Boundedness of global solutions of a supercritical parabolic equation*

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Abstract

We study radially symmetric classical solutions of the Dirichlet problem for a heat equation with a supercritical nonlinear source. We give a sufficient condition under which blow-up in infinite time cannot occur. This condition involves only the growth rate of the source term at infinity. We do not need the homogeneity property which played a key role in previous proofs of similar results. We also establish the blow-up rate for a class of solutions which blow up in finite time.

Key words and phrases. global solutions, supercritical parabolic equation.

1 Introduction

Consider the heat equation with a superlinear source

$$\begin{cases} u_t = \Delta u + f(u), & x \in \Omega, \quad t > 0, \\ u = D \geq 0, & x \in \partial\Omega, \ t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbf{R}^n$, D a constant, and $u_0 \in C(\bar{\Omega})$. Here by a **superlinear source** we mean that for some positive constants ε and A,

$$uf(u) \geq (2+\varepsilon) \int_0^u f(v)\, dv > 0 \qquad \forall\, u \geq A.$$

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This condition guarantees that there are initial data u_0 such that u becomes unbounded. There are two different ways how a solution may become unbounded:

Blow-up (in finite time): There is a finite positive T such that $\lim_{t\nearrow T}\|u(\cdot,t)\|_{L^\infty(\Omega)}=\infty$;

Grow-up (in infinite time): $u\in C(\bar\Omega\times[0,\infty))$, $\quad \limsup_{t\nearrow\infty}\|u(\cdot,t)\|_{L^\infty(\Omega)}=\infty$.

It is shown in [4] that grow-up does not occur if the growth of f is **subcritical**:

(1) $n=1$, f is locally Lipschitz;

(2) $n=2$, $|f'(u)|\le e^{u^q}$ for all $u\ge 0$ and some constant $q\in(0,2)$;

(3) $n\ge 3$, f is locally Lipschitz and $f(u)\le u^p$ for all $u\ge 1$ and some $p\in(1,p_S)$, where

$$p_S:=\frac{n+2}{n-2}\,;$$

see also [1].

For critical/supercritical growth, so far all results have been restricted to radially symmetric solutions and to two particular nonlinearities $f(u)=u^p$ and $f(u)=e^u$. Assume that

$$\Omega=B(R):=\{x\in\mathbf{R}^n \mid |x|<R\},\qquad u_0(x)=U_0(|x|). \tag{1.2}$$

It is known that grow-up does not occur in the following two **supercritical** cases:

(4) $f(u)=e^u,\quad 3\le n\le 9,\quad D=0$ (see [5]);

(5) $f(u)=u^p,\quad n\ge 3,\quad p>p_S,\quad D=0$ (see [7, 12]).

On the other hand, grow-up does occur in the following critical/supercritical cases:

(a) $f(u)=u^p,\quad n\ge 3,\quad p=p_S,\quad D=0$ (cf. [7]);

(b) $f(u)=u^p,\quad n\ge 11,\quad p>1+\frac{4}{n-4-2\sqrt{n-1}},\quad D=R^{-\frac{2}{p-1}}\left[\frac{2[(n-2)p-n]}{(p-1)^2}\right]^{\frac{1}{p-1}}$ (cf. [3]);

(c) $f(u)=2(n-2)e^u,\quad n\ge 10,\quad R=1,\quad D=0$ (cf. [10, 15]).

These results in [5, 7, 12] are built upon a scaling invariance of the problem: for each $\lambda>0$, the equation $u_t-\Delta u=f(u)$ is unchanged under the variable change

$$u^\lambda(x,t):=\begin{cases}\lambda^{2/(p-1)}u(\lambda x,\lambda^2 t) & \text{if } f(s)=s^p,\\ 2\lambda+u(e^\lambda x,e^{2\lambda}t) & \text{if } f(s)=e^s.\end{cases}$$

Here we are interested mainly in the question **what happens if the scaling invariance is removed.** For example, what happens when

$$f(u)=u^p+\varepsilon_1+\varepsilon_2\sin e^{u^2}+\varepsilon_3 u^q,\qquad \varepsilon_1\ge\varepsilon_2\ge 0,\quad \varepsilon_3\ge 0,\quad p>q.$$

We shall assume the following:

$$\textbf{(A)}\quad f\in C^1,\quad f(\cdot)\ge 0 \text{ in } [0,\infty),\quad \lim_{u\to\infty}u^{-p}f(u)\;=\;1,\quad n\ge 3,\quad p_S<p<p_{JL}.$$

Here p_{JL}, referred to as the Joseph-Lundgren exponent [9], is defined by

$$p_{JL}:=\begin{cases}1+\frac{4}{n-4-2\sqrt{n-1}} & \text{if } n\ge 11,\\ \infty & \text{if } n\le 10.\end{cases}$$

We shall prove the following.

Theorem 1 *Assume* **(A)**. *There is no grow-up for* (1.1)-(1.2)*; that is, any radially symmetric global classical solution to* (1.1) *is uniformly bounded.*

The proof of Theorem 1 can be modified to yield a result on the blow-up rate:

Theorem 2 *Assume* **(A)**. *If u is a solution of* (1.1)-(1.2) *which blows up in a finite time T and satisfies $u(0,t) = \max_\Omega u(\cdot,t)$ for all t close to T, then*

$$\limsup_{t\nearrow T}\ (T-t)^{1/(p-1)}\|u(\cdot,t)\|_{L^\infty(\Omega)} < \infty. \tag{1.3}$$

It was shown in [11] that (1.3) holds for $D=0$, $f(u)=|u|^{p-1}u$, $p_S<p<p_{JL}$ and all radially symmetric (even sign-changing) solutions. On the other hand, it is well-known (cf. [8, 13]) that there are positive radially decreasing solutions of

$$u_t = \Delta u + u^p, \qquad x\in \mathbf{R}^N, \quad p > p_{JL},$$

such that

$$\limsup_{t\nearrow T}\ (T-t)^{1/(p-1)}u(0,t) = \infty.$$

As we have already mentioned, many results on blow-up of solutions of (1.1) are derived under the assumption that $f(u)$ behaves like u^p near zero (which happens to be the same power u^p near infinity). From the physical point of view, this assumption should not play any role in such results because near blow-up the solution is large and the behavior of $f(u)$ for small values of u should not have any influence on blow-up.

On the other hand, extensions of blow-up results to more general nonlinearities, e.g., those that do not have any scaling invariance, may be challenging mathematically. In particular, when $p>p_S$, the lack of the imbedding $W^{1,2}\subset L^{p+1}$ makes it difficult to use functional analytic methods, whereas other methods often rely heavily on the specific form of the nonlinearity, cf. [6, 7, 8, 11, 12, 13], for instance.

The aim of this paper is to extend results obtained previously for $f(u)=u^p$ to a large class of nonlinearities which may not have any scaling invariance, by a relatively simple argument based on intersection properties of solutions.

Remarks.

1. Theorem 1 and the energy identity

$$\frac{d}{dt}\int_\Omega\Big\{\frac{1}{2}|\nabla u|^2 - F(u)\Big\}dx + \int_\Omega u_t^2 dx = 0, \qquad F(u) := \int_0^u f(s)ds,$$

imply that if u is a global classical solution to (1.1)-(1.2), then the ω-limit set consists of classical solutions to

$$\Delta v + f(v) = 0 \quad \text{in } \Omega, \qquad v = D \text{ on } \partial\Omega. \tag{1.4}$$

Using radial symmetry and non-increase of the zero-number (cf. [2]) one can further show that

$$v := \lim_{t\to\infty} u(\cdot,t) \quad \text{in} \;\; C^2(\bar\Omega)$$

where v is a solution of (1.4).

2. Suppose every maximal classical solution to (1.4) is stable. Then Theorem 1 also implies the existence of global non-classical solutions to (1.1)-(1.2), i.e., the existence of solutions that blow up in finite time but still can be extended beyond blow-up time. Indeed, let $\zeta \in C^2(\bar\Omega)$ be any non-negative non-trivial radially symmetric function satisfying $\zeta = 0$ on $\partial\Omega$. For each $h > 0$, let u^h be the solution to (1.1) with $u_0 = D + h\zeta$. One can show that u^h is strictly monotone in h, that for small positive h, u^h is a global classical solution, and that for large positive h, u^h blows up in a finite time T^h. Let $h^* = \sup\{h > 0 \mid u^h \in C(\bar\Omega \times [0,\infty))\}$. Then the function $u^* = \lim_{h\nearrow h^*} u^h$ is a global non-classical solution to (1.1) with initial data $u_0 = D + h^*\zeta$. In fact, if u^* is classical, then $v = \lim_{t\to\infty} u^*(\cdot,t)$ is a classical solution to (1.4). From which, one can show that u^h is also a global classical solution to (1.1) when h is slightly bigger than h^*, which is impossible. See [7, 10, 14] for more details. One observes that in particular, as $h \searrow h^*$, $T^h \nearrow T^{h^*} < \infty$.

3. In Theorem 2, we conjecture that the condition

$$u(0,t) = \max_{\bar\Omega}\{u(\cdot,t)\} \quad \text{for all } t \text{ close to } T \tag{1.5}$$

can be removed, based on the following evidence. (a) Let $\hat M(t) = \max\{u(\cdot,t)\} = u(\hat R(t),t)$. Our proof is still valid if the condition (1.5) is replaced by $\liminf_{t\nearrow T} \; u(0,t)/\hat M(t) > 0$ or by $\limsup_{t\nearrow T} \; \hat R(t)/\sqrt{T-t} < \infty$. (b) If $\liminf_{t\nearrow T} \; \hat R(t)/\sqrt{T-t} = \infty$, then blow-up should be similar to one-dimensional blow-up in which case any exponent p is subcritical and therefore the assertion of Theorem 2 is also true. We leave such an important relaxation as an open problem. Note that the restriction (1.5) is not needed in the case $f(u) = u^p$ (cf. [11]).

In the next section, we establish the existence of a singular steady state of (1.1)-(1.2). Using this singular steady state and an intersection-comparison argument, we then prove Theorems 1 and 2 in Section 3.

2 Singular Steady States

Here we consider solutions to

$$v_{rr} + \frac{n-1}{r} v_r + f(v) = 0, \quad v_r \le 0 < v \qquad \text{in} \;\; (0,\epsilon) \tag{2.1}$$

where $n \ge 3$ and ϵ is a small positive constant. We study the existence of a singular solution.

Lemma 2.1 *Suppose $f \in C(\mathbf{R})$ and $\lim_{u\to\infty} u^{-p} f(u) = 1$ where $p > p_S$ and $n \ge 3$. Then for a small positive ϵ, (2.1) admits a singular solution $v = v^*$ satisfying*

$$\lim_{r\searrow 0} \; r^m v^*(r) = \gamma, \qquad m := \frac{2}{p-1}, \quad \gamma := \left[\frac{2[(n-2)p-n]}{(p-1)^2}\right]^{\frac{1}{p-1}}.$$

Proof. Set $s = \ln r$ and $W = r^m v - \gamma$. Then v is a singular solution to (2.1) if and only if

$$\ddot{W} + \alpha \dot{W} + \beta W = h(s, W) \quad \text{in} \quad (-\infty, \ln \epsilon), \qquad \lim_{s \to -\infty} W(s) = 0, \tag{2.2}$$

where $\alpha = \frac{n-2}{p-1}(p - \frac{n+2}{n-2})$, $\beta = \frac{2(n-2)}{p-1}(p - \frac{n}{n-2})$, and $h = h_1 + h_2$ with

$$h_1 := \gamma^p + p\gamma^{p-1}W - [\gamma + W]^p, \qquad h_2 := [\gamma + W]^p - e^{mps} f(e^{-ms}[\gamma + W]).$$

Denote by λ_+ and λ_- the two roots to the characteristic equation

$$\lambda^2 + \alpha\lambda + \beta = 0, \quad \text{i.e.} \quad \lambda_\pm = \frac{1}{2}(-\alpha \pm \sqrt{\alpha^2 - 4\beta}). \tag{2.3}$$

Though $\lambda_\pm$ may not be real, their real parts are always negative. By a variation of constants,

$$W(s) \quad = \quad \int_{-\infty}^{s} \frac{e^{\lambda_+(s-\tau)} - e^{\lambda_-(s-\tau)}}{\lambda_+ - \lambda_-} h(\tau, W(\tau)) d\tau, \quad s \in (-\infty, \ln \epsilon]. \tag{2.4}$$

Here we use the notation $\frac{e^{\lambda_+ z} - e^{\lambda_- z}}{\lambda_+ - \lambda_-} := z e^{-\alpha z/2}$ when $\lambda_+ = \lambda_-$.

If one is willing to assume that $f \in C^1(\mathbf{R})$ and $f'(u) \sim p u^{p-1}$ as $u \to \infty$, the existence of a unique solution to (2.4) follows from a standard Picard's iteration.

Under the assumptions of the Lemma, we can establish the existence of a solution to this integral equation by using Schauder's fixed point theorem. For this, we set

$$\mathcal{X} := \Big\{ \Phi \in C((-\infty, \ln \epsilon]) \;\Big|\; \|\phi\|_{\mathcal{X}} := \sup\nolimits_{s \in (-\infty, \ln \epsilon]} |\Phi(s)| < \infty \Big\},$$
$$\mathcal{B}(\delta) := \Big\{ \Phi \in \mathcal{X} \;\Big|\; \|\Phi\|_{\mathcal{X}} \leq \delta \Big\},$$

where $\epsilon > 0$ and $\delta \in (0, \gamma/4]$ are small constants to be chosen later. Note that $\mathcal{B}(\delta)$ is a closed convex subset of the Banach space $\mathcal{X}$. For each $\Phi \in \mathcal{B}(\delta)$, define, for $i = 1, 2$,

$$\mathbf{T}_i \Phi(s) = \int_{-\infty}^{s} \frac{e^{\lambda_+(s-\tau)} - e^{\lambda_-(s-\tau)}}{\lambda_+ - \lambda_-} h_i(\tau, \Phi(\tau)) d\tau \qquad \forall\, s \leq \ln \epsilon.$$

Note that W is a solution to (2.4) if $W = (\mathbf{T}_1 + \mathbf{T}_2)W$ or $W = (\mathbf{I} - \mathbf{T}_1)^{-1}\mathbf{T}_2 W$ where $\mathbf{I}$ denotes the identity operator. We shall show that $\mathbf{I} - \mathbf{T}_1$ is invertible and $(\mathbf{I} - \mathbf{T}_1)^{-1}\mathbf{T}_2$ has a fixed point.

First consider $\mathbf{T}_1$. Note that $\mathbf{T}_1 \mathbf{0} = \mathbf{0}$. Also, when $|w_1| + |w_2| \leq 2\delta$,

$$|h_1(s, w_1) - h_1(s, w_2)| \leq p(p-1) \max\{[\gamma - 2\delta]^{p-2}, [\gamma + 2\delta]^{p-2}\} 2\delta\, |w_1 - w_2|.$$

This implies that when δ is small, $\|\mathbf{T}_1\Phi_1 - \mathbf{T}_1\Phi_2\|_{\mathcal{X}} \leq \frac{1}{2}\|\Phi_1 - \Phi_2\|_{\mathcal{X}}$ for every $\Phi_1, \Phi_2 \in \mathcal{B}(\delta)$. We now fix such a small $\delta \in (0, \gamma/4]$. Then from $\mathcal{B}(\delta/2)$ to $\mathcal{X}$, $(\mathbf{I} - \mathbf{T}_1)^{-1}$ exists and

$$\|(\mathbf{I} - \mathbf{T}_1)^{-1}\Psi_1 - (\mathbf{I} - \mathbf{T}_1)^{-1}\Psi_2\|_{\mathcal{X}} \leq 2\|\Psi_1 - \Psi_2\|_{\mathcal{X}} \qquad \forall\, \Psi_1, \Psi_2 \in \mathcal{B}(\delta/2).$$

Next consider $\mathbf{T}_2$. Define

$$\omega(s) := \sup_{v \geq \gamma e^{-ms}/2} \left| \frac{f(v)}{v^p} - 1 \right|.$$

Note that $\omega(\cdot)$ is non-negative and non-decreasing. For any $\Phi \in \mathcal{B}(\delta)$ and $\tau \le s \le \ln\epsilon$,

$$|h_2(\tau,\Phi(\tau))| = (\gamma+\Phi)^p \Big|\frac{f(e^{-m\tau}(\gamma+\Phi))}{[e^{-m\tau}(\gamma+\Phi)]^p} - 1\Big| \le (2\gamma)^p\omega(s),$$
$$|\mathbf{T}_2\Phi(s)| \le \omega(s)\int_0^\infty \frac{(2\gamma)^p|e^{\lambda_+\tau} - e^{\lambda_-\tau}|}{|\lambda_+ - \lambda_-|}d\tau =: K_1\omega(s),$$
$$\Big|\frac{d}{ds}\mathbf{T}_2\Phi(s)\Big| \le \omega(s)\int_0^\infty \frac{(2\gamma)^p|\lambda_+e^{\lambda_+\tau} - \lambda_-e^{\lambda_-\tau}|}{|\lambda_+ - \lambda_1|}d\tau =: K_2\omega(s).$$

Thus,

$$\mathbf{T}_2\mathcal{B}(\delta) \subset \hat{\mathcal{B}} = \Big\{\Phi \in \mathcal{X} \;\Big|\; |\Phi(s)| + |\Phi'(s)| \le (K_1+K_2)\omega(s) \quad \forall\, s \le \ln\epsilon\Big\}.$$

Since $f \in C(\mathbf{R})$ and $\lim_{s\to-\infty}\omega(s) = 0$, one sees that $\mathbf{T}_2$ is continuous. Fix $\epsilon > 0$ such that $\omega(\ln\epsilon) \le \delta/[2(K_1+K_2)]$. Then $\hat{\mathcal{B}}$ is a compact subset of $\mathcal{B}(\delta/2)$, so that $\mathbf{T}_2$ is a continuous compact operator from $\mathcal{B}(\delta)$ to $\mathcal{B}(\delta/2)$. Consequently, $(\mathbf{I} - \mathbf{T}_1)^{-1}\mathbf{T}_2$ is a continuous compact operator from $\mathcal{B}(\delta)$ to itself, so by Schauder's fixed point theorem it has a fixed point $W \in \mathcal{B}(\delta)$. One can further show that $|W(s)| \le 2K_1\omega(s)$ for all $s \le \ln\epsilon$, so that W is a solution of (2.2). □

Proposition 1 *Assume $n \ge 3$ and $p \in (p_S, p_{JL})$. Then there is a unique solution ϕ to*

$$\phi_{rr} + \frac{n-1}{r}\phi_r + \phi^p = 0 \quad on \;\; (0,\infty), \quad \phi_r(0) = 0, \quad \phi(0) = 1. \tag{2.5}$$

The solution satisfies $\phi > 0 > \phi_r$ in $(0,\infty)$ and for $\phi^*(r) := \gamma\, r^{-m}$ with $m = 2/(p-1)$, there are infinitely many roots to the equation $\phi - \phi^* = 0$ in $(0,\infty)$.

This is a well-known result (cf. [17]). We only sketch the proof for reader's convenience.

Note that when $f(u) \equiv u^p$, (2.2) is autonomous since $h_2 \equiv 0$. It has two equilibria $W \equiv -\gamma$ and $W \equiv 0$, with a unique trajectory connecting them. In terms of (2.1) with $f = v^p$, there are the singular solution $v = \gamma r^{-m}$ and the regular solution $v = \phi$ to (2.5). All other solutions are regular, given by $v = a\phi(a^{1/m}r)$, $a \in [0,\infty)$.

Finally, notice that when $p \in (p_S, p_{JL})$, the characteristic roots $\lambda_\pm$ to (2.3) are complex with negative real parts; namely, $(0,0)$ is a focus on the $(W,\dot{W})$ phase plane. It then follows that $\phi - \phi^* = 0$ has infinitely many roots in $(0,\infty)$.

3 Proof of the Main Results

Before the proof of Theorem 1 we introduce the following definition:

Definition 3.1 Let I be an interval (open, half-open or closed) with endpoints a, b. Let ψ be a continuous function on I. We define the **zero number** (**number of sign changes**) of ψ by

$$\mathcal{Z}_I(\psi) = \sup\Big\{n \in \mathbf{N} \;\Big|\; \exists\, a < x_0 < x_1 < \ldots < x_n < b,\;\; \psi(x_i)\psi(x_{i+1}) < 0 \;\forall\, i = 0,\cdots,n-1\Big\}$$

if ψ changes sign in I and $\mathcal{Z}_I(\psi) = 0$ otherwise.

In the sequel, $B = B(R)$ and u is a radially symmetric solution to (1.1)-(1.2), which, by the maximum principle, is non-negative. We let v^* be the singular solution to (2.1), extended to its maximum existence interval $(0, \epsilon^*]$ where either $\epsilon^* = \infty$ or $v^*(\epsilon^*) = 0$. We set $R^* = \min\{\epsilon^*, R\}$.

We identify $u(x,t)$ and $u(r,t)$, where $r = |x|$. We first recall a classical result on monotonicity of solutions (see [2, 16]):

Proposition 2 *Suppose u is a global radially symmetric non-negative classical non-stationary solution to* (1.1)-(1.2). *Then there exists a constant $T_1 \geq 0$ such that*

(i) $\mathcal{Z}_{[0,R]}(u_t(\cdot,t))$ *is a constant independent of* $t \geq T_1$*;*

(ii) *either* $u_t(0,\cdot) > 0$ *on* $[T_1,\infty)$ *or* $u_t(0,\cdot) < 0$ *on* $[T_1,\infty)$*;*

(iii) $u_r \leq 0$ *on* $[0,R] \times [T_1,\infty)$*;*

(iv) $\mathcal{Z}_{[0,R]}(u(\cdot,t) - v^*(\cdot)) =: N^*$ *is a constant independent of* $t \geq T_1$.

To prove Theorem 1, we use a contradiction argument. Suppose, on the contrary, that there is a grow-up solution; i.e., there exists a solution $u \in C(\bar{B} \times [0,\infty))$ to (1.1)-(1.2) satisfying

$$\limsup_{t\to\infty} \|u(\cdot,t)\|_{L^\infty(B)} = \infty.$$

By a time translation, we can assume that all assertions of Proposition 2 hold with $T_1 = 0$.

1. First we claim that

$$\liminf_{t\to\infty} \frac{\dot{u}_t(0,t)}{u^p(0,t)} = 0.$$

Indeed, if the assertion is not true, then there exists $\delta > 0$ such that $u_t(0,t) \geq \delta u^p(0,t)$ for all $t \geq 0$, which implies that $u(0,\cdot)$ blows up in finite time since $p > 1$.

2. Now let $\{t_i\}$ be a sequence in $(0,\infty)$ such that

$$\lim_{i\to\infty} u(0,t_i) = \infty, \qquad \lim_{i\to\infty} \frac{u_t(0,t_i)}{u^p(0,t_i)} = 0.$$

Set

$$M_i = u(0,t_i), \quad R_i = M_i^{-1/m}, \qquad w_i(\rho,\tau) = R_i^m\, u(R_i\rho, t_i + R_i^2\tau).$$

Then

$$w_{i\tau} - \Delta w_i = w_i^p g_i \qquad \text{in } \ B(R/R_i) \times (-t_i/R_i^2, \tau^*],$$

where

$$g_i = \frac{f(M_i w_i)}{(M_i w_i)^p} = \frac{f(u)}{u^p}, \quad \tau^* := 1/[2(p-1)g^*], \quad g^* = \sup_{s\geq 1} \frac{f(s)}{s^p}.$$

First observe that

$$w_{i\tau}(0,0) = \frac{R_i^2}{M_i}\, u_t(0,t_i) = \frac{u_t(0,t_i)}{u^p(0,t_i)} \to 0 \text{ as } \ i \to \infty.$$

Next observe that $w_i \le 1$ for all $\tau \le 0$. Also, comparing w_i with the solution to the ode $\dot{M} = g^* M^p, M(0) = 1$, we see that

$$\max_{B(R/R_i)} w_i(\cdot,\tau) \le [1-(p-1)g^*\tau]^{-1/(p-1)} \le 2^{1/(p-1)} \qquad \forall\, \tau \in [0,\tau^*].$$

It then follows by a standard parabolic estimate that $\{w_i\}$ is a locally uniformly bounded family in $C^{2,1}$, and, along a subsequence, it converges uniformly in any compact subset of $\mathbf{R}^n \times (-\infty,\tau^*]$ to a non-negative radially symmetric limit w satisfying

$$w_\tau - \Delta w = w^p \quad \text{in } \ \mathbf{R}^n \times (-\infty,\tau^*], \qquad w(0,0)=1, \quad w_\tau(0,0)=0.$$

Here the $\lim_{i\to\infty} w_i^p g_i = w^p$ is obtained as follows: Fix any $(\rho,\tau) \in [0,\infty)\times(-\infty,\tau^*]$: (a) If $w(\rho,\tau) > 0$, then, at (ρ,τ),

$$\lim_{j\to\infty} g_j = \lim_{j\to\infty} \frac{f(M_j w_j)}{(M_j w_j)^p} = \lim_{s\to\infty} \frac{f(s)}{s^p} = 1.$$

(b) If $w(\rho,\tau)=0$, (ρ,τ) is a local minimum of w so that at (ρ,τ), $w_\tau - \Delta w \le 0$. This implies that $0 \le \lim_{i\to\infty} w_i^p g_i = w_\tau - \Delta w = 0$ at (ρ,τ).

3. We claim that $w_\tau \equiv 0$. Suppose not. Then $\mathcal{Z}_{[0,\infty)}(w_\tau(\cdot,\tau))$ decreases at least by 1 when τ passes by 0. Indeed, since $\mathcal{Z}_{[0,\infty)}(w_\tau(\cdot,0)) < \infty$ by Proposition 2(i), there is $\delta > 0$ such that either $w_\tau(\cdot,0) > 0$ in $(0,\delta]$ or $w_\tau(\cdot,0) < 0$ in $(0,\delta]$. Then there is $\tau_1 > 0$ such that either $w_\tau(\delta,\cdot) > 0$ in $[-\tau_1,\tau_1]$ or $w_\tau(\delta,\cdot) < 0$ in $[-\tau_1,\tau_1]$. Hence it follows from the strong maximum principle that $\mathcal{Z}_{[0,\delta]}(w_\tau(\cdot,\tau_1)) = 0$. But, $\mathcal{Z}_{[0,\delta]}(w_\tau(\cdot,-\tau_1)) \ge 1$. Consequently, $\mathcal{Z}_{[0,\infty)}(u_t(\cdot,t))$ decreases at least by one near $t = t_j$ for every sufficiently large j (in the subsequence), which is impossible. Thus, $w_\tau \equiv 0$; namely, $w(\cdot,\tau) = \phi(\cdot)$ for all τ, where ϕ is the unique solution to (2.5).

We know that, for $\phi^*(\rho) = \gamma\rho^{-m}$, $\mathcal{Z}_{[0,\infty)}(\phi-\phi^*) = \infty$, so that $\mathcal{Z}_{[0,\rho^*]}(\phi-\phi^*) = N^*+1$ for some $\rho^* > 0$. Also, we know that

$$1 = \lim_{r\searrow 0} \frac{r^m v^*(r)}{\gamma} = \lim_{\tilde{r}\searrow 0} \frac{\tilde{r}^m v^*(\tilde{r}\rho)}{\phi^*(\rho)} \qquad \text{uniformly in } \rho \in (0,\rho^*].$$

Since all zeros of $\phi - \phi^* = 0$ are non-degenerate and $\lim_{i\to\infty} w_i(\cdot,0) = \phi$ in $C^1([0,\rho^*])$, we see that, setting $v_i^*(\rho) = R_i^m v^*(R_i\rho)$,

$$\liminf_{i\to\infty} \mathcal{Z}_{[0,\rho^*]}(w_i - v_i^*) \ge \mathcal{Z}_{[0,\rho^*]}(\phi-\phi^*) = N^*+1.$$

Transferring back to the original variables, we see that for all $i \gg 1$,

$$\mathcal{Z}_{[0,\rho^* R_i]}(u(\cdot,t_i) - v^*(\cdot)) \ge N^*+1,$$

contradicting (iv) of Proposition 2. This contradiction shows that $\sup_{t>0} \|u(\cdot,t)\|_{L^\infty(B)} < \infty$. The assertion of Theorem 1 thus follows. $\square$

Proof of Theorem 2. We use a similar idea as above. Suppose u is a solution to (1.1)-(1.2) that blows up at a finite time T and satisfies $u(0,t) = \max_B u(\cdot,t)$ for all t close to T. Note that $u_t(0,t) > 0$ near $t = T$. The assertions of Proposition 2 can be modified as follows:

Both $\mathcal{Z}_{[0,R]}(u_t(\cdot,t))$ and $\mathcal{Z}_{[0,R]}(u(\cdot,t)-v^*)$ are non-increasing in $t \in [0,T)$ so that they are constants for all $t \in [T_1,T)$ for some $T_1 \in [0,T)$.

By assumption, $u(0,t) = \max_B u(\cdot,t)$ for all $t \in [T_1,T)$ and $\lim_{t \nearrow T} u(0,t) = \infty$. Set

$$M(t) = u(0,t), \quad \delta = \liminf_{t \nearrow T} \frac{u_t(0,t)}{u^p(0,t)} = \liminf_{t \nearrow T} \frac{\dot{M}(t)}{M^p(t)}.$$

We claim that $\delta > 0$. Suppose not. Then along a sequence $\{t_i\}$ that approaches monotonically to T as $i \to \infty$, there holds $\lim_{i\to\infty} \frac{\dot{M}(t_i)}{M^p(t_i)} = 0$. Defining w_i and following line by line the arguments in the previous proof, we obtain a contradiction. This contradiction shows that $\delta > 0$. Consequently there exists $T_2 \in [T_1,T)$ such that

$$\frac{\dot{M}(t)}{M^p(t)} \geq \frac{\delta}{2} \qquad \forall t \in [T_2,T).$$

Integrating this inequality over (t,T) gives $M(t)(T-t)^{1/(p-1)} \leq [(p-1)\delta/2]^{1/(1-p)}$ for all $t \in [T_2,T)$. The assertion of Theorem 2 follows.

References

[1] T. Cazenave and P.L. Lions, Solutions globales d'équations de la chaleur semilinéaires, *Comm. Partial Diff. Equations* **9** (1984), 955–978.

[2] X.-Y. Chen and P. Poláčik, Asymptotic periodicity of positive solutions of reaction-diffusion equations on a ball, *J. Reine Angew. Math.* **472** (1996), 17–51.

[3] J.W. Dold, V.A. Galaktionov, A.A. Lacey and J.L. Vázquez, Rate of approach to a singular steady state in quasilinear reaction-diffusion equations, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **26** (1998), 663–687.

[4] M. Fila, Boundedness of global solutions of nonlinear diffusion equations, *J. Diff. Equations* **98** (1992), 226–240.

[5] M. Fila and P. Poláčik, Global solutions of a semilinear parabolic equation, *Adv. Diff. Equations* **4** (1999), 163–196.

[6] A. Friedman, J.B. McLeod, Blow-up of positive solutions of semilinear heat equations, *Indiana Univ. Math. J.* **34** (1985), 425–447.

[7] V.A. Galaktionov and J.L. Vázquez, Continuation of blow-up solutions of nonlinear heat equations in several space dimensions, *Comm. Pure Appl. Math.* **50** (1997), 1–67.

[8] M.A. Herrero and J.J.L. Velázquez, A blow up result for semilinear heat equations in the supercritical case, preprint.

[9] D.D Joseph and T.S. Lundgren, Quasilinear Dirichlet problems driven by positive sources, *Arch. Rat. Mech. Anal.* **49** (1972), 241–269.

[10] A.A. Lacey and D. Tzanetis, Global existence and convergence to a singular steady state for a semilinear heat equation, *Proc. Roy. Soc. Edinburgh Sect. A* **105** (1987), 289–305.

[11] H. Matano and F. Merle, On non-existence of type II blow-up for a supercritical nonlinear heat equation, *Comm. Pure Appl. Math.* **57** (2004), 1494–1541.

[12] N. Mizoguchi, Boundedness of global solutions for a supercritical semilinear heat equation and its application, *Indiana Univ. Math. J.* **54** (2005), 1047–1059.

[13] N. Mizoguchi, Type II blowup for a semilinear heat equation, *Adv. Diff. Equations*, **9** (2004), 1279–1316.

[14] W.-M. Ni, P.E. Sacks and J. Tavantzis, On the asymptotic behavior of solutions of certain quasilinear parabolic equations, *J. Diff. Equations* **54** (1984), 97–120.

[15] I. Peral and J.L. Vazquez, On the stability or instability of the singular solution of the semilinear heat equation with exponential reaction term, *Arch. Rat. Mech. Anal.* **129** (1995), 201–224.

[16] W.-M. Ni and P. Sacks, The number of peaks of positive solutions of semilinear parabolic equations, *SIAM J. Math. Anal.* **16** (1985), 460–471.

[17] X. Wang, On the Cauchy problem for reaction-diffusion equations, *Trans. Amer. Math. Soc.* **337** (1993), 549–590.