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BLOW-UP BEHAVIOR FOR A QUASILINEAR PARABOLIC EQUATION WITH NONLINEAR BOUNDARY CONDITION

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Abstract. In this paper, we study the solution of an initial boundary value problem for a quasilinear parabolic equation with a nonlinear boundary condition. We first show that any positive solution blows up in finite time. For a monotone solution, we have either the single blow-up point on the boundary, or blow-up on the whole domain, depending on the parameter range. Then, in the single blow-up point case, the existence of a unique self-similar profile is proven. Moreover, by constructing a Lyapunov function, we prove the convergence of the solution to the unique self-similar solution as t approaching the blow-up time.

1. **Introduction.** In this paper, we study the following initial boundary value problem (P):

$$u_t = u^{1+\gamma} u_{xx},\ 0 < x < 1,\ t > 0, \tag{1.1}$$

$$u_x(0,t) = -u^q(0,t),\ u_x(1,t) = 0,\ t > 0, \tag{1.2}$$

$$u(x,0) = u_0(x),\ 0 \le x \le 1, \tag{1.3}$$

where $\gamma > 0$ and $q > 0$ are given constants, and u_0 is a positive bounded smooth function defined on $[0,1]$ such that $u_0'(0) = -u_0^q(0)$ and $u_0'(1) = 0$.

The local existence and uniqueness of positive solution of (P) can be derived by the standard theory of parabolic equation. We say that a solution u blows up in finite time T, if $\limsup_{t\to T^-}\{\max_{x\in[0,1]} u(x,t)\} = \infty$. The study of blow-up has attracted much attentions for past years. The typical questions are concerned about blow-up criteria, blow-up locations, blow-up rates, blow-up profiles, and so on. We refer the reader to the survey papers of Levine [18] and Deng-Levine [7], and the book by Samarskii-Galaktionov-Kurdyumov-Mikhailov [19]. Problem (P) with $\gamma = 0$ was studied by Ferreira-de Pablo-Rossi [8] for both the bounded interval and semi-infinite interval cases. For blow-up on the boundary, we refer the reader to the survey papers by Chlebík-Fila [4] and Fila-Filo [10].

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In [9], Ferreira-de Pablo-Quirós-Rossi studied the following initial boundary value problem $(\hat{P})$:

$$\hat{u}_\tau = (\hat{u}^{m-1}\hat{u}_\xi)_\xi,\ 0 < \xi < l,\ \tau > 0, \tag{1.4}$$
$$(\hat{u}^{m-1}\hat{u}_\xi)(0,\tau) = \hat{u}^m(0,\tau),\ (\hat{u}^{m-1}\hat{u}_\xi)(l,\tau) = 0,\ \tau > 0, \tag{1.5}$$
$$\hat{u}(\xi,0) = \hat{u}_0(\xi) > 0,\ 0 \le \xi \le l, \tag{1.6}$$

where $m < 0$ and $l > 0$ are constants. It is shown that the solution of (1.4)-(1.6) *quenches*, i.e., its minimum reaches zero in finite time. The first time when the minimum of the solution reaches zero is called the *quenching time*. Note that the phenomenon of quenching is different from the following dead-core problem (cf., e.g., [16] and the references cited therein):

$$u_t = u_{xx} - u^p,\ -1 < x < 1,\ t > 0,$$
$$u(\pm 1, t) = k,\ t > 0,$$
$$u(x,0) = u_0(x) > 0,\ -1 \le x \le 1,$$

where $0 < p < 1$ and k is a positive constant. For quenching, some derivative of the solution becomes singular at the quenching time. Indeed, for the problem (1.4)-(1.6), we have the time derivative blows up at the quenching time. On the other hand, in the dead-core problem, the solution stays regular whenever its minimum reaches zero in finite time and the solution can be continued for all time.

It is well-known that quenching problem is related to blow-up problem. Indeed, by setting

$$u = \hat{u}^m,\ \gamma = -1/m,\ \xi = \gamma x,\ \tau = \gamma^2 t,$$

the problem (1.4)-(1.6) becomes (1.1)-(1.3) with $q = 1$ and spatial domain $[0, l/\gamma]$. Therefore, in this paper we shall only consider the case when $q \neq 1$. Another related problem to (P) is about the blow-up behavior of the solution of the Cauchy problem for the equation

$$u_t = u^\sigma(\Delta u + u^p),\ x \in \mathbb{R}^n,\ t > 0,$$

where $\sigma \ge 1$ and $p > 1$. We refer the reader to [13, 14, 15] and the references cited therein. On the other hand, the Cauchy problem for the equation (1.1) in higher spatial dimension has been studied by Bertsch-Ughi [2] and Bertsch-Dal Passo-Ughi for nonnegative initial data. See also [20] for a more general equation.

In studying the blow-up behavior near the blow-up time, it is crucial to analyze the so-called (backward) self-similar solutions of (P). Let $T < \infty$ be the blow-up time and assume that $x = 0$ is a blow-up point. For $q > 1$, we introduce the following self-similar change of variables:

$$v(y,s) := (T-t)^\alpha u(x,t),\ y := \frac{x}{(T-t)^\beta},\ s := -\ln(T-t), \tag{1.7}$$

where the similarity exponents are given by

$$\alpha := \frac{1}{\gamma + 2q - 1},\ \beta := (q-1)\alpha.$$

Note that $\alpha > 0$ and $\beta > 0$, if $q > 1$. It follows that u satisfies (1.1)-(1.3) if and only if v satisfies

$$v_s = v^{1+\gamma} v_{yy} - \beta y v_y - \alpha v,\ 0 < y < R(s) := e^{\beta s}, s > s_0 := -\ln T, \tag{1.8}$$
$$v_y(0,s) = -v^q(0,s),\ v_y(R(s),s) = 0,\ s > s_0, \tag{1.9}$$
$$v(y,s_0) = v_0(y) := T^\alpha u_0(yT^\beta),\ 0 \le y \le 1/T^\beta. \tag{1.10}$$

Note that $s \to \infty$ and $R(s) \to \infty$ as $t \uparrow T^-$. We expect that, as $s \to \infty$, the solution of (1.8)-(1.10) is stabilized. In this paper, we shall call a global solution of the following problem as a self-similar profile of (P):

$$g'' - \beta y g^{-\gamma-1} g' - \alpha g^{-\gamma} = 0,\ 0 < y < \infty, \tag{1.11}$$

$$g'(0) = -g^q(0). \tag{1.12}$$

Since we are interested in the behavior of u as $t \uparrow T^-$, we shall be concerned with the positive global solution of (1.11)-(1.12). In particular, we are looking for a monotone decreasing positive global solution of (1.11)-(1.12).

This paper is organized as follows. In §2, we shall derive a blow-up criterion, prove the single point blow-up for monotone solutions when $q > 1$, and study the case when $q \in (0, 1)$. Motivated by a recent work [8], we shall prove in §3 that for $q > 1$ the self-similar profile exists and is unique, by using a phase plane analysis approach. Finally, by using a method of Zelenyak [21] (see also [15]), in §4, we shall prove the convergence of v, as $s \to \infty$, to the unique self-similar profile for $q > 1$.

2. **Blow-up Criterion and Location.** In this section, we first prove that the solution of the problem (P) always blows up in finite time.

Theorem 2.1. *Suppose that $q > 0$. Then for every positive bounded smooth initial data u_0, there exists a finite time $T > 0$ such that*

$$\limsup_{t \to T^-} \{ \max_{x \in [0,1]} u(x, t) \} = \infty. \tag{2.1}$$

Proof. By assumption, there is a positive constant δ such that $u_0 \geq \delta$ in $[0, 1]$. Then, by the maximum principle, $u(x, t) \geq \delta$ for the corresponding solution u of (P).

We introduce the following quantity

$$N(t) := \int_0^1 u^{-\gamma}(x, t) dx.$$

By differentiating $N(t)$ and using (1.1)-(1.2), we get

$$N'(t) = -\gamma u^q(0, t). \tag{2.2}$$

Since $q > 0$, we get

$$N'(t) \leq -\eta$$

for some constant $\eta > 0$. Thus $N(t)$ should vanish at some finite time. Therefore, the solution u cannot be bounded for all $t > 0$. This implies that there exists a finite $T > 0$ such that (2.1) holds and the theorem is proved. □

In the following, we shall always assume that the solution u of (P) blows up at time $T < \infty$. For simplicity, from now on we shall further assume that

$$u_0' \leq 0,\ u_0'' \geq 0 \quad \text{in } [0, 1]. \tag{2.3}$$

Using (2.3), it is easily seen by the strong maximum principle that $u_x < 0$, $u_{xx} > 0$ and $u_t > 0$.

We say that a point $x = a$ is a blow-up point, if there is a sequence $\{(x_n, t_n)\}$ such that $x_n \to a$, $t_n \to T^-$, and $u(x_n, t_n) \to \infty$ as $n \to \infty$. Note that $x = 0$ is always a blow-up point.

Theorem 2.2. *Suppose that $q > 1$. Under the assumption (2.3), $x = 0$ is the only blow-up point.*

Proof. Suppose, for contradiction, that there exists another blow-up point $a \in (0, 1]$. Then any point $b \in [0, a]$ is also a blow-up point, since $u_x < 0$ and $u_t > 0$.

Now we fix any number $b \in (0, a)$. Following [11], we consider the function

$$J(x, t) := u_x(x, t) + \varepsilon h(x) u^q(x, t),\ h(x) := (x - b)^2,\ \varepsilon > 0.$$

Then it is easy to compute that

$$\begin{aligned} & J_t - u^{1+\gamma} J_{xx} - (1+\gamma) u^\gamma u_x J_x \\ = \;& -\varepsilon(\gamma + q) q h u^{\gamma+q-1} u_x^2 - \varepsilon(1 + \gamma + 2q) h' u^{\gamma+q} u_x - \varepsilon h'' u^{\gamma+q+1} \\ \leq \;& 0 \text{ in } (0, b) \times (0, T), \end{aligned}$$

by using the properties of h and the fact that $u_x < 0$. Clearly, $J(b, t) < 0$ for all $t \in (0, T)$. Moreover, $J(0, t) = -u^q(0, t)(1 - \varepsilon b^2) \leq 0$ for all $t \in (0, T)$, if $\varepsilon < 1/b^2$. By choosing ε small enough and using $u_x(x, T/2) < 0$ in $[0, b]$, we have $J(x, T/2) \leq 0$ for all $x \in [0, b]$. Therefore, it follows from the maximum principle that $J \leq 0$ in $[0, b] \times [T/2, T)$, i.e.,

$$-u^{-q}(x, t) u_x(x, t) \geq \varepsilon(x - b)^2,\ x \in [0, b],\ t \in [T/2, T). \tag{2.4}$$

Integrating (2.4) from 0 to b, we obtain that

$$[u^{1-q}(b, t) - u^{1-q}(0, t)]/(q - 1) \geq \varepsilon \int_0^b (x - b)^2 dx = \varepsilon b^3/3 \ \ \forall\, t \in (T/2, T).$$

Letting $t \uparrow T^-$, we reach a contradiction. Thus the theorem follows. $\square$

For $0 < q < 1$, since $u_{xx} > 0$, we have $u_x(x, t) \geq u_x(0, t) = -u^q(0, t)$ and so

$$u(x, t) \geq u(0, t) - x u^q(0, t) = u(0, t)[1 - x u^{q-1}(0, t)] \tag{2.5}$$

for all $x \in (0, 1]$. Since $u(0, t) \to \infty$ as $t \to T^-$ and $0 < q < 1$, we conclude that $u(x, t) \to \infty$ as $t \to T^-$ for any $x \in [0, 1]$. This means that we have the blow-up in the whole domain.

Moreover, we can estimate the blow-up rate for the case $q \in (0, 1)$ as follows.

Theorem 2.3. *Suppose that $0 < q < 1$. Then, under the assumption (2.3), there are positive constants c_1 and c_2 such that*

$$c_1(T - t)^{-1/(q+\gamma)} \leq u(0, t) \leq c_2(T - t)^{-1/(q+\gamma)}. \tag{2.6}$$

for all $t \in [0, T)$.

Proof. First, we choose $t_0 \in (0, T)$ such that $u^{q-1}(0, t_0) \leq 1/2$. Then, using $u_x < 0$ and (2.5), we have

$$u(0, t)/2 \leq u(x, t) \leq u(0, t) \ \ \forall\, x \in [0, 1],\ t \in [t_0, T).$$

Hence we obtain

$$u^{-\gamma}(0, t) \leq N(t) \leq 2^\gamma u^{-\gamma}(0, t) \ \ \forall\, t \in [t_0, T),$$

i.e.,

$$N^{-1/\gamma}(t) \leq u(0, t) \leq 2N^{-1/\gamma}(t) \ \ \forall\, t \in [t_0, T). \tag{2.7}$$

It follows from (2.2) that

$$-2^q \gamma N^{-q/\gamma}(t) \leq N'(t) \leq -\gamma N^{-q/\gamma}(t) \ \ \forall\, t \in [t_0, T). \tag{2.8}$$

Then the estimate (2.6) follows by an integration of (2.8) from t to T and using (2.7). $\square$

3. **Self-similar Profile for $q > 1$.** In this section, we shall study the solution of the initial value problem (1.11)-(1.12):

$$g'' - \beta y g^{-\gamma-1} g' - \alpha g^{-\gamma} = 0, \ y > 0, \tag{3.1}$$

$$g'(0) = -g^q(0). \tag{3.2}$$

From the local existence and uniqueness theorem of ordinary differential equations, it follows that there is a unique positive local solution g of (3.1)-(3.2) for each given initial value $g(0) > 0$. For convenience, let $[0, R)$ be the maximum existence interval of g. Note that $g > 0$ in $[0, R)$ and $0 < R \leq \infty$.

Since $g'' = \alpha g^{-\gamma} > 0$ when $g' = 0$, we see that any critical point of g must be a local minimum point. Hence there is at most one critical point of g. Moreover, if g has a critical point $y_0 > 0$, then $g'(y) > 0$ for any $y \in (y_0, R)$ and $g''(y) > 0$ for any $y \in [y_0, R)$.

For a given solution g, define

$$\rho(y) = \exp\left\{-\beta \int_0^y s g^{-\gamma-1}(s) ds\right\}.$$

From (3.1) it follows that

$$(\rho g')'(y) = \alpha \rho(y) g^{-\gamma}(y)$$

and so

$$g'(y) = \frac{g'(0) + \alpha \int_0^y g^{-\gamma}(s)\rho(s) ds}{\rho(y)}. \tag{3.3}$$

Later on in §4, we shall need the following property.

Lemma 3.1. *If there exists $R < \infty$ such that $g(R^-) = 0$, then $g'(y) \to -\infty$ as $y \to R^-$.*

Proof. Note that g must be monotone decreasing to zero, under the assumption of the lemma. Integrating (3.1) from 0 to y, we get

$$g'(y) = g'(0) - (\beta/\gamma) \int_0^y z (g^{-\gamma})'(z) dz + \alpha \int_0^y g^{-\gamma}(z) dz.$$

Using integration by parts, we have

$$g'(y) = g'(0) - (\beta/\gamma) y g^{-\gamma}(y) + (\alpha + \beta/\gamma) \int_0^y g^{-\gamma}(z) dz. \tag{3.4}$$

Taking $K = [1 + \alpha/(\alpha + \beta/\gamma)]R/2$, from (3.4) it follows that

$$\begin{aligned}
g'(y) &= g'(0) - (\beta/\gamma) y g^{-\gamma}(y) + (\alpha + \beta/\gamma) \left[\int_0^K g^{-\gamma}(z) dz + \int_K^y g^{-\gamma}(z) dz\right] \\
&\leq g'(0) - (\beta/\gamma) y g^{-\gamma}(y) \\
&\quad + (\alpha + \beta/\gamma) \int_0^K g^{-\gamma}(z) dz + (\alpha + \beta/\gamma) g^{-\gamma}(y)(y - K) \\
&= g'(0) + (\alpha + \beta/\gamma) \int_0^K g^{-\gamma}(z) dz + [\alpha y - (\alpha + \beta/\gamma) K] g^{-\gamma}(y) \\
&\to -\infty \ \text{ as } \ y \to R^-.
\end{aligned}$$

The lemma follows. $\square$

We shall need the asymptotic behavior as $y \to \infty$ of any monotone decreasing positive global solution of (3.1)-(3.2) as follows.

Lemma 3.2. *For any monotone decreasing positive global solution g of* (3.1)-(3.2), *we have $g(y) \to 0$, $g'(y) \to 0$ and $[yg'(y)/g(y)] \to -\alpha/\beta$ as $y \to \infty$.*

Proof. By assumption, we see that $g(y) \to L$ as $y \to \infty$ for some $L \in [0, \infty)$. We claim that $L = 0$. If $L \in (0, \infty)$, then there exists $\{y_n\} \to \infty$ such that $g'(y_n) \to 0$ as $n \to \infty$. Dividing (3.1) by y and integrating the resulting equation from 1 to y_n, we obtain

$$\int_1^{y_n} \frac{g''(s)}{s} ds + \frac{\beta}{\gamma} \int_1^{y_n} (g^{-\gamma})'(s) ds = \alpha \int_1^{y_n} \frac{g^{-\gamma}(s)}{s} ds. \tag{3.5}$$

We compute that

$$\begin{aligned} &\int_1^{y_n} \frac{g''(s)}{s} ds = \frac{g'(y_n)}{y_n} - g'(1) + \int_1^{y_n} \frac{g'(s)}{s^2} ds, \\ &0 > \int_1^{y_n} \frac{g'(s)}{s^2} ds \geq \int_1^{y_n} g'(s) ds = g(y_n) - g(1), \\ &\int_1^{y_n} (g^{-\gamma})'(s) ds = g^{-\gamma}(y_n) - g^{-\gamma}(1). \end{aligned}$$

Hence the left-hand side of (3.5) is uniformly bounded for all n. But, for K large enough, we have

$$\int_1^{y_n} \frac{g^{-\gamma}(s)}{s} ds \geq \int_K^{y_n} \frac{(2L)^{-\gamma}}{s} ds \to \infty \text{ as } n \to \infty,$$

a contradiction. Hence $L = 0$.

Next, we claim that $g'(y) \to 0$ as $y \to \infty$. For this, we set

$$I := \alpha \int_0^\infty g^{-\gamma}(s) \rho(s) ds.$$

We claim that $g'(0) + I = 0$. Since $g' < 0$, by (3.3), $g'(0) + I \leq 0$. Since $g(y) \to 0$ as $y \to \infty$, there exists a sequence $\{y_n\}$ such that $y_n \to \infty$ and $g'(y_n) \to 0$ as $n \to \infty$. Since $\rho(y) \to 0$ as $y \to \infty$, by (3.3), we must have $g(0) + I = 0$. Now, by L'Hôpital's Rule, we compute from (3.3) that

$$\begin{aligned} \lim_{y\to\infty} g'(y) &= \lim_{y\to\infty} \frac{g'(0) + \alpha \int_0^y g^{-\gamma}(s)\rho(s) ds}{\rho(y)} \\ &= -\lim_{y\to\infty} \frac{\alpha g(y)}{\beta y} \\ &= 0. \end{aligned}$$

Finally, by applying L'Hôpital's Rule to (3.3) again, we obtain that

$$\begin{aligned} \lim_{y\to\infty} \frac{yg'(y)}{g(y)} &= \lim_{y\to\infty} \frac{g'(0) + \alpha \int_0^y g^{-\gamma}(s)\rho(s) ds}{y^{-1}\rho(y) g(y)} \\ &= \lim_{y\to\infty} \frac{\alpha}{-y^{-2} g^{\gamma+1}(y) + y^{-1} g^{\gamma}(y) g'(y) - \beta} \\ &= -\frac{\alpha}{\beta}. \end{aligned}$$

This completes the proof. $\square$

Following [1, 17, 8], we introduce the following variables:

$$U := \frac{yg'(y)}{g(y)},\ V := y^2 g^{-\gamma-1}(y),\ z := \ln y \tag{3.6}$$

for any solution g of (3.1). Then it is easily to check that (U, V) satisfies the first order autonomous system (Q):

$$\frac{dU}{dz} = U(1-U) + V(\alpha + \beta U), \tag{3.7}$$

$$\frac{dV}{dz} = V[2 - (1+\gamma)U]. \tag{3.8}$$

Note that there are two finite critical points $A := (0, 0)$ and $B := (1, 0)$ for the system (Q). Since the linearization of (Q) around A gives the matrix

$$\begin{bmatrix} 1 & \alpha \\ 0 & 2 \end{bmatrix},$$

which has eigenvalues $\{1, 2\}$ and corresponding eigenvectors $\{(1, 0), (\alpha, 1)\}$, we see that A is an unstable improper node. In particular, it follows from an easy phase plane analysis that every orbit near A in the second quadrant of (U, V)-plane leaves A horizontally (see, e.g., [6]). Notice that orbits corresponding to monotone decreasing positive solutions of (3.1)-(3.2) lie in the second quadrant.

From Lemma 3.2 we see that a monotone decreasing positive global solution g of (3.1)-(3.2) corresponding to an orbit connecting from A to the point $D := (-\alpha/\beta, \infty)$ in (U, V)-plane. To learn the behavior near D, we choose the following new dependent variable (U, W), $W := 1/V$, and independent variable $\tau := \int_0^z V(s)ds$. Then the system (Q) becomes the system (R):

$$\frac{dU}{d\tau} = WU(1-U) + (\alpha + \beta U), \tag{3.9}$$

$$\frac{dW}{d\tau} = -W^2[2 - (1+\gamma)U]. \tag{3.10}$$

Note that the critical point D of (Q) becomes the critical point $E := (-\alpha/\beta, 0)$ of (R) in the (U, W)-plane. It is easy to see that the linearization of (R) around E gives the matrix

$$\begin{bmatrix} \beta & -(1+\alpha/\beta)(\alpha/\beta) \\ 0 & 0 \end{bmatrix},$$

which has eigenvalues $\lambda_1 = \beta > 0, \lambda_2 = 0$, and corresponding eigenvectors $\mathbf{v}_1 = (1, 0), \mathbf{v}_2 = ((1+\alpha/\beta)(\alpha/\beta), \beta)$. Hence the horizontal line is tangent to the unstable manifold of E. Since the center manifold is tangent to the eigenspace spanned by $\mathbf{v}_2$ and $dW/d\tau < 0$ for $(U, W) \in S$, where

$$S := \{(U, W) \mid W > 0,\ -\alpha/\beta < U < 0\},$$

by a standard technique (see, e.g., [5]), there exists a unique orbit of the system (R) tending to E as $\tau \to \infty$. This shows that there exists a unique orbit, call it as Γ^*, of the system (Q) tending to the critical point D as $z \to \infty$. Note that Γ^* lies in the strip S for all large z. Since $dU/dz < 0$ on $\{V > 0, U = -\alpha/\beta\}$, $dU/dz > 0$ on $\{V > 0, U = 0\}$, and $dV/dz > 0$ in the second quadrant, the orbit Γ^* must tend to A as $z \to -\infty$.

We thus have proved the following existence theorem.

Theorem 3.3. *There exists a monotone decreasing positive global solution of* (3.1)-(3.2).

We continue to prove the uniqueness of such solution. Note first that any orbit tending to A has the behavior $V = bU^2 + O(U^3)$ as $U \to 0^-$ for some positive constant b (which depending on each orbit). Let b^* be the constant corresponding to the orbit Γ^*. Therefore, by the phase plane analysis, for each $b > b^*$ the corresponding orbit shall reach the positive V-axis in finite time and continue to stay in the first quadrant. These orbits are those solutions of (3.1)-(3.2) with exactly one critical point.

On the other hand, for each $b \in (0, b^*)$ the corresponding orbit shall reach the half-line $L := \{V > 0, U = -\alpha/\beta\}$ in finite time. We claim that these orbits are those solutions of (3.1)-(3.2) which tend to zero in finite time. Let $V = V(U)$ be an orbit from A such that $(U, V)(z_0) = (-\alpha/\beta, c)$ for some $c > 0$. Note that $V > c$ and $U < -\alpha/\beta$ for $z > z_0$. Hence

$$\frac{dU}{dz} \le -U^2 \quad \text{for } z > z_0. \tag{3.11}$$

It follows from (3.11) that $U(z) \to -\infty$ as $z \to z_1^-$ for some finite $z_1 > z_0$. Set $y_1 := e^{z_1}$. Suppose for contradiction that $g(y) > 0$ for all $y \in [0, y_1]$. Then $g'(y_1)$ is finite by (3.3). This implies that $U(z_1^-)$ is finite, a contradiction. Hence we have proved that $g(y) \to 0^+$ as $y \to y_1^-$.

Therefore, we are ready to prove the following uniqueness theorem.

Theorem 3.4. *There exists a unique monotone decreasing positive global solution of* (3.1)-(3.2).

Proof. Since we have a unique orbit in (U, V)-plane connecting the critical points A and D, it remains to show the one-to-one correspondence of orbits with the positive solutions of (3.1)-(3.2). This is equivalent to show that different values of $g(0)$ give different orbits in S leaving from A. Given a positive constant b (which corresponding to an orbit in S leaving from A). Since

$$b = \lim_{z \to -\infty} \frac{V(z)}{U^2(z)} = \lim_{y \to 0} \frac{g^{-\gamma+1}(y)}{(g')^2(y)} = g^{1-\gamma-q}(0),$$

by using (3.2), we obtain the one-to-one correspondence between b and $g(0)$. Hence the theorem follows. □

In the following, we shall denote g^* to be the unique monotone decreasing positive global solution of (3.1)-(3.2) and let $\mu^* := g^*(0)$.

4. **Asymptotic Behavior Near Blow-up Time for $q > 1$.**

In this section, we shall study the asymptotic behavior of the solution u of (P) near the blow-up time T. This is equivalent to study the stabilization, as $s \to \infty$, of the solution v of (1.8)-(1.10). More precisely, we shall prove the following main theorem of this section.

Theorem 4.1. *Let v be the solution of (1.8)-(1.10) and g^* be the unique self-similar profile obtained in Theorem 3.4. Then, under the assumption (2.3), as $s \to \infty$, $v(y, s) \to g^*(y)$ uniformly for any compact subset of* $[0, \infty)$.

To prove this theorem, we shall divide our discussions into a few subsections as follows.

4.1. **Some a priori bounds.** In this subsection, we shall derive some a priori bounds for v. First, we derive the following blow-up rate estimate.

Lemma 4.2. *Under the assumption (2.3), there are positive constants a, κ such that*

$$a(T-t)^{-\alpha} \le u(0,t) \le \kappa(T-t)^{-\alpha} \tag{4.1}$$

for all $t \in (0,T)$.

Proof. The proof of this lemma is based on the so-called intersection comparison principle (cf. [19, 8]). We first recall from §3 that there exists a positive constant μ^* such that the solution g of (3.1)-(3.2) with $g(0) \in (\mu^*, \infty)$ satisfying $g' < 0$ in $[0,R)$ and $g(y) \to 0$ as $y \to R^-$ for some finite R depending on $g(0)$. Moreover, if $g(0) \in (0, \mu^*)$, then there exists a unique $y_0 > 0$ such that $g' < 0$ in $[0, y_0)$, $g' > 0$ in (y_0, ∞), and $g(y) \to \infty$ as $y \to \infty$.

For the lower bound, we compare u with the function

$$U_1(x,t) := (T-t)^{-\alpha} g_1(x/(T-t)^\beta),$$

where g_1 is the solution of (3.1)-(3.2) with a very small $g_1(0)$ such that $u_0(0) > U_1(0,0)$ and u_0 has at most one intersection with $U_1(x,0)$ in $[0,1]$. This is possible, since u_0 is positive and monotone decreasing. We claim that $u(0,t) > U_1(0,t)$ for all $t \in (0,T)$. Suppose, for contradiction, that $u(0,t) > U_1(0,t)$ for $t \in (0,t_0)$ and $u(0,t_0) = U_1(0,t_0)$ for some $t_0 \in (0,T)$. Since the number of intersections is non-increasing, we must have $u(x,t_0) < U_1(x,t_0)$ for all $x \in (0,1]$. Then we have $u < U_1$ in the set

$$\{(x,t) \mid x(t) < x < 1,\ 0 < t \le t_0\},$$

where $x(t)$ is the intersection point for each t. But, this leads to a contradiction with the Hopf Lemma. We thus have derived the lower bound that $u(0,t) > g_1(0)(T-t)^{-\alpha}$ for all $t \in (0,T)$.

For the upper bound, we compare u with the function

$$U_2(x,t) := (T-t)^{-\alpha} g_2(x/(T-t)^\beta),$$

where g_2 is the solution of (3.1)-(3.2) with $g_2(0)$ very large so that g_2 is decreasing to zero at some finite R and u_0 has at most one intersection with $U_2(x,0)$. Note that this is possible, since, by Lemma 3.1, $g_2'(y) \to -\infty$ as $y \to R^-$. Note also that U_2 is defined only in the set $\{(x,t) \mid 0 \le x < R(T-t)^\beta,\ 0 \le t < T\}$. Similar argument as above gives that $u(0,t) < g_2(0)(T-t)^{-\alpha}$ for all $t \in (0,T)$. The lemma follows. □

As a consequence of (4.1), we obtain the following estimate

$$0 < a \le v(0,s) \le \kappa < \infty \ \text{ for all } \ s > s_0. \tag{4.2}$$

Since $u_{xx} > 0$, we have $v_{yy} > 0$. Hence, using (1.9) and (4.2), we obtain

$$v_y(y,s) \ge v_y(0,s) = -v^q(0,s) \ge -\kappa^q. \tag{4.3}$$

Also, $u_x < 0$ implies that $v_y < 0$ and so $v \le \kappa$. Using (4.2) and (4.3), it is easy to see that there is a positive constant $\delta \in (0,1)$ such that

$$v(y,s) \ge a/2 \quad \text{for } \ 0 \le y \le \delta,\ s > s_0. \tag{4.4}$$

Moreover, we claim that

$$v(y,s) \ge \frac{a}{2}\Big(\frac{y}{\delta}\Big)^{-\alpha/\beta} \quad \text{for } \ \delta < y \le e^{\beta(s-s_0^+)},\ s > s_0,\ s_0^+ := \max(s_0, 0). \tag{4.5}$$

Indeed, given (y,s) with $y \in (\delta, e^{\beta(s-s_0^+)})$, $s > s_0$, we can find an $l \in (s_0^+, s)$ such that $y = \delta e^{\beta(s-l)}$. Since $v_{yy} > 0$, it follows from (1.8) that $v_s + \beta y v_y + \alpha v > 0$. Then

$$\frac{d}{d\tau} v(z,\tau) = v_s(z,\tau) + \beta z v_y(z,\tau) \geq -\alpha v(z,\tau), \quad z := y e^{\beta(\tau - s)}.$$

Hence (4.5) follows by an integration of the above inequality from $\tau = l$ to $\tau = s$. From (4.4) and (4.5), we can derive that polynomial growth estimates in y for v_s and v_{yy}, by applying the interior parabolic estimates to (1.8). More precisely, we have the following.

Lemma 4.3. *Under the assumption (2.3), there is a positive constant C such that*

$$|v_s(y,s)| \leq C(1 + y^{2+\gamma/(q-1)}) \ \ \forall\, y \in [0, e^{\beta s}/2], \ s \geq s_0. \tag{4.6}$$

Proof. First, we derive the following estimate

$$v(y,s) \leq C(1+y)^{-\alpha/\beta}, \ y \in [0, e^{\beta s}/2], \ s > s_0, \tag{4.7}$$

for some positive constant C. Note that $\alpha/\beta = 1/(q-1)$. We consider the function

$$J(x,t) := u_x(x,t) + c u^q(x,t), \ c > 0.$$

Then we have

$$J_t - u^{1+\gamma} J_{xx} - (1+\gamma) u^\gamma u_x J_x = -q(q+\gamma) c u^{q+\gamma-1} u_x^2.$$

Also, $J(0,t) = (-1+c) u^q(0,t) < 0$ if $c < 1$. Since $x = 0$ is the only blow-up point and $u_x < 0$ for $x < 1$ and $0 < t < T$, we have $J(1/2,t) < 0$ for $t \in [T/2, T)$ and $J(x, T/2) < 0$ for $x \in [0, 1/2]$, if $c \ll 1$. It follows from the maximum principle that $J < 0$ in $[0,1/2] \times [T/2, T)$. Therefore, we obtain that

$$v_y(y,s) \leq -c v^q(y,s), \quad y \in [0, e^{\beta s}/2], \ s \gg 1. \tag{4.8}$$

By integrating (4.8), the estimate (4.7) follows.

To estimate v_s for a given $(\bar{y}, \bar{s})$ with $\bar{y} \gg 1$, as in [15], we make the following change of variables:

$$\begin{aligned} &V(y,s) := K v(\mu y + \bar{y}, \mu^2 K^{1+\gamma} s + \bar{s}), \ |y| \leq 1, \ -1 < s \leq 0, \\ &K := k^{\alpha/\beta} = k^{1/(q-1)}, \quad \mu := k^{-1-(1+\gamma)/(q-1)}, \end{aligned}$$

where $k \geq 1$ is chosen so that $2k \leq \bar{y} \leq 4k$. Then V satisfies the equation

$$V_s = V^{1+\gamma} V_{yy} - \mu K^{1+\gamma} (\mu y + \bar{y}) \beta V_y - \mu^2 K^{1+\gamma} \alpha V.$$

Note that, by the choices of K and μ, we have

$$\begin{aligned} &0 < \mu K^{1+\gamma} (\mu y + \bar{y}) \leq 4 \mu K^{1+\gamma} k \leq 4 \ \text{ for } |y| \leq 1, \\ &0 < \mu^2 K^{1+\gamma} \leq 1. \end{aligned}$$

Also, by using (4.7) and (4.5), we have

$$0 < c_0 \leq V \leq C_0 < \infty, \quad |y| \leq 1, \ -1 < s \leq 0,$$

for some constants c_0 and C_0 which are independent of $(\bar{y}, \bar{s})$. By applying the interior Schauder estimate, we see that $V_s(0,0)$ is bounded by a constant which is independent of $(\bar{y}, \bar{s})$. This gives the estimate (4.6) and the lemma is proved. □

From (1.8) and combining all the above estimates, the polynomial growth estimate in y for v_{yy} can also be derived.

4.2. **Backward problem.** To derive the convergence result, we need to construct a Lyapunov function. In constructing a suitable Lyapunov function, we first study the following backward initial value problem for a given (y, v, ξ) with $y > 0, v > 0, \xi \in \mathbb{R}$:

$$g'' - \beta z g^{-\gamma-1} g' - \alpha g^{-\gamma} = 0,\ z < y, \tag{4.9}$$

$$g(y) = v, \quad g'(y) = \xi. \tag{4.10}$$

The local existence and uniqueness of the solution of (4.9)-(4.10) near y is trivial. We call this local backward solution as $g(z; y, v, \xi)$ or simply $g(z)$. As before, we define

$$\rho(z) := \exp\left\{\beta \int_z^y s g^{-\gamma-1}(s; y, v, \xi) ds\right\}.$$

Then $\rho > 1$, $\rho' < 0$, and

$$g'(z) = \frac{1}{\rho(z)}\left\{\xi - \alpha \int_z^y \rho(s) g^{-\gamma}(s) ds\right\}. \tag{4.11}$$

We first prove that this backward solution always stays bounded in $[0, y]$. Otherwise, if $g(z) \to \infty$ as $z \to z_0^+$ for some $z_0 \in [0, y]$, then $g'(z) \to -\infty$ as $z \to z_0^+$. On the other hand, since $g \geq \delta$ in $(z_0, y]$ for some constant $\delta > 0$, ρ is uniformly bounded in $[z_0, y]$. It then follows from (4.11) that $g'(z_0^+)$ is finite, a contradiction. Hence g remains bounded.

In particular, if $\xi \leq 0$, then, by (4.11), $g' < 0$ in $[0, y)$ and so $g(z; y, v, \xi) \geq v$ for all $z \in [0, y]$. We conclude that any local solution $g(z; y, v, \xi)$ can be continued backward beyond $z = 0$ as a positive solution of (4.9)-(4.10) defined in $[0, y]$ for any given (y, v, ξ) with $y > 0, v > 0, \xi \leq 0$.

We claim that

$$g(z; y, v, \xi) \leq v + \alpha y^2 v^{-\gamma} - \xi y \quad \text{for} \quad z \in [0, y], \tag{4.12}$$

if $\xi \leq 0$. Indeed, from (4.11) it follows that

$$\begin{aligned} g'(z) &= \xi/\rho(z) - \alpha \int_z^y [\rho(s)/\rho(z)] g^{-\gamma}(s) ds \\ &\geq \xi - \alpha \int_z^y g^{-\gamma}(s) ds \\ &\geq \xi - \alpha y v^{-\gamma} \end{aligned}$$

for $z \in [0, y]$, by using $\xi \leq 0$, $\rho > 1$, and $\rho' < 0$. Then for $\xi \leq 0$ we have

$$g(z) = v - \int_z^y g'(s) ds \leq v - y(\xi - \alpha y v^{-\gamma}) \quad \text{for} \quad z \in [0, y].$$

The estimate (4.12) follows.

4.3. **Lyapunov function.** In this subsection, we shall construct a Lyapunov function by using a method of Zelenyak [21].

First, we define

$$E[v](s) := \int_0^s \Phi(y, v(y, s), v_y(y, s)) dy - \frac{v^{q+1}(0, s)}{q + 1}, \tag{4.13}$$

where $\Phi = \Phi(y, v, \xi)$ is to be determined later. Then, using (1.8) and an integration by parts, we compute that

$$\frac{d}{ds} E[v](s) = J_0(s) + J_1(s) + J_2(s), \tag{4.14}$$

where

$$
\begin{aligned}
J_0(s) &= -\int_0^s \Phi_{\xi\xi}(y,v(y,s),v_y(y,s))v^{-\gamma-1}(y,s)v_s^2(y,s)dy,\\
J_1(s) &= \Phi_\xi(s,v(s,s),v_y(s,s))v_s(s,s)-\Phi_\xi(0,v(0,s),v_y(0,s))v_s(0,s)\\
&\quad+\Phi(s,v(s,s),v_y(s,s))-v^q(0,s)v_s(0,s),\\
J_2(s) &= \int_0^s \Big\{\Phi_v-\Phi_{\xi y}-\Phi_{\xi v}v_y-\Phi_{\xi\xi}\Big[\beta y v^{-\gamma-1}v_y+\alpha v^{-\gamma}\Big]\Big\}v_s(y,s)dy\\
&:= \int_0^s K(y,v(y,s),v_y(y,s))v_s(y,s)dy.
\end{aligned}
$$

Next, we introduce

$$
\begin{aligned}
\Phi(y,v,\xi) &:= \int_0^\xi (\xi-\sigma)P(y,v,\sigma)d\sigma+\int_\kappa^v \alpha\mu^{-\gamma}P(y,\mu,0)d\mu,\\
P(y,v,\sigma) &:= \exp\Big\{-\beta\int_0^y zg^{-\gamma-1}(z;y,v,\sigma)dz\Big\}
\end{aligned}
$$

with the constant κ defined in (4.2) and $g(z;y,v,\sigma)$ defined in §4.2. Then

$$
\Phi_\xi(y,v,\xi)=\int_0^\xi P(y,v,\sigma)d\sigma,\quad \Phi_{\xi\xi}(y,v,\xi)=P(y,v,\xi).
$$

Moreover, we compute that

$$
\begin{aligned}
K(y,v,\xi) &= \int_0^\xi \Big\{-\sigma P_v(y,v,\sigma)-P_y(y,v,\sigma)\\
&\qquad +\frac{\partial}{\partial\sigma}[P(y,v,\sigma)(-\beta y v^{-\gamma-1}\sigma-\alpha v^{-\gamma})]\Big\}d\sigma\\
&= \int_0^\xi \Big\{-\beta P(y,v,\sigma)\Big[\int_0^y(-\gamma-1)zg^{-\gamma-2}(z;y,v,\sigma)\cdot\\
&\qquad \Big(-\sigma g_v(z;y,v,\sigma)-g_y(z;y,v,\sigma)\\
&\qquad +(-\beta y v^{-\gamma-1}\sigma-\alpha v^{-\gamma})g_\sigma(z;y,v,\sigma)\Big)dz\Big]\Big\}d\sigma.
\end{aligned}
$$

Now, using (4.9)-(4.10), we can derive (cf., e.g., [15]) that

$$
g_y(z;y,v,\sigma)=-\sigma g_v(z;y,v,\sigma)+(-\beta y v^{-\gamma-1}\sigma-\alpha v^{-\gamma})g_\sigma(z;y,v,\sigma). \tag{4.15}
$$

It follows from (4.15) that $K(y,v,\xi)\equiv 0$ and hence $J_2=0$.

Using (4.12), we find that

$$
P(y,v,\sigma)\le \exp\Big[-(\beta/2)y^2(v+\alpha y^2 v^{-\gamma}-\sigma y)^{-\gamma-1}\Big] \tag{4.16}
$$

for $y \in [0,\infty), v > 0, \sigma \le 0$. Since $P(0,v,\sigma) \equiv 1$, we have $\Phi_\xi(0,v,\xi) = \xi$. Also, it follows from (4.16) that for $\xi \le 0$ and $v \in (0,\kappa]$

$$\begin{aligned}
|\Phi_\xi(y,v,\xi)| &= \Big|\int_0^\xi P(y,v,\sigma)d\sigma\Big| \le |\xi| \exp\Big[-(\beta/2)y^2(v+\alpha y^2 v^{-\gamma} - \xi y)^{-\gamma-1}\Big],\\
|\Phi(y,v,\xi)| &\le \frac{\xi^2}{2}\exp\Big[-(\beta/2)y^2(v+\alpha y^2 v^{-\gamma}-\xi y)^{-\gamma-1}\Big]\\
&\qquad +\alpha v^{-\gamma}\kappa \exp\Big[-(\beta/2)y^2(\kappa+\alpha y^2 v^{-\gamma})^{-\gamma-1}\Big].
\end{aligned}$$

Note that from (1.9) it follows that

$$J_1(s) = \Phi_\xi(s, v(s,s), v_y(s,s))v_s(s,s) + \Phi(s, v(s,s), v_y(s,s)).$$

Since $-\kappa^q < v_y < 0$, $0 < v < \kappa$, and v_s is bounded by polynomial in y, we have

$$|J_1(s)| \le C\exp(-\lambda s^2)$$

for some small $\lambda > 0$. Combining all the above estimates, we obtain

$$\int_{s_0}^\infty \int_0^s P(y, v(y,s), v_y(y,s))v^{-\gamma-1}v_s^2 dyds < \infty.$$

Taking any sequence $\{s_n\}$ with $s_n \to \infty$ as $n \to \infty$, by the standard arguments (e.g., [12]), we conclude that a subsequence of the sequence $\{v_n(y,s) := v(y, s+s_n)\}$ converges to the unique monotone decreasing positive global solution $g^*(y)$ of (3.1)-(3.2) as $n \to \infty$. Since this limit is independent of the choice of $\{s_n\}$, we conclude that $v(y,s) \to g^*(y)$ as $s \to \infty$ uniformly for any compact subset of $[0,\infty)$. This completes the proof of Theorem 4.1.

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