used to define them.

3.3. **An Analytic Proof of the Abresch and Langer Result** [1].

In [1], Abresch and Langer proved with the help of a computer that $\omega(c)$, the period of $\gamma(c)$, is a strictly decreasing function. Here we provide a purely analytical proof. We remark that this monotonicity result has been established before by Andrews [6] for a larger class of equations, namely, $V = \kappa^\alpha$ for $\alpha > 0$. Our proof here is different from that in [6].

Theorem 4. *(1) The function $\omega(c)$ defined in* (3.6) *satisfies*

$$\lim_{c\searrow 1} \omega(c) = \sqrt{2}\,\pi, \qquad \omega'(c) < 0 \quad \forall\, c \in (1,\infty), \qquad \lim_{c\nearrow\infty} \omega(c) = \pi.$$

(2) The function $\omega_1(c)$ defined in (3.7) *satisfies*

$$\lim_{c\searrow 1} \omega_1(c) = \frac{\sqrt{2}\,\pi}{4}, \qquad \omega_1'(c) > 0 \quad \forall\, c \in (1,\infty), \qquad \lim_{c\nearrow\infty} \omega_1(c) = \frac{\pi}{2}.$$

Proof. **(i)** For smooth f and $a < b$ satisfying $f(a) = f(b)$ and for every $x \in [a,b]$,

$$f(x) - f(a) = -\frac{(b-x)\int_a^x (y-a) f''(y)dy + (x-a)\int_x^b (b-y) f''(y)dy}{b-a}\,.$$

Estimating the lower and upper bounds of the integral by replacing $f''(y)$ with $\max_{y\in[a,b]} f''(y)$ and $\min_{y\in[a,b]} f''(y)$ respectively, we conclude that

$$f(x) - f(a) = -\frac{f''(\xi)}{2}(x-a)(b-x)$$

for some $\xi = \xi(x) \in (a,b)$. Hence, setting $f(\rho) = \rho e^{1-\rho}$, we have

$$\begin{aligned}
\omega(c) &= \int_{G_1(c)}^{G_2(c)} \frac{\sqrt{2}\,d\rho}{\rho\sqrt{-c\, f''(\xi(\rho))[\rho - G_1(c)][G_2(c)-\rho]}} \\
&= \frac{\sqrt{2}}{\eta\sqrt{-cf''(\xi)}} \int_{G_1}^{G_2} \frac{d\rho}{\sqrt{[\rho - G_1][G_2-\rho]}} = \frac{\sqrt{2}\,\pi}{\eta\sqrt{-c\, f''(\xi)}}
\end{aligned}$$

for some $\xi, \eta \in [G_1(c), G_2(c)]$, by the same estimation technique as above. Since $f''(1) = -1$ and $G_1(c), G_2(c) \to 1$ as $c \to 1$, $\lim_{c\searrow 1} \omega(c) = \sqrt{2}\,\pi/\sqrt{-f''(1)} = \sqrt{2}\,\pi$.

Similarly, we have $\omega_1(c) \to \sqrt{2}\pi/4$ as $c \searrow 1$.

(ii) Note that $G_1(c) \to 0$ and $G_2(c) \to \infty$ as $c \to \infty$. The change of variable $\rho = G_1(c) r$ gives

$$\omega(c) = \int_1^{G_2(c)/G_1(c)} \frac{dr}{r\sqrt{r\, e^{G_1(c)(1-r)} - 1}} \longrightarrow \int_1^\infty \frac{dr}{r\sqrt{r-1}} = \pi \quad \text{as} \quad c \to \infty;$$

here we omit the details of the limit taken process. Analogously, $\omega_1(c) \to \pi/2$ as $c \to \infty$.

(iii) The most difficult part is showing the monotonicity of $\omega(\cdot)$ and $\omega_1(\cdot)$. We provide a method that may have other applications.

For each $c > 1$, we denote by $(\rho(c;\varsigma), \psi(c;\varsigma))$ the solution to (3.1) subject to the initial condition

$$\rho(c;0) = G_1(c), \qquad \psi(c;0) = \tfrac{1}{2}\pi.$$

We shall use (3.4). Note that when $\varsigma \in (0, \omega(c)/2)$, $\psi \in (0, \pi/2)$. We use $'$ for differentiation with respect to ς and subscript c for differentiation with respect to c. Then, differentiating $\psi' = \rho - 1$ and $e^{\rho-1} = c\rho \sin^2\psi$ with respect to c, we obtain $\psi'_c = \rho_c$ and $(1 - 1/\rho)\rho_c = 1/c + 2\psi_c \cot\psi$. As $\rho' = 2\rho\cot\psi$, this implies that $(\rho-1)\psi'_c - (\rho-1)'\psi_c = \rho/c$. Using the integrating factor $(\rho-1)^{-2}$ and $d\varsigma = d\rho/[2\rho\sqrt{c\rho e^{1-\rho} - 1}]$, we derive that, for $\varsigma \in (0, \omega_1(c))$,

$$\frac{\psi_c(c;\varsigma)}{\rho(c;\varsigma)-1} = \int_0^{\varsigma} \frac{\rho(c;\hat{\varsigma})\, d\hat{\varsigma}}{c\,[\rho(c;\hat{\varsigma})-1]^2} = \int_{G_1(c)}^{\rho(c;\varsigma)} \frac{ds}{2c\,[s-1]^2\sqrt{c\,s\,e^{1-s}-1}}\,.$$

Denote $(\rho,\psi) = (\rho(c;\varsigma), \psi(c;\varsigma))$. For $\varsigma \in (0, \omega_1(c))$,

$$\begin{aligned} \frac{2c(c-1)}{\rho-1}\psi_c &= \int_{G_1}^{\rho} \frac{\sqrt{cse^{1-s}-1}}{[1-s]^2}ds + \int_{G_1}^{\rho} \frac{\frac{c-1}{\sqrt{cse^{1-s}-1}} - \sqrt{cse^{1-s}-1}}{[1-s]^2}ds \\ &= \frac{\sqrt{c\rho e^{1-\rho}-1}}{1-\rho} - \int_{G_1}^{\rho} \frac{ce^{1-s}}{2\sqrt{cse^{1-s}-1}}ds + \int_{G_1}^{\rho} \frac{c - cse^{1-s}}{[1-s]^2\sqrt{cse^{1-s}-1}}ds; \end{aligned}$$

here we used integration by parts for the first integral and $\sqrt{cse^{1-s}-1}|_{s=G_1(c)} = 0$. Hence,

$$\begin{aligned} \frac{2c(c-1)}{\rho-1}\psi_c &= \frac{\sqrt{c\rho e^{1-\rho}-1}}{1-\rho} + c\int_{G_1}^{\rho} \frac{1 - se^{1-s} - \frac{1}{2}[1-s]^2 e^{1-s}}{[1-s]^2\sqrt{cse^{1-s}-1}}\, ds \\ &= \frac{\sqrt{c\rho e^{1-\rho}-1}}{1-\rho} + \int_{G_1}^{\rho} \frac{2e^{s-1} - 2s - [1-s]^2}{[1-s]^3} d\sqrt{cse^{1-s}-1} \\ &= \frac{\sqrt{c\rho e^{1-\rho}-1}}{1-\rho} + \frac{2e^{\rho-1} - 2\rho - [1-\rho]^2}{[1-\rho]^3}\sqrt{c\rho e^{1-\rho}-1} \\ &\quad - \int_{G_1}^{\rho} \sqrt{cse^{1-s}-1}\, \frac{d}{ds}\Big(\frac{2e^{s-1} - 2s - [1-s]^2}{[1-s]^3}\Big) ds \\ &= \frac{2[e^{\rho-1} - \rho]}{(1-\rho)^3}\sqrt{c\rho e^{1-\rho}-1} + \int_{G_1}^{\rho} \frac{J(s)\,\sqrt{cse^{1-s}-1}}{(1-s)^4}ds, \end{aligned} \tag{3.8}$$

where

$$J(s) := 2(s-4)e^{s-1} + s^2 + 2s + 3 = \frac{1}{3}\int_1^s t(s-t)^3 e^{t-1} dt > 0 \quad \forall s \in (0,1)\cup(1,\infty).$$

Therefore,

$$\psi_c(c;\varsigma) = \frac{[\rho - e^{\rho-1}]\sqrt{c\rho e^{1-\rho}-1}}{c(c-1)(1-\rho)^2} + \frac{\rho-1}{2c(c-1)}\int_{G_1}^{\rho} \frac{J(s)\,\sqrt{cse^{s-1}-1}}{(1-s)^4}ds \tag{3.9}$$

for $\varsigma \in (0, \omega_1(c))$. It is easy to check that the functions on both sides in (3.9) are real analytic in $[0, \omega(c)/2]$. Hence by the unique continuation property the above formula (3.9) is valid for all $\varsigma \in [0, \omega(c)/2]$.

Also, using $\rho_c = [1/c + 2\psi_c \cot\psi]/[1 - 1/\rho]$, $\cot\psi = \sqrt{c\rho e^{1-\rho} - 1}$ and (3.8) we obtain

$$\rho_c(c;\varsigma) = \frac{\rho[1 - \rho e^{1-\rho}]}{(c-1)(\rho-1)} + \frac{\rho\cot\psi}{(c-1)}\int_{G_1}^{\rho} \frac{1 - se^{1-s} - \frac{1}{2}[1-s]^2 e^{1-s}}{[1-s]^2\sqrt{cse^{1-s}-1}}\, ds. \tag{3.10}$$

Since $1 - \rho e^{1-\rho} = O([\rho-1]^2)$, the first function on the right-hand side of (3.10) is smooth.

Now, differentiating $\psi(c; \omega(c)/2) = \frac{1}{2}\pi$ with respect to c, using $\psi' = \rho - 1$ and (3.9), we obtain

$$\frac{d\omega(c)}{dc} = -\frac{2\psi_c(c;\omega(c)/2)}{\psi'(c;\omega/2)} = -\int_{G_1(c)}^{G_2(c)} \frac{J(s)\sqrt{cs\,e^{1-s}-1}}{c(c-1)(1-s)^4}ds < 0.$$

Finally, differentiating $\rho(c;\omega_1(c)) = 1$ with respect to c, using $\rho' = 2\rho\cot\psi$ and (3.10), we obtain

$$\frac{d\omega_1(c)}{dc} = -\frac{\rho_c(c;\omega_1)}{\rho'(c;\omega_1)} = \frac{-1}{2(c-1)}\int_{G_1}^1 \frac{e^{1-s}(e^{s-1}-s-\frac{1}{2}[1-s]^2)}{[1-s]^2\sqrt{cse^{1-s}-1}}\,ds > 0,$$

since the integrand is negative. This completes the proof. □

3.4. **The Concavity of** $\ell_2(\cdot,\varphi)$.

Here we investigate the function $\ell_2(c,\varphi)$ defined in (3.5), which represents the second time that $\gamma(c)$ (stating from the leftmost point) intersects the line $\psi = \varphi$.

Assume $\varphi \in (0,\pi/2)$ is fixed and $c > 1/\sin^2\varphi$. We begin with differentiating $\sin^{-2}\varphi = c\rho(c;\ell_2)e^{1-\rho(c;\ell_2)}$ to derive

$$\frac{d}{dc}\rho(c;\ell_2(c,\varphi)) = \frac{\rho}{c(\rho-1)}.$$

Next, differentiating the relation $\psi(c;\ell_2(c,\varphi)) = \varphi$ with respect to c and using (3.9), we obtain

$$\begin{aligned}\frac{d}{dc}\ell_2(c,\varphi)) &= -\frac{\psi_c}{\psi'}\Big|_{\varsigma=\ell_2(c,\varphi)} = \frac{\psi_c}{1-\rho}\Big|_{\varsigma=\ell_2(c,\varphi)}\\ &= \frac{1}{c(c-1)}\Big\{\frac{(\rho-e^{\rho-1})}{(1-\rho)^3}\cot\varphi - \frac{1}{2}\int_{G_1}^{\rho}\frac{J(s)\,\sqrt{cse^{1-s}-1}}{(1-s)^4}ds\Big\}\Big|_{\rho=\rho(c;\ell_2(c,\varphi))},\end{aligned}$$

where we have used $\cot\psi = \sqrt{c\rho e^{1-\rho}-1}$. Note that when $s \in [1,\rho(c;\ell_2(c,\varphi))]$, $\sqrt{cse^{1-s}-1} \geqslant \cot\varphi$. Also, when $c \gg 1$, $\rho = \rho(c;\ell_2(c,\varphi)) \approx \ln c \gg 1$ and

$$\frac{e^{\rho-1}}{(\rho-1)^3} \approx \int_4^{\rho}\frac{(s-4)e^{s-1}}{(1-s)^4}ds.$$

It then follows from the expression of J that

$$\frac{d}{dc}\ell_2(c,\varphi) < 0 \qquad \forall\, c \gg 1.$$

We continue to investigate the second derivative of ℓ_2:

$$\begin{aligned}\frac{d}{dc}\Big(c(c-1)\frac{d}{dc}\ell_2(c,\varphi)\Big) &= \frac{[1-\rho][1-e^{\rho-1}]+3[\rho-e^{\rho-1}]-\frac{1}{2}J(\rho)}{[1-\rho]^4}\cot\varphi\frac{d\rho(c;\ell_2(c,\varphi))}{dc}\\ &\quad -\frac{1}{4}\int_{G_1}^{\rho}\frac{J(s)\;se^{1-s}}{(1-s)^4\sqrt{cse^{1-s}-1}}ds.\\ &= -\frac{\rho\cot\varphi}{2c(\rho-1)^3} - \frac{1}{4}\int_{G_1(c)}^{\rho}\frac{J(s)\;se^{1-s}}{(1-s)^4\sqrt{cse^{1-s}-1}}ds\Big|_{\rho=\rho(c;\ell_2(c,\varphi))}.\end{aligned}$$

Thus, we have the following lemma:

Lemma 3.1. *For every $\varphi \in (0,\pi/2]$ and $c > 1/\sin^2\varphi$, let $\ell_2(c,\varphi) \in (\omega_1(c),\frac{1}{2}\omega(c))$ be the unique "time" such that $\psi(c;\ell_2) = \varphi$. Then $\ell_2(\infty,\varphi) = \pi/2$ and*

$$\frac{d\ell_2(c,\varphi)}{dc} < 0 \quad \forall\, c \gg 1, \qquad\qquad \frac{d}{dc}\Big(c(c-1)\frac{d}{dc}\ell_2(c,\varphi)\Big) < 0 \quad \forall\, c > \frac{1}{\sin^2\varphi}.$$

Remark 3.1. If we make a new parameter $\alpha := \ln[1-1/c]$ so $d\alpha = dc/[c(c-1)]$, then we have

$$\frac{d^2\ell_2}{d\alpha^2} = c(c-1)\frac{d}{dc}\Big[c(c-1)\frac{d\ell_2}{dc}\Big] < 0.$$

That is, as a function of α, ℓ_2 is a concave function. This observation is the key to our analysis.

3.5. **Proof of Theorem 3, Part I.**

Now consider (3.1)–(3.2). Since $\omega(c) > \pi$ for every $c > 1$, any solution trajectory γ to (3.1)-(3.2) must be within one period of a trajectory $\gamma(c)$ for some $c > 1$. We consider the four cases described earlier. For simplicity, we assume that

$$0 < \psi_+ \leqslant \psi_- \leqslant \pi/2.$$

Henceforth $c_* = 1/\sin^2\psi_+$.

Case (1): $A_1 \neq A_2$ and γ is the part of $\gamma(c)$ from A_2 to B_2 (counterclockwise).

By our definition of ℓ_2, it is necessary and sufficient to find a solution c to

$$\pi = h_I(c) := \ell_2(c, \psi_+) + \ell_2(c, \psi_-), \quad c > c_*. \tag{3.11}$$

It follows from Lemma 3.1 that

$$h_I(\infty) = \pi, \qquad \frac{dh_I(c)}{dc} < 0 \quad \forall\, c \gg 1, \qquad \frac{d}{dc}\Big(c(c-1)\frac{dh_I(c)}{dc}\Big) < 0 \quad \forall\, c > c_*.$$

This implies that there exists $\hat{c} \in [c_*, \infty)$ such that $h'_I < 0$ in $(\hat{c}, \infty)$ and $h'_I > 0$ in $[c_*, \hat{c})$. Consequently, $h_I > \pi$ in $(\hat{c}, \infty)$. Note that we may have $h'_I(c) < 0$ for all $c > c_*$ (so that $\hat{c} = c_*$). In this case, (3.11) does not have any solution. In any case, (3.11) admits at least a solution if and only if

$$h_I(c_*) < \pi.$$

In addition, under the above condition, the solution c to (3.11) is unique.

Case (2): γ is the par of $\gamma(c)$ from A_1 to B_2 (counterclockwise).

What we need is to find c such that

$$\pi = h_{II}(c) := \ell_1(c, \psi_+) + \ell_2(c, \psi_-), \qquad c \geqslant c_*. \tag{3.12}$$

First of all, it is easy to calculate

$$h_{II}(\infty) = \pi - \psi_+, \qquad h_{II}(c_*) = h_I(c_*).$$

If $h_I(c_*) \geqslant \pi$, then there exists at least one solution to (3.12). In the next subsection we shall show that if (3.12) admits a solution, then it is unique and $h_{II}(c_*) = h_I(c_*) \geqslant \pi$.

Thus, exactly one of (3.11) and (3.12) admits a solution, and the solution is unique.

Case (3): γ is the par of $\gamma(c)$ from A_2 to B_1 (counterclockwise).

What we need is to find c such that

$$\pi = h_{III}(c) := \ell_1(c, \psi_-) + \ell_2(c, \psi_+), \qquad c \geqslant c_*. \tag{3.13}$$

We shall show in the next subsection that if (3.13) admits a solution, then the solution is unique and $h_{III}(c_*) \geqslant \pi$. However, since $c_* = 1/\sin^2\psi_+$, we have $\ell_2(c_*, \psi_+) = \ell_1(c_*, \psi_+)$, so $h_{III}(c_*) \leqslant 2\omega_1(c) < \pi$. Thus, (3.13) has no solution.

Case (4): γ is the part of $\gamma(c)$ from A_1 to B_1 (counterclockwise).

This is equivalent to find c such that

$$\ell_1(c, \psi_-) + \ell_1(c, \psi_+) = \pi, \qquad c \geqslant c_*.$$

This equation has no solution, since $\max\{\ell_1(c, \varphi_-), \ell_1(c, \varphi_+\} \leqslant \omega_1(c) < \pi/2$ for all $c \geqslant 1$.

This proves the assertion of Theorem 3.

Remark 3.2. The discussion for case (3) in particular implies that when $\psi_+ = \psi_- \in (0, \pi/2]$, there is no solution trajectory to (3.1)–(3.2) that connects either from A_1 to B_2 or from A_2 to B_1.

3.6. **Proof of Theorem 3, Part II.**

It remains to consider solutions in cases (2) and (3). Here we drop the assumption on the order $\psi_+ \leqslant \psi_-$, so both cases (2) and (3) are essentially the same. We shall introduce a different method.

Note from (3.1) that $\psi'' = \rho' = 2\rho\cot\psi = 2(1+\psi')\cot\psi$. This suggests that we study, for each parameter $\alpha \in (0,\infty)$, the initial value problem

$$\begin{cases} \Psi''(\alpha;\varsigma) = 2(1+\Psi')\cot\Psi, \quad \varsigma \in \mathbb{R}, \\ \Psi(\alpha;0) = \pi - \psi_+, \qquad \Psi'(\alpha;0) = \alpha - 1. \end{cases} \tag{3.14}$$

This problem has a unique solution. Given such a solution Ψ, the function $(\psi,\rho) := (\Psi, 1+\Psi')$ solves (3.1). Solving (3.1)–(3.2) is equivalent to find $\alpha^* > 0$ such that

$$\Psi(\alpha^*;\pi) = \psi_-. \tag{3.15}$$

To address the leftover discussion for the cases (2) and (3) from the previous subsection, we need only consider the case $\alpha \in (0,1]$. We divide the analysis into a few steps.

1. First of all, the solution $\Psi(\alpha;\cdot)$ corresponds to $\gamma(c)$ with

$$c = \frac{e^{\alpha-1}}{\alpha\sin^2\psi_+}.$$

We now investigate the number of sign change of $\Psi_\alpha(\alpha;\cdot)$ in one period $(0,\omega(c))$.

Note that both Ψ_α and Ψ' satisfy the second order linear ODE

$$\mathcal{L}\Psi_\alpha = 0 = \mathcal{L}\Psi' \quad \text{in } \mathbb{R}, \qquad \mathcal{L}\phi := \phi'' - (2\cot\Psi)\phi' + \frac{2(1+\Psi')}{\sin^2\Psi}\phi.$$

When $\alpha = 1$, Ψ_α is a constant multiple of Ψ'. When $\alpha \neq 1$, Ψ' and Ψ_α are linearly independent so their zeros interlace. We denote the first zero of Ψ_α by $\varsigma_1(\alpha)$ and its second zero by $\varsigma_2(\alpha)$. Also, we denote by $\hat\omega_1(\alpha)$ and $\hat\omega_2(\alpha)$ the first and second time that $\Psi' = R - 1 = 0$ respectively. Then

$$0 < \hat\omega_1(\alpha) \leqslant \varsigma_1(\alpha) < \hat\omega_2(\alpha) \leqslant \varsigma_2(\alpha) \quad \forall \alpha \in (0,1]. \tag{3.16}$$

2. Next, we estimate the location of the first zero $\varsigma_1(\alpha)$ of Ψ_α. Set $R = 1 + \Psi'$. We find

$$R > 0, \quad (R^{-1/2})' = -R^{-1/2}\cot\Psi, \quad (R^{-1/2})'' = R^{-1/2}\Big\{\frac{R}{\sin^2\Psi} - 1\Big\}.$$

It then follows that

$$\begin{aligned} \Big(R^{-1/2}\Psi_\alpha\Big)'' &= R^{-1/2}\Psi_\alpha'' + 2(R^{-1/2})'\Psi_\alpha' + (R^{-1/2})''\Psi_\alpha \\ &= R^{-1/2}\Psi_\alpha'' - 2R^{-1/2}\Psi_\alpha'\cot\Psi + R^{-1/2}\Big\{\frac{R}{\sin^2\Psi} - 1\Big\}\Psi_\alpha \\ &= -\Big\{\frac{R}{\sin^2\Psi} + 1\Big\}R^{-1/2}\Psi_\alpha. \end{aligned}$$

Since $R>0$, by the Liouville theorem, the distance of two consecutive zeros of $R^{-1/2}\Psi_\alpha$ is strictly smaller than π. Hence,

$$\varsigma_1(\alpha)\in(0,\pi). \tag{3.17}$$

3. Now we consider the case when $\alpha=R(\alpha;0)\in(0,1]$. First of all, since $\omega(\infty)=\pi$ and $\omega(c)>\pi$ for all $c>1$, we have

$$\lim_{\alpha\searrow 0}\Psi(\alpha;\pi)=\pi-\psi_+,\qquad \Psi(\alpha;\pi)>\pi-\psi_+\ \text{ when }\ 0<\alpha\ll 1.$$

Hence, we can define

$$a_1:=\max\{a>0\mid \Psi(\cdot;\pi)>\pi-\psi_+\ \text{ in }\ (0,a]\}.$$

Since $\omega(c)>\pi$, we must have $a_1\in(0,1)$. Since $\psi_-<\pi-\psi_+$, there is no solution to $\Psi(\alpha;\pi)=\psi_-$ when $\alpha\in(0,a_1]$.

To consider the situation when $\alpha\in(a_1,1]$ we define

$$a_2:=\sup\{a\in(a_1,1]\mid \Psi_\alpha(\alpha;\pi)<0\ \forall\,\alpha\in[a_1,a]\}.$$

Note that $\Psi(a_1;\pi)=\pi-\psi_+$ and $R(a_1;\pi)>1=R(a_1;\hat\omega_2(a_1))$. We must have $\pi<\hat\omega_2(a_1)$ so by (3.16) and (3.17), $\pi\in(\varsigma_1(a_1),\varsigma_2(a_1))$. This implies that $\Psi_\alpha(a_1;\pi)<0$. Hence, a_2 is well-defined.

Now in $[a_1,a_2)$, $\Psi_\alpha(\cdot;\pi)<0$ so $\Psi(\cdot;\pi)<\pi-\psi_+$ in $(a_1,a_2]$. Consequently, $\hat\omega_2(\alpha)>\pi$ for all $\alpha\in[a_1,a_2]$. Hence,

$$\pi\in(\varsigma_1(\alpha),\hat\omega_2(\alpha))\subset(\varsigma_1(\alpha),\varsigma_2(\alpha))\quad\forall\,\alpha\in[a_1,a_2].$$

This implies that $\Psi_\alpha(\alpha;\pi)<0$ for all $\alpha\in[a_1,a_2]$. Hence, we must have $a_2=1$.

That $\Psi_\alpha(\alpha;\pi)>0$ for all $\alpha\in[a_1,1]$ implies that the equation

$$\Psi(\alpha;\pi)=\psi_-,\qquad a\in(0,1]$$

admits at most one solution. In addition, there exists a solution if and only if $\Psi(1;\pi)\leqslant\psi_-$. Since $\psi_-=\Psi(1;h_{II}(c_*))$, we conclude that if (3.12) admits a solution, then it is unique and $h_{II}(c_*)\geqslant\pi$.

This completes the proof of Theorem 3.

4. Asymptotic Behavior

In this section, we consider the asymptotic behavior, as $t\nearrow T$, of solutions to problem **(P)**, where $[0,T)$ is the maximum existence interval. We want to show that as $t\nearrow T$, $\Gamma(t)$ shrinks to a single point. In addition, after magnification, the solution approaches a self-similar profile. We focus on the formulation (2.4), so necessarily we need to assume that $\Gamma(0)$ is a graph and

$$0<\psi_\pm<\frac{\pi}{2}.$$

When the upper-half plane is replaced by a sector of open angle strictly less than π, the problem has been studied by Guo and Hu [28]. We shall use the techniques developed in [28] and also ideas from Grayson [27] and Angenent [7].

For notational simplicity, we denote

$$a(s)=\arctan s,\qquad \gamma_\pm=\tan\psi_\pm.$$

Thus, we consider, for unknown $(u, l_\pm, T)$, to

$$(4.1)\qquad \begin{cases} u_t = (a(u_x))_x, & x \in (l_-(t), l_+(t)), \quad t \in (0,T), \\ u(l_\pm(t),t) = 0, \quad u_x(l_\pm(t),t) = \mp\gamma_\pm, & t \in [0,T), \\ u(x,0) = u^0(x), & x \in [l_-(0), l_+(0)] := [l_-^0, l_+^0], \\ \lim_{t\nearrow T} \max_{x\in[l_-(t),l_+(t)]} |u_{xx}(x,t)| = \infty. \end{cases}$$

Without loss of generality, we assume that

$$(4.2)\qquad u^0 \in C^\infty([l_-^0, l_+^0]), \quad u^0(l_\pm^0) = 0, \quad u^0(\cdot) > 0 \text{ in } (l_-^0, l_+^0), \quad \mp u_x^0(l_\pm^0) = \gamma_\pm > 0.$$

We extend the problem to the case where a is a generic function satisfying

$$(4.3)\qquad a(\cdot) \in C^\infty(\mathbb{R}), \qquad a' > 0 \text{ in } \mathbb{R}, \qquad a(0) = 0.$$

In studying the asymptotic behavior of the solution, we need one of the following assumptions

$$(4.4)\qquad \text{(i) either } sa''(s) \leqslant 0 \quad \forall\, s \in \mathbb{R} \qquad \text{(ii) or } u_{xx}^0 \leqslant 0 \text{ in } (l_-^0, l_+^0).$$

Clearly, the first alternative is satisfied when $a(s) = \arctan s$.

We use the following notations, for each $t \in [0,T)$,

$$\begin{aligned} &I(t) := (l_-(t), l_+(t)), \quad \Gamma(t) := \{(x, u(x,t)) \mid x \in I(t)\}, \\ &\Omega(t) := \{(x,y) \mid x \in I(t), 0 < y < u(x,t)\}, \quad A(t) := |\Omega(t)| = \int_{I(t)} u(x,t)dx, \\ &Q(t) := \{(x,\tau) \mid x \in I(\tau), 0 < \tau < t\}. \end{aligned}$$

Since $u(l_\pm(t),t) = 0$, it is easy to calculate

$$\frac{d}{dt}A(t) = \int_{I(t)} u_t dx = a(u_x)\Big|_{l_-(t)}^{l_+(t)} = a(-\gamma_+) - a(\gamma_-) < 0.$$

Hence,

$$0 < T \leqslant T_{\max} := \frac{A(0)}{a(\gamma_-) - a(-\gamma_+)}, \qquad A(t) = [a(\gamma_-) - a(-\gamma_+)](T_{\max} - t).$$

4.1. **The Maximum Existence Interval.**

Theorem 5. *Assume that $a(\cdot)$ satisfies* (4.3) *and u^0 satisfies* (4.2). *Then* (4.1) *admits a unique solution. In addition, $T = T_{\max}$, and as $t \nearrow T$, $\Gamma(t)$ shrinks to a point.*

When $a(s) = \arctan s$, the existence of a unique solution is shown as in the previous section. For a generic a satisfying (4.3), the existence for $C^{1+\alpha}$ ($\alpha > 0$) initial data was shown in [13] by using a semi-group theory. Here for completeness, we provide an alternative proof.

Proof. The idea is to work on the function $w := u_x$. Fix a small $T_1 > 0$. For each pair of $C^{(2+\alpha)/2}([0,T_1])$ functions $l_\pm(\cdot)$ satisfying $l_+ > l_-$, define w as the unique solution to the (non-degenerate) porous medium problem

$$\begin{aligned} &w_t = (a(w))_{xx} \quad \text{in } Q(T_1) = \{(x,t) \mid x \in (l_-(t), l_+(t)), t \in (0,T_1)\}, \\ &w(l_\pm(t),t) = \mp\gamma_\pm \quad \forall\, t \in [0,T_1], \quad w(x,0) = u_x^0(x) \quad \forall\, x \in I(0). \end{aligned}$$

This problem has a unique solution $w \in C^{2+\alpha,(2+\alpha)/2}(\bar{Q}(T_1))$; see DiBenedetto [19]. Define

$$\tilde{l}_\pm(t) = l^0_\pm \pm \frac{a'(\mp\gamma_\pm)}{\gamma_\pm} \int_0^t w_x(l_\pm(\tau),\tau)d\tau \quad \forall\, t \in [0,T_1]. \tag{4.5}$$

Then $\tilde{l}^\pm \in C^{(3+\alpha)/2}([0,T_1])$, since $w_x \in C^{1+\alpha,(1+\alpha)/2}$. This improvement of regularity from $l_\pm \in C^{(2+\alpha)/2}$ to $\tilde{l}_\pm \in C^{(3+\alpha)/2}$ for the time evolution problem allows us to apply the Schauder's fixed point theorem to conclude that there is a fixed point $l_\pm = \tilde{l}_\pm$, provided that T_1 is suitably small. Regularity of the fixed point follows by a bootstrap argument. Uniqueness can be proven by showing that the map from $l_\pm$ to $\tilde{l}_\pm$ is a contraction (under a suitable norm). Existence on a maximal interval follows by a step by step extension.

Finally, u can be recovered from w by setting

$$u(x,t) = \int_{l_-(t)}^x w(\varsigma,t)\,d\varsigma \quad \forall\, x \in [l_-(t), l_+(t)],\ t \in [0,T_1].$$

From (4.5) with $\tilde{l}_\pm = l_\pm$ and the system satisfied by w, it is easy to verify that u solves (4.1).

We remark that weak solutions for the porous medium equation for w can be defined for L^∞ initial data, and if $|a(\pm\infty)| < \infty$, even for L^1 initial data. We shall not elaborate on this topic. We refer interested readers to the book of DiBenedetto [19].

To show that $T = T_{\max}$, we need only estimate the $C^2(I(t))$ norm of $u(\cdot,t)$, since we know from [13] that short time extension exists as long as $u(\cdot,t) \in C^{1+\alpha}$. The following two lemmas originate from [28]. For completeness, we shall repeat, with simplification, the proofs in [28].

The first lemma concerns the upper bound of u_{xx}.

Lemma 4.1. *Let*

$$M_1 := \max_{x\in\bar{I}(0)} |u^0_x(x)|, \qquad M_2 = \max_{x\in\bar{I}(0)} a(u^0_x(x))_x, \qquad M_0 = \max_{x\in\bar{I}(0)} \frac{1}{a'(u^0_x)}.$$

Then for every $t \in [0,T)$ and $x \in [l_-(t), l_+(t)]$,

$$|u_x(x,t)| \leqslant M_1, \quad u_t(x,t) \leqslant M_2, \quad u_{xx} \leqslant M_0M_2.$$

Proof. The function $w = u_x$ satisfies the parabolic equation $(w)_t = (a'(u_x)w_x)_x$. By the maximum principle, $\max_{Q(T)} |u_x| \leqslant \max\{M_1, \gamma_+, \gamma_-\} = M_1$.

Similarly, the function $w = u_t$ satisfies $w_t = (a'(u_x)w_x)_x$ in $Q(T)$ so the maximum and minimum of u_t are attained on the parabolic boundary. Initially, $u_t(x,0) = (a(u^0_x))_x \leqslant M_2$. To find boundary values of u_t, we differentiate $u_x(l_-(t),t) = \gamma_-$ and $u(l_-(t),t) = 0$ with respect to t to obtain $u_{xt} + \dot{l}_-u_{xx} = 0$ and $u_t + u_x\dot{l}_- = 0$. Hence, using $a'(u_x)u_{xx} = u_t$, we obtain

$$a'(u_x)u_{tx}\Big|_{x=l_-(t)} \quad = \quad -a'(u_x)u_{xx}\dot{l}_- = -\,u_t\dot{l}_- = (\dot{l}_-)^2u_x = \gamma_-\ \dot{l}^2_- \geqslant 0.$$

Similarly, one can show that $u_{tx}(l_+(t),t) \leqslant 0$. Thus, the maximum of u_t in $Q(T)$ is attained at the initial boundary, so $u_t \leqslant M_2$. Upon using $u_{xx} = u_t/a'(u_x)$, we also obtain the maximum estimate for u_{xx}. □

To find a lower bound for u_{xx}, we introduce notation $\ell(t)$, the width, and $h(t)$, the maximum height of $\Gamma(t)$:

$$\ell(t) := l_+(t) - l_-(t), \qquad h(t) := \max_{x\in I(t)} u(x,t) \quad \forall\, t \in [0,T).$$

Then we have the following lemma for a lower bound of u_{xx}.

Lemma 4.2. *There exists a constant C that depends only on u^0 such that for every $(x,t)\in Q(T)$,*

$$u_t(x,t)\geqslant -\frac{Ch(0)}{h(t)},\qquad u_{xx}(x,t)\geqslant -\frac{Ch(0)}{h(t)}.$$

Proof. Let ξ_0 and ξ^0 be the minimum and maximum roots to $u^0_x(\cdot)=0$ in $[l^0_-,l^0_+]$. Set

$$K_+:=\sup_{x\in(\xi_0,l^0_+)}\frac{a(u^0_x)-a(-\gamma_+)}{u^0(x)},\qquad K_-:=\sup_{x\in(l^0_-,\xi^0)}\frac{a(\gamma_-)-a(u^0_x)}{u^0(x)}.$$

Since $u^0_x(l^0_+)<0<u^0_x(l^0_-)$, both K_+ and K_- are bounded.

Fix an arbitrary $\tau\in(0,T)$. Consider the function

$$w(x,t)=\frac{K_+h(0)}{h(\tau)}u(x,t)-a(u_x(x,t))+a(-\gamma_+).$$

Using $u_t=a'(u_x)u_{xx}$ and $u_{xt}=[a(u_x)]_{xx}$ we derive that

$$w_t-a'(u_x)w_{xx}\quad=\quad\frac{K_+h(0)}{h(\tau)}\Big\{u_t-a'(u_x)u_{xx}\Big\}-a'(u_x)\Big\{[u_{xt}-[a(u_x)]_{xx}\Big\}=0.$$

Hence, w satisfies the maximum principle.

Next we find an appropriate parabolic domain for our application of the maximum principle. Since critical points of $u(\cdot,t)$ cannot be created in time evolution, roots to $u_x=0$ form continuous curves connected to $\bar I(0)\times\{0\}$. Hence, there exits a continuous function $\xi(\cdot)$ defined on $[0,\tau]$ such that $u(\xi(t),t)=h(t)$ and

$$\xi(t)\in(l_-(t),l_+(t)),\quad u_x(\xi(t),t)=0,\quad u_{xx}(\xi(t),t)\leqslant 0\quad \forall\, t\in[0,\tau].$$

As $u_t(x,t)|_{x=\xi(t)}=a'(u_x)u_{xx}\leqslant 0$, $u(\xi(t),t)$ is a decreasing function, so that $u(\xi(t),t)\geqslant h(\tau)$ for all $t\in[0,\tau]$. Now consider w on the parabolic domain $\{(x,t)\mid x\in[\xi(t),l_+(t)],t\in[0,\tau]\}$.

Initially, by the definition of K_+, $w(x,0)\geqslant 0$ for all $x\in[\xi(0),l_+(0)]\subset[\xi_0,l_+(0)]$;

On the boundary $x=\xi(t)$, $w\geqslant K_+h(0)+a(-\gamma_+)>0$;

On the boundary $x=l_+(t), w=0$.

Hence, $w\geqslant 0$ for all $x\in[\xi(t),l_+(t)],t\in[0,\tau]$. Consequently, as $w(l_+(t),t)=0$, we have

$$0\geqslant w_x(x,t)\Big|_{x=l_+(t)}=\frac{K_+h(0)}{h(\tau)}u_x-[a(u_x)]_x=-\frac{K_+h(0)\gamma_+}{h(\tau)}-u_t,$$

i.e.,

$$u_t(l_+(t),t)\geqslant-\frac{K_+h(0)\gamma_+}{h(\tau)}\qquad\forall\, t\in[0,\tau].$$

An analogous estimate can also be found at $x=l_-(t)$. As u_t in $Q(\tau)$ can only attain its minimum on the parabolic boundary, we see that for every $(x,t)\in\bar Q(\tau)$,

$$u_t(x,t)\geq-\frac{Ch(0)}{h(\tau)},\qquad C=\max\{M_2,K_+\gamma_+,K_-\gamma_-\}.$$

Setting $t=\tau$ we obtain the estimate for u_t. Using the equation $u_{xx}=u_t/a'(u_x)$, we also obtain the lower bound for u_{xx}. □

Proof of Theorem 5. So far we have $\|u_{xx}(\cdot,t)\|_{C^0(\bar I(t))}\leqslant C/h(t)$ for every $t\in[0,T)$. It follows from the short-time existence theorem that $\lim_{t\nearrow T}h(t)=0$.

Next we estimate the width $\ell(t)$. As $u(l_\pm(t),t)=0$ implies that $u_x(l_\pm,t)\dot l_\pm+u_t(l_\pm,t)=0$, we have $\dot l_\pm(t)=\pm u_t(l_\pm,t)/\gamma_\mp$. It follows from the upper bound of u_t that

$$\frac{d}{dt}\Big\{\frac{M_2t}{\gamma_-}+l_-(t)\Big\}=\frac{M_2}{\gamma_-}-\frac{u_t}{\gamma_-}\geqslant 0,\qquad \frac{d}{dt}\Big\{l_+(t)-\frac{M_2t}{\gamma_+}\Big\}=\frac{u_t}{\gamma_+}-\frac{M_2}{\gamma_+}\leqslant 0.$$

It then follows that $l_\pm(T) := \lim_{t\nearrow T} l_\pm(t)$ exist and are finite. As $\Omega(t) \subset [l_-(t), l_+(t)] \times [0, h(t)]$, we have $A(t) < [l_+(t) - l_-(t)]h(t)$ so that $\lim_{t\nearrow T} A(t) = 0$. Thus $T = T_{\max}$.

Finally, we show that $\Gamma(t)$ shrinks to a point, i.e., $l_+(T) = l_-(T)$. We use a contradiction argument. Suppose, on the contrary that $l_+(T) > l_-(T)$. The function

$$d(x,t) := \frac{a(u_x(x,t))}{u_x(x,t)} \quad \text{if} \quad u_x(x,t) \neq 0, \quad d(x,t) = a'(0) \qquad \text{otherwise}$$

is uniformly bounded from above and uniformly bounded away from zero. Hence, we can apply the Harnack inequality (cf. [35]) to $u_t = (d(x,t)u_x)_x$ to conclude that $u(x,T) > 0$ for all $x \in (l_-(T), l_+(T))$. This contradicts the known fact that $h(T) := \lim_{t\nearrow T} h(t) = 0$. Thus, $l_+(T) = l_-(T)$ and $\Gamma(t)$ approaches a single point as $t \nearrow T$. This completes the proof. □

4.2. **The Grim Reaper Solution.**

To study the asymptotic behavior, as $t \nearrow T_{\max}$, the essential difficulty is to exclude that the blow-up solution has the shape of a long needle. Indeed, in a related problem, Mantegazza, Novaga and Tortorelli proved in [39] that asymototically the solution is either a self-similar solution or a **grim reaper** solution of the form $y = \pi/2 - \arcsin(e^{-x})$ (see also [5]). Here, we shall provide a generalization for such a solution.

For each constant $c > 0$, there is a special solution of the form

$$l_+(t) = \infty, \quad l_-(t) = ct, \qquad u(x,t) = U(x - ct) \quad \forall\, x \in (ct, \infty), t \in \mathbb{R}.$$

Then the equation for the profile U is

$$-cU' = (a(U'))' \ \text{ in } \ (0,\infty), \qquad U(0) = 0, \quad U'(0) = \gamma > 0, \quad U' > 0 \text{ in } \ (0,\infty).$$

This gives $a(U') = a(\gamma) - cU$ and

$$U(\infty) =: m = \frac{a(\gamma)}{c}, \quad \text{or} \quad c = \frac{a(\gamma)}{m}.$$

Denote by a^{-1} the inverse function of $a(\cdot)$, we have $U' = a^{-1}(a(\gamma) - cU)$ so the profile U is implicitly given by

$$\xi \quad = \quad \int_0^{U(\xi)} \frac{ds}{a^{-1}(a(\gamma) - cs))} = \frac{1}{c}\int_{a(\gamma)-cU(\xi)}^{a(\gamma)} \frac{dv}{a^{-1}(v)} \quad \forall\, \xi \geqslant 0.$$

After substituting c by m, U can be expressed as the implicit function of

$$\xi = \frac{m}{a(\gamma)} \int_{a(\gamma)[1-U(\xi)/m]}^{a(\gamma)} \frac{dv}{a^{-1}(v)}.$$

In the special case of motion by curvature, i.e., $a(s) = \arctan s$, we have $a^{-1}(v) = \tan v$. Writing $\psi = a(\gamma) = \arctan\gamma$, we have

$$\ln \frac{\sin\psi}{\sin[\psi(1 - U(\xi)/m)]} = \frac{\xi\psi}{m}.$$

That is,

$$U(\xi) = m\Big\{1 - \frac{1}{\psi}\arcsin[e^{-\xi\psi/m}\sin\psi]\Big\}, \qquad \xi \geqslant 0.$$

Note that the original grim reaper solution [5] corresponds to the profile U with $c = 1$ and $m = \psi = \frac{\pi}{2}$. With the flexibility of m (or $c = a(\gamma)/m$), the grim reaper solutions can be used as supersolutions in many applications.

4.3. **Scaling Invariance.** The problem (4.1) is scaling invariant. Namely, for each $\lambda > 0$, define

$$u^\lambda(x,t) = \frac{u(\lambda x, T + \lambda^2(t-1))}{\lambda}, \quad l^\lambda_\pm(t) = \lambda l_\pm(T + \lambda^2(t-1)).$$

Then $(u^\lambda, l^\lambda_\pm)$ solves the same set of problem in time interval $[1 - T/\lambda^2, 1)$. Note that

$$\{(x, u^\lambda(x,0)) \mid x \in [l^\lambda_-(0), l^\lambda_+(0)]\} = \lambda\, \Gamma(T-\lambda^2) := \{(\lambda x, \lambda y) \mid (x,y) \in \Gamma(T-\lambda^2)\},$$
$$\Omega^\lambda(0) = \lambda\, \Omega(T-\lambda^2) := \{(\lambda x, \lambda y) \mid (x,y) \in \Omega(T-\lambda^2)\}.$$

By the above scaling, any time interval $[t_0, T)$ for solution u is transferred to the time interval $[0,1)$ for u^λ by taking $\lambda = \sqrt{T-t_0}$. Note that the constants in Lemma 4.1 become

$$M_0^\lambda \leqslant M_0, \qquad M_1^\lambda \leqslant M_1, \qquad M_2^\lambda \leqslant \lambda M_2 = \sqrt{T-t_0}\, M_2.$$

Thus, when t_0 is sufficiently close to T, M_2^λ with $\lambda = \sqrt{T-t_0}$ is very small; this is why the curve is asymptotically concave.

4.4. **Eventual Concavity.**

For a curvature flow of a closed curve, Angenent [7] provided an elegant method to show the eventual convexity of curves; see also the earlier result of Grayson [27]. As we are working on segments of curves, periodicity cannot be used. We can only use Angenent's method to obtain a partial result.

Lemma 4.3. *There exists $t_* \in [0,T)$ and $\xi \in C^1([t_*,T))$ such that for each $t \in [t_*,T)$,*

$$u_x(\cdot,t) > 0 \text{ in } [l_-(t), \xi(t)), \qquad u_x(\xi(t),t) = 0 > u_{xx}(\xi(t),t), \qquad u_x(\cdot,t) < 0 \text{ in } (\xi(t), l_+(t)].$$

In addition,

$$\dot{l}_-(t) > 0, \qquad \dot{l}_+(t) < 0 \quad \forall\, t \in [t_*,T).$$

Proof. The function $w = u_x$ satisfies $w_t = (a'(u_x)w_x)_x$ in $Q(T)$. As $w \neq 0$ on the parabolic boundary of $Q(T)$, the number of roots of $w(\cdot,t) = 0$ is finite and non-increasing in time. Passing each time at which there is a root of multiplicity $k > 1$, the number of roots decreases by at least $k-1$. Hence, there exists $t_0 > 0$ such that the number of roots to $u_x(\cdot,t) = 0$ in $\bar{I}(t)$ is independent of $t \in [t_0,T)$ and every root is simple. We claim that there is indeed exactly one root. Suppose for contradiction that this is not true. Then the function

$$h^*(t) := \min\Big\{u(x,t) \mid x \in \bar{I}(t), u_x(x,t) = 0 < u_{xx}(x,t)\Big\}$$

is well–defined and is continuous for all $t \in [t_0,T)$. Now from $u_t = a'(u_x)u_{xx}$, one sees that $h^*(t)$ is a strictly increasing function, so $h^*(t) \geqslant h^*(t_0)$ for all $t \geqslant t_0$. On the other hand, we know that $h^*(t) \leqslant h(t)$ and $\lim_{t \nearrow T} h(t) = 0$, so there is a contradiction. Thus, in $[t_0,T)$, $u_x(\cdot,t)$ changes sign only once.

In a similar manner, $w := u_t$ satisfies $w_t = (a'(u_x)w_x)_x$ in $Q(T)$ and the mixed type boundary condition $w_x = -[\dot{l}_\pm/a'(\mp\gamma_\pm)]w$, obtained by differentiating $u_x(l_\pm(t),t) = \gamma_\pm$ with respect to t and using $u_{xx} = w/a'(\mp\gamma_\pm)$. This implies that the roots to $u_t = 0$ does not increase with time, and passing each time at which there is a non-simple root the number of roots decreases at least by one. Note in particular that any root on the boundary is non-simple. Hence, there is a time, which we still denote by t_0, such that in $[t_0,T)$, the number of roots to $u_t(\cdot,t) = 0$ does not change with t and every root is simple. Hence, $u_t \neq 0$ on the boundary. As the solution shrinks to a single point, we must have $u_t < 0$ on the boundary. Thus, $\dot{l}_\pm = -u_t/u_x = \pm u_t/\gamma_\pm$ has the required sign for all $t \in [t_0, T)$. □

To continue, we need a refinement on the upper bound on u_{xx} given in Lemma 4.1.

Lemma 4.4. *Assume that either (i) $u^0_{xx} \leqslant 0$ on $I(0)$ or (ii) $sa''(s) \leqslant 0$ on $\mathbb{R}$. Then there exists a constant M and a time $t_* \in [0, T)$ such that*

$$u_{xx}(x,t) \leqslant Mu(x,t) \quad \forall\, x \in I(t), \quad t \in [t_*, T).$$

Proof. **(i)** Suppose $u^0_{xx} \leqslant 0$ on $\bar{I}(0)$. Then $u_t(\cdot, 0) = a'(u^0_x)u^0_{xx} \leqslant 0$ on $\bar{I}(0) \times \{0\}$. It then follows from the maximum principle that $u_t \leqslant 0$ in $Q(T)$. Consequently, $u_{xx} = u_t/a'(u_x) \leqslant 0$.

(ii) Suppose $sa''(s) \leqslant 0$ on $\mathbb{R}$. Let t_* be as in the previous lemma. If $u_t(\cdot, t^*) \leqslant 0$ on $I(t_*)$, then by (i), we have $u_t \leqslant 0$ in $Q(T) \setminus Q(t_*)$. Hence, consider the case that $u_t(\cdot, t_*)$ is not everywhere non-positive in $I(t_*)$. Let

$$M = \max_{x \in I(t_*)} \frac{u_t(x, t_*)}{u(x, t_*)}.$$

Consider the function $w := u_t - Mu$ on $Q(T) \setminus Q(t_*)$. By the definition of M, we have $w(\cdot, t_*) \leqslant 0$ on $I(t_*)$. Also, as $u_t(l_\pm(t), t) < 0$ for all $t \in [t_*, T)$, we have $w(l_\pm(t), t) < 0$ for all $t \in [t_*, T)$. Finally, we calculate

$$\begin{aligned}
& w_t - [a'(u_x)w_x]_x - M\frac{a''(u_x)u_x}{a'(u_x)} w \\
= \;& \Big\{u_{tt} - [a'(u_x)u_{tx}]_x\Big\} - M\Big\{u_t - [a'(u_x)u_x]_x\Big\} - M\frac{a''(u_x)u_x}{a'(u_x)}(u_t - Mu) \\
= \;& M^2 \frac{uu_x a''(u_x)}{a'(u_x)} \leqslant 0.
\end{aligned}$$

It then follows from the maximum principle that $w \leqslant 0$ in $Q(T) \setminus Q(t_*)$. This completes the proof. □

4.5. **The Aspect Ratio.**

In order to use a blow-up technique showing that $\Gamma(t)$ shrinks to a point in an asymptotically self-similar manner, we need to estimate the aspect ratio $h(t)/\ell(t)$. Since u is Lipschitz, the upper bound is trivial,

$$\frac{h(t)}{\ell(t)} \leqslant \|u_x(\cdot, t)\|_\infty \leqslant M_1.$$

The difficulty here is to find a positive lower bound for $h(t)/\ell(t)$; namely, to exclude the case that $\Gamma(t)$ evolves into long thin needle-like shape, such as that of a grim reaper solution [5, 39].

In [28], the curve $\Gamma(t)$ is restricted to a sector of open angle strictly less than π, which guarantees a positive lower bounded of $h(t)/\ell(t)$, since the graph $y = u(x,t)$ is asymptotically concave so it is asymptotically above a line segment with positive contact angles. This argument does not work in the current case since the line segment is reduced to the x-axis. Here we use a modification of Grayson's idea [27]. For convenience, we denote

$$e(t) := \frac{\ell(t)}{h(t)}, \qquad q(t) := \frac{\ell(t)^2}{A(t)}.$$

As explained by Grayson, finding an upper bound for $e(t)$ is equivalent to finding an upper bound for $q(t)$. Indeed, since $A(t) \leqslant h(t)\ell(t)$, we have

$$q(t) = \frac{\ell^2(t)}{A(t)} \geqslant \frac{\ell^2(t)}{h(t)\ell(t)} = e(t).$$

4.6. **The Grayson's Technique** [27]**.**

In the sequel, we show that $q(t)$ is uniformly bounded.

Step 1. This is a step that we specifically develop here to accommodate the Grayson's technique. It is vital to the application of Grayson's original ideas.

Let $t \in [t_*, T)$ be any fixed constant. Consider the function

$$v(x,t) = u(x,t) + \tfrac{1}{2}Mh(t)\,[x - l_-(t)]\,[l_+(t) - x].$$

By Lemma 4.4, we have

$$v_{xx} = u_{xx} - Mh(t) \leqslant u_{xx} - Mu(x,t) \leqslant 0.$$

Thus, $v(\cdot,t)$ is concave. We denote by $\tilde\Omega(t)$ the region bounded by $y = v(x,t)$ and the x-axis:

$$\tilde\Omega(t) := \{(x,y) \mid l_-(t) < x < l_+(t), 0 < y < v(x,t)\}.$$

Then

$$\tilde A(t) \quad := \quad |\tilde\Omega(t)| = \int_{I(t)} v(x,t)dt = A(t) + \tfrac{1}{6}Mh(t)\ell^3(t).$$

Since $v(\cdot,t)$ is concave, we have

$$\tilde A(t) \geqslant \tfrac{1}{2}\ell(t)\|v(\cdot,t)\|_{L^\infty} \geqslant \tfrac{1}{2}\ell(t)h(t).$$

It then follows that

$$A(t) \geqslant \tfrac{1}{2}h(t)\ell(t) - \tfrac{1}{6}Mh(t)\ell^3(t) = \tfrac{1}{2}h(t)\ell(t)\Big\{1 - \tfrac{1}{3}M\ell^2(t)\Big\}. \tag{4.6}$$

Hence

$$1 < \frac{\tilde A(t)}{A(t)} = \frac{A(t) + \frac{1}{6}Mh(t)\ell^3(t)}{A(t)} \leqslant 1 + \frac{M\ell^2(t)}{3 - M\ell^2(t)}.$$

Here the fundamental difference in using the estimation $u_{xx} \leqslant Mu$ from Lemma 4.4 and in using $u_{xx} \leqslant M_0M_2$ from Lemma 4.1 is profound. Since a priori it is unknown that the ratio $\ell^2(t)/h(t)$ is small, without the extra factor $h(t)$, the resulting estimate between A and $\tilde A$ is almost useless.

Step 2. Referring to Figure 3, set $q_0(t) = l_-(t)$ and $q_4(t) = l_+(t)$. Let $q_1(t), q_2(t), q_3(t)$ be real numbers such that $q_0 < q_1 < q_2 < q_3 < q_4$ and

$$\int_{q_0}^{q_1} v(x,t)dx = \int_{q_1}^{q_2} v(x,t)dx = \frac{\theta}{2}\tilde A(t), \qquad \int_{q_2}^{q_3} v(x,t)dx = \int_{q_3}^{q_4} v(x,t)dx = \frac{1-\theta}{2}\tilde A(t),$$

where

$$\theta := \frac{a(\gamma_-)}{a(\gamma_-) + |a(-\gamma_+)|} \in (0,1), \qquad 1 - \theta = \frac{|a(-\gamma_+)|}{a(\gamma_-) + |a(-\gamma_+)|}.$$

One of the most important ideas of Grayson [27] is to show that at time $t_1 \approx t + \frac{1}{2}\sqrt{T-t}$, $I(t_1)$ is within a tiny neighborhood of $[q_1(t), q_3(t)]$. Since the area from time t to time t_1 shrinks

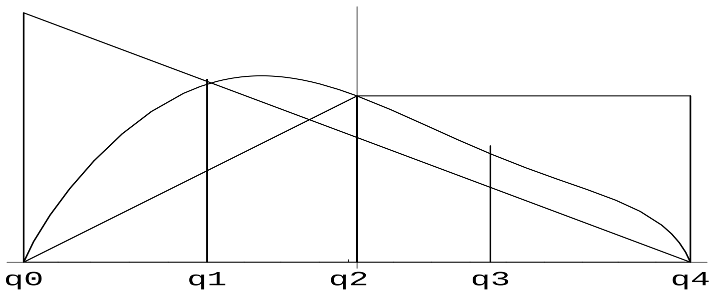

FIGURE 3. Division of $\tilde{\Omega}(t)$ into four pieces by vertical lines and comparison of areas by a rectangle, a triangle and trapezoids.

about by half, to improve the aspect ratio, one would like to show that if $e(t)$ is large, then $\ell^2(t_1)/\ell^2(t) < 1/2$. Hence, we would like to show $[q_3(t) - q_1(t)] < \frac{1}{\sqrt{2}}\ell(t)$.

Let $\xi \in I(t)$ be the number such that $v(\xi, t) = \max_{I(t)} v(\cdot, t)$. By symmetry, we assume without loss of generality that $\xi \in [q_0, q_2]$. From left to right, we denote by $\Omega_i = \Omega_i(t)$ the region bounded by $y = v(x, t)$, $y = 0$, $x = q_{i-1}$ and $x = q_i$. Also, we set $l_i := q_i - q_{i-1}$ the base width of each Ω_i.

(i) First of all, $v(\cdot, t)$ is decreasing in $[q_2, q_4]$. As the area of Ω_3 equals that of Ω_4, we have

$$\begin{aligned} &l_3 = q_3 - q_2 < l_4 = q_4 - q_3, \\ &(1-\theta)\tilde{A}(t) = |\Omega_3 \cup \Omega_4| \leqslant v(q_2, t)(q_4 - q_2). \end{aligned}$$

(ii) Since v is concave, the curve $y = v(x, t)$ is above the line $y = v(q_2, t)(x - q_0)/(q_2 - q_0)$ in $[q_0, q_2]$. Hence,

$$\tfrac{1}{2} v(q_2, t)(q_2 - q_0) \leqslant |\Omega_1 \cup \Omega_2| = \theta \tilde{A}(t).$$

Using the conclusion of (i), we obtain

$$\begin{aligned} & \tfrac{1}{2}(1-\theta) v(q_2, t)(q_2 - q_0) \leqslant (1-\theta)\, \theta\, \tilde{A}(t) \leqslant \theta v(q_2, t)(q_4 - q_2), \\ \Rightarrow \quad & \tfrac{1}{2}(1-\theta)\{\ell(t) - (q_4 - q_2)\} \leqslant \theta(q_4 - q_2) \\ \Rightarrow \quad & l_3 + l_4 = q_4 - q_2 \geqslant \frac{1-\theta}{1+\theta}\, \ell(t). \end{aligned}$$

(iii) Here again, as $v(\cdot, t)$ is concave, the curve $y = v(\cdot, t)$ is above the line $y = v(q_1, t)(q_4 - x)/(q_4 - q_1)$ in $[q_1, q_4]$ and below the same line in $[q_0, q_1]$. Hence, estimating the areas $|\Omega_1|$ and $|\Omega_2|$ by the corresponding trapezoids, we obtain

$$\frac{q_1 - q_0}{2}\left\{\frac{q_4 - q_0}{q_4 - q_1} v(q_1, t) + v(q_1, t)\right\} \geqslant |\Omega_1| = |\Omega_2| \geqslant \frac{q_2 - q_1}{2}\left\{v(q_1, t) + \frac{q_4 - q_2}{q_4 - q_1} v(q_1, t)\right\}.$$

Using $l_i = q_i - q_{i-1}$ and cancelling the common factor $\frac{1}{2}v(q_1, t)$, we obtain

$$\begin{aligned} & l_1 \frac{\{\ell(t) + \ell(t) - l_1\}}{\ell(t) - l_1} \geqslant \{\ell(t) - l_1 - [l_3 + l_4]\} \frac{\{\ell(t) - l_1 + [l_3 + l_4]\}}{\ell(t) - l_1} \\ \Rightarrow \quad & \ell^2(t) - \{\ell(t) - l_1\}^2 \geqslant \{\ell(t) - l_1\}^2 - (l_3 + l_4)^2 \\ \Rightarrow \quad & \ell(t) - l_1 \leqslant \frac{1}{\sqrt{2}} \sqrt{\ell^2(t) + (l_3 + l_4)^2}. \end{aligned}$$

(iv) In conclusion, we have

$$\begin{aligned} q_3 - q_1 &= \ell(t) - l_1 - l_4 \leqslant \frac{1}{\sqrt{2}}\sqrt{\ell^2(t) + (l_3+l_4)^2} - l_4 \\ &\leqslant \frac{1}{\sqrt{2}}\sqrt{\ell^2(t) + (l_3+l_4)^2} - \frac{1}{2}(l_3+l_4). \end{aligned}$$

Since the function $\frac{1}{\sqrt{2}}\sqrt{\ell^2+x^2} - x/2$ is decreasing in $x \in [0,\ell]$, using (ii) we then obtain

$$q_3(t) - q_1(t) \leqslant \frac{1}{\sqrt{2}}\sqrt{\ell^2(t)+x^2} - \tfrac{1}{2}x\Big|_{x=\frac{1-\theta}{1+\theta}\ell(t)} = \ell(t)\nu_1(\theta),$$

where

$$\nu_1(\theta) := \frac{\sqrt{1+\theta^2} - (1-\theta)/2}{1+\theta} \in \Big(\frac{1}{2}, \frac{1}{\sqrt{2}}\Big).$$

Similarly, if $\xi \in [q_2, q_4]$, then

$$q_3(t) - q_1(t) \leqslant \ell(t)\nu_1(1-\theta).$$

In conclusion, we have

$$q_3(t) - q_1(t) \leqslant \ell(t)\nu(\theta) \quad \forall\, t \in [t_*, T), \qquad \nu(\theta) := \min\{\nu_1(\theta), \nu_1(1-\theta)\} \in \Big(\frac{1}{2}, \frac{1}{\sqrt{2}}\Big). \tag{4.7}$$

Step 3. Now let's fix a $t_0 \in [t_*, T)$. For some $t_1 \in (t_0 + \frac{1}{2}[T-t_0], T)$ and a small positive ε to be determined later, set

$$\begin{aligned} \hat q_1 &:= \min\{q_1(t_0), l_+(t_1)\}, \\ \hat q_3 &:= \max\{l_-(t_1), q_3(t_0)\}, \\ \tau_1 &:= \max\{t \in [t_0, t_1] \mid l_-(t) \leqslant \hat q_1 - h(t_0)/\varepsilon\}, \\ \tau_2 &:= \max\{t \in [t_0, t_1] \mid l_+(t) \geqslant \hat q_3 + h(t_0)/\varepsilon\}. \end{aligned}$$

It is possible that τ_1 is not well-defined, in which case $l_-(t_0) > \hat q_1 - h(t_0)/\varepsilon$, so by monotonicity, $l_-(t_1) > l_-(t_0) > \hat q_1 - h(t_0)/\varepsilon$ which is what we want. It τ_1 is well-defined, we want to show that $\tau_1 < t_1$ so that $l_-(\tau_1) = \hat q_1 - h(t_0)/\varepsilon$ and we still have $l_-(t_1) > \hat q_1 - h(t_0)/\varepsilon$. Similarly, either τ_2 is not well-defined or $\tau_2 < t_1$ so that $l_+(t_1) < \hat q_3 + h(t_0)/\varepsilon$.

Hence, suppose that $\tau_1 > t_0$. Then $[\hat q_1 - h(t_0)/\varepsilon, \hat q_1] \subset [l_-(t), l_+(t)]$ for all $t \in [t_0, \tau_1]$. For every $t \in [t_0, \tau_1]$, by the mean value theorem, there exists $\hat x \in (\hat q_1 - h(t_0)/\varepsilon, \hat q_1)$ such that

$$u_x(\hat x, t) = \frac{u(\hat q_1, t) - u(\hat q_1 - h(t_0)/\varepsilon, t)}{\hat q_1 - [\hat q_1 - h(t_0)/\varepsilon]} \leqslant \frac{\varepsilon u(\hat q_1, t)}{h(t_0)} \leqslant \varepsilon.$$

Consequently, using $u_{xx} \leqslant Mu$ we have

$$u_x(\hat q_1, t) = u_x(\hat x, t) + \int_{\hat x}^{\hat q_1} u_{xx}(x,t)dx \leqslant \varepsilon + Mh(t)[\hat q_1 - \hat x] \leqslant \varepsilon + \frac{Mh^2(t_0)}{\varepsilon} \leqslant 2\varepsilon,$$

if ε is taken to satisfy

$$\varepsilon \geqslant \sqrt{M}h(t_0).$$

By choosing t_* larger, we may also assume that $2\sqrt{M}h(t_*) < \min\{\gamma_-, \gamma_+\}$. Henceforth we require

$$\sqrt{M}h(t_0) \leqslant \varepsilon < \min\{\gamma_-, \gamma_+\}/2.$$

Denote by $A_1(t)$ the area bounded by the curve $y = u(x,t)$, the x axis, and the vertical line $x = \hat{q}_1$. Then, for all $t \in [t_0, \tau_1)$, using $u(l_-(t), t) = 0$ and the fact that $\hat{q}_1$ is a constant, we have

$$\begin{aligned}\frac{d}{dt}A_1(t) &= \frac{d}{dt}\int_{l_-(t)}^{\hat{q}_1} u(x,t)\,dt \\ &= \int_{l_-(t)}^{\hat{q}_1} u_t(x,t)dx = a(u_x(\hat{q}_1,t)) - a(\gamma_-) \leqslant a(2\varepsilon) - a(\gamma_-).\end{aligned}$$

Hence

$$A_1(\tau_1) \leqslant A_1(t_0) - [a(\gamma_-) - a(2\varepsilon)](\tau_1 - t_0).$$

Observe that

$$\begin{aligned}A_1(t_0) &\leqslant |\Omega_1(t_0)| = \tfrac{1}{2}\theta|\tilde{A}(t_0)| \leqslant \tfrac{1}{2}\theta A(t_0)\Big\{1 + \frac{M\ell^2(t_0)}{3 - M\ell^2(t_0)}\Big\} \\ &= \frac{a(\gamma_-)(T-t_0)}{2}\Big\{1 + \frac{M\ell^2(t_0)}{3 - M\ell^2(t_0)}\Big\},\end{aligned}$$

by using the definition of θ and the fact that $A(t) = [a(\gamma_-) - a(-\gamma_+)](T-t)$. It then follows that

$$\tau_1 - t_0 \leqslant \frac{A_1(t_0)}{a(\gamma_-) - a(2\varepsilon)} \leqslant \frac{a(\gamma_-)}{2a(\gamma_-) - 2a(2\varepsilon)}\Big\{1 + \frac{M\ell^2(t_0)}{3 - M\ell^2(t_0)}\Big\}(T - t_0).$$

Similarly, if $\tau_2 > t_0$, then we can show that

$$\tau_2 - t_0 \leqslant \frac{|a(-\gamma_+)|}{2|a(-\gamma_+)| - 2|a(-2\varepsilon)|}\Big\{1 + \frac{M\ell^2(t_0)}{3 - M\ell^2(t_0)}\Big\}(T - t_0).$$

Now set

$$\begin{aligned}t_1 &:= t_0 + \tfrac{1}{2}(T-t_0)\max\left\{\varepsilon + \frac{a(\gamma_-)\{1 + \frac{M\ell^2(t_0)}{3-M\ell^2(t_0)}\}}{a(\gamma_-) - a(2\varepsilon)}, \varepsilon + \frac{|a(-\gamma_+)|\{1 + \frac{M\ell^2(t_0)}{3-M\ell^2(t_0)}\}}{|a(-\gamma_+)| - |a(-2\varepsilon)|}\right\} \\ &= t_0 + \tfrac{1}{2}(T-t_0)[1 + O(\varepsilon + \ell^2(t_0))].\end{aligned}$$

Then we must have both $\tau_1 < t_1$ and $\tau_2 < t_1$. Therefore, we have

$$l_-(t_1) \geqslant \hat{q}_1 - h(t_0)/\varepsilon, \qquad l_+(t_1) \leqslant \hat{q}_3 + h(t_0)/\varepsilon.$$

Now if $\hat{q}_1 < q_1(t_0)$, then by definition of $\hat{q}_1$, we must have $\hat{q}_1 = l_+(t_1)$ so the first estimate implies $\ell(t_1) \leqslant h(t_0)/\varepsilon$. Similarly, if $\hat{q}_3 > q_3(t_0)$, then $\hat{q}_3 = l_-(t_1)$ and we still have $\ell(t_1) \leqslant h(t_0)/\varepsilon$. Finally, when $\hat{q}_1 = q_1(t_0)$ and $\hat{q}_3 = q_3(t_0)$, using (4.7) we have

$$\ell(t_1) \leqslant \hat{q}_3 - \hat{q}_1 + 2h(t_0)/\varepsilon = q_3(t_0) - q_1(t_0) + 2h(t_0)/\varepsilon \leqslant \ell(t_0)\nu(\theta) + 2h(t_0)/\varepsilon.$$

Hence, in any case we have $\ell(t_1) \leqslant \ell(t_0)\nu(\theta) + 2h(t_0)/\varepsilon$ and the ratio can be calculated as

$$\begin{aligned}\frac{q(t_1)}{q(t_0)} = \frac{A(t_0)}{A(t_1)}\frac{\ell^2(t_1)}{\ell^2(t_0)} &\leqslant \frac{T-t_0}{T-t_1}\left(\nu(\theta) + \frac{2h(t_0)}{\varepsilon\ell(t_0)}\right)^2. \\ &= \frac{2}{1 + O(\varepsilon + \ell^2(t_0))}\left(\nu(\theta) + \frac{2h(t_0)}{\varepsilon\ell(t_0)}\right)^2.\end{aligned}$$

The above estimate indeed holds also with t_1 replaced by $t_0 + (T - t_0)/2$. To see this, we introduce a parameter $k \in [0, \infty)$ and redefine q_1 and q_3 in terms of k by the relation

$$\tilde{A}_2(t_0) = (1+k)\tilde{A}_1(t_0), \quad \tilde{A}_3(t_0) = (1+k)\tilde{A}_4(t_0), \quad \tilde{A}_i(t) := \int_{q_{i-1}}^{q_i} v(x,t)dx.$$

As before, the point q_2 is also defined so that

$$\int_{q_0}^{q_2} v(x,t)dx = \theta\tilde{A}(t).$$

Then the quantity $q_3 - q_1$ differs from the original estimate by a factor of $1 + O(k)$. By taking $k = K[\varepsilon + M\ell^2(t_0)]$ for some large fixed constant K, we can have $t_1 = t_0 + (T - t_0)/2$. We leave the reevaluation of the preceding estimates to the readers.

Hence, there exists a positive constant K that is independent of ε and t_0 such that

$$\frac{q(t_0 + \frac{1}{2}[T - t_0])}{q(t_0)} \leqslant 2\nu^2(\theta) + K\Big\{\varepsilon + \ell^2(t_0) + \frac{2h(t_0)}{\varepsilon^2\ell(t_0)}\Big\} \quad \forall\, t_0 \in [t_*, T).$$

Here we used the fact that $h(t_0)/\ell(t_0)$ is bounded by M_1 so $(\frac{h}{\varepsilon\ell})^2 \leqslant M_1\frac{h}{\varepsilon^2\ell}$.

Finally, we fix

$$\varepsilon = \frac{1 - 2\nu^2(\theta)}{3K}.$$

Take t_* sufficiently close to T such that

$$\ell^2(t) \leqslant \frac{1 - 2\nu^2(\theta)}{3K} \quad \forall t \in [t_*, T].$$

We then obtain

$$\frac{q(t_0 + \frac{1}{2}[T - t_0])}{q(t_0)} \leqslant \frac{2 + 2\nu^2(\theta)}{3} + \frac{K_0 h(t_0)}{\ell(t_0)} \quad \forall\, t_0 \in [t_*, T)$$

for some positive constant K_0 independent of t_0. As $A(t) \geqslant h(t)\ell(t)$,

$$\frac{K_0 h(t)}{\ell(t)} = \frac{K_0 h(t)\ell(t)}{\ell^2(t)} \leqslant \frac{K_0 A(t)}{\ell^2(t)} = \frac{K_0}{q(t)} \quad \forall\, t \in [t_*, T).$$

It then follows that

$$\frac{q(t_0 + \frac{1}{2}[T - t_0])}{q(t_0)} \leqslant \frac{2 + 2\nu^2(\theta)}{3} + \frac{K_0}{q(t_0)} \quad \forall\, t_0 \in [t_*, T).$$

That is,

$$q(t_0 + \tfrac{1}{2}[T - t_0]) \leqslant \frac{2 + 2\nu^2(\theta)}{3} q(t_0) + K_0 \qquad \forall\, t_0 \in [t_*, T). \tag{4.8}$$

Step 4. Now we are ready to show that the quantity $q(t) := \ell^2(t)/A(t)$ is uniformly bounded. For this, we set

$$\tau_0 = t_*, \qquad \tau_j = T - \frac{T - t_*}{2^j} \quad \forall\, j = 1, 2, \cdots.$$

Then, by (4.8), we have

$$q(\tau_{j+1}) \leqslant \frac{2 + 2\nu^2(\theta)}{3} q(\tau_j) + K_0, \quad j = 0, 1, \cdots.$$

By a mathematical induction, one sees that

$$\sup_{j \geqslant 0} q(\tau_j) \leqslant \max\Big\{q(t_*), \frac{3K_0}{1 - 2\nu^2(\theta)}\Big\}.$$

Finally, using the monotonicity of $\ell(t)$, we have

$$q(t) \leqslant 2q(\tau_j) \qquad \forall\, t \in [\tau_j, \tau_{j+1}].$$

Therefore, $q(t)$ is uniformly bounded.

Note that

$$M_1\ell^2(t) \geqslant h(t)\ell(t) \geqslant A(t) = [a(\gamma_-) + |a(-\gamma_+)|](T-t) = \frac{\ell^2(t)}{q(t)} \geqslant \frac{\ell^2(t)}{\|q\|_\infty}.$$

We then conclude the following key estimate:

Lemma 4.5. *There exists a constant $C > 0$ such that*

$$\sqrt{T-t} \leqslant C\ell(t) \leqslant C^2 h(t) \leqslant C^3\sqrt{T-t} \quad \forall\, t \in [0,T).$$

By translation, we may assume that $l_\pm(T) := \lim_{t\nearrow T} l_\pm(T) = 0$. Since $\dot{l}_+(t) < 0 < \dot{l}_-(t)$ for $t \in [t_*, T]$, we have

$$|l_\pm(t)| < \ell(t) \leqslant C\sqrt{T-t} \quad \forall t \in [t_*, T).$$

4.7. **The Self-similar Transformation.**

Now we make the change of dependent and independent variables:

$$\begin{aligned} & z = \frac{x}{\sqrt{2(T-t)}}, \qquad s = -\ln\sqrt{2(T-t)}, \\ & U(z,s) := e^s u(ze^{-s}, T - \tfrac{1}{2}e^{-2s}), \qquad L_\pm(s) = e^s\, l_\pm(T - \tfrac{1}{2}e^{-2s}), \\ \Longleftrightarrow \quad & u(x,t) = \sqrt{2(T-t)}\, U\Big(\frac{x}{\sqrt{2(T-t)}}, -\ln\sqrt{2(T-t)}\Big). \end{aligned}$$

Set $s_0 = -\ln[\sqrt{2T}]$. Then the functions $(U, L_\pm)$ satisfies

$$\begin{cases} U_s = [a(U_z)]_z - zU_z + U, & z \in (L_-(s), L_+(s)),\ s > s_0, \\ U(L_\pm(s), s) = 0, & s > s_0, \\ U_z(L_\pm(s), s) = \mp\gamma_\pm, & s > s_0. \end{cases}$$

By the estimates established so far, we have

$$\begin{aligned} 0 \leqslant U(z,s) &= \frac{u(x,t)}{\sqrt{2(T-t)}} \leqslant C, \\ |U_z(z,s)| &= |u_x(x,t)| \leqslant M_1, \\ |U_{zz}(z,s)| &= |\sqrt{2(T-t)}u_{xx}| \leqslant C\frac{\sqrt{2(T-t)}}{h(t)} \leqslant C, \\ |L_\pm(s)| &= \frac{|l_\pm(t)|}{\sqrt{2(T-t)}} \leqslant C. \end{aligned}$$

Using these estimates and applying a standard blow-up technique used in [28], we can derive the convergence to the self-similar profile as $t \nearrow T$. The key is to find a Lyapunov functional (see also [49]). Since the proof is rather standard, we safely omit it here.

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