WAVE PROPAGATION FOR A TWO-COMPONENT LATTICE DYNAMICAL SYSTEM ARISING IN STRONG COMPETITION MODELS

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ABSTRACT. We study a Lokta-Volterra type competition system with bistable nonlinearity in which the habitat is divided into discrete niches. We show that there exist non-monotone stationary solutions when the migration coefficients are sufficiently small. Also, we prove that the propagation failure phenomenon occurs. Finally, we focus on the traveling wave with nonzero wave speed. By investigating the asymptotic behavior of tails of wave profiles, we show that nonzero speed wave profiles are monotone. Moreover, the nonzero wave speed is unique in the sense that the wave cannot propagate with two different nonzero wave speeds.

1. INTRODUCTION

This work is devoted to the study of the following lattice dynamical system with Lotka-Volterra type nonlinearity

$$\begin{cases} \dfrac{du_j}{dt} = d_1(u_{j+1}+u_{j-1}-2u_j)+r_1u_j(1-b_1u_j-kv_j),\ j\in\mathbb{Z}, \\ \dfrac{dv_j}{dt} = d_2(v_{j+1}+v_{j-1}-2v_j)+r_2v_j(1-b_2v_j-hu_j),\ j\in\mathbb{Z}, \end{cases} \tag{1.1}$$

where b_i, d_i, r_i, $i=1,2$, h and k are some positive constants. In mathematical ecology, this model describes that two species u and v living in a discrete habitat compete each other. The quantities $u_j(t)$ and $v_j(t)$ stand for the populations of two species at time t and position j, respectively; r_i is the net birth rate, d_i is the migration coefficient, and $1/b_i$ is the carrying capacity of species i for $i=1,2$. Here the index $i=1$ corresponds to species $u:=\{u_j\}_{j\in\mathbb{Z}}$ while the index $i=2$ is referred to the species $v:=\{v_j\}_{j\in\mathbb{Z}}$. Moreover, the parameters h,k are competition coefficients of u,v respectively.

To model biological problems, lattice dynamics have been extensively used, for example, see the books [8, 23, 21] or the survey paper [3]. It is interesting to understand that under what conditions one species will survive and the other will die out, or both species will coexist. The purpose of this paper is to study the case when both species can survive. It

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is known that the existence of stationary solutions, i.e., $du_j/dt = dv_j/dt = 0$ for all j, is relevant to the coexistence of two species. Since we are concerned about how the migration and competition coefficients influence the existence of the stationary solutions of (1.1), we shall assume without loss of generality that $r_i = b_i = 1$, $i = 1, 2$. Therefore, (1.1) is reduced to the system

$$\frac{du_j}{dt} = d_1(u_{j+1} + u_{j-1} - 2u_j) + u_j(1 - u_j - kv_j),\ j \in \mathbb{Z}, \tag{1.2}$$

$$\frac{dv_j}{dt} = d_2(v_{j+1} + v_{j-1} - 2v_j) + v_j(1 - v_j - hu_j),\ j \in \mathbb{Z}. \tag{1.3}$$

Note that our analysis works well even if r_i and b_i, $i = 1, 2$, are not equal to 1.

In this article, we shall focus on the strong competition case with bistable nonlinearity, i.e., $h, k > 1$. A sufficient condition for the existence of stationary solutions of (1.2)-(1.3) will be provided later. Here $\{(u_j, v_j)\}$ is a stationary solution of (1.2)-(1.3) if $\{(u_j, v_j)\}$ satisfies

$$0 = d_1(u_{j+1} + u_{j-1} - 2u_j) + f(u_j, v_j),\ j \in \mathbb{Z}, \tag{1.4}$$

$$0 = d_2(v_{j+1} + v_{j-1} - 2v_j) + g(u_j, v_j),\ j \in \mathbb{Z}, \tag{1.5}$$

where $f(u, v) := u(1 - u - kv)$ and $g(u, v) := v(1 - v - hu)$.

For one component lattice dynamical systems with bistable nonlinearity, it is shown in [16] that a weak coupling (or small migration coefficient) implies the existence of stationary solutions. This also gives a propagation failure phenomenon. See also [18] and [2]. For multiple component lattice dynamical systems, the authors of [17] showed steady states can be continued to steady states in weak coupling by using the Implicit Function Theorem that is a different approach from [16]. In [22], under some conditions, the author also proved that there exist time-independent solutions in the spatial disorder of coupled discrete nonlinear Schrödinger equations with piecewise-monotone nonlinearities. In contrast to the lattice dynamical system (1.2)-(1.3), positive stationary solutions of Lotka-Volterra competition PDE (partial differential equation) models have been studied extensively. We refer to [6, 1, 19, 14] and the references cited therein.

Besides the stationary solutions, traveling wave solution is also an important object to understand the competition mechanism. Recall a traveling wave solution of (1.2)-(1.3) has the form $(u_j(t), v_j(t)) = (U(\xi), V(\xi))$, where $\xi := j + ct$. Here $c \in \mathbb{R}$ is called the wave speed and U, V are wave profiles. For the existence and uniqueness of traveling wave solution of Lotka-Volterra lattice dynamical system with monostable nonlinearity, we refer to [10]. There are many works in corresponding PDE models, for example, see [25, 9, 7, 11, 15, 13] and the references cited therein.

We now describe the main results of this paper as follows.

Firstly, we establish the propagation failure phenomenon for the system (1.2)-(1.3) when the migration coefficients are sufficiently small. For related results in this direction, we refer the reader to, for example, [16, 17, 18, 2].

Theorem 1. *Given $h, k > 1$. When d_1 and d_2 are small enough, there is no traveling wavefront solution of* (1.2)-(1.3) *with nonzero speed connecting* $(0, 1)$ *and* $(1, 0)$.

Set two rectangles

$$I_1 := [0, x_1] \times [y_1, 1] \text{ and } I_2 := [x_2, 1] \times [0, y_2], \tag{1.6}$$

where $y_1 \in (\max\{1/2, 1/k\}, 1)$, $x_2 \in (\max\{1/2, 1/h\}, 1)$ such that

$$(2u - 1)(2v - 1) + hu(2u - 1) + kv(2v - 1) > 0 \quad \text{for } (u, v) \in I_1 \cup I_2. \tag{1.7}$$

Due to the restrictions of y_1 and x_2, (1.7) holds as long as $0 < x_1, y_2 \ll 1$. Moreover, we fix x_1 and y_2 satisfying

$$x_1 < \frac{1 - y_1}{h} \text{ and } y_2 < \frac{1 - x_2}{k}. \tag{1.8}$$

The choices of I_1, I_2 with the restrictions (1.7)-(1.8) are to guarantee the existence of invariant sets used in the proof of Theorem 1. Moreover, the inequality (1.7) is also critical in the construction of a suitable mapping Φ defined in §3. This mapping is used for the derivation of the existence of stationary solutions as described in the following theorem. Roughly speaking, we prove that stationary solutions of (1.2)-(1.3) exist when the coupling is sufficiently weak. For related results, we refer to, for example, [16, 17].

Theorem 2. *Given $h, k > 1$. Then there are infinitely many solutions of* (1.4)-(1.5)*, provided d_1 and d_2 are small enough. Indeed, if d_1 and d_2 are small enough, then* (1.4)-(1.5) *has a unique solution $\{(u_j, v_j)\}_{j\in\mathbb{Z}}$ such that $(u_j, v_j) \in I_{s_j}$ for all $j \in \mathbb{Z}$ for any given infinite sequence $\{s_j\}_{j\in\mathbb{Z}}$ with $s_j \in \{1, 2\}$ for all $j \in \mathbb{Z}$, where I_1 and I_2 are chosen so that the conditions* (1.7) *and* (1.8) *hold.*

This theorem tells us that there are infinitely many non-monotone solutions of (1.4)-(1.5). Besides, we can see the profiles of stationary solutions. Given a sequence $\{s_j\}_{j\in\mathbb{Z}}$, for example, $s_1 = 1$, then the corresponding solution $\{(u_j, v_j)\}_{j\in\mathbb{Z}}$ satisfies $u_1 \in [0, x_1]$ and $v_1 \in [y_1, 1]$. Since $x_1 < y_1$, this also tells us that in the position $j = 1$, the population of the species v is much more than the other species u. From the biological point of view, the solutions we constructed in this theorem has the property that if one species likes to stay in the niches j, then the other species will not like to stay there.

Finally, we focus on the traveling wave with nonzero wave speed. Let us recall the general system:

$$\begin{cases} \dfrac{du_j}{dt} = d_1(u_{j+1} + u_{j-1} - 2u_j) + r_1 u_j(1 - b_1 u_j - a_1 v_j), \ j \in \mathbb{Z}, \\ \dfrac{dv_j}{dt} = d_2(v_{j+1} + v_{j-1} - 2v_j) + r_2 v_j(1 - b_2 v_j - a_2 u_j), \ j \in \mathbb{Z}, \end{cases}$$

By the transformation

$$d_1 t \to t, \quad b_1 u_j \to u_j, \quad b_2 v_j \to v_j,$$

and by letting

$$a = r_1/d_1, \quad b = r_2/d_1, \quad d = d_2/d_1, \quad k = a_1/b_2, \ h = a_2/b_1,$$

the system is reduced to the following system

$$\begin{cases} \dfrac{du_j}{dt} = (u_{j+1} + u_{j-1} - 2u_j) + a u_j(1 - u_j - k v_j), \\ \dfrac{dv_j}{dt} = d(v_{j+1} + v_{j-1} - 2v_j) + b v_j(1 - v_j - h u_j). \end{cases} \tag{1.9}$$

Then a traveling wave solution (c, U, V) of (1.9) satisfies the problem (P):

$$cU'(\xi) = D_2[U(\xi)] + aU(\xi)[1 - U(\xi) - kV(\xi)], \ \xi \in \mathbb{R}, \tag{1.10}$$

$$cV'(\xi) = dD_2[V(\xi)] + bV(\xi)[1 - V(\xi) - hU(\xi)], \ \xi \in \mathbb{R}, \tag{1.11}$$

$$(U, V)(-\infty) = (0, 1), \ (U, V)(+\infty) = (1, 0), \tag{1.12}$$

$$0 \le U, V \le 1 \text{ on } \mathbb{R}, \tag{1.13}$$

where $D_2[w(\xi)] := w(\xi + 1) + w(\xi - 1) - 2w(\xi)$ for $w = U, V$.

By investigating the asymptotic behavior of tails of wave profiles, we have the monotonicity of wave profiles as follows.

Theorem 3. *Given $h, k > 1$ and $a, b, d > 0$. The wave profiles of any solution (c, U, V) of (P) with nonzero speed are strictly monotone, i.e., $U' > 0$ and $V' < 0$ in $\mathbb{R}$.*

Moreover, the nonzero wave speed is unique in the following sense.

Theorem 4. *Given $h, k > 1$ and $a, b, d > 0$. Let (c_i, U_i, V_i), $i = 1, 2$, be two arbitrary solutions of (P) with nonzero speeds. Then $c_1 = c_2$.*

We now describe the main ideas of proofs of the above results and the organization of this paper as follows. In next section, Theorem 1 will be proved by constructing two invariant sets and using the comparison principle. Although a similar result to Theorem 2 can be found in [17], our proof (based on the Smale horseshoe theory [24]) is different from the approach of MacKay and Sepulchre [17]. Moreover, our proof gives us more information on the behavior of stationary solutions. In § 3, we shall use some ideas from [16] and [22] to prove two so-called *Conley-Morse conditions* such that the horseshoe theory can be applied (cf. [20, 26])

and so that Theorem 2 can be proved. In § 4, we study the asymptotic behavior of wave tails of traveling wave solutions with nonzero speed. Besides a key lemma (Lemma 4.2 below) which is similar to [10, Lemma 3.4], we shall use a different method from the one used in [10] (for monostable case) to derive the asymptotic behavior of wave tails of traveling waves solutions in bistable case (Propositions 4.1 and 4.6). The main idea of this method is to construct some auxiliary functions to compare with the wave profiles. Such idea is from [4, section 5]. Using the asymptotic behaviors of wave tails we show that all wave profiles with nonzero speed are strictly monotone by applying the sliding method of [5]. Also, motivated by [11], we shall prove Theorem 4 by using the information of wave tails.

2. Propagation failure

We study in this section the propagation failure phenomenon for the competition model (1.2)-(1.3). Here propagation failure means that (1.2)-(1.3) have no traveling wavefront solution with *nonzero* speed. We remark that, in [16], propagation failure is meant by the existence of infinitely many stationary solutions which block solutions from propagating. When $d_1, d_2 \ll 1$, the species almost do not have migration tendencies. Intuitively, the phenomenon of propagation failure occurs.

The idea of proof is quite simple, as in [16], due to the comparison principle, we shall show that I_i defined in (1.6), $i = 1, 2$, such that (1.7) and (1.8) hold, are invariant sets in the following sense: if $\{(u_j(t), v_j(t))\}_{j\in\mathbb{Z}}$ is a solution of (1.2)-(1.3) with initial data $\{(u_j(0), v_j(0))\}_{j\in\mathbb{Z}}$ such that $(u_J(0), v_J(0))$ falls in I_i for some $J \in \mathbb{Z}$, then $(u_J(t), v_J(t))$ always stays in I_i for all $t \geq 0$. Since I_1 and I_2 are disjoint, this leads to the non-existence of traveling wavefront with nonzero speed.

Proof of Theorem 1. Let $\{(u_j(t), v_j(t))\}_{j\in\mathbb{Z}}$ be a solution of (1.2)-(1.3) for $t \geq 0$ with $0 \leq u_j(0), v_j(0) \leq 1$ for all j. By the comparison principle, we have $0 \leq u_j(t), v_j(t) \leq 1$ for all $t \geq 0$. Recall

$$I_1 = [0, x_1] \times [y_1, 1], \quad I_2 = [x_2, 1] \times [0, y_2]$$

and (1.7)-(1.8). We now claim that I_1 is an invariant set in the above sense.

Suppose that $(u_J(0), v_J(0)) \in I_1$ for some $J \in \mathbb{Z}$. We claim that $(u_J(t), v_J(t))$ stays in I_1 for all $t \geq 0$. For the u-component, there exists $l > 0$ such that $f(u, v) < -l < 0$ for all $(u, v) \in [x_1/2, x_1] \times [y_1, 1]$. If $u_J(t) \in [x_1/2, x_1]$ for a certain time t, then

$$u_J'(t) \leq 2d_1[1 - u_J(t)] + f(u_J(t), v_J(t)) < 2d_1[1 - u_J(t)] - l \leq 0$$

as long as $d_1 \leq l/[2(1 - x_1/2)]$. This implies that $u_J(t)$ stays in $[0, x_1]$ for all $t \geq 0$, if $d_1 \in (0, l/\{2(1 - x_1/2)\}]$.

We now turn to the v-component. Note that $(1-hx_1+y_1)/2 > y_1$, since $1-hx_1 > y_1$. Then there is $m>0$ such that

$$g(u,v) > m \text{ for all } (u,v) \in [0,x_1]\times[y_1,(1-hx_1+y_1)/2].$$

If $v_J(t) \in [y_1,(1-hx_1+y_1)/2]$ for some $t \geq 0$, then

$$v_J'(t) \geq -2d_2 v_J(t) + g(u_J(t),v_J(t)) > -2d_2 v_J(t) + m \geq 0$$

as long as $d_2 \leq m/(1-hx_1+y_1)$. We conclude that $(u_J(t),v_J(t))$ always stays in I_1 for all $t\geq 0$ if $(u_J(0),v_J(0)) \in I_1$, provided that

$$d_1 \in (0, l/\{2(1-x_1/2)\}] \text{ and } d_2 \in (0, m/(1-hx_1+y_1)].$$

The same argument can be used for I_2 and we conclude that I_1 and I_2 are invariant sets.

To show the propagation failure, we assume that there is a traveling wavefront solution with nonzero speed connecting $(0,1)$ and $(1,0)$. Then we can find a positive integer $J \gg 1$ such that $(u_J(0),v_J(0)) \in I_1$ and $(u_{-J}(0),v_{-J}(0)) \in I_2$ (or, $(u_J(0),v_J(0)) \in I_2$ and $(u_{-J}(0),v_{-J}(0)) \in I_1$). Since the wave speed is nonzero,

$$(u_J(+\infty),v_J(+\infty)) = (u_{-J}(+\infty),v_{-J}(+\infty)) \in \{(0,1),(1,0)\}.$$

This contradicts that I_1 and I_2 are disjoint. Hence we complete the proof of this theorem. $\square$

3. Existence of stationary solutions

This section is devoted to the study of stationary solutions. We first introduce some notation. A C^1-function $(w,z) = \eta(u,v) := (\eta_1(u,v),\eta_2(u,v))$ is called a *μ-horizontal slice* on $[0,1]\times[0,1]$ if $0 \leq w,z \leq 1$ and $||D\eta(u,v)|| \leq \mu$ for all $(u,v) \in [0,1]\times[0,1]$. Hereafter, $\|\cdot\|$ denotes the Euclidean norm. Given two *nonintersecting* μ-horizontal slices $\eta = (\eta_1,\eta_2)$ and $\bar\eta = (\bar\eta_1,\bar\eta_2)$ with $\eta_i(u,v) < \bar\eta_i(u,v)$, $i=1,2$, a *μ-horizontal strip* is defined as

$$H := \{(u,v,w,z) \in E \mid 0 \leq u,v \leq 1,\ \eta_1(u,v) \leq w \leq \bar\eta_1(u,v),\ \eta_2(u,v) \leq z \leq \bar\eta_2(u,v)\},$$

where $E := [0,1]\times[0,1]\times[0,1]\times[0,1]$. Similarly, we call a C^1-function $(u,v) = \zeta(w,z) := (\zeta_1(w,z),\zeta_2(w,z))$ a *μ-vertical slice* on $[0,1]\times[0,1]$ if $0 \leq u,v \leq 1$ and $||D\zeta(w,z)|| \leq \mu$ for all $(w,z) \in [0,1]\times[0,1]$. Given two *nonintersecting* μ-vertical slices with $\zeta_i(w,z) < \bar\zeta_i(w,z)$, $i=1,2$, a *μ-vertical strip* is defined as

$$V := \{(u,v,w,z) \in E \mid 0 \leq w,z \leq 1,\ \zeta_1(w,z) \leq u \leq \bar\zeta_1(w,z),\ \zeta_2(w,z) \leq v \leq \bar\zeta_2(w,z)\}.$$

The width of a strip H and V are defined as, respectively,

$$d(H) := \max_{0\leq u,v\leq 1} ||\eta(u,v) - \bar\eta(u,v)||, \quad d(V) := \max_{0\leq w,z\leq 1} ||\zeta(w,z) - \bar\zeta(w,z)||.$$

We also define vertical and horizontal boundaries as follows. The vertical boundary of μ-horizontal strip H is defined by

$$\partial_v H := \{(u, v, w, z) \in \partial H \mid u \in \{0, 1\} \text{ or } v \in \{0, 1\}\}.$$

The horizontal boundary of μ-horizontal strip H is defined as

$$\partial_h H := \partial H - \partial_v H.$$

The vertical and horizontal boundaries of μ-vertical strip V can be defined similarly.

Next, motivated by [16], we set $w_j = u_{j-1}$, $z_j = v_{j-1}$, $r = 1/d_1$ and $s = 1/d_2$. Then the system (1.4)-(1.5) can be reduced to the following iteration

$$\begin{cases} u_{j+1} = 2u_j - w_j - rf(u_j, v_j), \\ v_{j+1} = 2v_j - z_j - sg(u_j, v_j), \\ w_{j+1} = u_j, \\ z_{j+1} = v_j, \end{cases}$$

for all $j \in \mathbb{Z}$. Define the map $\Phi : \mathbb{R}^4 \to \mathbb{R}^4$ by

$$\Phi \begin{pmatrix} u \\ v \\ w \\ z \end{pmatrix} = \begin{pmatrix} 2u - w - rf(u, v) \\ 2v - z - sg(u, v) \\ u \\ v \end{pmatrix}.$$

Then the inverse map $\Phi^{-1} : \mathbb{R}^4 \to \mathbb{R}^4$ is defined by

$$\Phi^{-1} \begin{pmatrix} u \\ v \\ w \\ z \end{pmatrix} = \begin{pmatrix} w \\ z \\ 2w - u - rf(w, z) \\ 2z - v - sg(w, z) \end{pmatrix}.$$

Let us recall the Conley-Moser conditions (cf. [26]) as follows. Let $\widetilde{V}_i$, $i = 1, 2$, be two disjoint μ-vertical strips and $\widetilde{H}_i$, $i = 1, 2$, be two disjoint μ-horizontal strips.

Condition 1. $0 \le \mu < 1$ and $\Phi(\widetilde{V}_i) = \widetilde{H}_i$ homeomorphically for $i = 1, 2$. Moreover, the horizontal boundaries and the vertical boundaries of $\widetilde{V}_i$ map to the horizontal boundaries and the vertical boundaries of $\widetilde{H}_i$ respectively for $i = 1, 2$.

Condition 2. Let H be a μ-horizontal strip contained in $\widetilde{H}_1 \cup \widetilde{H}_2$. Then $\Phi(H) \cap \widetilde{H}_i$ is a μ-horizontal strip for $i = 1, 2$. Moreover, $d(\Phi(H) \cap \widetilde{H}_i) \le \nu d(H)$ for some $\nu \in (0, 1)$. Similarly, let V be a μ-vertical strip contained in $\widetilde{V}_1 \cup \widetilde{V}_2$. Then $\Phi^{-1}(V) \cap \widetilde{V}_i$ is a μ-vertical strip for $i = 1, 2$. Moreover, $d(\Phi^{-1}(V) \cap \widetilde{V}_i) \le \nu d(V)$ for some $\nu \in (0, 1)$.

We now define the following sets

$$H_i := \{(u, v, w, z) \in E \mid 0 \le u, v \le 1, \ (w, z) \in I_i\}, \ i = 1, 2,$$

$$V_i := \{(u, v, w, z) \in E \mid 0 \le w, z \le 1, \ (u, v) \in I_i\}, \ i = 1, 2,$$

for rectangles I_i, $i = 1, 2$, which are defined by (1.6) such that (1.7) and (1.8) hold. Due to the definition of Φ and Φ^{-1}, it is not hard to see that

$$\Phi(E \setminus V_i) \cap H_i = \emptyset, \quad \Phi^{-1}(E \setminus H_i) \cap V_i = \emptyset, \quad i = 1, 2.$$

Hence we have

$$\Phi(E) \cap H_i = \Phi(V_i) \cap H_i,\ i = 1, 2, \tag{3.1}$$

$$\Phi^{-1}(E) \cap V_i = \Phi^{-1}(H_i) \cap V_i,\ i = 1, 2. \tag{3.2}$$

We shall verify **Condition 1** and **Condition 2** for the sets

$$\widetilde{H}_i := \Phi(E) \cap H_i,\ i = 1, 2, \tag{3.3}$$

$$\widetilde{V}_i := \Phi^{-1}(E) \cap V_i,\ i = 1, 2, \tag{3.4}$$

when $d_1, d_2 \ll 1$.

Hereafter we choose a fixed number $\mu \in (0, (\sqrt{3} - 1)/2)$. The following three lemmas are to prove the Condition 2. At first, we should check that $\widetilde{H}_i$ and $\widetilde{V}_i$ defined in (3.3)-(3.4) are μ-horizontal strip and μ-vertical strip respectively for $i = 1, 2$. This can be seen in the proof of the following lemma.

Lemma 3.1. *Given $h, k > 1$. Suppose that H is a μ-horizontal strip, $i = 1, 2$. Then there exists $K = K(h, k, I_1, I_2) > 0$ such that for any $r, s \geq K$, $\Phi(H) \cap \widetilde{H}_i$ is also a μ-horizontal strip, $i = 1, 2$.*

Proof. Let Γ_i is a μ-horizontal slice defined on I_i, there exists two functions γ_1 and $\gamma_2 \in C^1(I_i)$ such that $(w, z) = (\gamma_1(u, v), \gamma_2(u, v))$ for all $(u, v) \in I_i$ and

$$\begin{pmatrix} \bar{u} \\ \bar{v} \\ \bar{w} \\ \bar{z} \end{pmatrix} = \Phi \begin{pmatrix} u \\ v \\ \gamma_1(u, v) \\ \gamma_2(u, v) \end{pmatrix} = \begin{pmatrix} 2u - \gamma_1(u, v) - rf(u, v) \\ 2v - \gamma_2(u, v) - sg(u, v) \\ u \\ v \end{pmatrix}.$$

We now prove that $\Phi(\Gamma_i) \cap E$ is contained in H_i and forms a μ-horizontal slice. By the definition of Φ, it is easy to see that $\Phi(\Gamma_i) \cap E \subset H_i$.

Next, for convenience we define

$$\begin{aligned} \phi_1(u, v) &:= 2u - \gamma_1(u, v) - rf(u, v), \\ \phi_2(u, v) &:= 2v - \gamma_2(u, v) - sg(u, v), \\ \phi(u, v) &:= (\phi_1(u, v), \phi_2(u, v)). \end{aligned}$$

Consider the Jacobian matrix

$$J_\phi(u,v) = \begin{pmatrix} \dfrac{\partial\phi_1}{\partial u} & \dfrac{\partial\phi_1}{\partial v} \\ \dfrac{\partial\phi_2}{\partial u} & \dfrac{\partial\phi_2}{\partial v} \end{pmatrix} = \begin{pmatrix} 2+2ru-r+rkv-\dfrac{\partial\gamma_1}{\partial u} & rku-\dfrac{\partial\gamma_1}{\partial v} \\ shv-\dfrac{\partial\gamma_2}{\partial u} & 2+2sv-s+shu-\dfrac{\partial\gamma_2}{\partial v} \end{pmatrix}.$$

By a simple calculation, we obtain that

$$\begin{aligned} \det J_\phi(u,v) &= rs(2u-1)(2v-1)+rshu(2u-1)+rskv(2v-1) \\ &\quad +4+2s(2v-1)+2r(2u-1)+2shu+2rkv \\ &\quad -[2+r(2u-1)+rkv]\frac{\partial\gamma_2}{\partial v}-[2+s(2v-1)+shu]\frac{\partial\gamma_1}{\partial u} \\ &\quad +rku\frac{\partial\gamma_2}{\partial u}+shv\frac{\partial\gamma_1}{\partial v}+\frac{\partial\gamma_1}{\partial u}\frac{\partial\gamma_2}{\partial v}-\frac{\partial\gamma_1}{\partial v}\frac{\partial\gamma_2}{\partial u}. \end{aligned}$$

Note that $|\partial\gamma_j/\partial u| < \mu$ and $|\partial\gamma_j/\partial v| < \mu$, $j = 1,2$. Also, due to (1.7), there exists $K_0 = K_0(h,k,I_1,I_2) > 0$ such that $\det J_\phi(u,v) > 0$ for all $r,s \geq K_0$ and $(u,v) \in I_i$, $i = 1,2$. So we can apply the Inverse Function Theorem, there exists $\psi := (\psi_1,\psi_2) \in C^1$ such that

$$(u,v) = (\psi_1(\bar{u},\bar{v}),\psi_2(\bar{u},\bar{v})) \tag{3.5}$$

locally. Moreover, $\phi = (\phi_1,\phi_2)$ is one-to-one and an open mapping.

We now prove that there exists $K_1 = K_1(h,k,I_1,I_2) > K_0$ such that for any $r,s \geq K_1$,

$$[0,1]\times[0,1] \subset \{(\bar{u},\bar{v})|\,(\bar{u},\bar{v}) = \phi(u,v),\ \forall\,(u,v)\in I_i\},\ i=1,2. \tag{3.6}$$

Indeed, it suffices to show that $\phi := (\phi_1,\phi_2)$ maps the boundary of I_i $(i = 1,2)$ onto a closed curve which surrounds the square $[0,1]\times[0,1]$. Divide the boundary of I_1 into

$$\begin{aligned} \Sigma_1 &:= \{(u,v)|\,u=0,\ y_1\leq v\leq 1\}, \\ \Sigma_2 &:= \{(u,v)|\,0\leq u\leq x_1,\ v=1\}, \\ \Sigma_3 &:= \{(u,v)|\,u=x_1,\ y_1\leq v\leq 1\}, \\ \Sigma_4 &:= \{(u,v)|\,0\leq u\leq x_1,\ v=y_1\}, \end{aligned}$$

where x_1 and y_1 are defined in (1.6). Note that

$$\phi_1(0,v) = -\gamma_1(0,v), \quad \phi_2(0,v) = 2v-\gamma_2(0,v)-sv(1-v)$$

for $y_1 \leq v \leq 1$. Recall the definition of μ-horizontal slice, $0\leq\gamma_i\leq 1$ for $i=1,2$, so we have $\phi_1(u,v)\leq 0$ for all $(u,v)\in\Sigma_1$ and $[0,1]\subset\phi_2(\Sigma_1)$ as long as $s\gg 1$. For Σ_2, we have

$$\phi_1(u,1) = 2u-\gamma_1(u,1)-ru(1-u-k), \quad \phi_2(u,1) = 2-\gamma_2(u,1)+shu$$

for $0\leq u\leq x_1$. It is easy to see $[0,1]\subset\phi_1(\Sigma_2)$ and $\phi_2(u,v)\geq 1$ for all $(u,v)\in\Sigma_2$ as long as $r\gg 1$. For Σ_3, we have

$$\begin{aligned} \phi_1(x_1,v) &= 2x_1-\gamma_1(x_1,v)-rx_1(1-x_1-kv), \\ \phi_2(x_1,v) &= 2v-\gamma_2(x_1,v)-sv(1-v-hx_1) \end{aligned}$$

for $y_1 \leq v \leq 1$. Recall (1.6) that $y_1 > 1/k$ implies $1 - x_1 - ky_1 < 0$, so we have $\phi_1(x) \geq 1$ for all $x \in \Sigma_3$ if $r \gg 1$. Also, due to (1.8), $1 - y_1 - hx_1 > 0$ such that $[0,1] \subset \phi_2(\Sigma_3)$ as long as $s \gg 1$. For Σ_4, we note that

$$\phi_1(u, y_1) = 2u - \gamma_1(u, y_1) - ru(1 - u - ky_1),$$
$$\phi_2(u, y_1) = 2y_1 - \gamma_2(u, y_1) - sy_1(1 - y_1 - hu),$$

for $0 \leq u \leq x_1$. Then $1 - x_1 - ky_1 < 0$ implies $[0,1] \subset \phi_1(\Sigma_4)$ and $1 - y_1 - hx_1 > 0$ implies $\phi_2(u, v) \leq 0$ for all $(u, v) \in \Sigma_4$ as $r, s \gg 1$. From the above discussions, (3.6) holds for $i = 1$.

Similar reasoning can be applied to $i = 2$. This implies that ψ_i can be defined on $[0,1] \times [0,1]$, $i = 1, 2$. Recall the definition of Φ, $\bar{w} = u$ and $\bar{z} = v$. Thus (3.5) can be written as $(\bar{w}, \bar{z}) = (\psi_1(\bar{u}, \bar{v}), \psi_2(\bar{u}, \bar{v}))$ for all $(\bar{u}, \bar{v}) \in [0,1] \times [0,1]$, which implies that $\Phi(\Gamma_i) \cap E$ forms a C^1 horizontal slice.

To show that the horizontal slice is a μ-horizontal slice, we need to prove that $||D\psi|| \leq \mu$. Note that

$$||D\psi|| \leq \frac{O(r) + O(s)}{O(rs)} \text{ as } r, s \to +\infty,$$

we can find $K_2 = K_2(h, k, I_1, I_2) > 0$ such that $||D\psi|| \leq \mu$ as long as $r, s \geq K_2$.

Finally, we choose $r, s \geq K := \max\{K_1, K_2\}$, then $\Phi(\Gamma_i) \cap E$ is contained in H_i and forms a μ-horizontal slice. It follows that $\widetilde{H}_i := \Phi(V_i) \cap H_i$ forms a μ-horizontal strip, $i = 1, 2$. By the definition of Φ,

$$\Phi(H) \cap \widetilde{H}_i = \Phi(H \cap V_i) \cap H_i, i = 1, 2.$$

It is not hard to see that $\Phi(H) \cap \widetilde{H}_i$ is a μ-horizontal strip, $i = 1, 2$ as long as $r, s \geq K$. □

By the same argument as the above lemma, we can obtain the following lemma.

Lemma 3.2. *Given $h, k > 1$. Let V be a μ-vertical strip, $i = 1, 2$. Then there exists $\overline{K} = \overline{K}(h, k, I_1, I_2) > 0$ such that for any $r, s \geq \overline{K}$, $\Phi^{-1}(V) \cap \widetilde{V}_i$ is also a μ-vertical strip, $i = 1, 2$.*

Lemma 3.3. *Given $h, k > 1$ and suppose that $r, s \geq \max\{K, \overline{K}\}$, where $K, \overline{K}$ are the constants given in Lemmas 3.1 and 3.2, respectively. Let H and V be a μ-horizontal strip contained in H_i and a μ-vertical strip contained in V_i, $i = 1, 2$, respectively. Then there exists $\nu \in (0, 1)$ such that $d(\Phi(H) \cap \widetilde{H}_i) \leq \nu d(H)$ and $d(\Phi^{-1}(V) \cap \widetilde{V}_i) \leq \nu d(V)$, $i = 1, 2$.*

Proof. Since the proofs for both cases are the same, we only prove that $d(\Phi(H) \cap \widetilde{H}_i) \leq \nu d(H)$ for $i = 1, 2$. Set $\bar{P}(\bar{u}, \bar{v}, w^+, z^+)$ and $\bar{Q}(\bar{u}, \bar{v}, w^-, z^-)$ such that

$$d(\Phi(H) \cap \widetilde{H}_i) = d(\bar{P}, \bar{Q}) = ||(w^+, z^+) - (w^-, z^-)||. \tag{3.7}$$

Let the μ-horizontal strip

$$H := \{(u,v,w,z) \in E |\, (w,z) \text{ is between } \gamma(u,v) \text{ and } \widehat{\gamma}(u,v), (u,v) \in [0,1]\times[0,1]\},$$

for some C^1-functions $\gamma(u,v)$ and $\widehat{\gamma}(u,v)$.

By the definition of Φ, we can find two points $P(w^+,z^+,w_1,z_1)$ and $Q(w^-,z^-,w_2,z_2)$ such that $\Phi(P)=\bar{P}$ and $\Phi(Q)=\bar{Q}$.

Since $\bar{P}$ and $\bar{Q}$ are contained in a μ-vertical slice, P and Q are also contained in a μ-vertical slice. Therefore, there exists C^1-function $\zeta=(\zeta_1,\zeta_2)$ such that

$$(w^+,z^+)=(\zeta_1(w_1,z_1),\zeta_2(w_1,z_1)),\ (w^-,z^-)=(\zeta_1(w_2,z_2),\zeta_2(w_2,z_2)), \tag{3.8}$$

$$||\zeta(w_1,z_1)-\zeta(w_2,z_2)|| \le \mu||(w_1,z_1)-(w_2,z_2)||. \tag{3.9}$$

It follows from (3.7)-(3.9) that

$$\begin{aligned} d(\Phi(H)\cap\widetilde{H}_i) &\le |\zeta_1(w_1,z_1)-\zeta_1(w_2,z_2)| + |\,\zeta_2(w_1,z_1)-\zeta_2(w_2,z_2)| \\ &\le 2\mu||(w_1,z_1)-(w_2,z_2)|| \\ &\le 2\mu||\,\gamma(w^+,z^+)-\widehat{\gamma}(w^-,z^-)|| \\ &\le 2\mu[\,d(H)+\mu d(\Phi(H)\cap\widetilde{H}_i)\,]. \end{aligned}$$

This implies that

$$d(\Phi(H)\cap\widetilde{H}_i) \le \frac{2\mu}{1-2\mu^2} d(H) := \nu d(H).$$

Since $\mu \in (0,(\sqrt{3}-1)/2)$ is a fixed number, we have $\nu\in(0,1)$ and the lemma follows. $\square$

By Lemmas 3.1-3.3, we have established **Condition 2**. Next, **Condition 1** is confirmed by the following lemma.

Lemma 3.4. *Given $h,k>1$ and $r,s\ge \max\{K,\overline{K}\}$, where $K,\overline{K}$ are defined in lemmas 3.1 and 3.2. Then Φ maps $\widetilde{V}_i$ homeomorphically onto $\widetilde{H}_i$, $i=1,2$. Moreover, the horizontal boundaries of $\widetilde{V}_i$ map to the horizontal boundaries of $\widetilde{H}_i$ and the vertical boundaries of $\tilde{V}_i$ map to the vertical boundaries of $\widetilde{H}_i$, $i=1,2$.*

Proof. It is easy to see that both Φ and Φ^{-1} are one to one and continuous. From (3.1) and (3.2) it follows that

$$\Phi(\widetilde{V}_i)=\Phi(\Phi^{-1}(H_i)\cap V_i)=H_i\cap\Phi(V_i)=\widetilde{H}_i,$$

for $i=1,2$. Thus, Φ maps $\widetilde{V}_i$ homeomorphically onto $\widetilde{H}_i$, $i=1,2$.

Next, since $\det J_\Phi(u,v,w,z)=1$ for all (u,v,w,z), Φ is an open mapping. Also, by $\Phi(\widetilde{V}_i)=\widetilde{H}_i$ and the same reasoning as that of Lemma 3.1 the horizontal boundaries of $\widetilde{V}_i$ map to the horizontal boundaries of $\widetilde{H}_i$ and the vertical boundaries of $\tilde{V}_i$ map to the vertical boundaries of $\widetilde{H}_i$, $i=1,2$. Hence this lemma follows. $\square$

From Lemmas 3.1-3.4, we have verified the Conley-Moser conditions so that the following proposition can be readily proved. We define a full shift map σ on $S := \{1, 2\}$ by

$$\sigma(\{s_j\}_{j\in\mathbb{Z}}) = \{t_j\}_{j\in\mathbb{Z}}, \quad t_j := s_{j+1} \in S \quad \forall j \in \mathbb{Z},$$

i.e., $(\sigma(\{s_j\}_{j\in\mathbb{Z}}))_i = s_{i+1}$ for all i. Then the following proposition can be proved by modifying the proof of [26, Theorem 25.1.5].

Proposition 3.5. *Given $h, k > 1$ and assume that $r, s \geq \max\{K, \overline{K}\}$, where $K, \overline{K}$ are defined in lemmas 3.1 and 3.2. Then Φ has an invariant set Λ and Φ is topological conjugate to a full shift map σ on $S = \{1, 2\}$ in the sense*

$$\varphi \circ \Phi = \sigma \circ \varphi,$$

where φ is a homeomorphism mapping Λ onto Σ^2 with

$$\Lambda := \bigcap_{j=-\infty}^{\infty} \Phi^j(\widetilde{H}_1 \cup \widetilde{H}_2), \quad \Sigma^2 := \prod_{j=-\infty}^{+\infty} S_j, \quad S_j = S \ \forall\ j. \tag{3.10}$$

Proof. We divide the proof into three steps.

Step1. Construct Λ. Firstly, we define

$$\Lambda_{-1} := \widetilde{H}_1 \cup \widetilde{H}_2, \quad \Lambda_{-2} := \Phi(\Lambda_{-1}) \cap \Lambda_{-1}.$$

Note that

$$\begin{aligned}
\Lambda_{-2} &= \left[\Phi(\bigcup_{s_{-2}\in S} \widetilde{H}_{s_{-2}})\right] \cap \left[\bigcup_{s_{-1}\in S} \widetilde{H}_{s_{-1}}\right] = \bigcup_{s_{-i}\in S, i=1,2} \left[\Phi(\widetilde{H}_{s_{-2}}) \cap \widetilde{H}_{s_{-1}}\right] \\
&:= \bigcup_{s_{-i}\in S, i=1,2} \widetilde{H}_{s_{-1}s_{-2}},
\end{aligned}$$

where

$$\widetilde{H}_{s_{-1}s_{-2}} = \{x \in E \mid x \in \widetilde{H}_{s_{-1}},\ \Phi^{-1}(x) \in \widetilde{H}_{s_{-2}}\}.$$

Thus $\widetilde{H}_{s_{-1}s_{-2}} \subset \widetilde{H}_{s_{-1}}$. By **Conditions 1** and **2**, Λ_{-2} consists of 4 μ-horizontal strips, $\widetilde{H}_{11}$, $\widetilde{H}_{12}$, $\widetilde{H}_{21}$ and $\widetilde{H}_{22}$. Moreover, $d(\widetilde{H}_{s_{-1}s_{-2}}) \leq \nu d(\widetilde{H}_{s_{-1}})$ for $s_{-1}, s_{-2} \in S$. We continue this procedure and define for any $k \geq 2$

$$\begin{aligned}
\Lambda_{-k} &:= \Phi(\Lambda_{-(k-1)}) \cap \Lambda_{-1} \\
&= \bigcup_{s_{-i}\in S, i=1,\cdots,k} \left[\Phi^{k-1}(\widetilde{H}_{s_{-k}}) \cap \cdots \cap \Phi(\widetilde{H}_{s_{-2}}) \cap \widetilde{H}_{s_{-1}}\right] \\
&:= \bigcup_{s_{-i}\in S, i=1,\cdots,k} \widetilde{H}_{s_{-1}s_{-2}\cdots s_{-k}},
\end{aligned}$$

where

$$\widetilde{H}_{s_{-1}s_{-2}\cdots s_{-k}} = \{x \in E \mid \Phi^{-i+1}(x) \in \widetilde{H}_{s_{-i}}, i = 1, ..., k\} \tag{3.11}$$

with Φ^0 the identity mapping. Note that

$$\widetilde{H}_{s_{-1}s_{-2}\cdots s_{-k}} \subset \widetilde{H}_{s_{-1}s_{-2}\cdots s_{-(k-1)}} \subset \cdots \subset \widetilde{H}_{s_{-1}s_{-2}} \subset \widetilde{H}_{s_{-1}}.$$

By **Conditions 1** and **2** again, we have that Λ_{-k} consists of 2^k μ-horizontal strips and

$$d(\widetilde{H}_{s_{-1}s_{-2}\cdots s_{-k}}) \le \nu^{k-1} d(\widetilde{H}_{s_{-1}}).$$

Letting $k \to +\infty$, then $\Lambda_{-k} \to \Lambda_{-\infty}$, where $\Lambda_{-\infty}$ consists of an infinite number of μ-horizontal slices. Note that these μ-horizontal slices may not be C^1 slices, but at least they are Lipschitz continuous, more details see Lemma 25.1.3 in [26].

Next, for each $k \in \mathbb{N}$ we define

$$\begin{aligned}
\Lambda_0 &:= \widetilde{V}_1 \cup \widetilde{V}_2, \\
\Lambda_k &:= \Phi^{-1}(\Lambda_{k-1}) \cap \Lambda_0 = \bigcup_{s_i \in S, i=0,\cdots,k} \left[\Phi^{-k}(\widetilde{V}_{s_k}) \cap \cdots \cap \Phi^{-1}(\widetilde{V}_{s_1}) \cap \widetilde{V}_{s_0}\right] \\
&:= \bigcup_{s_i \in S, i=0,\cdots,k} \widetilde{V}_{s_0 s_1 \cdots s_k},
\end{aligned}$$

where

$$\widetilde{V}_{s_0 s_1 \cdots s_k} := \{x \in E \mid \Phi^i(x) \in \widetilde{V}_{s_i}, i = 0, 1, \cdots, k\}. \tag{3.12}$$

Then we can conclude that Λ_k forms 2^{k+1} μ-vertical strips and

$$d(\widetilde{V}_{s_0 s_1 \cdots s_k}) \le \nu^k d(\widetilde{V}_{s_0}).$$

Letting $k \to +\infty$, $\Lambda_k \to \Lambda_\infty$, which forms an infinite number of μ-vertical (Lipschitz) slices.

Finally, set $\Lambda := \Lambda_{-\infty} \cap \Lambda_\infty$. We need to show that $\Lambda \neq \emptyset$. Indeed, it suffices to show that the intersection of a μ-vertical slice and a μ-horizontal slice is a unique point. Define that a μ-vertical slice by $x = \zeta(y)$ and a μ-horizontal slice by $y = \eta(x)$, where $y = (w, z)$ and $x = (u, v)$. By the contraction mapping theorem, we can show that the equation $y = \eta(\zeta(y))$ has a unique solution by using $0 < \mu < 1$ and

$$||\eta(\zeta(y_1)) - \eta(\zeta(y_2))|| \le \mu ||\zeta(y_1) - \zeta(y_2)|| \le \mu^2 ||y_1 - y_2||,$$

for all $y_1, y_2 \in [0,1] \times [0,1]$.

Step2. Define $\varphi : \Lambda \to \Sigma^2$. Since the intersection of a horizontal slice and vertical slice is a unique point, we can define a map φ from Λ to bi-infinite sequences $\{s_k\}_{k=-\infty}^{\infty}$ with $s_k \in S$ for all k by

$$(\varphi(x))_i := s_i \quad \forall\, i$$

for $x \in \widetilde{H}_{s_{-1}s_{-2}\cdots s_{-k}\cdots} \cap \widetilde{V}_{s_0 s_1 \cdots s_k \cdots} \subset \Lambda$. Then φ is a homeomorphism (cf. [26, p.599-p.601]).

Step 3. Prove that $\varphi \circ \Phi = \sigma \circ \varphi$. Pick $x \in \Lambda$. Assume that $(\varphi(x))_i := s_i$ for all i. By the definition of the shift map σ,

$$(\sigma \circ \varphi(x))_i = (\varphi(x))_{i+1} = s_{i+1} \text{ for all } i.$$

On the other hand, recall that $\Phi(\widetilde{V}_i) = \widetilde{H}_i$, $i = 1, 2$, we have

$$\begin{aligned}(\varphi \circ \Phi(x))_i &= (\varphi \circ \Phi(\widetilde{H}_{s_{-1}s_{-2}\cdots s_{-k}\cdots} \cap \widetilde{V}_{s_0 s_1 \cdots s_k \cdots}))_i \\ &= (\varphi(\widetilde{H}_{s_0 s_{-1}s_{-2}\cdots s_{-k}\cdots} \cap \widetilde{V}_{s_1 \cdots s_k \cdots}))_i \\ &= s_{i+1}\end{aligned}$$

for all i. Hence we complete the proof of the proposition. □

Then Theorem 2 is just a corollary of the above proposition.

Proof of Theorem 2. Since Λ defined in (3.10) form an invariant set under the mapping Φ, there exists a solution of (1.4)-(1.5). Given an infinite sequence $\{s_j\}_{j\in\mathbb{Z}}$, by Proposition 3.5, we can find a unique point $x \in \Lambda$ such that

$$x := (u_0, v_0, w_0, z_0) \in \widetilde{H}_{s_{-1}s_{-2}\cdots s_{-k}\cdots} \cap \widetilde{V}_{s_0 s_1 \cdots s_k \cdots}.$$

Therefore, $(u_0, v_0) \in \widetilde{V}_{s_0} \subset V_{s_0}$, and so $(u_0, v_0) \in I_{s_0}$. It follows from $\varphi \circ \Phi = \sigma \circ \varphi$ that

$$(\varphi \circ \Phi(u_0, v_0, w_0, z_0))_i = s_{i+1} \quad \forall\ i.$$

Then we have $(u_1, v_1, w_1, z_1) \in \widetilde{V}_{s_1}$ which implies $(u_1, v_1) \in I_{s_1}$. Consequently, we can obtain that $(u_j, v_j) \in I_{s_j}$ for all $j \in \mathbb{Z}$.

For uniqueness, assume that there are two solutions $\{(u_j, v_j)\}_{j\in\mathbb{Z}}$ and $\{(\bar{u}_j, \bar{v}_j)\}_{j\in\mathbb{Z}}$ such that

$$(u_j, v_j) \in I_{s_j}, \quad (\bar{u}_j, \bar{v}_j) \in I_{s_j} \quad \forall\ j \in \mathbb{Z}. \tag{3.13}$$

From (3.13), we see that $(u_0, v_0, u_{-1}, v_{-1}) \in \widetilde{V}_{s_0}$ and $(u_0, v_0, u_{-1}, v_{-1}) \in \widetilde{H}_{s_{-1}}$. Recall from (3.11) and (3.12) that

$$(u_0, v_0, u_{-1}, v_{-1}) \in \widetilde{H}_{s_{-1}s_{-2}\cdots s_{-k}\cdots} \cap \widetilde{V}_{s_0 s_1 \cdots s_k \cdots}.$$

The same reasoning for $(\bar{u}_0, \bar{v}_0, \bar{u}_{-1}, \bar{v}_{-1})$, we obtain

$$(\bar{u}_0, \bar{v}_0, \bar{u}_{-1}, \bar{v}_{-1}) \in \widetilde{H}_{s_{-1}s_{-2}\cdots s_{-k}\cdots} \cap \widetilde{V}_{s_0 s_1 \cdots s_k \cdots}.$$

Since $\widetilde{H}_{s_{-1}s_{-2}\cdots s_{-k}\cdots} \cap \widetilde{V}_{s_0 s_1 \cdots s_k \cdots}$ is a singleton, we obtain that

$$(u_0, v_0, u_{-1}, v_{-1}) = (\bar{u}_0, \bar{v}_0, \bar{u}_{-1}, \bar{v}_{-1}).$$

Hence, by the definition of Φ, these two solutions must be identical. This completes the proof of Theorem 2. □

4. Monotonicity and uniqueness

In this section, we shall always assume that a traveling wavefront (U,V) of (1.9) with a nonzero wave speed c exists. We first study the asymptotic behavior of wave tails of traveling wave solutions. In this section, we always define $W(\xi) := 1 - V(\xi)$. Note that, by (1.11), W satisfies the equation

$$cW' = dD_2[W] + b(1-W)(hU - W). \tag{4.1}$$

For any fixed $c \neq 0$, $a > 0$, $b > 0$, $h > 1$ and $k > 1$, let $\lambda_1 = \lambda_1(c) > 0$ and $\lambda_2 = \lambda_2(c) < 0$ be two real roots of

$$c\lambda = (e^{\lambda} + e^{-\lambda} - 2) + a(1-k). \tag{4.2}$$

Also, let $\nu_1 = \nu_1(c) > 0$ and $\nu_2 = \nu_2(c) < 0$ be two real roots of

$$c\lambda = d(e^{\lambda} + e^{-\lambda} - 2) - b. \tag{4.3}$$

We first state the following main result on the asymptotic behaviors of wave tails at $\xi = -\infty$.

Proposition 4.1. *Let (c,U,V) be a solution of (P) with $c \neq 0$. Then there exist constants $C_i > 0$, $i = 1,2$, such that*

$$\lim_{\xi\to-\infty} \frac{U(\xi)}{e^{\lambda_1 \xi}} = C_1, \quad \lim_{\xi\to-\infty} \frac{1-V(\xi)}{|\xi|^m e^{\alpha\xi}} = C_2,$$

where $m = 0$ if $\lambda_1 \neq \nu_1$, $m = 1$ if $\lambda_1 = \nu_1$ and $\alpha := \min\{\lambda_1, \nu_1\}$.

The following lemma plays an important role to show Proposition 4.1.

Lemma 4.2. *Let (c,U,V) be a solution of (P) with $c \neq 0$. Then we have the following two alternatives.*

(i) *If $\liminf_{\xi\to-\infty} U(\xi)/W(\xi) = 0$, then*

$$\lim_{\xi\to-\infty} \frac{U(\xi)}{W(\xi)} = 0, \quad \lim_{\xi\to-\infty} \frac{W'(\xi)}{W(\xi)} = \nu_1 \leq \lambda_1 = \lim_{\xi\to-\infty} \frac{U'(\xi)}{U(\xi)}.$$

(ii) *If $\liminf_{\xi\to-\infty} U(\xi)/W(\xi) > 0$, then*

$$\lim_{\xi\to-\infty} \frac{U(\xi)}{W(\xi)} = \frac{1}{bh}\left\{(1-d)(e^{\lambda_1} + e^{-\lambda_1} - 2) + a(1-k)\right\} + \frac{1}{h} > 0,$$

$$\lim_{\xi\to-\infty} \frac{W'(\xi)}{W(\xi)} = \lambda_1 = \lim_{\xi\to-\infty} \frac{U'(\xi)}{U(\xi)}.$$

Proof. Firstly, by using $U(-\infty) = 0$, $V(-\infty) = 1$ and Theorem 4 in [4], we obtain that

$$\lim_{\xi\to-\infty} [U'(\xi)/U(\xi)] = \lambda_1. \tag{4.4}$$

Recall that $W := 1 - V$. Then (4.1) can be rewritten as

$$c\frac{W'}{W} = d\frac{D_2[W]}{W} + b(1-W)\left(\frac{hU}{W} - 1\right). \tag{4.5}$$

We now prove that

$$\sup_{\xi\in\mathbb{R}}\{U(\xi)/W(\xi)\} < +\infty. \tag{4.6}$$

If the conclusion is not true, then either $\lim_{\xi\to-\infty}[U(\xi)/W(\xi)] = +\infty$ or there exists a sequence $\{\xi_n\}$ of extreme points of U/W such that $\xi_n \to -\infty$, $U(\xi_n)/W(\xi_n) \nearrow +\infty$ as $n \to +\infty$. If the latter case occurs, then

$$0 = \left(\frac{U}{W}\right)'(\xi_n) = \left[\frac{U'(\xi_n)}{U(\xi_n)} - \frac{W'(\xi_n)}{W(\xi_n)}\right]\frac{U(\xi_n)}{W(\xi_n)}. \tag{4.7}$$

Letting $n \to +\infty$, we obtain that $W'(\xi_n)/W(\xi_n) \to \lambda_1$ as $n \to +\infty$. However, from (4.5), this is impossible. Hence the former case happens, i.e., $\lim_{\xi\to-\infty}[U(\xi)/W(\xi)] = +\infty$. In this case, we have $\lim_{\xi\to-\infty}[cW'(\xi)/W(\xi)] = \infty$, by letting $\xi \to -\infty$ in (4.5). If $c < 0$, then we obtain that $\lim_{\xi\to-\infty}[W'(\xi)/W(\xi)] = -\infty$, which contradicts with $W(-\infty) = 0$. If $c > 0$, then we have $\lim_{\xi\to-\infty}[W'(\xi)/W(\xi)] = +\infty$ and so that

$$\frac{W(\xi+1)}{W(\xi)} = \exp\left\{\int_{\xi}^{\xi+1}\frac{W'(s)}{W(s)}ds\right\} \to +\infty \quad \text{as } \xi \to -\infty. \tag{4.8}$$

On the other hand, by choosing $\mu \gg 1$, we have $W' + \mu W > 0$ in $\mathbb{R}$. Integrating over $[\xi - s, \xi]$, $s > 0$, we have

$$W(\xi - s) \le W(\xi)e^{\mu s} \text{ for all } \xi \in \mathbb{R}. \tag{4.9}$$

Thus,

$$W(\xi + \frac{1}{2}) \le W(\eta + 1)e^{\mu/2} \text{ for all } \eta \in [\xi - \frac{1}{2}, \xi]. \tag{4.10}$$

Due to $U(\xi)/W(\xi) \to +\infty$ as $\xi \to -\infty$, there exists $N \gg 1$ such that

$$(1 - W)(hU - W) > 0 \text{ on } (-\infty, -\mathrm{N}]. \tag{4.11}$$

Integrating (4.1) over $(-\infty, \xi)$ for $\xi \le -N$ and using (4.9)-(4.11) gives

$$\begin{aligned}
\frac{c}{d}W(\xi) &\ge \int_{-\infty}^{\xi} D_2[W](s)ds = \int_{\xi-1}^{\xi} W(s+1)ds - \int_{\xi-1}^{\xi} W(s)ds \\
&\ge \int_{\xi-\frac{1}{2}}^{\xi} W(s+1)ds - e^{\mu}W(\xi) \\
&\ge \frac{1}{2}e^{-\mu/2}W(\xi + \frac{1}{2}) - e^{\mu}W(\xi).
\end{aligned}$$

Hence we obtain $W(\xi + 1/2)/W(\xi) \le 2e^{\mu/2}(c/d + e^{\mu})$ for all $\xi \in (-\infty, -N]$, this contradicts with (4.8), so that $\lim_{\xi\to-\infty}[U(\xi)/W(\xi)] = +\infty$ can not happen. Therefore, (4.6) holds.

We now start to prove the part (ii). We divide it into two cases.

Case 1. U/W has infinitely many extreme points for $\xi < 0$. Let

$$M := \limsup_{\xi\to-\infty} \frac{U(\xi)}{W(\xi)}, \quad m := \liminf_{\xi\to-\infty} \frac{U(\xi)}{W(\xi)}.$$

Note that $0 < m \leq M < +\infty$ because of (4.6) and the assumption in (ii). We now choose a sequence $\{x_n\}$ ($\{y_n\}$) of local maximal (minimal, respectively) points of U/W such that $x_n \to -\infty$ ($y_n \to -\infty$, resp.) and $U(x_n)/W(x_n) \to M$ as $n \to +\infty$ ($U(y_n)/W(y_n) \to m$ as $n \to +\infty$, resp.). For any given $\varepsilon > 0$,

$$\frac{W(x_n \pm 1)}{W(x_n)} = \frac{W(x_n \pm 1)}{U(x_n \pm 1)} \frac{U(x_n \pm 1)}{U(x_n)} \frac{U(x_n)}{W(x_n)} \geq \frac{1}{M+\varepsilon} \frac{U(x_n \pm 1)}{U(x_n)} \frac{U(x_n)}{W(x_n)}$$

for all large enough n. Using (4.7), we know that $U'(x_n)/U(x_n) = W'(x_n)/W(x_n)$ for all $n \in \mathbb{N}$. Thus, it follows from (4.5) that

$$\begin{aligned} c\lambda_1 &= \lim_{n\to\infty} \left\{ d\frac{D_2[W](x_n)}{W(x_n)} \right\} + b(hM-1) \\ &\geq d[\frac{M}{M+\varepsilon}(e^{\lambda_1} + e^{-\lambda_1}) - 2] + b(hM-1), \end{aligned}$$

Because $\varepsilon > 0$ is arbitrary,

$$c\lambda_1 \geq d(e^{\lambda_1} + e^{-\lambda_1} - 2) + b(hM-1), \tag{4.12}$$

Similarly, we can obtain

$$c\lambda_1 \leq d(e^{\lambda_1} + e^{-\lambda_1} - 2) + b(hm-1). \tag{4.13}$$

Using (4.12), (4.13) and the fact of $M \geq m$, we see that $M = m$. Thus, by (4.2) we obtain that

$$\lim_{\xi\to-\infty} \frac{U(\xi)}{W(\xi)} = \frac{1}{bh} \left\{ (1-d)(e^{\lambda_1} + e^{-\lambda_1} - 2) + a(1-k) \right\} + \frac{1}{h} > 0.$$

Finally, by (4.5) and noting that

$$\frac{W(\xi \pm 1)}{W(\xi)} = \frac{W(\xi \pm 1)}{U(\xi \pm 1)} \frac{U(\xi \pm 1)}{U(\xi)} \frac{U(\xi)}{W(\xi)} \to e^{\pm\lambda_1} \quad \text{as } \xi \to -\infty,$$

it follows that $\lim_{\xi\to-\infty}[W'(\xi)/W(\xi)] = \lambda_1$.

Case 2. U/W is monotone for $-\xi \gg 1$. Thus, the limit $l := \lim_{\xi\to-\infty}[U(\xi)/W(\xi)]$ exists and $l > 0$. Note that

$$\frac{W(\xi \pm 1)}{W(\xi)} = \frac{W(\xi \pm 1)}{U(\xi \pm 1)} \frac{U(\xi \pm 1)}{U(\xi)} \frac{U(\xi)}{W(\xi)} \to \frac{1}{l} \cdot e^{\pm\lambda_1} \cdot l = e^{\pm\lambda_1} \text{ as } \xi \to -\infty.$$

We see from (4.5) that $\lim_{\xi\to-\infty}[W'(\xi)/W(\xi)]$ exists. Using the equality

$$\frac{U(\xi+1)}{W(\xi+1)} = \frac{U(\xi)}{W(\xi)} \exp\left\{ \int_{\xi}^{\xi+1} \left[\frac{U'(s)}{U(s)} - \frac{W'(s)}{W(s)} \right] ds \right\}.$$

and letting $\xi \to -\infty$, we have $\lim_{\xi\to-\infty}[W'(\xi)/W(\xi)] = \lambda_1$. Then it is easy to deduce that

$$l = \frac{1}{bh} \left\{ (1-d)(e^{\lambda_1} + e^{-\lambda_1} - 2) + a(1-k) \right\} + \frac{1}{h}.$$

Hence, we complete the proof of (ii).

We now start to show (i). The proof will be also divided into two cases as above.

Case 1. U/W has infinitely many extreme points for $\xi < 0$. Since $U(\xi)/W(\xi) \to 1$ as $\xi \to +\infty$, we can choose a local minimal point $\xi_0 \in \mathbb{R}$ such that

$$\frac{U(\xi_0)}{W(\xi_0)} \leq \frac{U(\xi)}{W(\xi)} \text{ for all } \xi \in [\xi_0, \xi_0 + 1].$$

Let $\{\xi_n\}$ be the sequence of local minimal points of U/W in $(-\infty, \xi_0)$ such that $\xi_n < \xi_{n-1}$ for $n \in \mathbb{N}$ and

$$\frac{U(\xi_n)}{W(\xi_n)} < \frac{U(\xi_{n-1})}{W(\xi_{n-1})},\ n = 1, 2, \cdots .$$

Then $\lim_{n\to+\infty} \xi_n = -\infty$ and $\lim_{n\to+\infty}[U(\xi_n)/W(\xi_n)] = 0$. Moreover,

$$\frac{U(\xi_n)}{W(\xi_n)} \leq \frac{U(\xi_n + 1)}{W(\xi_n + 1)},\ \forall\ n. \tag{4.14}$$

Due to $(U/W)'(\xi_n) = 0$ for all $n \in \mathbb{N}$, it follows from (4.7) that

$$\lim_{n\to+\infty} \frac{W'(\xi_n)}{W(\xi_n)} = \lim_{n\to+\infty} \frac{U'(\xi_n)}{U(\xi_n)} = \lambda_1 \tag{4.15}$$

Next, we shall focus on the condition:

$$\frac{U(\xi_n)}{W(\xi_n)} > \frac{U(\xi_n - 1)}{W(\xi_n - 1)},\ \forall\ n \gg 1. \tag{4.16}$$

If (4.16) dose not hold, then (i) can be proved as follows. Choosing a subsequence $\{\xi_{n_j}\}$ of $\{\xi_n\}$ such that

$$\frac{U(\xi_{n_j})}{W(\xi_{n_j})} \leq \frac{U(\xi_{n_j} - 1)}{W(\xi_{n_j} - 1)},\ \forall\ j. \tag{4.17}$$

Thus, from (4.5) we obtain

$$\begin{aligned} c\frac{W'(\xi_{n_j})}{W(\xi_{n_j})} &= d\frac{W(\xi_{n_j}+1)}{U(\xi_{n_j}+1)}\frac{U(\xi_{n_j}+1)}{U(\xi_{n_j})}\frac{U(\xi_{n_j})}{W(\xi_{n_j})} + d\frac{W(\xi_{n_j}-1)}{U(\xi_{n_j}-1)}\frac{U(\xi_{n_j}-1)}{U(\xi_{n_j})}\frac{U(\xi_{n_j})}{W(\xi_{n_j})} \\ &\quad -2d + b[1 - W(\xi_{n_j})]\left(\frac{hU(\xi_{n_j})}{W(\xi_{n_j})} - 1\right) \\ &\leq d\frac{U(\xi_{n_j}+1)}{U(\xi_{n_j})} + d\frac{U(\xi_{n_j}-1)}{U(\xi_{n_j})} - 2d + b[1 - W(\xi_{n_j})]\left(\frac{hU(\xi_{n_j})}{W(\xi_{n_j})} - 1\right). \end{aligned}$$

Letting $j \to +\infty$, we obtain

$$c\lambda_1 \leq d(e_1^{\lambda} + e^{-\lambda_1} - 2) - b. \tag{4.18}$$

Now, set $M := \limsup_{\xi\to-\infty}[U(\xi)/W(\xi)] \in [0, +\infty)$. We now claim that $M = 0$. Suppose, on the contrary, that $M > 0$. We choose a sequence $\{x_n\}$ of local maximal points of U/W such that $x_n \to -\infty$ and $U(x_n)/W(x_n) \to M$ as $n \to +\infty$. For any $\varepsilon > 0$, we have

$$\frac{W(x_n \pm 1)}{W(x_n)} = \frac{W(x_n \pm 1)}{U(x_n \pm 1)}\frac{U(x_n \pm 1)}{U(x_n)}\frac{U(x_n)}{W(x_n)} \geq \frac{1}{M + \varepsilon}\frac{U(x_n \pm 1)}{U(x_n)}\frac{U(x_n)}{W(x_n)}$$

for all large enough n. Using (4.7) again, we have $U'(x_n)/U(x_n) = W'(x_n)/W(x_n)$ for all n. Then by (4.5) and letting $n \to +\infty$, we obtain

$$c\lambda_1 \geq d[\frac{M}{M+\varepsilon}(e^{\lambda_1} + e^{-\lambda_1}) - 2] + b(hM - 1).$$

Letting $\varepsilon \to 0$, it follows from the assumption $M > 0$ that

$$c\lambda_1 \geq d[(e^{\lambda_1} + e^{-\lambda_1}) - 2] + b(hM - 1), \tag{4.19}$$

From (4.18) and (4.19) we see $M = 0$, a contradiction with $M > 0$. Thus, we obtain that $\lim_{\xi\to-\infty}[U(\xi)/W(\xi)] = 0$. Then applying Theorem 4 in [4] we obtain that

$$\lim_{\xi\to-\infty}[W'(\xi)/W(\xi)] = \nu_1.$$

In this case, we obtain $\nu_1 = \lambda_1$ by (4.15).

It remains to deal with the case when condition (4.16) holds. The aim is to show that $M = 0$. Assume that $M > 0$. By the definition of ξ_n and (4.16), we can see that

$$(U/W)'(\xi) \geq 0 \quad \text{for } \xi \in (\xi_{n+1} + 1, \xi_n - 1), n \in \mathbb{N} \tag{4.20}$$

if $\xi_{n+1} + 1 \leq \xi_n - 1$. This implies that we can choose a sequence $\{y_n\}$ such that $y_n \in [\xi_n - 1, \xi_n + 1]$ for $n \in \mathbb{N}$, $\lim_{n\to+\infty} y_n = -\infty$ and $\lim_{n\to+\infty}[U(y_n)/W(y_n)] = M > 0$ (if necessary by passing to a subsequence). Using (4.4) and $y_n \in [\xi_n-1, \xi_n+1]$, there is a constant $\beta > 0$ such that $U(\xi_n)/U(y_n) \geq \beta > 0$ for all n. Also, by $\lim_{n\to+\infty}[U(\xi_n)/W(\xi_n)] = 0$, it follows that

$$\frac{W(\xi_n)}{W(y_n)} = \frac{W(\xi_n)}{U(\xi_n)}\frac{U(\xi_n)}{U(y_n)}\frac{U(y_n)}{W(y_n)} \to +\infty \quad \text{as } n \to +\infty, \tag{4.21}$$

On the other hand, we shall prove actually that $W(\xi_n)/W(y_n)$ is bounded in n, which leads to a contradiction with (4.21), so that $M = 0$. It suffices to show that W'/W is bounded in $\mathbb{R}$. Here the proofs for the cases $c > 0$ and $c < 0$ are a little bit different. We first assume that $c > 0$. If W'/W is unbounded, then we may choose a sequence $\{x_n\}$ such that $\lim_{n\to+\infty} x_n = -\infty$ and $\lim_{n\to+\infty} W'(x_n)/W(x_n) = +\infty$. Since $c > 0$, there exists $\mu > 0$ satisfying (4.9). In particular, $W(\xi - 1)/W(\xi) \leq e^{\mu}$ for $\xi \in \mathbb{R}$. Thus, we see from (4.5) that

$$\lim_{n\to+\infty}[W(x_n + 1)/W(x_n)] = +\infty. \tag{4.22}$$

Next, we can conclude that $x_n \in [\xi_m - 1, \xi_m + 1]$ for some $m = m(n)$. Indeed, by (4.20),

$$[(U'/U)(\xi) - (W'/W)(\xi)](U/W)(\xi) = (U/W)'(\xi) \geq 0 \text{ if } \xi \in (\xi_j + 1, \xi_{j-1} - 1) \ \forall\ j.$$

It follows that $W'(\xi)/W(\xi) \leq \sup_{\mathbb{R}}[U'(\xi)/U(\xi)] < +\infty$ for $\xi \in (\xi_j + 1, \xi_{j-1} - 1)$ and $j \in \mathbb{N}$. Thus, by the definition of x_n, we see that $x_n \in [\xi_m - 1, \xi_m + 1]$ for some $m = m(n)$.

For sufficiently large $n \in \mathbb{N}$, by (4.16) and the definition of ξ_m, we can have

$$\frac{U(\xi_m - 1)}{W(\xi_m - 1)} \leq \frac{U(\xi_m)}{W(\xi_m)} \leq \frac{U(\xi_m + 2)}{W(\xi_m + 2)},$$

which implies that

$$\exp\left\{\int_{\xi_m-1}^{\xi_m+2}\left[\frac{U'(s)}{U(s)}-\frac{W'(s)}{W(s)}\right]ds\right\}\geq 1.$$

Set $E := (\xi_m-1,\xi_m+2)\backslash(x_n,x_n+1)$. Then by (4.9) we have

$$\begin{aligned}3\{\sup_{\xi\in\mathbb{R}}[U'(\xi)/U(\xi)]\} &\geq \int_{\xi_m-1}^{\xi_m+2}\frac{U'(s)}{U(s)}ds \geq \int_{\xi_m-1}^{\xi_m+2}\frac{W'(s)}{W(s)}ds \\ \geq \int_{x_n}^{x_n+1}\frac{W'(s)}{W(s)}ds &+ \int_E \frac{W'(s)}{W(s)}ds \geq \ln\frac{W(x_n+1)}{W(x_n)} - 3\mu,\end{aligned}$$

this contradicts with (4.22). Thus, we have proved that the boundedness of W'/W under the condition (4.16) and $c>0$. It follows that $W(\xi+s)/W(\xi)$ is uniformly bounded in $\xi\in\mathbb{R}$ and $s\in[-1,1]$. Thus, $W(\xi_n)/W(y_n)$ is bounded in n, since $|y_n-\xi_n|\leq 1$ for all n. By (4.21), we reach a contradiction so that $M=0$.

If $c<0$, we still can prove the boundedness of W'/W. Again, we use a contradiction argument, assume that W'/W is unbounded in $\mathbb{R}$. Since $c<0$, there exists a constant $L\gg 1$ such that

$$W(\xi+s)/W(\xi)\leq L \quad \forall s\in[0,1],\xi\in\mathbb{R}. \tag{4.23}$$

From (4.5) and using the boundedness of U/W in $\mathbb{R}$, we see that $W(\xi-1)/W(\xi)$ is unbounded in $\xi\in\mathbb{R}$.

Since $\limsup_{\xi\to-\infty}[U(\xi)/W(\xi)] := M>0$, similar to (4.12), we have the inequality

$$c\lambda_1 \geq d(e^{\lambda_1}+e^{-\lambda_1}-2)+b(hM-1),$$

where $\lambda_1>0$. It follows from $c<0$ that $hM-1<0$. Hence we can find $N\gg 1$ such that

$$(1-W(\xi))[hU(\xi)-W(\xi)]<0,\ \forall\ \xi\leq -N.$$

We now choose $z_0<-N$ such that $W(z_0-1)/W(z_0)>L$. Due to $W(-\infty)=0$, there exists $x_0\leq z_0$ such that

$$W(x_0)=\max\{W(\xi)|\ \xi\in(-\infty,z_0]\ \}.$$

If $x_0\in(z_0-1,z_0)$, then by (4.23) we obtain that $W(x_0)\geq W(z_0-1)\geq LW(z_0)\geq W(x_0+1)$, which implies

$$\begin{aligned}0=cW'(x_0) &= d[W(x_0+1)-W(x_0)]+d[W(x_0-1)-W(x_0)] \\ &\quad +b(1-W(x_0))[hU(x_0)-W(x_0)]<0,\end{aligned}$$

a contradiction. Thus, W'/W is bounded in $\mathbb{R}$. Similar to the case $c>0$, we can conclude that $M=0$.

Hence we have proved that $M = 0$ if $c \neq 0$. By using Theorem 4 in [4], we obtain that $\lim_{\xi\to-\infty} W'(\xi)/W(\xi) = \nu_1$. Moreover, by (4.15), we see that $\nu_1 = \lambda_1$ and so the proof of (i) is completed when **Case 1** occurs.

Case 2. U/W is monotone for $-\xi \gg 1$. Then $\lim_{\xi\to-\infty} U(\xi)/W(\xi)$ exists and is equal to 0. Again, Theorem 4 in [4] implies that $\lim_{\xi\to-\infty} W'(\xi)/W(\xi) = \nu_1$. Moreover, note that

$$(U'/U - W'/W)(U/W)(\xi) = (U/W)'(\xi) \geq 0 \quad \text{for} \quad -\xi \gg 1,$$

then $U'/U \geq W'/W$ for all $-\xi \gg 1$. Thus, we obtain that $\lambda_1 \geq \nu_1$. Therefore, we complete the proof of the lemma. □

Remark 4.1. *From the above lemma, we see that (ii) must happen if* $\nu_1 > \lambda_1$.

Concerning about the behavior at $\xi = \infty$, we let $\mu_1 > 0$ and $\mu_2 < 0$ be two real roots of

$$c\lambda = d(e^{\lambda} + e^{-\lambda} - 2) + b(1-h). \tag{4.24}$$

Also, let $\sigma_1 > 0$ and $\sigma_2 < 0$ be two real roots of

$$c\lambda = (e^{\lambda} + e^{-\lambda} - 2) - a. \tag{4.25}$$

Then, similar to Lemma 4.2, we have the following asymptotic behavior of the wave tails at $\xi = +\infty$.

Lemma 4.3. *Let* (c, U, V) *be a solution of* (P) *with* $c \neq 0$. *Then we have the following two alternatives.*

(i) *If* $\liminf_{\xi\to+\infty} V(\xi)/[1-U(\xi)] = 0$, *then*

$$\lim_{\xi\to+\infty} \frac{V(\xi)}{1-U(\xi)} = 0, \quad \lim_{\xi\to+\infty} \frac{U'(\xi)}{U(\xi)-1} = \sigma_2 \geq \mu_2 = \lim_{\xi\to+\infty} \frac{V'(\xi)}{V(\xi)}.$$

(ii) *If* $\liminf_{\xi\to+\infty} V(\xi)/[1-U(\xi)] > 0$, *then*

$$\lim_{\xi\to+\infty} \frac{V(\xi)}{1-U(\xi)} = \frac{1}{ak}\left\{(d-1)(e^{\mu_2} + e^{-\mu_2} - 2) + b(1-h)\right\} + \frac{1}{k} > 0,$$

$$\lim_{\xi\to+\infty} \frac{U'(\xi)}{U(\xi)-1} = \mu_2 = \lim_{\xi\to+\infty} \frac{V'(\xi)}{V(\xi)}.$$

With these two lemmas, we are ready to prove the monotonicity of wave profiles.

Proof of Theorem 3. The proof is by using the sliding method used in [5]. Indeed, it follows from Lemma 4.2 and Lemma 4.3 that there exists $N \gg 1$ such that $U' > 0$ and $W' > 0$ in $\mathbb{R} \setminus [-N, N]$. Also, by (1.12), we know the set

$$A := \{\eta > 0 |\ U(\xi + s) \geq U(\xi),\ W(\xi + s) \geq W(\xi),\ \forall\ s \geq \eta,\ \xi \in \mathbb{R}\}$$

is non-empty. Since we have the strong comparison principle (see also [10, Lemma 4.1]), we can derive that $\inf A = 0$. This implies that $U' \geq 0$ and $W' \geq 0$ in $\mathbb{R}$. Moreover, we can derive that $U' > 0$ and $W' > 0$ in $\mathbb{R}$. The proof is the same as that of [10, Theorem 3]. □

In the following two lemmas we shall focus on the asymptotic behavior of U at $\xi = -\infty$.

Lemma 4.4. *Let (c, U, V) be a solution of (P) with $c \neq 0$. Then there exists two positive constants k_1 and k_2 such that*

$$k_1 e^{\lambda_1 \xi} \leq U(\xi) \leq k_2 e^{\lambda_1 \xi} \text{ for all } \xi \in (-\infty, 0].$$

Proof. First, it follows from Lemma 4.2 that there exist constants $\gamma > 0$ and $M > 0$ such that

$$W(\xi) \leq M e^{\gamma \xi} \tag{4.26}$$

for all $\xi \in (-\infty, 0]$. We now define the function

$$\phi(\xi) := \varepsilon + e^{\lambda_1 \xi} - \delta e^{(\lambda_1+\gamma)\xi},$$

where $\varepsilon \geq 0$ and $\delta > 0$ are two free parameters. For all $\xi < 0$ such that $\phi > 0$, using (4.26) and by a direct calculation, we get

$$\begin{aligned}
& c\phi'(\xi) - D_2[\phi(\xi)] - a\phi(\xi)(1 - \phi(\xi) - k(1 - W(\xi))) \\
= \;& \delta A e^{(\lambda_1+\gamma)\xi} + a(k-1)\varepsilon - ak\phi(\xi)W(\xi) + a[\phi(\xi)]^2 \\
\geq \;& \delta A e^{(\lambda_1+\gamma)\xi} + a(k-1)\varepsilon - akM(\varepsilon + e^{\lambda_1\xi} - \delta e^{(\lambda_1+\gamma)\xi})e^{\gamma\xi} \\
\geq \;& \delta e^{(\lambda_1+\gamma)\xi}[A - akM/\delta] + a\varepsilon[(k-1) - Mke^{\gamma\xi}],
\end{aligned}$$

where

$$A := -c(\lambda_1 + \gamma) + [e^{\lambda_1+\gamma} + e^{-(\lambda_1+\gamma)} - 2 + a(1-k)] > 0.$$

Note that $k > 1$. Then, by choosing $\delta > akM/A$, we conclude that there is $x_0 \gg 1$ such that

$$c\phi' \geq D_2[\phi] + a\phi(1 - \phi - k(1 - W)) \quad \text{in } (-\infty, x_0] \tag{4.27}$$

and $\phi' > 0$ in $(-\infty, -x_0]$ for all $\varepsilon \geq 0$.

By virtue of the property of ϕ, we are ready to derive $U(\xi) \leq k_2 e^{\lambda_1 \xi}$ for all $\xi \in (-\infty, 0]$. To emphasis the dependence on ε, we write ϕ as ϕ_ε. For a suitable translation we can choose $\xi_1 > 0$ such that

$$\overline{U}(\xi) := U(\xi - \xi_1) \leq \phi_0(\xi) := e^{\lambda_1\xi} - \delta e^{(\lambda_1+\gamma)\xi} \tag{4.28}$$

for all $\xi \in (-x_0 - 1, -x_0]$. For $\varepsilon = 1$, we have

$$\overline{U}(\xi) < \phi_1(\xi) \tag{4.29}$$

for all $\xi \in (-\infty, -x_0]$, since $\overline{U}(\cdot) < 1 \leq \phi_1(\cdot)$ in $(-\infty, -x_0]$.

We now claim that

$$\overline{U}(\xi) \leq \phi_0(\xi) \text{ for all } \xi \in (-\infty, -x_0]. \tag{4.30}$$

If not, by (4.28) and (4.29) there exists $\varepsilon_1 \in (0,1)$ such that $\overline{U}(\xi) \leq \phi_{\varepsilon_1}(\xi)$ for all $\xi \in (-\infty, -x_0]$ and $\overline{U}(z) = \phi_{\varepsilon_1}(z)$ for some $z \in (-\infty, -x_0 - 1]$. Note that $\overline{U}'(z) = \phi'_{\varepsilon_1}(z)$. Using (4.27), we obtain

$$\begin{aligned} & D_2[\overline{U}(z)] + a\overline{U}(z)(1 - \overline{U}(z) - k(1 - \overline{W}(z))) \\ \geq \ & D_2[\phi_{\varepsilon_1}(z)] + a\phi_{\varepsilon_1}(z)(1 - \phi_{\varepsilon_1}(z) - k(1 - W(z))), \end{aligned}$$

where $\overline{W}(\xi) := W(\xi - \xi_1)$. This implies that

$$\overline{U}(z+1) + \overline{U}(z-1) + ak\overline{U}(z)\overline{W}(z) \geq \phi_{\varepsilon_1}(z+1) + \phi_{\varepsilon_1}(z-1) + ak\phi_{\varepsilon_1}(z)W(z).$$

But, $\overline{U}(\xi) \leq \phi_{\varepsilon_1}(\xi)$ and $W' > 0$ in $\mathbb{R}$, we reach a contradiction. Hence we have proved (4.30) and therefore there exists $k_2 > 0$ such that

$$U(\xi) \leq k_2 e^{\lambda_1 \xi} \text{ for all } \xi \in (-\infty, 0].$$

The proof of the other inequality is quite similar to the above case. First, we set

$$B := -c(2\lambda_1) + (e^{2\lambda_1} + e^{-2\lambda_1} - 2) + a(1-k) > 0$$

and define the function

$$\psi(\xi) = \psi_\varepsilon(\xi) := -\varepsilon + \kappa e^{\lambda_1 \xi} + e^{2\lambda_1 \xi},$$

where $0 < \kappa < \sqrt{B/4a}$ is fixed and $0 \leq \varepsilon \leq (k-1)/2$.

Next, we choose $y_0 \gg 1$ such that

$$W(\xi) < \frac{k-1}{2k}, \quad e^{2\lambda_1 \xi} < \frac{B}{4a}, \quad \text{for all } \xi \in (-\infty, -y_0].$$

Then

$$k - 1 - kW - \varepsilon \geq 0 \text{ in } \xi \in (-\infty, -y_0].$$

Hence for $\xi \in (-\infty, -y_0]$,

$$\begin{aligned} & c\psi' - D_2[\psi] - a\psi(1 - \psi - k(1 - W)) \\ \leq \ & -Be^{2\lambda_1 \xi} - a(k-1)\varepsilon + a(-\varepsilon + \kappa e^{\lambda_1 \xi} + e^{2\lambda_1 \xi})^2 + a\varepsilon k W \\ \leq \ & -Be^{2\lambda_1 \xi} - a(k-1)\varepsilon + a\varepsilon^2 + 2a\kappa^2 e^{2\lambda_1 \xi} + 2ae^{4\lambda_1 \xi} + a\varepsilon k W \\ \leq \ & e^{2\lambda_1 \xi}[-B + 2a\kappa^2 + 2ae^{2\lambda_1 \xi}] - a\varepsilon(k - 1 - kW - \varepsilon) \\ \leq \ & 0, \end{aligned}$$

i.e.,

$$c\psi' \leq D_2[\psi] + a\psi(1 - \psi - k(1 - W)) \text{ in } (-\infty, -y_0]. \tag{4.31}$$

We are ready to prove that there exists $k_1 > 0$ such that $U(\xi) \geq k_1 e^{\lambda_1 \xi}$ for all $\xi \in (-\infty, 0]$. By a suitable translation, we may assume without loss of generality that

$$U(\xi) \geq \psi_0(\xi) = \kappa e^{\lambda_1 \xi} + e^{2\lambda_1 \xi} \text{ for all } \xi \in (-y_0 - 1, -y_0],$$
$$U(\xi) \geq \psi_{\varepsilon_2}(\xi) = -\varepsilon_2 + \kappa e^{\lambda_1 \xi} + e^{2\lambda_1 \xi} \text{ for all } \xi \in (-\infty, -y_0],\ \varepsilon_2 := (k-1)/2.$$

Then we claim that $U \geq \psi_0$ in $\in (-\infty, -y_0]$. Otherwise, there exists $\varepsilon_3 \in (0, \varepsilon_2)$ such that $U \geq \psi_{\varepsilon_3}$ in $(-\infty, -y_0]$ and $U(z) = \psi_{\varepsilon_3}(z)$ for some $z \in (-\infty, -y_0 - 1]$. Since $\psi_{\varepsilon_3}(\xi) \to -\varepsilon_3 < 0$ as $\xi \to -\infty$, we may assume that $U(\xi) > \psi_{\varepsilon_3}(\xi)$ for all $\xi < z$. It follows from $U'(z) = \psi'_{\varepsilon_3}(z)$ and (4.31) that

$$U(z+1) + U(z-1) \leq \psi_{\varepsilon_3}(z+1) + \psi_{\varepsilon_3}(z-1).$$

This is a contradiction. Thus $U \geq \psi_0$ in $\in (-\infty, -y_0]$ and so there is $k_1 > 0$ such that $U(\xi) \geq k_1 e^{\lambda_1 \xi}$ for all $\xi \in (-\infty, 0]$. Then the lemma follows. ☐

Lemma 4.5. *There exists a positive constant C_1 such that*

$$\lim_{\xi \to -\infty} \frac{U(\xi)}{e^{\lambda_1 \xi}} = C_1. \tag{4.32}$$

Proof. Firstly, set $R(\xi) := U(\xi)/e^{\lambda_1 \xi}$. Due to Lemma 4.2, we have

$$\lim_{\xi \to -\infty} R'(\xi) = \lim_{\xi \to -\infty} R(\xi) \left\{ \frac{U'(\xi)}{U(\xi)} - \lambda_1 \right\} = 0. \tag{4.33}$$

Define

$$l =: \liminf_{\xi \to -\infty} R(\xi) \leq \limsup_{\xi \to -\infty} R(\xi) := L.$$

We see from Lemma 4.4 that $0 < l \leq L < +\infty$. To prove $l = L$, we use a contradiction argument. Assume that $L > l$. Then we divide our discussion into three cases as follows.

Case 1. $\lim_{\xi \to -\infty} [U(\xi)/W(\xi)] \in [0, k)$. Then there exist constants $\theta_1 \in (0,1)$ and $y_1 > 0$ such that

$$U(\xi)/[kW(\xi)] \leq \theta_1 \text{ for all } \xi \in (-\infty, -y_1 + 1). \tag{4.34}$$

It follows from (4.34) that there exists $\alpha \in (l, L)$ such that $kW(\xi) > \alpha e^{\lambda_1 \xi}$ for all $\xi \in (-\infty, -y_1 + 1)$. Define $\phi(\xi) := \alpha e^{\lambda_1 \xi}$. It is easy to calculate that

$$c\phi' \leq D_2[\phi] + a\phi(1 - \phi - k(1 - W)) \text{ in } (-\infty, -y_1). \tag{4.35}$$

By (4.33), we may choose $z_1 < z_2 < -y_1 - 2$ such that $R(\cdot) \geq \alpha$ in $[z_1 - 1, z_1] \cup [z_2, z_2 + 1]$ and $R(\eta) < \alpha$ for some $\eta \in (z_1, z_2)$. This is equivalent to $U \geq \phi$ in $[z_1 - 1, z_1] \cup [z_2, z_2 + 1]$ and $U(\eta) < \phi(\eta)$. Thus, we can find $\xi_0 > 0$ such that

$$\overline{U}(\xi) := U(\xi + \xi_0) \geq \phi(\xi) \text{ for all } \xi \in [z_1 - 1, z_2 + 1] \tag{4.36}$$

and $\overline{U}(z_3) = \phi(z_3)$ for some $z_3 \in [z_1, z_2]$. Note that $z_3 \in [z_1, z_2]$ is due to $\xi_0 > 0$ and $U' > 0$ in $\mathbb{R}$. Also, noting $\overline{U}'(z_3) = \phi'(z_3)$ and using (4.35) we have

$$\overline{U}(z_3+1) + \overline{U}(z_3-1) + ak\overline{U}(z_3)\overline{W}(z_3) \le \phi(z_3+1) + \phi(z_3-1) + ak\phi(z_3)W(z_3)$$

which contradicts with (4.36). Hence we conclude that $l = L$; namely, (4.32) holds for some positive constant C_1.

Case 2. $\lim_{\xi\to-\infty}[U(\xi)/W(\xi)] \in (k, \infty)$. Then there exists $\theta_2 > 0$ and $y_2 > 0$ such that

$$U(\xi)/[kW(\xi)] \ge \theta_2 > 1 \text{ for all } \xi \in (-\infty, -y_2).$$

Following the argument in the previous case, we just change the inequality sign reversely. Choose $\beta \in (l, L)$ such that $kW(\xi) < \beta e^{\lambda_1\xi}$ for all $\xi \in (-\infty, -y_2 + 1)$. Define $\phi(\xi) := \beta e^{\lambda_1\xi}$. Note that

$$c\phi' \ge D_2[\phi] + a\phi(1 - \phi - k(1 - W)) \text{ in } (-\infty, -y_2).$$

Also, we can choose $x_1 < x_2 < -y_2 - 2$ such that $R(\xi) \le \beta$ in $[x_1 - 1, x_1] \cup [x_2, x_2 + 1]$ and $R > \beta$ for some point in (x_1, x_2). Thus, we can find $\xi_0 > 0$ such that

$$\begin{aligned} &\overline{U}(\xi) := U(\xi - \xi_0) \le \phi(\xi) \text{ for all } \xi \in [x_1 - 1, x_2 + 1]; \\ &\overline{U}(x_3) = \phi(x_3) \text{ for some } x_3 \in [x_1, x_2]. \end{aligned} \tag{4.37}$$

Note that $x_3 \in [x_1, x_2]$, since $\xi_0 > 0$ and $U' > 0$ in $\mathbb{R}$. It follows from $\overline{U}'(x_3) = \phi'(x_3)$ that

$$\overline{U}(x_3+1) + \overline{U}(x_3-1) + ak\overline{U}(x_3)\overline{W}(x_3) \ge \phi(x_3+1) + \phi(x_3-1) + ak\phi(x_3)W(x_3)$$

which contradicts with (4.37). Hence we conclude that $l = L$

Case 3. $\lim_{\xi\to-\infty}[U(\xi)/W(\xi)] = k$. In this case we have from Lemma 4.2 the equality

$$c\lambda_1 = d(e^{\lambda_1} + e^{-\lambda_1} - 2) + b(hk - 1). \tag{4.38}$$

Let $Q(\xi) := W(\xi)/e^{\lambda_1\xi}$. Then $\liminf_{\xi\to-\infty}[kQ(\xi)] = l$ and $\limsup_{\xi\to-\infty}[kQ(\xi)] = L$. Choose $\gamma \in (l, L)$ and define the function $\psi(\xi) := [\gamma e^{\lambda_1\xi}]/k$. A direct calculation gives us that for some $z_0 \gg 1$

$$c\psi' \ge dD_2[\psi] + b(1 - \psi)(hk\psi - \psi) \text{ for all } (-\infty, -z_0), \tag{4.39}$$

by using (4.38).

Note that $Q'(\xi) \to 0$ as $\xi \to -\infty$, so that we may choose $z_1 < z_2 < -z_0 - 2$ such that

$$R(\cdot),\ kQ(\cdot) \le (\gamma + l)/2 < \gamma \text{ in } [z_1 - 1, z_1] \cup [z_2, z_2 + 1] \tag{4.40}$$

and $kQ > \gamma$ at some point in (z_1, z_2). Now we again consider the translation

$$\overline{W}(\xi) := W(\xi - \xi_0),$$

for some $\xi_0 > 0$ so that the following hold

$$\begin{aligned}
&\overline{W}(\xi) \le \psi(\xi) \text{ for all } \xi \in [z_1 - 1, z_2 + 1];\\
&\overline{W}(z_3) = \psi(z_3) \text{ for some } z_3 \in [z_1, z_2], \text{ (since } W' > 0 \text{ in } \mathbb{R}),
\end{aligned}$$

where $z_3 \in [z_1, z_2]$ is the minimal value such that the equality holds. Since $\overline{W}'(z_3) = \psi'(z_3)$, by (4.39) we obtain

$$\begin{aligned}
&\overline{W}(z_3 + 1) + \overline{W}(z_3 - 1) + b(1 - \overline{W}(z_3))(h\overline{U}(z_3) - \overline{W}(z_3))\\
\ge\ &\psi(z_3 + 1) + \psi(z_3 - 1) + b(1 - \psi(z_3))(hk\psi(z_3) - \psi(z_3)),
\end{aligned}$$

which implies that $\overline{U}(z_3) > k\psi(z_3)$.

On the other hand, from (4.40) we know $\overline{U}(\xi) < U(\xi) < k\psi(\xi)$ in $[z_1 - 1, z_1] \cup [z_2, z_2 + 1]$. Hence there exists $\xi_1 > 0$ such that $\hat{U}(\xi) := \overline{U}(\xi - \xi_1) \le k\psi(\xi)$ for all $\xi \in [z_1 - 1, z_2 + 1]$ and $\hat{U}(z_4) = k\psi(z_4)$ for some $z_4 \in [z_1, z_2]$. Moreover, we have

$$\hat{W}(\xi) < \overline{W}(\xi) \le \psi(\xi) \text{ for all } \xi \in [z_1 - 1, z_2 + 1]. \tag{4.41}$$

Here $z_4 \in [z_1, z_2]$ can be chosen as the left-most point such that $\hat{U} = k\psi$. Then $k\psi'(z_4) = \hat{U}'(z_4)$ and

$$c\psi' \ge D_2[\psi] + a\psi(1 - k\psi - k(1 - \psi)).$$

This gives $\hat{W}(z_4) > \psi(z_4)$, a contradiction to (4.41). Thus (4.32) holds for some positive constant C_1. Hence the lemma follows. □

Proof of Proposition 4.1. By Lemma 4.5, it remains to show that W has the desired exponential decay. We divide the proof into three cases. The arguments are quite similar to the previous two lemmas, by constructing suitable ϕ and ψ.

Case 1. $\lambda_1 < \nu_1$. We see from Remark 4.1 that

$$\lim_{\xi \to -\infty} \frac{U(\xi)}{W(\xi)} := A > 0.$$

By Lemma 4.5, we have

$$\lim_{\xi \to -\infty} \frac{W(\xi)}{e^{\lambda_1 \xi}} = \lim_{\xi \to -\infty} \frac{W(\xi)}{U(\xi)} \frac{U(\xi)}{e^{\lambda_1 \xi}} = \frac{C_1}{A}.$$

Case 2. $\lambda_1 > \nu_1$. We first fix $\tau \in (\nu_1, \min\{\lambda_1, 2\nu_1\})$ and define two functions

$$\begin{aligned}
\phi(\xi) &:= \varepsilon + e^{\nu_1 \xi} - \delta_1 e^{\tau \xi},\ \varepsilon \in [0, 1/2],\\
\psi(\xi) &:= -\varepsilon + \delta_2 e^{\nu_1 \xi} + e^{(\nu_1 + \lambda_1)\xi},\ \varepsilon \ge 0,
\end{aligned}$$

where two positive constants $\delta_1 \gg 1$ and $\delta_2 \ll 1$ are to be determined.

For a given solution (U, W), we now show

$$c\phi' \geq dD[\phi] + b(1-\phi)(hU-\phi) \quad \text{in } (-\infty, -y_1] \tag{4.42}$$

$$c\psi' \leq dD[\psi] + b(1-\psi)(hU-\psi) \quad \text{in } (-\infty, -y_2], \tag{4.43}$$

for some $y_1, y_2 \gg 1$. Set

$$A := -c\tau + d(e^{\tau} + e^{-\tau} - 2) - b.$$

Note that $A > 0$, since $\tau > \nu_1$. Then for a fixed constant $\delta_1 \geq (2b + bhM)/A$, there exists $y_1 \gg 1$ such that

$$e^{\nu_1 \xi} - \delta_1 e^{\tau \xi} > 0 \text{ in } (-\infty, -y_1]. \tag{4.44}$$

Pick $M > 0$ such that

$$U(\xi) \leq M e^{\lambda_1 \xi} \tag{4.45}$$

for all $\xi \in (-\infty, -y_1]$. Hence, by (4.44) and (4.45), we have

$$\begin{aligned}
& c\phi' - dD[\phi] - b(1-\phi)(hU-\phi) \\
\geq\ & \delta_1 A e^{\tau\xi} - bhMe^{\lambda_1\xi} + b\varepsilon - b(\varepsilon + e^{\nu_1\xi} - \delta_1 e^{\tau\xi})^2 \\
\geq\ & \delta_1 A e^{\tau\xi} - bhMe^{\lambda_1\xi} + b\varepsilon - 2b\varepsilon^2 - 2be^{2\nu_1\xi} \\
\geq\ & \delta_1 e^{\tau\xi}[A - 2b/\delta_1 - bhM/\delta_1] + 2b\varepsilon(1/2 - \varepsilon) \geq 0
\end{aligned}$$

for all $\xi \in (-\infty, 0]$, since $\delta_1 \geq (2b + bhM)/A$ and $\varepsilon \in [0, 1/2]$. Thus we have (4.42) for a fixed $\delta_1 \geq (2b + bhM)/A$ and for any $\varepsilon \in [0, 1/2]$.

For (4.43), we can obtain that for all $\xi \in (-\infty, 0]$,

$$\begin{aligned}
& c\psi' - dD[\psi] - b(1-\psi)(hU-\psi) \\
\leq\ & -Be^{(\nu_1+\lambda_1)\xi} + \delta_2 bhUe^{\nu_1\xi} + bhUe^{(\nu+\lambda_1)\xi} \\
\leq\ & e^{(\nu_1+\lambda_1)\xi}[-B + \delta_2 bhM + hbMe^{\lambda_1\xi}],
\end{aligned}$$

where $B := -c(\nu_1 + \lambda_1) + [d(e^{\nu_1+\lambda_1} + e^{-\nu_1-\lambda_1} - 2) - b] > 0$ and $M > 0$ is defined as in (4.45). Hence, as long as we choose $\delta_2 < B/bhM$, there exist $z_0 \gg 1$ such that (4.43) holds for all $\varepsilon \geq 0$.

By using the same argument in Lemma 4.4, we can derive

$$h_1 e^{\nu_1\xi} \leq W(\xi) \leq h_2 e^{\nu_1\xi} \quad \text{for all } \xi \in (-\infty, 0].$$

for some h_1 and $h_2 > 0$. Next we shall follow the steps of Case 1 in Lemma 4.5 to derive that

$$\lim_{\xi \to -\infty} \frac{W(\xi)}{e^{\nu_1\xi}} = C_2,$$

for some $C_2 > 0$. Set $Q(\xi) := W(\xi)/e^{\nu_1\xi}$ and

$$l =: \liminf_{\xi\to-\infty} Q(\xi) \le \limsup_{\xi\to-\infty} Q(\xi) := L,$$

Note that $0 < l \le L < +\infty$.

Assume that $l \neq L$. Pick $\alpha \in (l, L)$, and consider the function

$$\pi(\xi) := \alpha e^{\nu_1\xi},$$

then there exists $-x_0 < 0$ such that

$$c\pi' \le dD[\pi] + b(1-\pi)(hU - \pi) \text{ in } (-\infty, -x_0].$$

On the other hand, since $Q'(\xi) \to 0$ as $\xi \to -\infty$, there exist $x_1 < x_2 < -x_0 + 2$ such that $Q(\xi) > \alpha$ for all $[x_1 - 1, x_2 + 1]$, while $Q(z) < \alpha$ for some $z \in (x_1, x_2)$.

Then, we can find $\xi_1 > 0$ such that $\overline{W}(\xi) := W(\xi + \xi_1)$ such that

$$\begin{aligned} &\overline{W}(\xi) := W(\xi+\xi_1) \ge \pi(\xi) \;\text{ for all } \xi \in [x_1 - 1, x_2 + 1]; \\ &\overline{W}(z) = \pi(z) \;\text{ for some } z \in [x_1, x_2]. \end{aligned} \tag{4.46}$$

Note that $z \in [x_1, x_2]$, since $W' > 0$ in $\mathbb{R}$. It follows from $\overline{U}'(z) = \pi'(z)$ that

$$\begin{aligned} &d\overline{W}(z+1) + d\overline{W}(z-1) + b(1-\overline{W})(h\overline{U} - \overline{W}) \\ &\le d\pi(z+1) + d\pi(z-1) + b(1-\pi)(hU-\pi) \end{aligned}$$

which contradicts with (4.46). Hence we conclude that $l = L$.

Case 3. $\lambda_1 = \nu_1$. Firstly, we note that $\theta := c - de^{\lambda_1} + de^{-\lambda_1} < 0$. Indeed, it is easy to observe that $\theta = \Psi'(\lambda_1) < 0$, since $\lambda_1 = \nu_1$ and ν_1 is the only positive root of

$$0 = \Psi(\lambda) := c\lambda - d(e^{\lambda} + e^{-\lambda} - 2) + b.$$

Define

$$\phi(\xi) := \varepsilon - \delta_1\xi e^{\lambda_1\xi}, \quad \psi(\xi) := -\varepsilon - \delta_2\xi e^{\lambda_1\xi}$$

where $\varepsilon \ge 0$ and $\delta_1 \gg 1$ and $0 < \delta_2 \ll 1$ are to be determined. We shall prove that there exists $\eta_i > 0$, $i = 1, 2$ such that

$$c\phi' \ge dD[\phi] + b(1-\phi)(hU - \phi) \;\text{ in } (-\infty, -\eta_1] \tag{4.47}$$

$$c\psi' \le dD[\psi] + b(1-\psi)(hU - \psi) \;\text{ in } (-\infty, -\eta_2]. \tag{4.48}$$

For (4.47) and consider $\xi \in (-\infty, 0]$, direct calculation implies

$$\begin{aligned} & c\phi' - dD[\phi] - b(1-\phi)(hU - \phi) \\ \ge\; & -\theta\delta_1 e^{\lambda_1\xi} - bhU(1-\phi) + b\varepsilon - b\phi^2 \\ \ge\; & \delta_1 e^{\lambda_1\xi}[-\theta - bhM/\delta_1 - 2\delta_1 b|\xi|^2 e^{\lambda_1\xi}] + b\varepsilon(1-2\varepsilon), \end{aligned}$$

where $M > 0$ is defined as in (4.45). Thus when we fix $\delta_1 > -bhM/\theta$, there is $\eta_1 > 0$ such that (4.47) holds for any $0 \le \varepsilon \le 1/2$.

For (4.48), let $m > 0$ such that $U(\xi) \ge me^{\lambda_1 \xi}$ for all $\xi \in (-\infty, 0]$. Consider $\xi < 0$, we have

$$\begin{aligned} & c\psi' - dD[\psi] - b(1-\psi)(hU - \psi) \\ \le\ & -\theta\delta_2 e^{\lambda_1 \xi} - bhU(1-\psi) - b\varepsilon, \\ \le\ & e^{\lambda_1 \xi}[-\theta\delta_2 - bhm + bhm\delta_2|\xi|e^{\lambda_1 \xi}] - b\varepsilon. \end{aligned}$$

Thus by fixing $\delta_2 > 0$ small enough, we can find $\eta_2 > 0$ such that (4.48) holds for all $\varepsilon > 0$. Therefore, by the same argument of Lemma 4.4 we can establish

$$h_3|\xi|e^{\lambda_1 \xi} \le W(\xi) \le h_4|\xi|e^{\lambda_1 \xi} \ \text{ for all } \xi \in (-\infty, 0].$$

for some h_3 and $h_4 > 0$.

Finally, we shall derive

$$\lim_{\xi \to -\infty} \frac{W(\xi)}{|\xi|e^{\lambda_1 \xi}} = C, \tag{4.49}$$

for some $C > 0$. To prove this, set

$$l =: \liminf_{\xi \to -\infty} Q(\xi) \le \limsup_{\xi \to -\infty} Q(\xi) := L,$$

where $Q(\xi) := W(\xi)/|\xi|e^{\lambda_1 \xi}$. Claim $l = L$. If not, we can chose $\alpha \in (l, L)$ and

$$\alpha \ne bhC_1/\omega, \quad \omega := -c + de^{\lambda_1} - de^{-\lambda_1} > 0,$$

where $C_1 := \lim_{\xi \to -\infty} U(\xi)/e^{\lambda_1 \xi}$.

If $\alpha > bhC_1/\omega$, we use the same argument in Case 2 of Lemma 4.5. Set $\phi(\xi) := \alpha|\xi|e^{\lambda_1 \xi}$, Note that for $\xi < 0$,

$$\begin{aligned} & c\phi' - dD_2[\phi] - b(1-\phi)(hU - \phi) \\ \ge\ & \alpha\omega e^{\lambda_1 \xi} - bhU - b\alpha^2|\xi|^2 e^{2\lambda_1 \xi} \\ =\ & e^{\lambda_1 \xi}[\alpha\omega - bhU(\xi)/e^{\lambda_1 \xi} - b\alpha^2|\xi|^2 e^{\lambda_1 \xi}]. \end{aligned}$$

Since $\alpha > bhC_1/\omega$, we have

$$\alpha\omega - bhU(\xi)/e^{\lambda_1 \xi} - b\alpha^2|\xi|^2 e^{\lambda_1 \xi} \to \alpha\omega - bhC_1 > 0 \ \text{ as } \xi \to -\infty.$$

Thus we can find $y_1 \gg 1$ such that

$$c\phi' \ge dD_2[\phi] + b(1-\phi)(hU - \phi) \text{ in } (-\infty, -y_1].$$

Next, by choosing $x_1 < x_2 < y_1 - 2$ such that $Q(\xi) \le \alpha$ in $[x_1 - 1, x_1] \cup [x_2, x_2 + 1]$ and $Q > \alpha$ for some point in (x_1, x_2). Using the translation likes (4.37), we can get a contradiction such that $l = L$.

If $\alpha < bhC_1/\omega$, we then use the argument in Case 1 of Lemma 4.5. Let $\psi(\xi) := \alpha|\xi|e^{\lambda_1\xi}$. Then

$$\begin{aligned} & c\psi' - dD_2[\psi] - b(1-\psi)(hU-\psi) \\ \le\ & \alpha\omega e^{\lambda_1\xi} - bhU + bhU\psi \\ =\ & e^{\lambda_1\xi}[\alpha\omega - bhU(\xi)/e^{\lambda_1\xi} + bh\psi U/e^{\lambda_1\xi}]. \end{aligned}$$

Using $\alpha < bhC_1/\omega$, then

$$\alpha\omega - bhU(\xi)/e^{\lambda_1\xi} + bh\psi U/e^{\lambda_1\xi} \to \alpha\omega - bhC_1 < 0 \text{ as } \xi \to -\infty.$$

Thus we can find $y_2 \gg 1$ such that

$$c\psi' \le dD_2[\psi] + b(1-\psi)(hU-\psi) \text{ in } (-\infty, -y_2].$$

Finally, by using the argument in Case 1 of Lemma 4.5, it is not hard to derive $l = L$, namely (4.49) holds. Therefore, we have completed the proof of the proposition. □

Similarly, we can prove the following asymptotic behavior of the wave tails at $\xi = +\infty$.

Proposition 4.6. *Let (c, U, V) be a solution of (P) with $c \neq 0$. Then there exist $C_i > 0$, $i = 3, 4$, such that*

$$\lim_{\xi\to+\infty} \frac{1-U(\xi)}{|\xi|^p e^{\beta\xi}} = C_3, \quad \lim_{\xi\to+\infty} \frac{V(\xi)}{e^{\mu_2\xi}} = C_4,$$

where $p = 0$ if $\sigma_2 \neq \mu_2$, $p = 1$ if $\sigma_2 = \mu_2$ and $\beta := \max\{\mu_2, \sigma_2\}$.

We are ready to prove Theorem 4.

Proof of Theorem 4. Let (c_i, U_i, V_i), $i = 1, 2$, be two arbitrary solutions of (P) with $c_i \neq 0$ for $i = 1, 2$. Without loss of generality we may assume that $U_1(0) = U_2(0) = 1/2$ by suitable translations.

To prove that $c_1 = c_2$, we may assume that $c_1 \le c_2$ without loss of generality. For a contradiction, suppose that $c_1 < c_2$. From the characteristic equations (4.2), (4.3), (4.24) and (4.25), we can see that $\lambda_1(c)$, $\nu_1(c)$, $\mu_2(c)$ and $\sigma_2(c)$ are strictly increasing in c. Thus, applying Propositions 4.1 and 4.6 we can find $x_0 \gg 1$ such that

$$U_1(\cdot) > U_2(\cdot),\ W_1(\cdot) > W_2(\cdot) \text{ on } \mathbb{R}\backslash[-x_0, x_0],$$

where $W_i := 1 - V_i$, $i = 1, 2$.

Since $U_1(0) = U_2(0) = 1/2$ and both U_i and W_i are strictly increasing in $\mathbb{R}$, we can find $\xi_0 \in [-x_0, x_0]$ and $\eta \ge 0$ such that one of the following two cases will occur:

$$U_1(\xi_0) = U_2(\xi_0 - \eta),\ U_1(\cdot) \ge U_2(\cdot - \eta),\ W_1(\cdot) \ge W_2(\cdot - \eta) \text{ on } \mathbb{R}. \tag{4.50}$$

$$W_1(\xi_0) = W_2(\xi_0 - \eta),\ U_1(\cdot) \ge U_2(\cdot - \eta),\ W_1(\cdot) \ge W_2(\cdot - \eta) \text{ on } \mathbb{R}. \tag{4.51}$$

If (4.50) occurs, then $U_1'(\xi_0) = U_2'(\xi_0 - \eta)$ and $D_2[U_1(\xi_0)] - D_2[U_2(\xi_0 - \eta)] \geq 0$. But, from the equation (1.10), we have

$$\begin{aligned} 0 &\leq c_1 U_1'(\xi_0) - c_2 U_2'(\xi_0 - \eta) - aU_1(\xi_0)[1 - U_1(\xi_0) - k(1 - W_1(\xi_0))] \\ &\quad + aU_2(\xi_0 - \eta)[1 - U_2(\xi_0 - \eta) - k(1 - W_2(\xi_0 - \eta))] \\ &= -(c_2 - c_1)U_2'(\xi_0 - \eta) - akU_2(\xi_0 - \eta)[W_1(\xi_0) - W_2(\xi_0 - \eta)] < 0, \end{aligned}$$

a contradiction. Similarly, if (4.51) occurs, due to $W_1'(\xi_0) = W_2'(\xi_0 - \eta)$ and $D_2[W_1(\xi_0)] - D_2[W_2(\xi_0 - \eta)] \geq 0$, the equation (1.11) gives us that

$$\begin{aligned} 0 &\leq c_1 W_1'(\xi_0) - c_2 W_2'(\xi_0 - \eta) - bh(1 - W_1(\xi_0))[U_1(\xi_0) - U_2(\xi_0 - \eta)] \\ &\leq -(c_2 - c_1)W_2'(\xi_0 - \eta) < 0, \end{aligned}$$

a contradiction again. Thus, we conclude that $c_1 = c_2$. □

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